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CONJUGACY CLASS AND DISCRETENESS IN $SL(2, \mathbb{C})$

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Abstract

In this note we establish a new discreteness criterion for a non-elementary group G in $SL(2, \mathbb{C})$. Namely, G is discrete if all the two-generator subgroups are discrete, where one generator is a non-trivial element f in G, and the other is in the conjugacy class of f.

1. Introduction

The discreteness of Möbius groups is a fundamental problem, which has been discussed by many authors. By using the well-known Jørgensen's inequality, Jørgensen [6] proved that a non-elementary subgroup G of Möbius transformations acting on $\bar{\mathbb{R}}^2$ is discrete if and only if for each f and g in G, the group $\langle f, g\rangle$ is discrete. This important result has become standard in literature. It shows that to test the discreteness of a non-elementary Möbius group, it is enough to test the discreteness of all its subgroups of rank two. Then a natural problem arises: whether the discreteness of the whole group can be determined by the discreteness of a part of all rank two subgroups? There are many further discussions in this direction (e.g. [2, 4, 5, 9, 10, 11, 12]). Among them, we cite here the following two results. Gilman [4] and Isochenko [5] showed that the discreteness of all two-generator subgroups, where each generator is loxodromic, is enough to secure the discreteness of the group. This is also a direct consequence of Rosenberger's result [7] about minimal generating system of a non-elementary Möbius group. From another perspective, Chen Min [2] showed that given a non-elementary Möbius group G and a non-trivial Möbius transformation f, if each group generated by f and an element in G is discrete, then G is discrete. A novel feature of this discreteness criterion is that the test map f need not be in G, which suggests that the discreteness is not a totally interior affair of the involved group.

The purpose of this note is to discuss the aforementioned problem from a different view. Our aim is to show that the discreteness of all two-generator subgroups, where both generators are in the conjugacy class of a fixed element, is enough to determine the discreteness of the whole group. The main result is the following:

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Theorem 1.1. *Let G be a non-elementary subgroup of $SL(2,\mathbb{C})$ and f a fixed non-trivial element in G. If for each element $g \in G$ the group $\langle f, gfg^{-1}\rangle$ is discrete, then G is discrete.*

Note that in the theory of Kleinian groups, there are some other places where the role of conjugacy classes is crucial. A typical example is the arguments of the proof of Jørgensen's inequality. It is well-known that Jørgensen's inequality says if $\langle f, g\rangle$ in $SL(2,\mathbb{C})$ generate a discrete and non-elementary group, then

$$|\mathrm{tr}^2(f)-4| + |\mathrm{tr}(fgf^{-1}g^{-1})-2| \geq 1.$$

Consider the dynamic of $g_{n+1} = g_n f g_n^{-1}$ in the conjugacy class of one generator f, where $g_0 = g$. If the above inequality fails, a calculation shows that one can find some N such that $g_N = f$. However, this will implies that $\langle f, g\rangle$ is elementary (except for the simple case that f is order 2), which is the desired contradiction.

We shall prove the main theorem by dividing into three cases (see Theorems 3.1, 3.2 and 3.3 in Section 3), according to that the fixed map f is elliptic, loxodromic or parabolic. Note that our proof also applies to the situation where the the fixed map f is not in the group G. This shares the same feature as in [2].

In practice, the applications of our theorem are possible if one can find a "good" test map f, such that its conjugacy class $\{gfg^{-1}\colon g \in G\}$ have some additional features. For instance, the size of the conjugacy class of f, or equivalently, the index of its centralizer, is finite. The following is a simple example. Let f be loxodromic or elliptic, and g elliptic of order two which exchanges the fixed points of f. Denote by G the group generated by f and g. It can be easily obtained that the conjugacy class of f consists of two elements, that is, f and f^{-1}. Then our theorem gives the discreteness of G.

2. Preliminaries

We begin with some elementary notations about Möbius groups. The reader is referred to [1] for more details.

Denote by Möb(2) the group of all (orientation-preserving) Möbius transformations of the extended complex plane $\overline{\mathbb{C}} = \mathbb{R}^2 \cup \infty$. Recall, any matrix $A \in SL(2,\mathbb{C})$ as the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ induces a Möbius transformation $f_A(z) = (az+b)/(cz+d)$. Then Möb(2) is isomorphic to $SL(2,\mathbb{C})/\{\pm I\}$, where I is the identity matrix. Let $\mathrm{tr}^2(f_A) = \mathrm{tr}^2(A)$, where tr denotes the trace of A. It is easy to see $\mathrm{tr}^2(f_n) \to \mathrm{tr}^2(f)$ when f_n converges to f in $SL(2,\mathbb{C})$. Non-trivial elements of $SL(2,\mathbb{C})$, or equivalently of Möb(2), can be classified into three types considering the Jordan normal forms.

(i) Elliptic elements are diagonalizable and have two distinct eigenvalues with absolute value 1, that is, those are conjugated to $\begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}$ with $|r| = 1$. In this case, $\mathrm{tr}^2(f)$ is real and $0 \leq \mathrm{tr}^2(f) < 4$.

(ii) Parabolic elements are not diagonalizable. They are conjugated to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then $\mathrm{tr}^2(f) = 4$ if f is parabolic.
(iii) Loxodromic elements are diagonalizable and the eigenvalues do not have absolute value 1, that is, those are conjugated to $\begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}$ with $|r| > 1$. If $\mathrm{tr}^2(f)$ is real and $\mathrm{tr}^2(f) > 4$, then f is called *hyperbolic* and if $\mathrm{tr}^2(f)$ is not in the interval $[0, +\infty)$, then f is termed *strictly loxodromic*. We use the term loxodromic to include both hyperbolic and strictly loxodromic elements. Since $\mathrm{tr}^2(f_n) \to \mathrm{tr}^2(f)$ when f_n converges to f in $SL(2, \mathbb{C})$, the set consisting of all loxodromic elements is open in $SL(2, \mathbb{C})$;

Recall that Möbius transformations are a finite composition of inversions in spheres and planes of the extended complex plane. Through Poincaré's extension, the action of $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can be extended to an action on the hyperbolic 3-space $\mathbb{H}^3 = \{\omega = z + tj : z \in \mathbb{C},\ t > 0\}$ by the formula $f(\omega) = (a\omega + b)/(c\omega + d)$. A subgroup G of Möb(2) is called elementary if there exists a finite G-orbit in the closure of $\mathbb{H}^3$ in Euclidean 3-space. Otherwise, the group is referred as non-elementary.

For each f and g in Möb(2), let $[f, g]$ denote the commutator $fgf^{-1}g^{-1}$. Gehring and Martin introduced the following three parameters for the two generator subgroup $\langle f, g\rangle$:

$$\beta(f) = \mathrm{tr}^2(f) - 4, \quad \beta(g) = \mathrm{tr}^2(g) - 4,$$
$$\gamma(f, g) = \mathrm{tr}(fgf^{-1}g^{-1}) - 2.$$

In terms of those parameters, the well-known Jørgensen's inequality gives a sharp lower bound for $|\gamma(f, g)|$ when $|\beta(f)| < 1$ or $|\beta(g)| < 1$. In [3], Gehring and Martin obtained the following result.

Lemma 2.1. *Let $\langle f, g\rangle$ be a discrete and non-elementary group of $SL(2, \mathbb{C})$ with $\beta(f) = \beta(g)$. Then $|\gamma(f, g)| > 0.193$.*

G is referred to be an elementary group of elliptic type if G contains only elliptic elements and the identity. It is well known that the elements of an elementary group of elliptic type have a common fixed point in $\mathbb{H}^3$ (cf. Theorem 4.3.7 of [1]). In [12] the authors give a characterization of such a groups in terms of the above parameter $\gamma(f, g)$. For the completeness, we include its proof as the following.

Lemma 2.2. *Let $\langle f, g\rangle$ be an elementary group of elliptic type in $SL(2, \mathbb{C})$. Then $\gamma(f, g) < 0$.*

Proof. We may assume, up to conjugation, that $f = \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}$ and g fixes the point (0, 0, 1) in the upper half-space model of $\mathbb{H}^3$. Hence g has the matrix form as $\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ with $|a|^2 + |b|^2 = 1$ (cf. Theorem 2.5.1 of [1]).

Recall that $r = e^{i\theta_0}$ for some $\theta_0 \neq 0 \pmod{2\pi}$, it follows that

$$\beta(f) = \left(r + \frac{1}{r}\right)^2 - 4 = e^{2i\theta_0} + e^{-2i\theta_0} - 2 = 2[\cos(2\theta_0) - 1] < 0.$$

Therefore, we have $\gamma(f, g) = \mathrm{tr}(fgf^{-1}g^{-1}) - 2 = |b|^2\beta(f) < 0$. □

We also need the following lemma, which is a direct consequence of the well-known proposition in [8, Section 1].

Lemma 2.3. *Let G be a non-elementary and non-discrete subgroup of $SL(2, \mathbb{C})$. After replacing G by its subgroups of index 2 if necessary, G is*
(a) *dense in $SL(2, \mathbb{C})$, or*
(b) *conjugate to a dense group of $SL(2, \mathbb{R})$.*

3. Main results

Theorem 3.1. *Let G be a non-elementary subgroup of $SL(2, \mathbb{C})$ containing an elliptic element f. If for each element $g \in G$ the group $\langle f, gfg^{-1}\rangle$ is discrete, then G is discrete.*

Proof. Suppose to the contrary that G is not discrete. Then we may assume that G is either dense in $SL(2, \mathbb{R})$, or dense in $SL(2, \mathbb{C})$ by Lemma 2.3.

Normalize the group G by possible conjugations such that f is represented by the matrix $\begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}$, and such that $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ with $b = 0 \neq c$. This is possible since G is non-elementary. By setting $h = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, we get $hgh^{-1} = \begin{pmatrix} a + ct & -ct^2 + (d - a)t + b \\ c & d - ct \end{pmatrix}$. Since G is dense, there exists a sequence $\{h_n\}$ in G which converges to h.

We denote $h_n g h_n^{-1}$ by $\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$. Let $l_n = h_n g h_n^{-1} f h_n g^{-1} h_n^{-1}$. By direct calculation, we explicitly obtain

$$\begin{aligned} l_n &= \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}\begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}\begin{pmatrix} d_n & -b_n \\ -c_n & a_n \end{pmatrix} \\ &= \begin{pmatrix} r a_n d_n - \dfrac{1}{r} b_n c_n & -a_n b_n \left(r - \dfrac{1}{r}\right) \\ c_n d_n \left(r - \dfrac{1}{r}\right) & \dfrac{1}{r} a_n d_n - r b_n c_n \end{pmatrix}. \end{aligned}$$

From the assumption, it follows the groups $\langle f, l_n\rangle$ are discrete for all n.

Now we divide our proof into two cases.

CASE 1. G is dense in $SL(2, \mathbb{C})$.

From the above, we obtain $\gamma(f, l_n) = a_n b_n c_n d_n |r - 1/r|^4$, which converges to $|r - 1/r|^4 c(ct + a)(ct - d)[ct^2 + (a - d)t - b]$ as $n \to \infty$. Appealing to the fundamental theorem of algebra, we can take the value of $t \in \mathbb{C}$ such that $|r - 1/r|^4 c(ct + a)(ct - d)[ct^2 + (a - d)t - b]$ is sufficiently small and positive, say,

$$(*) \qquad \left| r - \frac{1}{r} \right|^4 c(ct + a)(ct - d)[ct^2 + (a - d)t - b] = 0.1.$$

By Lemma 2.1, we see that the discrete groups $\langle f, l_n \rangle$ must be elementary for large n. Furthermore, Lemma 2.2 shows that it is of either parabolic or loxodromic type. Notice that the third entry $c_n d_n (r - 1/r)$ of l_n is close to $c(d - ct)(r - 1/r)$, which is not zero from our assumption $c \neq 0$ and the equation $(*)$. This implies that the elliptic elements f and l_n can't be in the same cyclic group. Then the only possibility is that $\langle f, l_n \rangle$ is of loxodromic type, where one of f and l_n exchanges the fixed points of the other (cf. pp. 87–89 of [1]). After normalization, a direct calculation shows that $f l_n f^{-1} l_n^{-1}$ must be elliptic. This is the desired contradiction to that $\gamma(f, l_n) = \operatorname{tr}(f l_n f^{-1} l_n^{-1}) - 2$ is close to 0.1.

CASE 2. G is dense in $SL(2, \mathbb{R})$.

From the assumption that $b = 0$, we obtain that $\gamma(f, l_n) = (r - 1/r)^4 ct(ct + a)(ct - d)[ct + (a - d)]$. It is easy to see that $\gamma(f, l_n)$ is a continuous real function with respect to t. Note that $\lim_{t \to \infty} \gamma(f, l_n) = +\infty$, and $\gamma(f, l_n) = 0$ when $t = 0$. Then we can also choose $t \in \mathbb{R}$ such that $\gamma(f, l_n) = (r - 1/r)^4 ct(ct + a)(ct - d)[ct + (a - d)]$ is sufficiently small and positive. Again we get the desired contradiction. □

Theorem 3.2. *Let G be a non-elementary subgroup of $SL(2, \mathbb{C})$ with a loxodromic element $f \in G$. If for each element $g \in G$ the group $\langle f, g f g^{-1} \rangle$ is discrete, then G is discrete.*

Proof. Suppose to the contrary that G is dense. Then we can find a sequence $\{g_n\}_{n=1}^{\infty}$ of distinct loxodromic elements in G such that $g_n \to I$. In fact, it is obvious to see that there exists a sequence $\{g_n'\}$ of loxodromic elements converging to the identity. Since G is dense, there is $g_n \in G$ arbitrarily close to g_n' for each n. Then g_n is also loxodromic.

By Jørgensen's inequality we may assume that $\langle f, g_n f g_n^{-1} \rangle = \langle g_n f g_n^{-1} f^{-1}, f \rangle$ are discrete and elementary for all n. Then f and g_n share the same fixed points. Since G is non-elementary, there is $g \in G$ which has distinct fixed points from that of f. Note that $g g_n g^{-1} \to I$. Similarly, f and $g g_n g^{-1}$ must share the same fixed points for large n. This is the desired contradiction. □

Theorem 3.3. *Let G be a non-elementary subgroup of $SL(2, \mathbb{C})$ containing a parabolic element f. If for each element $g \in G$ the group $\langle f, g f g^{-1} \rangle$ is discrete, then G is discrete.*

Proof. Normalize G such that $f(z) = z + 1$ is in G.

First we claim that the stabilizer of ∞ in G, denoted by $Stab_\infty$, is discrete. Suppose to the contrary that there is a sequence $\{g_n\}_{n=1}^{\infty}$ in $Stab_\infty$ such that $g_n \to I$. If g_n is not parabolic, then $g_n f g_n^{-1} f^{-1}$ is parabolic by [1, Theorem 4.3.5]. This implies that one can always find a sequence of parabolic elements, denoted by $\{h_n\}$, which fixes ∞ and converges to the identity. Since G is non-elementary, there is a parabolic $h \notin Stab_\infty$. According to Jørgensen's inequality, the subgroup $\langle f, hh_nh^{-1} f hh_n^{-1}h^{-1}\rangle = \langle hh_nh^{-1} f hh_n^{-1}h^{-1} f^{-1}, f\rangle$ is discrete and elementary of parabolic type for large n. This deduce that $hh_nh^{-1} f hh_n^{-1}h^{-1}(\infty) = \infty$. Then $hh_nh^{-1}(\infty) = \infty$ and hence $h(\infty) = \infty$. This is the desired contradiction.

Second we show the horoball $\{(z, t) \in \mathbb{H}^3 : t > 1\}$ is precisely invariant under $Stab_\infty$. For any $g(z) = (az + b)/(cz + d)$ in G with $c \neq 0$, gfg^{-1} is parabolic with $1/|c^2|$ as the radius of its isometric sphere. Applying Jørgensen's inequality to the discrete and non-elementary subgroup $\langle f, gfg^{-1}\rangle$, we obtain $1/|c^2| < 1$, and then $1/|c| < 1$. Note that the left term represents the radius of the isometric sphere of g. This implies that $\{(z, t) \in \mathbb{H}^3 : t > 1\}$ is precisely invariant under $Stab_\infty$ by viewing elements in $SL(2, \mathbb{C})$ as isometries of $\mathbb{H}^3$. Now the discreteness of G follows from combining the above two aspects. □

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