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# THE HOMOTOPY FIXED POINT SETS OF SPHERES ACTIONS ON RATIONAL COMPLEXES

YANLONG HAO, XIUGUI LIU\* and QIANWEN SUN

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## Abstract

In this paper, we describe the homotopy type of the homotopy fixed point sets of  $S^3$ -actions on rational spheres and complex projective spaces, and provide some properties of  $S^1$ -actions on a general rational complex.

## 1. Introduction

An action of a group  $G$  on a space  $M$  gives rise to two natural spaces, the fixed point set  $M^G$  and the homotopy fixed point set  $M^{hG}$ . It is crucially important that there is an injection

$$k: M^G \rightarrow M^{hG}.$$

Indeed, one version of the *generalized Sullivan conjecture* asserts that, when  $G$  is a finite  $p$ -group, and  $M$  is a  $G$ -CW-complex, then the  $p$ -completion of  $k$  is a homotopy equivalence. This conjecture was proved in the case when  $M$  is a finite complex by Miller [7].

For a finite group  $G$ , the rational homotopy theory of  $M^{hG}$  has been studied by Goyo [5].

In [1, 2], the authors studied the homotopy type of  $M^{hG}$  for a compact Lie group  $G$  with particular emphasis when  $G$  is the circle.

From now on, and unless explicitly stated otherwise,  $G$  will denote a compact connected Lie group and by a topological  $G$ -space we mean a nilpotent  $G$ -space with the homotopy type of a CW-complex of finite type and  $M^G \neq \emptyset$ . Then the action of  $G$  on  $M$  induces an action of  $G$  on  $M_{\mathbb{Q}}$ .

We then start by setting a sufficiently general context in which  $M_{\mathbb{Q}}^{hG}$  has the homotopy type of a nilpotent CW-complex. Identifying the homotopy fixed point set

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with the space  $\text{Sec}(\xi)$  of sections of the corresponding Borel fibration

$$\xi: M \rightarrow M_{hG} \rightarrow BG,$$

we have that if  $\pi_{>n}(M)$  are torsion groups for a certain  $n > 1$ , then  $M_{\mathbb{Q}}^{hG}$  is a rational nilpotent complex with the homotopy type of a CW-complex [1].

In this paper, we explicitly describe the rational homotopy type of the homotopy fixed point sets of certain  $S^3$ -actions.

**Theorem 1.1.** *Given an  $S^3$ -action on the rational  $n$ -sphere  $S_{\mathbb{Q}}^n$ .*

(1) *When  $n$  is odd,  $S_{\mathbb{Q}}^{n hS^3}$  has the rational homotopy type of products of odd dimensional spheres, precisely, we have*

$$S_{\mathbb{Q}}^{n hS^3} \simeq_{\mathbb{Q}} S^a \times S^{a+4} \times \cdots \times S^n,$$

where

$$a = \begin{cases} 1, & n = 4k + 1, \\ 3, & n = 4k + 3. \end{cases}$$

(2) *If  $n = 4k$ ,  $S_{\mathbb{Q}}^{n hS^3}$  is either path connected, and of the rational homotopy type of  $S^3 \times K_k$ , where  $K_k$  has the minimal Sullivan model*

$$(\Lambda((x_s)_{1 \leq s \leq k}, (y_r)_{2 \leq r \leq 2k}), d)$$

with  $|x_s| = 4s$ ,  $|y_r| = 4r - 1$ ,  $dx_s = 0$  ( $1 \leq s \leq k$ ),  $dy_r = \sum_{s+t=r} x_s x_t$  ( $2 \leq r \leq 2k$ ), or else, it has 2 components, each of them has the rational homotopy type of

$$S^{4k+3} \times S^{4k+7} \times \cdots \times S^{8k-1}.$$

(3) *If  $n = 4k + 2$ ,  $S_{\mathbb{Q}}^{n hS^3}$  is path connected, and of the rational homotopy type of  $S^3 \times S^7 \times T_k$ , where  $T_k$  has the minimal Sullivan model*

$$(\Lambda((x_s)_{1 \leq s \leq k}, (y_r)_{3 \leq r \leq 2k+1}), d)$$

with  $|x_s| = 4s + 2$ ,  $|y_r| = 4r - 1$ ,  $dx_s = 0$  ( $1 \leq s \leq k$ ),  $dy_r = \sum_{s+t=r-1} x_s x_t$  ( $3 \leq r \leq 2k + 1$ ).

**Theorem 1.2.** *Given an  $S^3$ -action in the rational complex projective space  $\mathbb{C}P_{\mathbb{Q}}^n$ .*

(1) If  $n$  is odd,  $\mathbb{C}P_{\mathbb{Q}}^{n hS^3}$  is path connected, and has the rational homotopy type of one of the following spaces:

$$\begin{aligned} & \mathbb{C}P^1 \times S^7 \times S^{11} \times \cdots \times S^{2n+1}, \\ & S^3 \times \mathbb{C}P^3 \times S^{11} \times \cdots \times S^{2n+1}, \\ & S^3 \times S^7 \times \mathbb{C}P^5 \times \cdots \times S^{2n+1}, \\ & \cdots, \\ & S^3 \times S^7 \times \cdots \times S^{2n-3} \times \mathbb{C}P^n. \end{aligned}$$

(2) If  $n$  is even,  $\mathbb{C}P_{\mathbb{Q}}^{n hS^3}$  is path connected, and has the rational homotopy type of one of the following spaces:

$$\begin{aligned} & * \times S^5 \times S^9 \times \cdots \times S^{2n+1}, \\ & S^1 \times \mathbb{C}P^2 \times S^9 \times \cdots \times S^{2n+1}, \\ & S^1 \times S^5 \times \mathbb{C}P^4 \times \cdots \times S^{2n+1}, \\ & \cdots, \\ & S^1 \times S^5 \times \cdots \times S^{2n-3} \times \mathbb{C}P^n. \end{aligned}$$

In [1, Corollary 2], they give a criterion of an elliptic  $S^1$ -space. We first show that the condition  $M$  is a finite complex is necessary by the following example: there is a nilpotent  $S^1$ -complex  $M$  which is not an elliptic space, such that each component of  $M_{\mathbb{Q}}^{hS^1}$  is elliptic. We also observe that an  $S^1$ -finite nilpotent complex  $M$  is elliptic if and only if one of the component of  $M_{\mathbb{Q}}^{hS^1}$  is elliptic, complementing the mentioned result.

Finally, we show that the injection  $k$  is generally not a rational homotopy equivalence.

**Theorem 1.3.** *For an  $S^1$ -complex  $M$  which is simply connected with*

$$\dim \pi_*(M) \otimes \mathbb{Q} < \infty.$$

*Then*

$$k: M_{\mathbb{Q}}^{S^1} \hookrightarrow M_{\mathbb{Q}}^{hS^1}$$

*is a rational homotopy equivalence if and only if  $M$  is rational homotopy equivalent to a product of  $\mathbb{C}P^{\infty}$ .*

In the next section we prove Theorems 1.1 and 1.2. In Section 3 we prove Theorem 1.3.

## 2. $S^3$ -rational spheres and complex projective spaces

Our results heavily depend on known facts and techniques arising from rational homotopy theory. All of them can be found with all details in [4]. We simply remark a few facts.

We recall that when  $M$  is path connected, the Sullivan model of  $M$  is a quasi-isomorphism

$$m: (\Lambda V_M, d) \rightarrow A_{PL}(M),$$

where  $(\Lambda V_M, d)$  is a Sullivan algebra.

We also recall that a space  $M$  is elliptic if both  $H^*(M; \mathbb{Q})$  and  $\pi_*(M) \otimes \mathbb{Q}$  are finite dimensional vector spaces over  $\mathbb{Q}$ .

For a  $G$ -space  $M$ , we have the corresponding Borel fibration

$$\xi: M \rightarrow M_{hG} \rightarrow BG,$$

where  $M_{hG} = (M \times EG)/G$ . It is a classical fact that the homotopy fixed point set

$$M^{hG} = \text{map}_G(EG, M)$$

is homotopy equivalent to the section space  $\text{Sec}(\xi)$  of this fibration.

Each fixed point gives rise to a trivial section of the product bundle

$$M^G \rightarrow BG \times M^G \rightarrow BG.$$

Composing with the injection  $M_G \times BG \hookrightarrow EG \times M/G = M_{hG}$  gives a section of the Borel fibration. Thus we have a natural injection:

$$k: M^G \hookrightarrow M^{hG}.$$

For any  $G$ -CW complex  $M$ , there is an equivariant rationalization  $m: M \rightarrow M_{\mathbb{Q}}$ , that is,  $M_{\mathbb{Q}}$  is also a  $G$ -CW complex,  $m$  is an equivariant map, and  $(M_{\mathbb{Q}})^G \simeq (M^G)_{\mathbb{Q}}$ . Moreover, we have

**Proposition 2.1** ([1, Proposition 12]). *If  $M$  is a Postnikov piece, that is,  $\pi_{>N}(M) = 0$  for some  $N$ , then*

- (i)  $M^{hG}$  has the homotopy type of a nilpotent CW-complex of finite type.
- (ii)  $(M^{hG})_{\mathbb{Q}} \simeq (M_{\mathbb{Q}})^{hG}$ .

Note that if  $M_{\mathbb{Q}}$  is a Postnikov piece, then  $(M_{\mathbb{Q}})^{hG}$  makes sense and is a rational space.

Now, we determine the homotopy type of the homotopy fixed point sets of certain  $S^3$ -actions.

Proof of Theorem 1.1. (1) CASE 1:  $n$  is odd.

We only prove the case  $n = 4k + 3$ , the case  $n = 4k + 1$  is similar, so we omit it.

As in the proof of [1, Theorem 19], it is not hard to get the model of the corresponding Borel fibration

$$\xi: (A, 0) \hookrightarrow ((\Lambda e) \otimes A, D) \rightarrow (\Lambda e, 0),$$

where  $(A, 0) = (\Lambda x/x^k, 0)$  and  $|x| = 4$ ,  $|e| = n$ . This fibration is trivial, so  $\text{Sec}(\xi) \simeq \text{Map}(\mathbb{H}P^k, S^n)$ .

By [1, Theorem 9], the model of  $S_{\mathbb{Q}}^{n \times S^3}$  is  $(\Lambda(x_1, x_2, \dots, x_{n+1/4}), 0)$ . It is exactly the model of  $S^3 \times S^7 \times \dots \times S^n$ . It follows that  $S_{\mathbb{Q}}^{n \times S^3} \simeq_{\mathbb{Q}} S^a \times S^{a+4} \times \dots \times S^n$ .

(2) CASE 2:  $n = 4k$ .

As  $\pi_{\geq 2n}(S^n) \otimes \mathbb{Q} = 0$ , a model of the Borel fibration is

$$\xi_{2n}: (A, 0) \hookrightarrow (\Lambda(e, e') \otimes A, D) \rightarrow (\Lambda(e, e'), d),$$

where  $A = \Lambda x/x^{2k+1}$ ,  $x, e, e'$  are of degree 4,  $n, 2n - 1$  respectively,  $De = 0$ ,  $De' = e^2 + \lambda x^{n/4}e$ ,  $de' = e^2$ .

(i) If  $\lambda = 0$ , then  $\xi_{2n}$  is trivial and

$$S_{\mathbb{Q}}^{n \times S^3} \simeq \text{Map}(\mathbb{H}P^{2k}, S^n)_{\mathbb{Q}}.$$

A straightforward computation shows that this mapping space has a model of the form

$$(\Lambda y_1, 0) \otimes (\Lambda((x_s)_{1 \leq s \leq k}, (y_r)_{2 \leq r \leq 2k}), d)$$

with  $|x_s| = 4s$ ,  $|y_r| = 4r - 1$ ,  $dx_s = 0$  ( $1 \leq s \leq k$ ),  $dy_r = \sum_{s+t=r} x_s x_t$  ( $r > 1$ ).

(ii) If  $\lambda \neq 0$ , then the fibration  $\xi_n$  has two non homotopic sections  $\sigma, \tau$  which correspond to the only two possible retractions of its model:

$$\varphi_{\sigma}, \varphi_{\tau}: (\Lambda(e, e') \otimes A, D) \rightarrow (A, 0), \quad \varphi_{\sigma}(e) = 0, \quad \varphi_{\tau}(e) = \lambda x^k.$$

By the same way in [1], we have that the model of  $\text{Sec}_{\sigma}(\xi_{2n})$  is of the form

$$(\Lambda((x_s)_{1 \leq s \leq k}, (y_r)_{1 \leq r \leq 2k}), \tilde{d})$$

with  $|x_s| = 4s$ ,  $|y_r| = 4r - 1$ . The linear part of  $\tilde{d}$  is:

$$\tilde{d}(y_r) = \lambda x_r$$

for  $1 \leq r \leq k$ , which shows that the minimal model of  $\text{Sec}_{\sigma}(\xi_{2n})$  is

$$(\Lambda(y_r)_{k+1 \leq r \leq 2k}, 0).$$

Replace  $\lambda$  by  $-\lambda$ , we have that the model of  $\text{Sec}_\tau(\xi_{2n})$  is the same.

(3) **Case 2:**  $n = 4k + 2$ .

As  $\pi_{\geq 2n}(S^n) \otimes \mathbb{Q} = 0$ , a model of the Borel fibration is

$$\xi_{2n}: (A, 0) \hookrightarrow (\Lambda(e, e') \otimes A, D) \rightarrow (\Lambda(e, e'), d),$$

where  $A = \Lambda x/x^{2k+1}$ ,  $x, e, e'$  are of degree 4,  $n, 2n - 1$  respectively,  $De = 0$ ,  $De' = e^2$ ,  $de' = e^2$ . It follows that the fibration  $\xi_{2n}$  is trivial, we have

$$S_{\mathbb{Q}}^{n, hG} \simeq \text{Map}(\mathbb{H}P^{2k}, S^n)_{\mathbb{Q}}.$$

The model of  $S_{\mathbb{Q}}^{n, hG}$  is

$$(\Lambda(y_1, y_2), 0) \otimes (\Lambda((x_s)_{1 \leq s \leq k}, (y_r)_{3 \leq r \leq 2k+1}), d)$$

with  $|x_s| = 4s + 2$ ,  $|y_r| = 4r - 1$ ,  $dx_s = 0$  ( $1 \leq s \leq k$ ),  $dy_r = \sum_{s+t=r-1} x_s x_t$  ( $3 \leq r \leq 2k + 1$ ).

The desired result follows.  $\square$

Proof of Theorem 1.2. First, we assume  $n = 2k + 1$ . As  $\pi_{\geq 4k+4}(\mathbb{C}P_{\mathbb{Q}}^n) = 0$ , it suffice to use the model of  $\xi_{2n+2}$

$$(A, 0) \rightarrow (\Lambda(e, e') \otimes A, D) \rightarrow (\Lambda(e, e'), d),$$

where  $A = (\Lambda x)/x^{k+2}$ ,  $|x| = 4$ ,  $|e| = 2$ ,  $|e'| = 4k + 3$ , and

$$De = 0, \quad De' = e^{n+1} + \sum_{j=1}^k \lambda_j e^j x^{n+1-2j}, \quad \lambda \in \mathbb{Q}, j = 1, \dots, n.$$

The retraction of this model of fibration is just  $\varphi(e) = 0$ . So we have  $\text{Sec}(\xi_{4k+4})$  is connected, and the model of it is

$$(\Lambda(e, (e'_r)_{1 \leq r \leq k+1}), \tilde{d})$$

with  $|e| = 2$ ,  $|e'_r| = 4r - 1$ ,  $\tilde{d}(e'_r) = \lambda_{k+1-r} e^{2r}$  for  $1 \leq r \leq k$  and  $\tilde{d}(e'_{k+1}) = e^{2k+2}$ .

If  $\lambda_1 \neq 0$  this is a model of

$$S^2 \times S^7 \times \dots \times S^{4k+3}.$$

If  $\lambda_1 = \dots = \lambda_{i-1} = 0$  and  $\lambda_i \neq 0$ , this is a model of

$$S^3 \times \dots \times S^{4k-4i-1} \times \mathbb{C}P^{2k+1-2i} \times S^{4k-4i+3} \times \dots \times S^{4k+3}.$$

Finally, if all  $\lambda_i = 0$ , then it is a model of

$$S^3 \times S^7 \times \cdots \times S^{4k-1} \times \mathbb{C}P^{2k+1}.$$

For  $n$  even, the proof is similar, so we omit it.  $\square$

### 3. The Inclusion $k: M^{S^1} \hookrightarrow M^{hS^1}$

We begin with some interesting observations on  $S^1$ -actions.

In [2, Example 12], there is an  $S^1$ -action on  $M = K(\mathbb{Z}, n) \times K(\mathbb{Z}, n+1)$ , such that the model of it's Borel fibration is

$$\eta_n: (\Lambda x, 0) \hookrightarrow (\Lambda x \otimes \Lambda(z, y), D) \rightarrow (\Lambda(z, y), d),$$

where  $|x| = 2$ ,  $|z| = n$ ,  $|y| = n+1$ ,  $D(z) = 0$ , and  $D(y) = xz$ . For  $n = 2k$ , there is only one retraction  $\sigma$ :  $\sigma(z) = \sigma(y) = 0$ . Thus  $\text{Sec}(\eta_{2k})$  is path connected.

By the same method used in [1], a model of  $\text{Sec}(\eta_{2k})$  is

$$(\Lambda((z_i)_{1 \leq i \leq k}, (y_j)_{1 \leq j \leq k+1}), d),$$

where  $|z_i| = 2i$ ,  $|y_j| = 2j-1$  and  $d(y_i) = z_i$ . Since the minimal model of  $\text{Sec}(\eta_{2k})$  is  $(\Lambda y_{k+1}, 0)$ ,  $\text{Sec}(\eta_{2k}) \simeq_{\mathbb{Q}} S^{2k+1}$  is an elliptic space. However,  $M$  is not an elliptic space.

Next we complement [1, Corollary 2] with the following

**Proposition 3.1.** *For an  $S^1$ -space  $M$  which is a nilpotent finite complex, the following conditions are equivalent:*

- 1)  $M$  is elliptic.
- 2) Each component of  $M_{\mathbb{Q}}^{hS^1}$  is elliptic.
- 3) One of the components of  $M_{\mathbb{Q}}^{hS^1}$  is elliptic.

Proof. 1)  $\Rightarrow$  2): [1, Theorem 15].

2)  $\Rightarrow$  3): Trivial.

3)  $\Rightarrow$  1): By [2, Theorem 13],  $2\dim \pi_*(\text{Sec}_\sigma(\xi) \otimes \mathbb{Q}) \geq \dim \pi_*(M) \otimes \mathbb{Q}$ . By  $\text{Sec}_\sigma(\xi)$  is elliptic,  $\dim \pi_*(\text{Sec}_\sigma(\xi)) \otimes \mathbb{Q}$  is finite, so  $\dim \pi_*(M) \otimes \mathbb{Q}$  is finite. Then  $M$  is elliptic.  $\square$

REMARK 3.2. The theorem holds also for  $G = S^3$ . The proof is similar.

The rest of the section is devoted to showing Theorem 1.3.

Let  $M$  be an  $S^1$ -space and  $M^G \neq \emptyset$ . Then the inclusion  $M^{S^1} \hookrightarrow M$  induces a map of Borel fibrations:

$$(3.1) \quad \begin{array}{ccc} M^{S^1} & \longrightarrow & M \\ \downarrow & & \downarrow \\ \mathbb{C}P^\infty \times M^{S^1} & \xrightarrow{\gamma} & M_{hS^1} \\ \eta \searrow & & \swarrow \xi \\ & \mathbb{C}P^\infty. & \end{array}$$

If there exists some  $N$  such that  $\pi_{\geq N}(M_{\mathbb{Q}}) = 0$  and  $\pi_{\geq N}(M_{\mathbb{Q}}^{S^1}) = 0$ . Then  $k$  is identified with the corresponding

$$M^{S^1} \hookrightarrow \text{Map}((\mathbb{C}P^\infty)^{(N)}, M^{S^1}) \rightarrow \text{Sec}(\xi_N) \cong M^{hS^1},$$

which can be obtained by truncating in the diagram (3.1):

$$\begin{array}{ccc} M^{S^1} & \longrightarrow & M \\ \downarrow & & \downarrow \\ F_N & \xrightarrow{\gamma_N} & E_N \\ \eta_N \searrow & & \swarrow \xi_N \\ & (\mathbb{C}P^\infty)^{(N)}. & \end{array}$$

Now let

$$(3.2) \quad \begin{array}{ccc} (A \otimes \Lambda V, D) & \longrightarrow & (\Lambda V, d) \\ (A, 0) \nearrow \swarrow & \downarrow \psi & \downarrow \varphi \\ (A, 0) \otimes (\Lambda Z, d) & \longrightarrow & (\Lambda Z, d) \end{array}$$

be a model of the above diagram, where  $(A, 0) = (\Lambda x / (\Lambda x)^{>N}, 0)$ ,  $(\Lambda V, d)$  and  $(\Lambda Z, d)$  are minimal Sullivan models of  $M$  and  $M^{S^1}$ , respectively.

Then we have the following

**Theorem 3.3.** [1, Theorem 21] *The composition*

$$(\Lambda(V \otimes A^\#), \tilde{d}) \xrightarrow{\phi} (\Lambda(Z \otimes A^\#), \tilde{d}) \xrightarrow{\gamma} (\Lambda Z, d)$$

is a model of  $k: M_{\mathbb{Q}}^{S^1} \hookrightarrow M_{\mathbb{Q}}^{hS^1}$ . The morphisms above are defined by

$$\begin{aligned}\phi(v \otimes \alpha) &= \rho^{-1}[\psi(v) \otimes \alpha], \quad v \otimes \alpha \in V \otimes A^\#, \\ \gamma(z \otimes \alpha) &= \begin{cases} z & \alpha = 1, \\ 0 & \alpha \neq 1, \end{cases} \quad z \otimes \alpha \in Z \otimes A^\#.\end{aligned}$$

Then we give some information about  $\psi$ . First, let  $(\Lambda x \otimes \Lambda V, D)$  be a model of the fibration  $\xi$ , we can decompose the differential  $D$  in  $A \otimes \Lambda V$  into

$$D = \sum_{i \leq 1} D_i, \quad D_i(V) \subset \Lambda x \otimes \Lambda^i V.$$

**Proposition 3.4.** [2, Lemma 14] *The vector space  $V$  can be decomposed into a direct sum  $W \oplus K \oplus S$  where*

- (1)  $W \oplus K = \ker D_1$ ,
- (2)  $K$  and  $S$  have the same dimension admitting bases  $\{v_i\}_{i \in I}$ ,  $\{s_i\}_{i \in I}$ , and for any  $i \in I$ , there exists  $n_i \geq 1$  such that  $D_1(s_i) = x^{n_i} v_i$ .

Let  $\mathbb{K} = Q(x)$ , the field of fractions of  $\Lambda x$ , we obtain a morphism of (ungraded) differential vector spaces

$$\bar{\psi}: (\mathbb{K} \otimes V, D_1) \rightarrow (\mathbb{K} \otimes Z, 0) = (Z_{\mathbb{K}}, 0).$$

If we assume  $\mathbb{K}$  concentrated in degree 0 and consider in  $V$  and  $Z$  the usual  $\mathbb{Z}_2$ -grading given by the parity of the generators, then the Borel localization theorem claim that:

**Theorem 3.5.** [1, Theorem 22] *The morphism*

$$\bar{\psi}: (\mathbb{K} \otimes V, D_1) \rightarrow (Z_{\mathbb{K}}, 0)$$

is a quasi-isomorphism.

By Proposition 3.4, we have

**Lemma 3.6.** (1)  $\dim W = \dim Z$ .  
 (2) *There are  $\{w_j\}_{j \in J}$ ,  $\{z_j\}_{j \in J}$  which are homogeneous basis of  $W$  and  $Z$  respectively, and non negative integers  $\{m_j\}_{j \in J}$  such that*

$$\psi(w_j) = x^{m_j} z_j + \Gamma_j, \quad \Gamma_j \in R \otimes \Lambda^{\geq 2} Z, \quad j \in J,$$

and

$$\psi(s_i) \in R \otimes \Lambda^{\geq 2} Z, \quad \psi(v_i) \in R \otimes \Lambda^{\geq 2} Z, \quad s_i \in S, \quad v_i \in K, \quad i \in I.$$

**Theorem 3.7.** *For an  $S^1$ -complex  $M$  which is simply connected with*

$$\dim \pi_*(M) \otimes \mathbb{Q} < \infty.$$

*Then the inclusion*

$$k: M^{S^1} \hookrightarrow M^{hS^1}$$

*is a rational homotopy equivalence if and only if  $M$  is rational homotopy equivalent to a product of  $\mathbb{C}P^\infty$ .*

Proof. By Theorem 3.3, the model of  $k$  is

$$\alpha: (\Lambda(V \otimes A^\#), \tilde{d}) \rightarrow (\Lambda(Z \otimes A^\#), \tilde{d}) \rightarrow (\Lambda Z, d).$$

By [1, Theorem 24],  $\pi_*(k) \otimes \mathbb{Q}$  is injective, so we only consider the surjective part.

By [1, Theorem 11],  $(\Lambda(V \otimes A^\#), \tilde{d})$  is a model of  $M_{\mathbb{Q}}^{hS^1}$ . Then we have

$$H^k(V \otimes A^\#, \tilde{d}_1) \cong \text{Hom}(\pi_k(M_{\mathbb{Q}}^{hS^1}), \mathbb{Q}),$$

where  $k \geq 1$ .

By Proposition 3.4,  $V = W \oplus K \oplus S$ . An easy computation shows that  $(W \otimes A^\#) \oplus S \subset H^*(V \otimes A^\#, \tilde{d}_1)$ . It is obvious that

$$\begin{aligned} \alpha(w_j) &= 0 \Leftrightarrow m_j \neq 0, \\ \alpha(w_j \otimes (x^i)^\#) &= 0 \Leftrightarrow m_j \neq i, \\ \alpha(s_j) &= 0. \end{aligned}$$

If there exists some  $j$  such that  $|w_j| \geq 2$  or  $S \neq \emptyset$ , then  $H(\alpha, \tilde{d}_1)$  is not injective, so  $k$  is not a rational homotopy equivalence.

If  $|w_j| = 2$ , for each  $j \in J$ , and  $S = \emptyset$ , we have  $(\Lambda W, d)$  is a model of a product of  $\mathbb{C}P^\infty$ . It is easy to show that  $k$  is a rational homotopy equivalence.  $\square$

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