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THE HOMOTOPY FIXED POINT SETS OF SPHERES ACTIONS ON RATIONAL COMPLEXES

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Abstract

In this paper, we describe the homotopy type of the homotopy fixed point sets of S^3-actions on rational spheres and complex projective spaces, and provide some properties of S^1-actions on a general rational complex.

1. Introduction

An action of a group G on a space M gives rise to two natural spaces, the fixed point set M^G and the homotopy fixed point set M^{hG}. It is crucially important that there is an injection

$$k\colon M^G \to M^{hG}.$$

Indeed, one version of the *generalized Sullivan conjecture* asserts that, when G is a finite p-group, and M is a G-CW-complex, then the p-completion of k is a homotopy equivalence. This conjecture was proved in the case when M is a finite complex by Miller [7].

For a finite group G, the rational homotopy theory of M^{hG} has been studied by Goyo [5].

In [1, 2], the authors studied the homotopy type of M^{hG} for a compact Lie group G with particular emphasis when G is the circle.

From now on, and unless explicitly stated otherwise, G will denote a compact connected Lie group and by a topological G-space we mean a nilpotent G-space with the homotopy type of a CW-complex of finite type and $M^G \neq \emptyset$. Then the action of G on M induces an action of G on $M_{\mathbb{Q}}$.

We then start by setting a sufficiently general context in which ${M_{\mathbb{Q}}}^{hG}$ has the homotopy type of a nilpotent CW-complex. Identifying the homotopy fixed point set

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with the space $\mathrm{Sec}(\xi)$ of sections of the corresponding Borel fibration

$$\xi : M \to M_{hG} \to BG,$$

we have that if $\pi_{>n}(M)$ are torsion groups for a certain $n > 1$, then ${M_{\mathbb{Q}}}^{hG}$ is a rational nilpotent complex with the homotopy type of a CW-complex [1].

In this paper, we explicitly describe the rational homotopy type of the homotopy fixed point sets of certain S^3-actions.

Theorem 1.1. *Given an S^3-action on the rational n-sphere $S^n_{\mathbb{Q}}$.*
(1) *When n is odd, ${S^n_{\mathbb{Q}}}^{hS^3}$ has the rational homotopy type of products of odd dimensional spheres, precisely, we have*

$${S^n_{\mathbb{Q}}}^{hS^3} \simeq_{\mathbb{Q}} S^a \times S^{a+4} \times \cdots \times S^n,$$

where

$$a = \begin{cases} 1, & n = 4k+1, \\ 3, & n = 4k+3. \end{cases}$$

(2) *If $n = 4k$, ${S^n_{\mathbb{Q}}}^{hS^3}$ is either path connected, and of the rational homotopy type of $S^3 \times K_k$, where K_k has the minimal Sullivan model*

$$(\Lambda((x_s)_{1\le s\le k}, (y_r)_{2\le r\le 2k}), d)$$

with $|x_s| = 4s$, $|y_r| = 4r-1$, $dx_s = 0$ $(1 \le s \le k)$, $dy_r = \sum_{s+t=r} x_s x_t$ $(2 \le r \le 2k)$, or else, it has 2 components, each of them has the rational homotopy type of

$$S^{4k+3} \times S^{4k+7} \times \cdots \times S^{8k-1}.$$

(3) *If $n = 4k+2$, ${S^n_{\mathbb{Q}}}^{hS^3}$ is path connected, and of the rational homotopy type of $S^3 \times S^7 \times T_k$, where T_k has the minimal Sullivan model*

$$(\Lambda((x_s)_{1\le s\le k}, (y_r)_{3\le r\le 2k+1}), d)$$

with $|x_s| = 4s+2$, $|y_r| = 4r-1$, $dx_s = 0$ $(1 \le s \le k)$, $dy_r = \sum_{s+t=r-1} x_s x_t$ $(3 \le r \le 2k+1)$.

Theorem 1.2. *Given an S^3-action in the rational complex projective space $\mathbb{C}P^n_{\mathbb{Q}}$.*

(1) *If n is odd, $\mathbb{C}P_{\mathbb{Q}}^{n\,hS^3}$ is path connected, and has the rational homotopy type of one of the following spaces*:

$$\begin{aligned}
&\mathbb{C}P^1 \times S^7 \times S^{11} \times \cdots \times S^{2n+1},\\
&S^3 \times \mathbb{C}P^3 \times S^{11} \times \cdots \times S^{2n+1},\\
&S^3 \times S^7 \times \mathbb{C}P^5 \times \cdots \times S^{2n+1},\\
&\ldots,\\
&S^3 \times S^7 \times \cdots \times S^{2n-3} \times \mathbb{C}P^n.
\end{aligned}$$

(2) *If n is even, $\mathbb{C}P_{\mathbb{Q}}^{n\,hS^3}$ is path connected, and has the rational homotopy type of one of the following spaces*:

$$\begin{aligned}
&* \times S^5 \times S^9 \times \cdots \times S^{2n+1},\\
&S^1 \times \mathbb{C}P^2 \times S^9 \times \cdots \times S^{2n+1},\\
&S^1 \times S^5 \times \mathbb{C}P^4 \times \cdots \times S^{2n+1},\\
&\ldots,\\
&S^1 \times S^5 \times \cdots \times S^{2n-3} \times \mathbb{C}P^n.
\end{aligned}$$

In [1, Corollary 2], they give a criterion of an elliptic S^1-space. We first show that the condition M is a finite complex is necessary by the following example: there is a nilpotent S^1-complex M which is not an elliptic space, such that each component of ${M_{\mathbb{Q}}}^{hS^1}$ is elliptic. We also observe that an S^1-finite nilpotent complex M is elliptic if and only if one of the component of ${M_{\mathbb{Q}}}^{hS^1}$ is elliptic, complementing the mentioned result.

Finally, we show that the injection k is generally not a rational homotopy equivalence.

Theorem 1.3. *For an S^1-complex M which is simply connected with*

$$\dim \pi_*(M) \otimes \mathbb{Q} < \infty.$$

Then

$$k\colon M_{\mathbb{Q}}^{S^1} \hookrightarrow M_{\mathbb{Q}}^{hS^1}$$

is a rational homotopy equivalence if and only if M is rational homotopy equivalent to a product of $\mathbb{C}P^\infty$.

In the next section we prove Theorems 1.1 and 1.2. In Section 3 we prove Theorem 1.3.

2. S^3-rational spheres and complex projective spaces

Our results heavily depend on known facts and techniques arising from rational homotopy theory. All of them can be found with all details in [4]. We simply remark a few facts.

We recall that when M is path connected, the Sullivan model of M is a quasi-isomorphism

$$m\colon (\Lambda V_M, d) \to A_{PL}(M),$$

where $(\Lambda V_M, d)$ is a Sullivan algebra.

We also recall that a space M is elliptic if both $H^*(M;\mathbb{Q})$ and $\pi_*(M)\otimes\mathbb{Q}$ are finite dimensional vector spaces over $\mathbb{Q}$.

For a G-space M, we have the corresponding Borel fibration

$$\xi : M \to M_{hG} \to BG,$$

where $M_{hG} = (M \times EG)/G$. It is a classical fact that the homotopy fixed point set

$$M^{hG} = \mathrm{map}_G(EG, M)$$

is homotopy equivalent to the section space $\mathrm{Sec}(\xi)$ of this fibration.

Each fixed point gives rise to a trivial section of the product bundle

$$M^G \to BG \times M^G \to BG.$$

Composing with the injection $M_G \times BG \hookrightarrow EG \times M/G = M_{hG}$ gives a section of the Borel fibration. Thus we have a natural injection:

$$k\colon M^G \hookrightarrow M^{hG}.$$

For any G-CW complex M, there is an equivariant rationalization $m\colon M \to M_{\mathbb{Q}}$, that is, $M_{\mathbb{Q}}$ is also a G-CW complex, m is an equivariant map, and $(M_{\mathbb{Q}})^G \simeq (M^G)_{\mathbb{Q}}$. Moreover, we have

Proposition 2.1 ([1, Proposition 12]). *If M is a Postnikov piece, that is, $\pi_{>N}(M) = 0$ for some N, then*
(i) *M^{hG} has the homotopy type of a nilpotent CW-complex of finite type.*
(ii) $(M^{hG})_{\mathbb{Q}} \simeq (M_{\mathbb{Q}})^{hG}$.

Note that if $M_{\mathbb{Q}}$ is a Postnikov piece, then $(M_{\mathbb{Q}})^{hG}$ makes sense and is a rational space.

Now, we determine the homotopy type of the homotopy fixed point sets of certain S^3-actions.

Proof of Theorem 1.1. (1) CASE 1: n is odd.

We only prove the case $n = 4k+3$, the case $n = 4k+1$ is similar, so we omit it.

As in the proof of [1, Theorem 19], it is not hard to get the model of the corresponding Borel fibration

$$\xi : (A, 0) \hookrightarrow ((\Lambda e) \otimes A, D) \to (\Lambda e, 0),$$

where $(A, 0) = (\Lambda x/x^k, 0)$ and $|x| = 4$, $|e| = n$. This fibration is trivial, so $\mathrm{Sec}(\xi) \simeq \mathrm{Map}(\mathbb{H}P^k, S^n)$.

By [1, Theorem 9], the model of $S_{\mathbb{Q}}^{n\ hS^3}$ is $(\Lambda(x_1, x_2, \ldots, x_{n+1/4}), 0)$. It is exactly the model of $S^3 \times S^7 \times \cdots \times S^n$. It follows that $S_{\mathbb{Q}}^{n\ hS^3} \simeq_{\mathbb{Q}} S^a \times S^{a+4} \times \cdots \times S^n$.

(2) CASE 2: $n = 4k$.

As $\pi_{\geq 2n}(S^n) \otimes \mathbb{Q} = 0$, a model of the Borel fibration is

$$\xi_{2n} : (A, 0) \hookrightarrow (\Lambda(e, e') \otimes A, D) \to (\Lambda(e, e'), d),$$

where $A = \Lambda x/x^{2k+1}$, x, e, e' are of degree 4, n, $2n-1$ respectively, $De = 0$, $De' = e^2 + \lambda x^{n/4} e$, $de' = e^2$.

(i) If $\lambda = 0$, then ξ_{2n} is trivial and

$$S_{\mathbb{Q}}^{n\ hS^3} \simeq \mathrm{Map}(\mathbb{H}P^{2k}, S^n)_{\mathbb{Q}}.$$

A straightforward computation shows that this mapping space has a model of the form

$$(\Lambda y_1, 0) \otimes (\Lambda((x_s)_{1\leq s\leq k}, (y_r)_{2\leq r\leq 2k}), d)$$

with $|x_s| = 4s$, $|y_r| = 4r-1$, $dx_s = 0$ $(1 \leq s \leq k)$, $dy_r = \sum_{s+t=r} x_s x_t$ $(r > 1)$.

(ii) If $\lambda \neq 0$, then the fibration ξ_n has two non homotopic sections σ, τ which correspond to the only two possible retractions of its model:

$$\varphi_\sigma, \varphi_\tau : (\Lambda(e, e') \otimes A, D) \to (A, 0), \quad \varphi_\sigma(e) = 0, \quad \varphi_\tau(e) = \lambda x^k.$$

By the same way in [1], we have that the model of $\mathrm{Sec}_\sigma(\xi_{2n})$ is of the form

$$(\Lambda((x_s)_{1\leq s\leq k}, (y_r)_{1\leq r\leq 2k}), \tilde{d})$$

with $|x_s| = 4s$, $|y_r| = 4r-1$. The linear part of $\tilde{d}$ is:

$$\tilde{d}(y_r) = \lambda x_r$$

for $1 \leq r \leq k$, which shows that the minimal model of $\mathrm{Sec}_\sigma(\xi_{2n})$ is

$$(\Lambda(y_r)_{k+1\leq r\leq 2k}, 0).$$

Replace λ by $-\lambda$, we have that the model of $\mathrm{Sec}_\tau(\xi_{2n})$ is the same.

(3) **Case 2**: $n = 4k+2$.

As $\pi_{\geq 2n}(S^n)\otimes\mathbb{Q} = 0$, a model of the Borel fibration is

$$\xi_{2n}\colon (A, 0) \hookrightarrow (\Lambda(e, e')\otimes A, D) \to (\Lambda(e, e'), d),$$

where $A = \Lambda x/x^{2k+1}$, x, e, e' are of degree 4, n, $2n-1$ respectively, $De = 0$, $De' = e^2$, $de' = e^2$. It follows that the fibration ξ_{2n} is trivial, we have

$$S^{n\ hG}_{\mathbb{Q}} \simeq \mathrm{Map}(\mathbb{H}P^{2k}, S^n)_{\mathbb{Q}}.$$

The model of $S^{n\ hG}_{\mathbb{Q}}$ is

$$(\Lambda(y_1, y_2), 0)\otimes(\Lambda((x_s)_{1\leq s\leq k}, (y_r)_{3\leq r\leq 2k+1}), d)$$

with $|x_s| = 4s+2$, $|y_r| = 4r-1$, $dx_s = 0$ $(1\leq s\leq k)$, $dy_r = \sum_{s+t=r-1} x_s x_t$ $(3\leq r\leq 2k+1)$.

The desired result follows. □

Proof of Theorem 1.2. First, we assume $n = 2k+1$. As $\pi_{\geq 4k+4}(\mathbb{C}P^n_{\mathbb{Q}}) = 0$, it suffice to use the model of ξ_{2n+2}

$$(A, 0) \to (\Lambda(e, e')\otimes A, D) \to (\Lambda(e, e'), d),$$

where $A = (\Lambda x)/x^{k+2}$, $|x| = 4$, $|e| = 2$, $|e'| = 4k+3$, and

$$De = 0, \quad De' = e^{n+1} + \sum_{j=1}^{k} \lambda_j e^j x^{n+1-2j}, \quad \lambda\in\mathbb{Q},\ j = 1, \ldots, n.$$

The retraction of this model of fibration is just $\varphi(e) = 0$. So we have $\mathrm{Sec}(\xi_{4k+4})$ is connected, and the model of it is

$$(\Lambda(e, (e'_r)_{1\leq r\leq k+1}, \tilde{d})$$

with $|e| = 2$, $|e'_r| = 4r-1$, $\tilde{d}(e'_r) = \lambda_{k+1-r}e^{2r}$ for $1\leq r\leq k$ and $\tilde{d}(e'_{k+1}) = e^{2k+2}$.

If $\lambda_1\neq 0$ this is a model of

$$S^2\times S^7\times\cdots\times S^{4k+3}.$$

If $\lambda_1 = \cdots = \lambda_{i-1} = 0$ and $\lambda_i\neq 0$, this is a model of

$$S^3\times\cdots\times S^{4k-4i-1}\times\mathbb{C}P^{2k+1-2i}\times S^{4k-4i+3}\times\cdots\times S^{4k+3}.$$

Finally, if all $\lambda_i = 0$, then it is a model of

$$S^3 \times S^7 \times \cdots \times S^{4k-1} \times \mathbb{C}P^{2k+1}.$$

For n even, the proof is similar, so we omit it. □

3. The Inclusion $k\colon M^{S^1} \hookrightarrow M^{hS^1}$

We begin with some interesting observations on S^1-actions.

In [2, Example 12], there is an S^1-action on $M = K(\mathbb{Z}, n) \times K(\mathbb{Z}, n+1)$, such that the model of it's Borel fibration is

$$\eta_n\colon (\Lambda x, 0) \hookrightarrow (\Lambda x \otimes \Lambda(z, y), D) \to (\Lambda(z, y), d),$$

where $|x| = 2$, $|z| = n$, $|y| = n + 1$, $D(z) = 0$, and $D(y) = xz$. For $n = 2k$, there is only one retraction $\sigma\colon \sigma(z) = \sigma(y) = 0$. Thus $\mathrm{Sec}(\eta_{2k})$ is path connected.

By the same method used in [1], a model of $\mathrm{Sec}(\eta_{2k})$ is

$$(\Lambda((z_i)_{1\le i\le k}, (y_j)_{1\le j\le k+1}), d),$$

where $|z_i| = 2i$, $|y_j| = 2j - 1$ and $d(y_i) = z_i$. Since the minimal model of $\mathrm{Sec}(\eta_{2k})$ is $(\Lambda y_{k+1}, 0)$, $\mathrm{Sec}(\eta_{2k}) \simeq_{\mathbb{Q}} S^{2k+1}$ is an elliptic space. However, M is not an elliptic space.

Next we complement [1, Corollary 2] with the following

Proposition 3.1. *For an S^1-space M which is a nilpotent finite complex, the following conditions are equivalent*:

1) *M is elliptic.*
2) *Each component of ${M_{\mathbb{Q}}}^{hS^1}$ is elliptic.*
3) *One of the components of ${M_{\mathbb{Q}}}^{hS^1}$ is elliptic.*

Proof. 1) ⇒ 2): [1, Theorem 15].

2) ⇒ 3): Trivial.

3) ⇒ 1): By [2, Theorem 13], $2\dim\pi_*(\mathrm{Sec}_\sigma(\xi)\otimes\mathbb{Q}) \ge \dim\pi_*(M)\otimes\mathbb{Q}$. By $\mathrm{Sec}_\sigma(\xi)$ is elliptic, $\dim\pi_*(\mathrm{Sec}_\sigma(\xi))\otimes\mathbb{Q}$ is finite, so $\dim\pi_*(M)\otimes\mathbb{Q}$ is finite. Then M is elliptic. □

REMARK 3.2. The theorem holds also for $G = S^3$. The proof is similar.

The rest of the section is devoted to showing Theorem 1.3.

Let M be an S^1-space and $M^G \neq \emptyset$. Then the inclusion $M^{S^1} \hookrightarrow M$ induces a map of Borel fibrations:

$$\begin{array}{ccc} M^{S^1} & \longrightarrow & M \\ \downarrow & & \downarrow \\ \mathbb{C}P^\infty \times M^{S^1} & \xrightarrow{\gamma} & M_{hS^1} \\ & {\scriptstyle \eta} \searrow \quad \swarrow {\scriptstyle \xi} & \\ & \mathbb{C}P^\infty . & \end{array} \tag{3.1}$$

If there exists some N such that $\pi_{\geq N}(M_{\mathbb{Q}}) = 0$ and $\pi_{\geq N}(M_{\mathbb{Q}}^{S^1}) = 0$. Then k is identified with the corresponding

$$M^{S^1} \hookrightarrow \mathrm{Map}((\mathbb{C}P^\infty)^{(N)}, M^{S^1}) \to \mathrm{Sec}(\xi_N) \cong M^{hS^1},$$

which can be obtained by truncating in the diagram (3.1):

$$\begin{array}{ccc} M^{S^1} & \longrightarrow & M \\ \downarrow & & \downarrow \\ F_N & \xrightarrow{\gamma_N} & E_N \\ & {\scriptstyle \eta_N} \searrow \quad \swarrow {\scriptstyle \xi_N} & \\ & (\mathbb{C}P^\infty)^{(N)} . & \end{array}$$

Now let

$$\begin{array}{ccccc} & & (A \otimes \Lambda V, D) & \longrightarrow & (\Lambda V, d) \\ & \nearrow & \downarrow {\scriptstyle \psi} & & \downarrow {\scriptstyle \varphi} \\ (A, 0) & & & & \\ & \searrow & & & \\ & & (A, 0) \otimes (\Lambda Z, d) & \longrightarrow & (\Lambda Z, d) \end{array} \tag{3.2}$$

be a model of the above diagram, where $(A,0) = (\Lambda x/(\Lambda x)^{>N}, 0)$, $(\Lambda V, d)$ and $(\Lambda Z, d)$ are minimal Sullivan models of M and M^{S^1}, respectively.

Then we have the following

Theorem 3.3. [1, Theorem 21] *The composition*

$$(\Lambda(V \otimes A^\#), \tilde{d}) \xrightarrow{\phi} (\Lambda(Z \otimes A^\#), \tilde{d}) \xrightarrow{\gamma} (\Lambda Z, d)$$

is a model of $k\colon M_{\mathbb{Q}}^{S^1} \hookrightarrow M_{\mathbb{Q}}^{hS^1}$. *The morphisms above are defined by*

$$\phi(v \otimes \alpha) = \rho^{-1}[\psi(v) \otimes \alpha], \quad v \otimes \alpha \in V \otimes A^{\#},$$

$$\gamma(z \otimes \alpha) = \begin{cases} z & \alpha = 1, \\ 0 & \alpha \neq 1, \end{cases} \quad z \otimes \alpha \in Z \otimes A^{\#}.$$

Then we give some information about ψ. First, let $(\Lambda x \otimes \Lambda V, D)$ be a model of the fibration ξ, we can decompose the differential D in $A \otimes \Lambda V$ into

$$D = \sum_{i \leq 1} D_i, \quad D_i(V) \subset \Lambda x \otimes \Lambda^i V.$$

Proposition 3.4. [2, *Lemma* 14] *The vector space* V *can be decomposed into a direct sum* $W \oplus K \oplus S$ *where*
(1) $W \oplus K = \ker D_1$,
(2) K *and* S *have the same dimension admitting bases* $\{v_i\}_{i\in I}$, $\{s_i\}_{i\in I}$, *and for any* $i \in I$, *there exists* $n_i \geq 1$ *such that* $D_1(s_i) = x^{n_i} v_i$.

Let $\mathbb{K} = Q(x)$, the field of fractions of Λx, we obtain a morphism of (ungraded) differential vector spaces

$$\bar{\psi}\colon (\mathbb{K} \otimes V, D_1) \to (\mathbb{K} \otimes Z, 0) = (Z_{\mathbb{K}}, 0).$$

If we assume $\mathbb{K}$ concentrated in degree 0 and consider in V and Z the usual $\mathbb{Z}_2$-grading given by the parity of the generators, then the Borel localization theorem claim that:

Theorem 3.5. [1, Theorem 22] *The morphism*

$$\bar{\psi}\colon (\mathbb{K} \otimes V, D_1) \to (Z_{\mathbb{K}}, 0)$$

is a quasi-isomorphism.

By Proposition 3.4, we have

Lemma 3.6. (1) $\dim W = \dim Z$.
(2) *There are* $\{w_j\}_{j\in J}$, $\{z_j\}_{j\in J}$ *which are homogeneous basis of* W *and* Z *respectively, and non negative integers* $\{m_j\}_{j\in J}$ *such that*

$$\psi(w_j) = x^{m_j} z_j + \Gamma_j, \quad \Gamma_j \in R \otimes \Lambda^{\geq 2} Z, \quad j \in J,$$

and

$$\psi(s_i) \in R \otimes \Lambda^{\geq 2} Z, \quad \psi(v_i) \in R \otimes \Lambda^{\geq 2} Z, \quad s_i \in S,\ v_i \in K,\ i \in I.$$

Theorem 3.7. *For an S^1-complex M which is simply connected with*

$$\dim \pi_*(M) \otimes \mathbb{Q} < \infty.$$

Then the inclusion

$$k\colon M^{S^1} \hookrightarrow M^{hS^1}$$

is a rational homotopy equivalence if and only if M is rational homotopy equivalent to a product of $\mathbb{C}P^\infty$.

Proof. By Theorem 3.3, the model of k is

$$\alpha\colon (\Lambda(V \otimes A^\#), \tilde{d}) \to (\Lambda(Z \otimes A^\#), \tilde{d}) \to (\Lambda Z, d).$$

By [1, Theorem 24], $\pi_*(k) \otimes \mathbb{Q}$ is injective, so we only consider the surjective part.

By [1, Theorem 11], $(\Lambda(V \otimes A^\#), \tilde{d})$ is a model of $M_{\mathbb{Q}}^{hS^1}$. Then we have

$$H^k(V \otimes A^\#, \tilde{d}_1) \cong \operatorname{Hom}(\pi_k(M_{\mathbb{Q}}^{hS^1}), \mathbb{Q}),$$

where $k \geq 1$.

By Proposition 3.4, $V = W \oplus K \oplus S$. An easy computation shows that $(W \otimes A^\#) \oplus S \subset H^*(V \otimes A^\#, \tilde{d}_1)$. It is obvious that

$$\begin{aligned} &\alpha(w_j) = 0 \Leftrightarrow m_j \neq 0, \\ &\alpha(w_j \otimes (x^i)^\#) = 0 \Leftrightarrow m_j \neq i, \\ &\alpha(s_j) = 0. \end{aligned}$$

If there exists some j such that $|w_j| \geq 2$ or $S \neq \emptyset$, then $H(\alpha, \tilde{d}_1)$ is not injective, so k is not a rational homotopy equivalence.

If $|w_j| = 2$, for each $j \in J$, and $S = \emptyset$, we have $(\Lambda W, d)$ is a model of a product of $\mathbb{C}P^\infty$. It is easy to show that k is a rational homotopy equivalence. □

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