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REPRESENTATION TYPE OF FINITE QUIVER HECKE ALGEBRAS OF TYPE $C_l^{(1)}$

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Abstract

We give a graded dimension formula described in terms of combinatorics of Young diagrams and a simple criterion to determine the representation type for the finite quiver Hecke algebras of type $C_l^{(1)}$.

Introduction

This is the fourth of our series on finite quiver Hecke algebras. The *quiver Hecke algebras*, or *affine quiver Hecke algebras*, were introduced by Khovanov–Lauda [18, 19] and Rouquier [26] for providing categorification of (the negative half of) quantum groups. Their certain quotient algebras, the *cyclotomic quiver Hecke algebras* $R^{\Lambda}(\beta)$, where Λ is fixed and β is varying, together with induction and restriction functors among their module categories, categorify the irreducible highest weight module $V(\Lambda)$ over the quantum group. When $\Lambda = \Lambda_0$, we call the algebras $R^{\Lambda_0}(\beta)$ the *finite quiver Hecke algebras*. As was explained in our previous papers [1, 2, 3] in the series, finite quiver Hecke algebras can be understood as vast generalization of the Iwahori–Hecke algebras associated with the symmetric group in the direction of Lie type.

In this paper, we study the representation type of finite quiver Hecke algebra $R^{\Lambda_0}(\beta)$ of affine type $C_l^{(1)}$. The main results are a graded dimension formula of $R^{\Lambda_0}(\beta)$ described in terms of combinatorics of Young diagrams (Theorem 2.6) and a criterion for the representation type of $R^{\Lambda_0}(\beta)$ in Lie theoretic terms (Theorem 5.5). Recall that we studied affine types $A_l^{(1)}$, $A_{2l}^{(2)}$ and $D_{l+1}^{(2)}$ in our previous papers, and proved that the patterns of the representation type followed natural generalization of Erdmann and Nakano's for the Iwahori–Hecke algebras associated with the symmetric group. However, the affine type $C_l^{(1)}$ shows a new pattern. In particular, we have an unexpected result that $R^{\Lambda_0}(\delta)$ is not of finite representation type.

Now, we explain in some detail the tools and the strategy to prove the results. Firstly, the q-deformed Fock space $\mathcal{F}$ of type $C_l^{(1)}$ [16] is a key ingredient for proving the graded dimension formula. This $C_l^{(1)}$-type Fock space $\mathcal{F}$ is constructed by folding

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the usual q-deformed $A^{(1)}_{2l-1}$-type Fock space. Namely, the basis is given by the set of all partitions as in the usual Fock space, but we change the residue pattern on the nodes of partitions via the folding map

$$\pi : \{0, 1, \dots, 2l-1\} \to \{0, 1, \dots, l\}$$

defined by $\pi(0) = 0$, $\pi(l) = l$ and $\pi(2l-i) = \pi(i) = i$ for $i = 1, \dots, l-1$. Investigating the action of $e_{\nu_1} \cdots e_{\nu_n} f_{\nu'_n} \cdots f_{\nu'_1}$ on the Fock space $\mathcal{F}$, we obtain the dimension formula. Thus, the formula is described in terms of combinatorics of Young diagrams, which is very similar to the graded dimension formula of affine type A in [5, Section 4.11]. We remark that the residue pattern (1.3) for type $C^{(1)}_l$ also appears as colors of arrows in the Kirillov–Reshetikhin crystal $B^{1,1}$ of type $C^{(1)}_l$, which is not a perfect crystal [10].

To achieve the second result, we follow the framework to determine the representation type given in [2]. Let $\max(\Lambda)$ denote the set of maximal weights of the irreducible highest weight module $V(\Lambda)$. In the three affine cases studied in our previous papers in the series, the set $\max(\Lambda_0)$ consists of a single Weyl group orbit. Thus, we may generalize the notion of cores and weights of Young diagrams. In the affine type $C^{(1)}_l$, $\max(\Lambda_0)$ consists of several Weyl group orbits and the representatives are given by the set $\max(\Lambda_0) \cap \mathsf{P}^+$. It is not difficult to calculate the set and the result is

$$\max(\Lambda_0) \cap \mathsf{P}^+ = \left\{ \Lambda_0 + \varpi_i - \frac{i}{2}\delta \;\middle|\; i \in I,\ i \text{ is even} \right\},$$

where $\varpi_0 = 0$ and if $i \neq 0$ then

$$\varpi_i = \alpha_1 + 2\alpha_2 + \cdots + (i-1)\alpha_{i-1} + i\left(\alpha_i + \alpha_{i+1} + \cdots + \alpha_{l-1} + \frac{1}{2}\alpha_l\right).$$

Thus, by the sl_2-categorification theorem, we have to investigate the representation type of $R^{\Lambda_0}(k\delta - \varpi_i)$ for $k \geq i/2$. We first consider the representation type of $R^{\Lambda_0}(\delta)$.

Recall that one of the ingredients in our series of papers was explicit construction of $R^{\Lambda_0}(\delta)$-modules or $R^{\Lambda_0}(2\delta)$-modules. Recently, an interesting paper by Kleshchev and Muth [21] appeared, and they constructed irreducible $R^{\Lambda_0}(\delta)$-modules for several untwisted affine types in the spirit of Kang, Kashiwara and Kim [15], which includes the affine type $C^{(1)}_l$. Thus, we use their construction and, combining with the dimension formula, we find the radical series of the indecomposable projective $R^{\Lambda_0}(\delta)$-modules, and determine the representation type of $R^{\Lambda_0}(\delta)$ (Theorem 3.7). The result is that $R^{\Lambda_0}(\delta)$ is a symmetric special biserial algebra if $l = 2$, and it is of wild representation type if $l \geq 3$.

Next task is to deal with the representation type of $R^{\Lambda_0}(2\delta - \varpi_4)$. In this case, we do not need explicit description of irreducible modules, and we may derive the radical series of the indecomposable projective modules from the categorification theorem and crystal properties. The result tells that $R^{\Lambda_0}(2\delta - \varpi_4)$ is of wild representation type (Theorem 4.2).

Using the same arguments in [2] with small modifications, we may handle the remaining cases, and we obtain the second main result (Theorem 5.5).

1. Quantum affine algebras

Let $I = \{0, 1, \dots, l\}$ be an index set, and A the *affine Cartan matrix* of type $C_l^{(1)}$ $(l \geq 2)$[1]

$$\mathsf{A} = (a_{ij})_{i,j \in I} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -2 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -2 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

An *affine Cartan datum* $(\mathsf{A}, \mathsf{P}, \Pi, \Pi^\vee)$ of type $C_l^{(1)}$ consists of
(1) the affine Cartan matrix A as above,
(2) a free abelian group P of rank $l + 2$, called the *weight lattice*,
(3) $\Pi = \{\alpha_i \mid i \in I\} \subset \mathsf{P}$, called the set of *simple roots*,
(4) $\Pi^\vee = \{h_i \mid i \in I\} \subset \mathsf{P}^\vee := \mathrm{Hom}(\mathsf{P}, \mathbb{Z})$, called the set of *simple coroots*,
which satisfy the following properties:
(a) $\langle h_i, \alpha_j \rangle = a_{ij}$ for all $i, j \in I$,
(b) Π and $\Pi^\vee$ are linearly independent sets.

The free abelian group $\mathsf{Q} = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ is called the *root lattice*, and $\mathsf{Q}^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$ is the *positive cone* of the root lattice. For $\beta = \sum_{i \in I} k_i\alpha_i \in \mathsf{Q}^+$, set $|\beta| = \sum_{i \in I} k_i$ to be the *height* of β. We denote by W the *Weyl group* associated with A, which is generated by $\{r_i\}_{i \in I}$ acting on P by $r_i\Lambda = \Lambda - \langle h_i, \Lambda \rangle \alpha_i$, for $\Lambda \in \mathsf{P}$. Let

$$\mathsf{P}^+ = \{\Lambda \in \mathsf{P} \mid \Lambda(h_i) \geq 0 \text{ for } i \in I\}.$$

For $i \in I$, let Λ_i be the ith *fundamental weight* in P^+. In particular, we have $\Lambda_i(h_j) = \delta_{i,j}$. The *null root* in the affine type $C_l^{(1)}$ is given by

$$\delta = \alpha_0 + 2\alpha_1 + \cdots + 2\alpha_{l-1} + \alpha_l.$$

Note that $\langle h_i, \delta \rangle = 0$ and $w\delta = \delta$, for $i \in I$ and $w \in \mathsf{W}$. Let $(\mathsf{d}_0, \mathsf{d}_1, \dots, \mathsf{d}_l) = (2, 1, \dots, 1, 2)$. Then the standard symmetric bilinear pairing $(\ |\ )$ on P satisfies

$$(\alpha_i \mid \Lambda) = \mathsf{d}_i \langle h_i, \Lambda \rangle \quad \text{for all} \quad \Lambda \in \mathsf{P}. \tag{1.1}$$

[1]If $l = 1$ then it becomes the affine type $A_1^{(1)}$, which was already studied in [1].

We set $\varpi_0 := 0$, and we define, for $i \in I \setminus \{0\}$,

$$\varpi_i := \alpha_1 + 2\alpha_2 + \cdots + (i-1)\alpha_{i-1} + i\left(\alpha_i + \alpha_{i+1} + \cdots + \alpha_{l-1} + \frac{1}{2}\alpha_l\right). \tag{1.2}$$

Note that if $i \neq 0$ then

$$\varpi_i(h_j) = \begin{cases} -1 & \text{if} \quad j = 0, \\ 1 & \text{if} \quad j = i, \\ 0 & \text{otherwise,} \end{cases}$$

and they form a basis for $\sum_{i \in I \setminus \{0\}} \mathbb{Q}\alpha_i$.

Let $\mathfrak{g}$ be the affine Kac–Moody algebra associated with the Cartan datum $(\mathsf{A}, \mathsf{P}, \Pi, \Pi^\vee)$ and let $U_q(\mathfrak{g})$ be its quantum group. The quantum group $U_q(\mathfrak{g})$ is a $\mathbb{C}(q)$-algebra generated by f_i, e_i $(i \in I)$ and q^h $(h \in \mathsf{P})$ with certain relations (see [12, Chapter 3]) for details). Let $\mathbb{A} = \mathbb{Z}[q, q^{-1}]$. We denote by $U_{\mathbb{A}}^-(\mathfrak{g})$ the subalgebra of $U_q(\mathfrak{g})$ generated by $f_i^{(n)} := f_i^n/[n]_i!$ for $i \in I$ and $n \in \mathbb{Z}_{\geq 0}$, where $q_i = q^{\mathsf{d}_i}$ and

$$[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i! = \prod_{k=1}^{n} [k]_i.$$

For a dominant integral weight $\Lambda \in \mathsf{P}^+$, let $V(\Lambda)$ be the irreducible highest weight $U_q(\mathfrak{g})$-module with highest weight Λ and $V_{\mathbb{A}}(\Lambda)$ the $U_{\mathbb{A}}^-(\mathfrak{g})$-submodule of $V(\Lambda)$ generated by the highest weight vector. As is usual, we denote by $B(\Lambda)$ the crystal associated with $V(\Lambda)$. We use standard notation $(\mathrm{wt}, \tilde{f}_i, \tilde{e}_i, \varepsilon_i, \varphi_i)$ $(i \in I)$ for crystal structure (see [12, Chapter 3] for details).

The Fock space representation for $U_q(C_l^{(1)})$ was constructed in [16] by folding the Fock space representation for $U_q(A_{2l-1}^{(1)})$ via the Dynkin diagram automorphism. Later, the combinatorial description for the Fock space and its crystal base were developed in [20, 24]. Let us recall the combinatorial realization for the Fock space.

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0)$ be a Young diagram of size $|\lambda| := \sum_{i=1}^{l} \lambda_i$. When $|\lambda| = n$, we write $\lambda \vdash n$. We consider the residue pattern

$$0, 1, 2, \ldots, l-1, l, l-1, \ldots, 2, 1. \tag{1.3}$$

We repeat the residue pattern in the first row, and shift it to the right by one in the next row. If b is a node of residue i at the (p, q)-position, b is called an i-node and $\mathrm{res}(p, q) = i$. For example, when $l = 4$ and $\lambda = (12, 10, 4, 2)$, we have $\mathrm{res}(2, 5) = 3$ and the residues are given as follows:

0	1	2	3	4	3	2	1	0	1	2	3
1	0	1	2	3	4	3	2	1	0		
2	1	0	1								
3	2										

Let $\mathsf{ST}(\lambda)$ be the set of all standard tableaux of shape $\lambda \vdash n$. For $T \in \mathsf{ST}(\lambda)$, we define the *residue sequence* of T by

$$\mathrm{res}(T) = (\mathrm{res}_1(T), \mathrm{res}_2(T), \ldots, \mathrm{res}_n(T)),$$

where $\mathrm{res}_k(T)$ is the residue of the node of entry k in T, for $1 \le k \le n$.

Let λ be a Young diagram. By an *addable* (resp. *removable*) node b of λ, we mean a node which can be added to (resp. removed from) λ to obtain another Young diagram $\lambda \swarrow b$ (resp. $\lambda \nearrow b$). For an addable or removable node b with $\mathrm{res}(b) = i$, we set

$$\begin{aligned} d_b(\lambda) &:= \mathsf{d}_i(\#\{\text{addable } i\text{-nodes of strictly below } b\} \\ &\qquad - \#\{\text{removable } i\text{-nodes of strictly below } b\}), \\ d^b(\lambda) &:= \mathsf{d}_i(\#\{\text{addable } i\text{-nodes of strictly above } b\} \\ &\qquad - \#\{\text{removable } i\text{-nodes of strictly above } b\}), \\ d_i(\lambda) &:= \#\{\text{addable } i\text{-nodes of } \lambda\} - \#\{\text{removable } i\text{-nodes of } \lambda\}, \end{aligned}$$

where d_i is given in (1.1). Let $\mathcal{F}$ be the $\mathbb{Q}(q)$-vector space generated by all Young diagrams, which is the *Fock space* concerned in this paper. For a Young diagram $\lambda \in \mathcal{F}$, we define

$$e_i\lambda = \sum_b q^{d_b(\lambda)} \lambda \nearrow b, \quad f_i\lambda = \sum_b q^{-d^b(\lambda)} \lambda \swarrow b, \tag{1.4}$$

where b runs over all removable i-nodes and all addable i-nodes respectively. Then, the actions e_i and f_i give a $U_q(\mathfrak{g})$-module structure on $\mathcal{F}$, and we have $q^{h_i}\lambda = q^{d_i(\lambda)}\lambda$, for $i \in I$.

We identify the crystal basis of the Fock space with the set of all Young diagrams. Its crystal structure can be described by considering the usual *i-signature*. Let λ be a Young diagram, and consider all addable or removable i-nodes $b_1, b_2, \ldots, b_m$ of λ from top to bottom. To each b_k of λ, we assign its signature s_k as $+$ (resp. $-$) if it is addable (resp. removable). We cancel out all possible $(-, +)$ pairs in the i-signature $(s_1, \ldots, s_m)$ so that a sequence of $+$'s is followed by $-$'s. We define $\tilde{f}_i\lambda$ to be a Young diagram obtained from λ by adding a node to the addable node corresponding to the right-most $+$ in the i-signature. Similarly, $\tilde{e}_i\lambda$ is defined to be a Young diagram obtained from λ by removing the removable node corresponding to the left-most $-$ in the i-signature. Then, the Young diagrams form a $U_q(\mathfrak{g})$-crystal.

We remark that the above description is obtained from the description in [24, Theorem 3.1] by flipping Young diagrams diagonally. This description matches with the description of the affine type A Fock space for a upper crystal base given in [5, Section 3.6].

2. Quiver Hecke algebras

Let $\mathbf{k}$ be an algebraically closed field and $(\mathsf{A}, \mathsf{P}, \Pi, \Pi^\vee)$ the affine Cartan datum in Section 1. We set polynomials $\mathcal{Q}_{i,j}(u, v) \in \mathbf{k}[u, v]$, for $i, j \in I$, of the form

$$\mathcal{Q}_{i,j}(u, v) = \begin{cases} \displaystyle\sum_{p(\alpha_i|\alpha_i)+q(\alpha_j|\alpha_j)+2(\alpha_i|\alpha_j)=0} t_{i,j;p,q} u^p v^q & \text{if} \quad i \neq j, \\ 0 & \text{if} \quad i = j, \end{cases}$$

where $t_{i,j;p,q} \in \mathbf{k}$ are such that $t_{i,j;-a_{ij},0} \neq 0$ and $\mathcal{Q}_{i,j}(u, v) = \mathcal{Q}_{j,i}(v, u)$. The symmetric group $\mathfrak{S}_n = \langle s_k \mid k = 1, \dots, n-1\rangle$ acts on I^n by place permutations.

DEFINITION 2.1. The *quiver Hecke algebra* $R(n)$ associated with polynomials $(\mathcal{Q}_{i,j}(u, v))_{i,j\in I}$ is the $\mathbb{Z}$-graded $\mathbf{k}$-algebra defined by three sets of generators

$$\{e(\nu) \mid \nu = (\nu_1, \dots, \nu_n) \in I^n\}, \quad \{x_k \mid 1 \le k \le n\}, \quad \{\psi_l \mid 1 \le l \le n-1\}$$

subject to the following relations:

$$e(\nu)e(\nu') = \delta_{\nu,\nu'}e(\nu), \quad \sum_{\nu\in I^n} e(\nu) = 1, \quad x_k e(\nu) = e(\nu)x_k, \quad x_k x_l = x_l x_k,$$

$$\psi_l e(\nu) = e(s_l(\nu))\psi_l, \quad \psi_k\psi_l = \psi_l\psi_k \quad \text{if} \quad |k-l| > 1,$$

$$\psi_k^2 e(\nu) = \mathcal{Q}_{\nu_k,\nu_{k+1}}(x_k, x_{k+1})e(\nu),$$

$$(\psi_k x_l - x_{s_k(l)}\psi_k)e(\nu) = \begin{cases} -e(\nu) & \text{if} \quad l = k \quad \text{and} \quad \nu_k = \nu_{k+1}, \\ e(\nu) & \text{if} \quad l = k+1 \quad \text{and} \quad \nu_k = \nu_{k+1}, \\ 0 & \text{otherwise}, \end{cases}$$

$$(\psi_{k+1}\psi_k\psi_{k+1} - \psi_k\psi_{k+1}\psi_k)e(\nu)$$
$$= \begin{cases} \dfrac{\mathcal{Q}_{\nu_k,\nu_{k+1}}(x_k, x_{k+1}) - \mathcal{Q}_{\nu_k,\nu_{k+1}}(x_{k+2}, x_{k+1})}{x_k - x_{k+2}} e(\nu) & \text{if} \quad \nu_k = \nu_{k+2}, \\ 0 & \text{otherwise}. \end{cases}$$

Using the isomorphism given in [26, p. 25] (cf. [1, Lemma 3.2]), we may assume that, for $i < j$,

$$\mathcal{Q}_{i,j}(u, v) = \begin{cases} u - v^2 & \text{if} \quad i = 0, \ j = 1, \\ u - v & \text{if} \quad j = i+1, \ i \neq 0, \ j \neq l, \\ u^2 - v & \text{if} \quad i = l-1, \ j = l, \\ 1 & \text{otherwise}. \end{cases}$$

$R(n)$ is a graded algebra by the $\mathbb{Z}$-grading given as follows:

$$\deg(e(\nu)) = 0, \quad \deg(x_k e(\nu)) = (\alpha_{\nu_k} \mid \alpha_{\nu_k}), \quad \deg(\psi_l e(\nu)) = -(\alpha_{\nu_l} \mid \alpha_{\nu_{l+1}}).$$

For an $R(m)$-module M and an $R(n)$-module N, we define an $R(m+n)$-module $M \circ N$ by

$$M \circ N = R(m+n) \otimes_{R(m)\otimes R(n)} (M \otimes N).$$

For a dominant integral weight $\Lambda \in \mathsf{P}^+$, let $R^\Lambda(n)$ be the quotient algebra of $R(n)$ by the ideal generated by the elements $\{x_1^{\langle h_{\nu_1}, \Lambda\rangle} e(\nu) \mid \nu \in I^n\}$, which is called the *cyclotomic quiver Hecke algebra*.

For $\beta \in \mathsf{Q}^+$ with $|\beta| = n$, we set $I^\beta = \left\{\nu = (\nu_1, \dots, \nu_n) \in I^n \;\middle|\; \sum_{k=1}^n \alpha_{\nu_k} = \beta\right\}$ and define

$$R^\Lambda(\beta) := R^\Lambda(n)e(\beta),$$

where $e(\beta) = \sum_{\nu \in I^\beta} e(\nu)$. We are interested in cyclotomic quiver Hecke algebras $R^{\Lambda_0}(\beta)$, which we call *finite quiver Hecke algebras of type* $C_l^{(1)}$. Let us recall some results which are valid for general $R^\Lambda(\beta)$.

Proposition 2.2 (cf. [2, Corollary 4.8]). *For $w \in \mathsf{W}$, $R^\Lambda(\beta)$ and $R^\Lambda(\Lambda - w\Lambda + w\beta)$ have the same number of simple modules and the same representation type.*

We denote the direct sum of the split Grothendieck groups of the categories $R^\Lambda(\beta)$-proj of finitely generated projective graded $R^\Lambda(\beta)$-modules by

$$K_0(R^\Lambda) = \bigoplus_{\beta \in \mathsf{Q}^+} K_0(R^\Lambda(\beta)\text{-proj}).$$

Note that $K_0(R^\Lambda)$ has a free $\mathbb{A}$-module structure induced from the $\mathbb{Z}$-grading on $R^\Lambda(\beta)$, i.e. $(qM)_k = M_{k-1}$ for a graded module $M = \bigoplus_{k\in\mathbb{Z}} M_k$. Let $e(\nu, i)$ be the idempotent corresponding to the concatenation of ν and (i), and set $e(\beta, i) = \sum_{\nu \in I^\beta} e(\nu, i)$ for $\beta \in \mathsf{Q}^+$. Then we define the induction functor $F_i\colon R^\Lambda(\beta)\text{-mod} \to R^\Lambda(\beta+\alpha_i)\text{-mod}$ and the restriction functor $E_i\colon R^\Lambda(\beta+\alpha_i)\text{-mod} \to R^\Lambda(\beta)\text{-mod}$ by

$$F_i(M) = R^\Lambda(\beta+\alpha_i)e(\beta,i) \otimes_{R^\Lambda(\beta)} M, \quad E_i(N) = e(\beta,i)N,$$

for an $R^\Lambda(\beta)$-module M and an $R^\Lambda(\beta+\alpha_i)$-module N.

Theorem 2.3 ([14, Theorem 5.2]). *Let $l_i = \langle h_i, \Lambda - \beta\rangle$, for $i \in I$. Then one of the following isomorphisms of endofunctors on $R^\Lambda(\beta)$-mod holds.*

(1) *If $l_i \geq 0$, then*

$$q_i^{-2} F_i E_i \oplus \bigoplus_{k=0}^{l_i - 1} q_i^{2k}\mathrm{id} \xrightarrow{\sim} E_i F_i.$$

(2) *If* $l_i \leq 0$, *then*

$$q_i^{-2} F_i E_i \xrightarrow{\sim} E_i F_i \oplus \bigoplus_{k=0}^{-l_i-1} q_i^{-2k-2}\mathrm{id}.$$

For the biadjointness of the functors, see [17]. If $i \neq j$, then $q^{-(\alpha_i|\alpha_j)} F_j E_i \xrightarrow{\sim} E_i F_j$ holds. Moreover, the functors $q_i^{1-\langle h_i, \Lambda-\beta\rangle} E_i$ and F_i make $K_0(R^\Lambda)$ into a $U_{\mathbb{A}}(\mathfrak{g})$-module and the next theorem shows that the module is $V_{\mathbb{A}}(\Lambda)$. For the latter half of the theorem, see also [22].

Theorem 2.4 ([14, Theorem 6.2]). *There exists a $U_{\mathbb{A}}(\mathfrak{g})$-module isomorphism between $K_0(R^\Lambda)$ and $V_{\mathbb{A}}(\Lambda)$. In particular, the number of isoclasses of irreducible $R^\Lambda(\beta)$-modules is equal to the size of $B(\Lambda)_{\Lambda-\beta}$, the weight $\Lambda - \beta$ part of the highest weight crystal $B(\Lambda)$.*

For a graded module $M = \bigoplus_{k\in\mathbb{Z}} M_k$, the *graded dimension* of M is defined by

$$\dim_q M = \sum_{k\in\mathbb{Z}} \dim(M_k) q^k.$$

Note that $\dim_q(q^t M) = q^t \dim_q M$. For an $R^\Lambda(\beta)$-module M, the *q-character* $\mathrm{ch}_q(M)$ and *character* $\mathrm{ch}(M)$ of M are defined by

$$\mathrm{ch}_q(M) = \sum_{\nu\in I^\beta} \dim_q(e(\nu)M)\nu, \quad \mathrm{ch}(M) = \sum_{\nu\in I^\beta} \dim(e(\nu)M)\nu.$$

For $\Lambda \in \mathsf{P}^+$ and $\beta \in \mathsf{Q}^+$, set

$$\mathrm{def}(\Lambda, \beta) = (\beta \mid \Lambda) - \frac{1}{2}(\beta \mid \beta).$$

Using $(\alpha_i \mid \alpha_i) = 2\mathsf{d}_i$, it is easy to check

$$\mathrm{def}(\Lambda, \beta - \alpha_i) + (\Lambda - \beta \mid \alpha_i) = \mathrm{def}(\Lambda, \beta) - \mathsf{d}_i.$$

Proposition 2.5 ([23, Proposition 3.3]). *Let $\nu = (\nu_1, \ldots, \nu_n)$, $\nu' = (\nu'_1, \ldots, \nu'_n) \in I^\beta$, and let v_Λ be the highest weight vector of the highest weight $U_q(\mathfrak{g})$-module $V(\Lambda)$. Then, we have*

$$e_{\nu_1} \cdots e_{\nu_n} f_{\nu'_n} \cdots f_{\nu'_1} v_\Lambda = q^{-\mathrm{def}(\Lambda,\beta)} (\dim_q e(\nu) R^\Lambda(\beta) e(\nu')) v_\Lambda.$$

We now consider the q-dimension $\dim_q R^{\Lambda_0}(\beta)$. Let $\lambda \vdash n$ and T be a standard tableau of shape λ. For $1 \leq k \leq n$, let $T_{<k}$ be a standard tableau obtained from T by

removing the nodes whose entries are greater than or equal to k. We define inductively

$$\deg(T) := \deg(T_{<n}) + d_b(\lambda), \quad \operatorname{codeg}(T) := \operatorname{codeg}(T_{<n}) + d^b(\lambda \nearrow b),$$

where b is the node of T containing entry n. We set $\deg(\emptyset) = \operatorname{codeg}(\emptyset) = 0$. Observe that if b is a removable i-node, then

$$d_b(\lambda) + d^b(\lambda \nearrow b) = \mathsf{d}_i d_i(\lambda) + \mathsf{d}_i.$$

One can prove the following identity by the same induction argument as [6, Lemma 3.12]:

$$\deg(T) + \operatorname{codeg}(T) = \operatorname{def}(\Lambda_0, \beta). \tag{2.1}$$

For $\nu \in I^n$, let

$$K_q(\lambda, \nu) := \sum_{T \in \mathsf{ST}(\lambda),\ \operatorname{res}(T)=\nu} q^{\deg(T)}, \quad K_q(\lambda) := \sum_{T \in \mathsf{ST}(\lambda)} q^{\deg(T)}.$$

Theorem 2.6. *For $\nu, \nu' \in I^\beta$, we have*

$$\dim_q e(\nu) R^{\Lambda_0}(\beta) e(\nu') = \sum_{\lambda \vdash n,\ \operatorname{wt}(\lambda)=\Lambda_0-\beta} K_q(\lambda, \nu) K_q(\lambda, \nu'),$$

$$\dim_q R^{\Lambda_0}(\beta) = \sum_{\lambda \vdash n,\ \operatorname{wt}(\lambda)=\Lambda_0-\beta} K_q(\lambda)^2,$$

$$\dim_q R^{\Lambda_0}(n) = \sum_{\lambda \vdash n} K_q(\lambda)^2.$$

Proof. Let $\nu = (\nu_1, \ldots, \nu_n)$ and $\nu' = (\nu'_1, \ldots, \nu'_n) \in I^\beta$. It follows from (1.4) and (2.1) that

$$\begin{aligned} &q^{\operatorname{def}(\Lambda_0,\beta)} e_{\nu_1} \cdots e_{\nu_n} f_{\nu'_n} \cdots f_{\nu'_1} \emptyset \\ &= q^{\operatorname{def}(\Lambda_0,\beta)} \sum_{\lambda \vdash n,\ \operatorname{wt}(\lambda)=\Lambda_0-\beta} \Bigg(\sum_{\substack{T \in \mathsf{ST}(\lambda),\\ \operatorname{res}(T)=\nu}} q^{\deg(T)} \Bigg) \Bigg(\sum_{\substack{T \in \mathsf{ST}(\lambda),\\ \operatorname{res}(T)=\nu'}} q^{-\operatorname{codeg}(T)} \Bigg) \emptyset \\ &= \sum_{\lambda \vdash n,\ \operatorname{wt}(\lambda)=\Lambda_0-\beta} K_q(\lambda, \nu) K_q(\lambda, \nu') \emptyset, \end{aligned}$$

which gives the first assertion by Proposition 2.5.

The remaining assertions follow from $R^{\Lambda_0}(\beta) = \bigoplus_{\nu,\nu' \in I^\beta} e(\nu) R^{\Lambda_0}(\beta) e(\nu')$ and $R^{\Lambda_0}(n) = \bigoplus_{|\beta|=n} R^{\Lambda_0}(\beta)$. □

The corollary below follows from Theorem 2.6 immediately.

Corollary 2.7. (1) *Let $\nu \in I^n$. Then, $e(\nu) \neq 0$ in $R^{\Lambda_0}(n)$ if and only if ν may be obtained from a standard tableau T as $\nu = \mathrm{res}(T)$.*
(2) *For a natural number n, we have* $\dim R^{\Lambda_0}(n) = n!$.

3. Representations of $R^{\Lambda_0}(\delta)$

In [21, Section 8.1], irreducible $R^{\Lambda_0}(\delta)$-modules for several non-simply laced affine types were constructed. Let us recall the construction for type $C_l^{(1)}$.

Let z be an indeterminate. For $k = 0,1,2,3$ and $1 \le i \le l$, except for $(k,i) = (2,1)$, let $L_{i,k}^z$ be the graded free 1-dimensional $\mathbf{k}[z]$-module with generator v_k, and set

$$\nu^{(k)} = \begin{cases} (0) & \text{if} \quad k = 0, \\ (1, 2, \dots, l-1, l, l-1, \dots, i+1) & \text{if} \quad k = 1,\ 1 \le i < l, \\ (1, 2, \dots, l-1) & \text{if} \quad k = 1,\ i = l, \\ (1, 2, \dots, i-1) & \text{if} \quad k = 2,\ 2 \le i \le l, \\ (i) & \text{if} \quad k = 3. \end{cases}$$

We set $\beta^{(k)} = \alpha_{\nu_1} + \cdots + \alpha_{\nu_t}$, where $\nu^{(k)} = (\nu_1, \nu_2, \dots, \nu_t)$. Define an $R^{\Lambda_0}(\beta^{(k)})$-module structure on $L_{i,k}^z$ by $e(\nu)v_k = \delta_{\nu,\nu^{(k)}} v_k$, $\psi_r v_k = 0$ and

$$x_s v_k = \begin{cases} z v_k & \text{if} \quad k = 1,\ s < l, \\ -z v_k & \text{if} \quad (k = 1,\ s > l) \text{ or } (k = 2) \text{ or } (k = 3,\ i < l), \\ z^2 v_k & \text{if} \quad (k = 0) \text{ or } (k = 1,\ s = l) \text{ or } (k = 3,\ i = l). \end{cases}$$

We set

$$L_i^z = \begin{cases} L_{i,0}^z \boxtimes L_{i,1}^z \boxtimes L_{i,3}^z & \text{if} \quad i = 1, \\ L_{i,0}^z \boxtimes L_{i,1}^z \circ L_{i,2}^z \boxtimes L_{i,3}^z & \text{if} \quad i > 1, \end{cases} \tag{3.1}$$

and declare that ψ_1 and ψ_{2l-1} act as 0 on L_i^z.

Proposition 3.1 ([21, Propositions 3.9.2, 8.1.3 and 8.1.6]). (1) *For $i = 1, \dots, l$, L_i^z is a $\mathbf{k}[z] \otimes R(\delta)$-module.*
(2) *The quotient $\mathcal{S}_i := L_i^z / z L_i^z$ is an irreducible $R^{\Lambda_0}(\delta)$-module.*
(3) *$\{\mathcal{S}_1, \mathcal{S}_2 \dots, \mathcal{S}_l\}$ is a complete list of irreducible $R^{\Lambda_0}(\delta)$-modules.*

Lemma 3.2. *If M is an irreducible $R(\beta)$-module with $\varepsilon_i(M) = 1$ then $E_i M$ is an irreducible $R(\beta - \alpha_i)$-module.*

Proof. It immediately follows from [18, Lemma 3.8]. □

By the definition of $\mathcal{S}_i$, we may enumerate basis elements of $L^z_{i,1} \circ L^z_{i,2}$ and we have the following description of the characters for $\mathcal{S}_i$.

$$\operatorname{ch} \mathcal{S}_i = \sum_{T \in \mathsf{ST}(\lambda^{(i)})} \operatorname{res}(T) * (i), \tag{3.2}$$

where $\lambda^{(i)} = (i, 1^{2l-1-i})$ and $\operatorname{res}(T) * (i)$ is the concatenation of $\operatorname{res}(T)$ and (i). Thus, we have $\varepsilon_j(\mathcal{S}_i) = \delta_{i,j}$, and Lemma 3.2 implies that

$$\mathcal{L}_i := E_i \mathcal{S}_i$$

is an irreducible $R^{\Lambda_0}(\delta - \alpha_i)$-module, for $i = 1, \dots, l$. Using (3.2) again, if $i \neq l$ then

$$\varepsilon_j(\mathcal{L}_i) = \begin{cases} 1 & \text{if} \quad j = i+1, i-1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

Thus, $E_{i\pm1}\mathcal{L}_i$ is an irreducible $R^{\Lambda_0}(\delta - \alpha_i - \alpha_{i\pm1})$-module, for $1 \le i \le l-1$, by Lemma 3.2.

Lemma 3.3. (1) *$R^{\Lambda_0}(\alpha_0 + \alpha_1)$ is isomorphic to $\mathbf{k}[x]/(x^2)$.*
(2) *For $1 \le i \le l-1$, $R^{\Lambda_0}(\delta - \alpha_i)$ is isomorphic to a matrix ring over $\mathbf{k}[x]/(x^2)$, and $\mathcal{L}_i$ is the unique irreducible $R^{\Lambda_0}(\delta - \alpha_i)$-module.*
(3) *For $1 \le i \le l-1$, $R^{\Lambda_0}(\delta - \alpha_i - \alpha_{i+1})$ is isomorphic to a matrix ring over $\mathbf{k}[x]/(x^2)$, and $E_{i+1}\mathcal{L}_i \simeq E_i\mathcal{L}_{i+1}$ is the unique irreducible $R^{\Lambda_0}(\delta - \alpha_i - \alpha_{i+1})$-module if $1 \le i \le l-2$, and $E_l\mathcal{L}_{l-1}$ is the unique irreducible $R^{\Lambda_0}(\delta - \alpha_{l-1} - \alpha_l)$-module.*
(4) *$R^{\Lambda_0}(\delta - \alpha_l)$ is a simple algebra and $\mathcal{L}_l$ is the unique irreducible $R^{\Lambda_0}(\delta - \alpha_l)$-module.*

Proof. The assertion (1) follows from Theorem 2.6. Indeed, $\dim_q R^{\Lambda_0}(2) = 1 + q^2$ implies that there is a homogeneous element $x \neq 0$ of degree 2 such that $x^2 = 0$. One can verify the following formulas, for $p = I \setminus \{0, l-1, l\}$ and $t = I \setminus \{0, l\}$, by direct computation.

$$\Lambda_0 - \delta + \alpha_p + \alpha_{p+1} = (r_{p-1}r_{p-2}\cdots r_1)(r_{p+2}\cdots r_{l-1}r_l r_{l-1}\cdots r_3 r_2)(\Lambda_0 - \alpha_0 - \alpha_1),$$
$$\Lambda_0 - \delta + \alpha_{l-1} + \alpha_l = (r_{l-2}r_{l-3}\cdots r_1)(r_{l-1}\cdots r_3 r_2)(\Lambda_0 - \alpha_0 - \alpha_1),$$
$$\Lambda_0 - \delta + \alpha_t = (r_{t-1}r_{t-2}\cdots r_1)(r_{t+1}\cdots r_{l-1}r_l r_{l-1}\cdots r_3 r_2)(\Lambda_0 - \alpha_0 - \alpha_1),$$
$$\Lambda_0 - \delta + \alpha_l = r_{l-1}\cdots r_2 r_1(\Lambda_0 - \alpha_0).$$

By [7, Theorem 6.4] (cf. [2, Theorem 4.5]), $R^{\Lambda_0}(\delta - \alpha_i)$ and $R^{\Lambda_0}(\delta - \alpha_i - \alpha_{i\pm1})$ are derived equivalent to $R^{\Lambda_0}(\alpha_0 + \alpha_1)$. Since $\mathbf{k}[x]/(x^2)$ is the unique Brauer tree algebra with one edge and no exceptional vertex, both $R^{\Lambda_0}(\delta - \alpha_i)$ and $R^{\Lambda_0}(\delta - \alpha_i - \alpha_{i\pm1})$ are Morita equivalent to $\mathbf{k}[x]/(x^2)$ by [25, Theorem 4.2]. In particular, they have a unique irreducible module. As we already know that $\mathcal{L}_i$ is an irreducible $R^{\Lambda_0}(\delta-\alpha_i)$-module, (2) follows. We also know that $E_{i+1}\mathcal{L}_i$, for $1\le i\le l-1$, and $E_i\mathcal{L}_{i+1}$, for $1\le i\le l-2$, are irreducible $R^{\Lambda_0}(\delta-\alpha_i-\alpha_{i+1})$-modules. Thus (3) follows. Finally, Proposition 2.2 tells that $R^{\Lambda_0}(\delta-\alpha_l)$ is a simple algebra, and we already know that $\mathcal{L}_l$ is an irreducible $R^{\Lambda_0}(\delta-\alpha_l)$-module, which proves (4). □

By Lemma 3.3 (4), $\mathcal{L}_l$ is a projective module. For $i\ne l$, we denote the projective cover of $\mathcal{L}_i$ by $\hat{\mathcal{L}}_i$. Then, we have a non-split exact sequence

$$0\to\mathcal{L}_i\to\hat{\mathcal{L}}_i\to\mathcal{L}_i\to 0. \tag{3.4}$$

We get indecomposable projective $R^{\Lambda_0}(\delta-\alpha_i)$-modules $\mathcal{M}_i$, for $1\le i\le l$, defined by

$$\mathcal{M}_i:=\begin{cases}\hat{\mathcal{L}}_i & \text{if} \quad i\ne l,\\ \mathcal{L}_l & \text{if} \quad i=l.\end{cases}$$

Lemma 3.4. *We have $E_j\mathcal{M}_i=0$ unless $j=i\pm 1$. If $j=i\pm 1$ then $E_i\mathcal{M}_j\simeq E_j\mathcal{M}_i$ is the unique indecomposable projective $R^{\Lambda_0}(\delta-\alpha_i-\alpha_j)$-module.*

Proof. If $j\ne i\pm 1$, then $E_j\mathcal{M}_i=0$ follows from (3.3). Computation of the characters implies $[E_{l-1}\mathcal{L}_l]=2[E_l\mathcal{L}_{l-1}]$, which is equal to $[E_l\hat{\mathcal{L}}_{l-1}]$. Since $E_{l-1}\mathcal{M}_l$ and $E_l\mathcal{M}_{l-1}$ are projective modules, $[E_{l-1}\mathcal{M}_l]=[E_l\mathcal{M}_{l-1}]$ implies that they are isomorphic. Suppose that $i\ne l$, $j\ne l$ and $j=i\pm 1$. Then we have the exact sequence

$$0\to E_j\mathcal{L}_i\to E_j\mathcal{M}_i\to E_j\mathcal{L}_i\to 0. \tag{3.5}$$

If $E_j\mathcal{L}_i$ was a projective module, it would contradict Lemma 3.3 (3). Thus, $E_j\mathcal{L}_i$ is not projective and (3.5) does not split. It implies that $E_j\mathcal{M}_i$ is an indecomposable projective $R^{\Lambda_0}(\delta-\alpha_i-\alpha_j)$-module. Interchanging the role of i and j, $E_i\mathcal{M}_j$ is also an indecomposable projective $R^{\Lambda_0}(\delta-\alpha_i-\alpha_j)$-module. As the indecomposable projective $R^{\Lambda_0}(\delta-\alpha_i-\alpha_j)$-module is unique by Lemma 3.3 (3), we conclude that they are isomorphic. □

We now consider the projective $R^{\Lambda_0}(\delta)$-modules $\mathcal{P}_i:=F_i\mathcal{M}_i$, for $1\le i\le l$. By the biadjointness of F_i and E_i and $\varepsilon_j(\mathcal{S}_i)=\delta_{i,j}$, we have

$$\dim\operatorname{Hom}(\mathcal{P}_i,\mathcal{S}_j)=\dim\operatorname{Hom}(\mathcal{M}_i,E_i\mathcal{S}_j)=\delta_{i,j}\dim\operatorname{Hom}(\mathcal{M}_i,\mathcal{L}_i)=\delta_{i,j},$$
$$\dim\operatorname{Hom}(\mathcal{S}_j,\mathcal{P}_i)=\dim\operatorname{Hom}(E_i\mathcal{S}_j,\mathcal{M}_i)=\delta_{i,j}\dim\operatorname{Hom}(\mathcal{L}_i,\mathcal{M}_i)=\delta_{i,j},$$

which tells that $\mathcal{P}_i$ is the projective cover of $\mathcal{S}_i$, for all i, and $R^{\Lambda_0}(\delta)$ is weakly symmetric. In particular, $\mathcal{P}_i$ are self-dual. It follows from Theorem 2.3 and Lemma 3.4

that, if $i \neq j$ then

$$\begin{aligned}\dim \mathrm{Hom}(\mathcal{P}_j, \mathcal{P}_i) &= \dim \mathrm{Hom}(\mathcal{M}_j, E_j F_i \mathcal{M}_i) = \dim \mathrm{Hom}(E_i \mathcal{M}_j, E_j \mathcal{M}_i) \\ &= 2\delta_{j,i\pm 1}.\end{aligned}$$

The similar argument shows that

$$\begin{aligned}\dim \mathrm{Hom}(\mathcal{P}_i, \mathcal{P}_i) &= \dim \mathrm{Hom}(\mathcal{M}_i, E_i F_i \mathcal{M}_i) \\ &= \dim \mathrm{Hom}(\mathcal{M}_i, \mathcal{M}_i^{\bigoplus \langle h_i, \Lambda_0 - \delta + \alpha_i \rangle}) \\ &= \begin{cases} 4 & \text{if} \quad i \neq l, \\ 2 & \text{if} \quad i = l. \end{cases}\end{aligned}$$

Thus, in the Grothendieck group, we have

$$\begin{aligned}&[\mathcal{P}_1] = 4[\mathcal{S}_1] + 2[\mathcal{S}_2], \quad [\mathcal{P}_i] = 2[\mathcal{S}_{i-1}] + 4[\mathcal{S}_i] + 2[\mathcal{S}_{i+1}], \\ &[\mathcal{P}_l] = 2[\mathcal{S}_{l-1}] + 2[\mathcal{S}_l],\end{aligned} \tag{3.6}$$

for $i = 2, \ldots, l-1$.

Define $\mathcal{Q}_i := F_i \mathcal{L}_i$, for $i \neq l$. By the same argument as above, we compute

$$\dim \mathrm{Hom}(\mathcal{Q}_i, \mathcal{S}_j) = \dim \mathrm{Hom}(\mathcal{S}_j, \mathcal{Q}_i) = \delta_{i,j}.$$

Applying the functor F_i to (3.4), and noting that $\mathcal{P}_i$ is indecomposable, we have the following non-split exact sequence, for $i = 1, \ldots, l-1$.

$$0 \to \mathcal{Q}_i \to \mathcal{P}_i \to \mathcal{Q}_i \to 0. \tag{3.7}$$

Since $\mathcal{P}_i$ is self-dual, and $\mathrm{Soc}(\mathcal{Q}_i) \simeq \mathcal{S}_i \simeq \mathrm{Top}(\mathcal{Q}_i)$, we conclude that

$$\mathcal{Q}_1 \simeq \begin{matrix} \mathcal{S}_1 \\ \mathcal{S}_2 \\ \mathcal{S}_1 \end{matrix}, \quad \mathcal{Q}_i \simeq \begin{matrix} \mathcal{S}_i \\ \mathcal{S}_{i-1} \oplus \mathcal{S}_{i+1} \\ \mathcal{S}_i \end{matrix} \quad (2 \leq i \leq l-1). \tag{3.8}$$

The radical series for $\mathcal{Q}_1$ is clear. Suppose that $\mathcal{Q}_i$, for some $2 \leq i \leq l-1$ is uniserial. If $\mathrm{Rad}(\mathcal{Q}_i)/\mathrm{Rad}^2(\mathcal{Q}_i) \simeq \mathcal{S}_{i\pm 1}$ then $\mathcal{S}_{i\pm 1}$ appears in $\mathrm{Rad}(\mathcal{P}_i)/\mathrm{Rad}^2(\mathcal{P}_i)$ and $\mathcal{S}_{i\mp 1}$ appears in $\mathrm{Soc}^2(\mathcal{P}_i)/\mathrm{Soc}(\mathcal{P}_i)$, which implies that $\mathcal{S}_{i\pm 1} \oplus \mathcal{S}_{i\mp 1}$ appears in $\mathrm{Rad}(\mathcal{P}_i)/\mathrm{Rad}^2(\mathcal{P}_i)$. On the other hand, either $2[\mathcal{S}_{i\pm 1}]$ or $2[\mathcal{S}_{i\mp 1}]$ all appear in $\mathrm{Rad}^2(\mathcal{P}_i)$. They contradict and we conclude that $\mathcal{Q}_i$ is not uniserial. We have the desired shape of the radical series for $\mathcal{Q}_i$.

Proposition 3.5. *The radical series of $\mathcal{P}_i$, for $1 \le i \le l$, are given as follows.*

$$\mathcal{P}_1 \simeq \begin{matrix} \mathcal{S}_1 \\ \mathcal{S}_1 \oplus \mathcal{S}_2 \\ \mathcal{S}_2 \oplus \mathcal{S}_1 \\ \mathcal{S}_1 \end{matrix}, \quad \mathcal{P}_i \simeq \begin{matrix} \mathcal{S}_i \\ \mathcal{S}_i \oplus \mathcal{S}_{i-1} \oplus \mathcal{S}_{i+1} \\ \mathcal{S}_{i+1} \oplus \mathcal{S}_{i-1} \oplus \mathcal{S}_i \\ \mathcal{S}_i \end{matrix} \quad (i \ne 1, l), \quad \mathcal{P}_l \simeq \begin{matrix} \mathcal{S}_l \\ \mathcal{S}_{l-1} \\ \mathcal{S}_{l-1} \\ \mathcal{S}_l \end{matrix}.$$

Proof. We set $\hat{\mathcal{S}}_1 := L_1^z/z^2 L_1^z$, where L_i^z is given in (3.1). By definition, x_1 acts as zero, and $\hat{\mathcal{S}}_1$ is an $R^{\Lambda_0}(\delta)$-module. On the other hand, x_2 acts as nonzero on $\hat{\mathcal{S}}_1$ by $l \ge 2$. It implies that $\hat{\mathcal{S}}_1$ is indecomposable and we have the radical series

$$\hat{\mathcal{S}}_1 \simeq \begin{matrix} \mathcal{S}_1 \\ \mathcal{S}_1 \end{matrix}.$$

Thus, $\mathrm{Rad}(\mathcal{P}_1)/\mathrm{Rad}^2(\mathcal{P}_1)$ has $\mathcal{S}_1$ as a direct summand. It follows from (3.7) and (3.8) that $\mathcal{P}_1$ has the radical series as follows.

$$\mathcal{P}_1 \simeq \begin{matrix} \mathcal{S}_1 \\ \mathcal{S}_1 \oplus \mathcal{S}_2 \\ \mathcal{S}_2 \oplus \mathcal{S}_1 \\ \mathcal{S}_1 \end{matrix}.$$

Let $\phi\colon \mathcal{P}_2 \to \mathcal{P}_1$ be a lift of the map $\mathcal{P}_2 \twoheadrightarrow \mathcal{S}_2 \hookrightarrow \mathrm{Rad}(\mathcal{P}_1)/\mathrm{Rad}^2(\mathcal{P}_1)$. From the shape of the radical series of $\mathcal{P}_1$, we know that $\mathrm{Rad}^2(\mathrm{Im}\,\phi) \simeq \mathcal{S}_1$. It implies that $\mathcal{S}_1$ appears in $\mathrm{Rad}^2(\mathcal{P}_2)/\mathrm{Rad}^3(\mathcal{P}_2)$. Under the projection $p_2\colon \mathcal{P}_2 \to \mathcal{Q}_2$, this $\mathcal{S}_1$ maps to zero. Namely, it appears in $\mathrm{Ker}(p_2) \simeq \mathcal{Q}_2$. Multiplying $\mathrm{Rad}(R^{\Lambda_0}(\delta))$ to this $\mathcal{S}_1$, we know that $\mathrm{Soc}(\mathcal{P}_2) = \mathrm{Rad}^3(\mathcal{P}_2)$. By (3.7) and (3.8), $\mathcal{S}_2$ appears in $\mathrm{Rad}^2(\mathcal{P}_2)/\mathrm{Rad}^3(\mathcal{P}_2)$. It follows that $\mathcal{P}_2$ has a uniserial submodule of length 2 with two $\mathcal{S}_2$ as composition factors. Hence, $\mathcal{S}_2$ appears in $\mathrm{Rad}(\mathcal{P}_2)/\mathrm{Rad}^2(\mathcal{P}_2)$. Then, this $\mathcal{S}_2$ must appear in $\mathrm{Ker}(p_2)$, which implies that $\mathcal{S}_3$ appears in $\mathrm{Rad}^2(\mathcal{P}_2)/\mathrm{Rad}^3(\mathcal{P}_2)$. We conclude that

$$\mathcal{P}_2 \simeq \begin{matrix} \mathcal{S}_2 \\ \mathcal{S}_2 \oplus \mathcal{S}_1 \oplus \mathcal{S}_3 \\ \mathcal{S}_3 \oplus \mathcal{S}_1 \oplus \mathcal{S}_2 \\ \mathcal{S}_2 \end{matrix}.$$

Applying the same argument to a lift of the map $\mathcal{P}_i \twoheadrightarrow \mathcal{S}_i \hookrightarrow \mathrm{Rad}(\mathcal{P}_{i-1})/\mathrm{Rad}^2(\mathcal{P}_{i-1})$, we obtain

$$\mathcal{P}_i \simeq \begin{matrix} \mathcal{S}_i \\ \mathcal{S}_i \oplus \mathcal{S}_{i-1} \oplus \mathcal{S}_{i+1} \\ \mathcal{S}_{i+1} \oplus \mathcal{S}_{i-1} \oplus \mathcal{S}_i \\ \mathcal{S}_i \end{matrix},$$

for $i = 2, \dots, l-1$. We now consider $\mathcal{P}_l$. Since $\mathcal{P}_l$ is self-dual, (3.6) implies that we have

$$\mathcal{P}_l \simeq \begin{matrix} \mathcal{S}_l \\ \mathcal{S}_{l-1} \\ \mathcal{S}_{l-1} \\ \mathcal{S}_l \end{matrix}$$

or

$$\mathcal{P}_l \simeq \begin{matrix} \mathcal{S}_l \\ \mathcal{S}_{l-1} \oplus \mathcal{S}_{l-1}. \\ \mathcal{S}_l \end{matrix}$$

Let $\psi : \mathcal{P}_l \to \mathcal{P}_{l-1}$ be a lift of the map $\mathcal{P}_l \twoheadrightarrow \mathcal{S}_l \hookrightarrow \mathrm{Rad}(\mathcal{P}_{l-1})/\mathrm{Rad}^2(\mathcal{P}_{l-1})$. It follows from the shape of the radical series of $\mathcal{P}_{l-1}$ that $\mathrm{Rad}^2(\mathrm{Im}\,\psi) \simeq \mathcal{S}_{l-1}$, which implies that $\mathcal{S}_{l-1}$ appears in $\mathrm{Rad}^2(\mathcal{P}_l)/\mathrm{Rad}^3(\mathcal{P}_l)$. Therefore, we have

$$\mathcal{P}_l \simeq \begin{matrix} \mathcal{S}_l \\ \mathcal{S}_{l-1} \\ \mathcal{S}_{l-1} \\ \mathcal{S}_l \end{matrix},$$

which completes the proof. □

Lemma 3.6. *If $l = 2$, then there is an isomorphism of algebras*

$$e(0121)R^{\Lambda_0}(4)e(0121) \simeq \mathbf{k}[x, y]/(x^2, y^2 - axy),$$

for some $a \in \mathbf{k}$.

Proof. We have $\delta = \alpha_0 + 2\alpha_1 + \alpha_2$, for $l = 2$. Theorem 2.6 gives

$$\dim e(012)R^{\Lambda_0}(\delta - \alpha_1)e(012) = \dim R^{\Lambda_0}(\delta - \alpha_1) = 2,$$
$$\dim e(0121)R^{\Lambda_0}(\delta)e(0121) = 4.$$

Since $\Lambda_0 - \delta + \alpha_1 = r_2(\Lambda_0 - \alpha_0 - \alpha_1)$, the argument in the proof of Lemma 3.3 shows

$$e(012)R^{\Lambda_0}(\delta - \alpha_1)e(012) = R^{\Lambda_0}(\delta - \alpha_1) \simeq \mathbf{k}[x]/(x^2).$$

Thus, it follows from Theorem 2.3 and $E_1 R^{\Lambda_0}(\delta - \alpha_1) = 0$ that we have an isomorphism of $R^{\Lambda_0}(\delta - \alpha_1)$-bimodules as follows.

$$(\mathbf{k} \oplus \mathbf{k}y) \otimes e(012)R^{\Lambda_0}(\delta - \alpha_1)e(012) \simeq e(0121)R^{\Lambda_0}(\delta)e(0121).$$

We conclude that $e(0121)R^{\Lambda_0}(\delta)e(0121) \simeq \mathbf{k}[x, y]/(x^2, y^2 - axy)$, for some $a \in \mathbf{k}$. □

Theorem 3.7. *If $l = 2$, then the algebra $R^{\Lambda_0}(\delta)$ is a symmetric special biserial algebra of tame representation type. When $l \geq 3$, $R^{\Lambda_0}(\delta)$ is of wild representation type.*

Proof. Suppose that $l = 2$. Proposition 3.5 gives

$$\mathcal{P}_1 \simeq \begin{matrix} \mathcal{S}_1 \\ \mathcal{S}_1 \oplus \mathcal{S}_2 \\ \mathcal{S}_2 \oplus \mathcal{S}_1 \\ \mathcal{S}_1 \end{matrix}, \quad \mathcal{P}_2 \simeq \begin{matrix} \mathcal{S}_2 \\ \mathcal{S}_1 \\ \mathcal{S}_1 \\ \mathcal{S}_2 \end{matrix},$$

which imply

$$\dim \mathrm{Hom}(\mathcal{P}_i, \mathrm{Rad}(\mathcal{P}_j)/\mathrm{Rad}^2(\mathcal{P}_j)) = \begin{cases} 1 & \text{if} \quad (i = 1) \text{ or } (i = 2,\ j = 1), \\ 0 & \text{if} \quad i = j = 2. \end{cases}$$

By (3.7), $\mathcal{P}_1$ has a submodule $\mathcal{Q}$ which is isomorphic to $\mathcal{Q}_1$. Let $\gamma : \mathcal{P}_1 \twoheadrightarrow \mathcal{Q} \hookrightarrow \mathcal{P}_1$ be the homomorphism induced from (3.7). Note that γ is a lift of $\mathcal{P}_1 \twoheadrightarrow \mathcal{S}_1 \hookrightarrow \mathrm{Rad}(\mathcal{P}_1)/\mathrm{Rad}^2(\mathcal{P}_1)$. Since $\mathrm{Im}(\gamma) = \mathrm{Ker}(\gamma) \simeq \mathcal{Q}_1$, we have $\gamma^2 = 0$. We set

$$\alpha = \text{a lift of } \mathcal{P}_1 \twoheadrightarrow \mathcal{S}_1 \hookrightarrow \mathrm{Rad}(\mathcal{P}_2)/\mathrm{Rad}^2(\mathcal{P}_2),$$
$$\beta = \text{a lift of } \mathcal{P}_2 \twoheadrightarrow \mathcal{S}_2 \hookrightarrow \mathrm{Rad}(\mathcal{P}_1)/\mathrm{Rad}^2(\mathcal{P}_1).$$

$\mathrm{Im}(\beta)$ is uniserial since $\mathcal{P}_2$ is. Considering the configuration of the radical series, we have

$$\mathrm{Im}(\alpha) = \mathrm{Rad}(\mathcal{P}_2), \quad \mathrm{Ker}(\alpha) \simeq \begin{matrix} \mathcal{S}_2 \\ \mathcal{S}_1 \\ \mathcal{S}_1 \end{matrix}, \quad \mathrm{Im}(\beta) \simeq \begin{matrix} \mathcal{S}_2 \\ \mathcal{S}_1 \\ \mathcal{S}_1 \end{matrix}, \quad \mathrm{Ker}(\beta) \simeq \mathcal{S}_2.$$

Thus, $\beta\alpha = 0$ and $\mathrm{Im}(\gamma\alpha\beta) = \mathrm{Soc}(\mathcal{P}_1) = \mathrm{Im}(\alpha\beta\gamma)$.

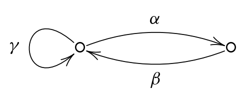
By Theorem 2.6, we have $\dim R^{\Lambda_0}(\alpha_0 + \alpha_1 + \alpha_2)e(012) = 2$. On the other hand, $\dim \mathcal{M}_1 = 2$ by $\dim \mathcal{L}_1 = |\mathsf{ST}(\lambda^{(1)})| = 1$ and we have a surjective homomorphism

$$R^{\Lambda_0}(\alpha_0 + \alpha_1 + \alpha_2)e(012) \to \mathcal{M}_1$$

by $e(012)\mathcal{L}_1 \neq 0$. Since $\mathcal{M}_1$ is projective, it is a split epimorphism. We have $\mathcal{M}_1 \simeq F_2F_1F_0\mathbf{1}$, where $\mathbf{1}$ is the trivial $R^{\Lambda_0}(0)$-module. Thus, we have $\mathcal{P}_1 \simeq F_1F_2F_1F_0\mathbf{1}$. Lemma 3.6 shows that $\mathrm{End}(\mathcal{P}_1) \simeq e(0121)R^{\Lambda_0}(\delta)e(0121)$ is commutative, which yields

$$\gamma\alpha\beta = \alpha\beta\gamma.$$

Therefore, the quiver of the basic algebra of $R^{\Lambda_0}(\delta)$ is given as

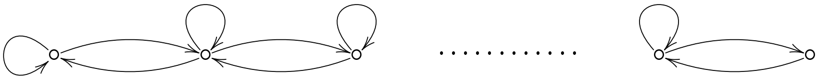


and the defining relations are

$$\beta\alpha = 0, \quad \gamma\alpha\beta = \alpha\beta\gamma, \quad \gamma^2 = 0.$$

The assertion follows by [1, Theorem 7.1 (2b)].

Suppose that $l \geq 3$. Considering the configuration of the radical series in Proposition 3.5, the quiver of the basic algebra of $R^{\Lambda_0}(\delta)$ has l vertices and it is given as follows.



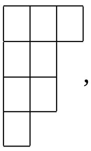
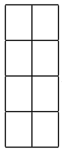
Then, the assertion follows by [8, I.10.8 (iv)]. □

4. Representations of $R^{\Lambda_0}(2\delta - \varpi_4)$

In this section, we assume that $l \geq 4$. Let

$$\beta_0 := 2\delta - \varpi_4 = 2\alpha_0 + 3\alpha_1 + 2\alpha_2 + \alpha_3.$$

Using the crystal of the Fock space in Section 1, $B(\Lambda_0)_{\Lambda_0-\beta_0}$ has two elements b_1, b_2, which are realized as the following Young diagrams:

$b_1 =$  , $b_2 =$  .

Note that

$$\varepsilon_i(b_1) = \begin{cases} 1 & \text{if} \quad i = 1, 3, \\ 0 & \text{otherwise}, \end{cases} \qquad \varepsilon_i(b_2) = \begin{cases} 1 & \text{if} \quad i = 2, \\ 0 & \text{otherwise}. \end{cases} \tag{4.1}$$

We denote by $\mathcal{T}_1$ and $\mathcal{T}_2$ the irreducible $R^{\Lambda_0}(\beta_0)$-modules which corresponds to b_1 and b_2 respectively.

On the other hand, $\Lambda_0 - \beta_0 + \alpha_0$ is not a weight of $V(\Lambda_0)$ by Theorem 2.6. Then, by direct computations, we have

$$\begin{aligned} \Lambda_0 - \beta_0 + \alpha_3 &= r_2r_1r_0r_1r_2(\Lambda_0 - \alpha_0 - \alpha_1), \\ \Lambda_0 - \beta_0 + \alpha_2 &= r_3r_1r_0r_1r_2(\Lambda_0 - \alpha_0 - \alpha_1), \\ \Lambda_0 - \beta_0 + \alpha_1 &= r_2r_3r_0r_1r_2(\Lambda_0 - \alpha_0 - \alpha_1), \end{aligned}$$

and the algebras $R^{\Lambda_0}(\beta_0-\alpha_k)$, for $k=1,2,3$, are derived equivalent to $R^{\Lambda_0}(\alpha_0+\alpha_1)$. Since $R^{\Lambda_0}(\alpha_0+\alpha_1)\simeq \mathbf{k}[x]/(x^2)$, $R^{\Lambda_0}(\beta_0-\alpha_k)$ are matrix rings over $\mathbf{k}[x]/(x^2)$ by the same argument as in Lemma 3.3. Similarly, it follows from

$$\Lambda_0-\beta_0+\alpha_1+\alpha_2=r_3r_0r_1r_2(\Lambda_0-\alpha_0-\alpha_1),$$
$$\Lambda_0-\beta_0+\alpha_2+\alpha_3=r_1r_0r_1r_2(\Lambda_0-\alpha_0-\alpha_1)$$

that $R^{\Lambda_0}(\beta_0-\alpha_1-\alpha_2)$ and $R^{\Lambda_0}(\beta_0-\alpha_2-\alpha_3)$ are isomorphic to matrix rings over $\mathbf{k}[x]/(x^2)$.

For $k=1,2,3$, let $\mathcal{U}_k$ be the unique irreducible $R^{\Lambda_0}(\beta_0-\alpha_k)$-module and $\hat{\mathcal{U}}_k$ its projective cover. Note that $\hat{\mathcal{U}}_k$ has the radical series

$$\hat{\mathcal{U}}_k\simeq \begin{matrix}\mathcal{U}_k\\ \mathcal{U}_k\end{matrix}. \tag{4.2}$$

By (4.1), we may apply Lemma 3.2 to $\mathcal{T}_1$ and $\mathcal{T}_2$. Then the uniqueness of the irreducible $R^{\Lambda_0}(\beta_0-\alpha_k)$-modules implies that

$$E_2(\mathcal{T}_2)\simeq\mathcal{U}_2,\quad E_1(\mathcal{T}_1)\simeq\mathcal{U}_1,\quad E_3(\mathcal{T}_1)\simeq\mathcal{U}_3.$$

We consider the following projective $R^{\Lambda_0}(\beta_0)$-modules

$$\mathcal{R}_i:=F_i\hat{\mathcal{U}}_i,\quad \text{for } i=1,2,3.$$

Then, by the biadjointness of E_i and F_i,

$$\dim\operatorname{Hom}(\mathcal{R}_i,\mathcal{T}_1)=\dim\operatorname{Hom}(\hat{\mathcal{U}}_i,E_i\mathcal{T}_1)=\begin{cases}1 & \text{if}\quad i=1,3,\\ 0 & \text{otherwise},\end{cases}$$
$$\dim\operatorname{Hom}(\mathcal{R}_i,\mathcal{T}_2)=\dim\operatorname{Hom}(\hat{\mathcal{U}}_i,E_i\mathcal{T}_2)=\begin{cases}1 & \text{if}\quad i=2,\\ 0 & \text{otherwise},\end{cases}$$
$$\dim\operatorname{Hom}(\mathcal{T}_1,\mathcal{R}_i)=\dim\operatorname{Hom}(E_i\mathcal{T}_1,\hat{\mathcal{U}}_i)=\begin{cases}1 & \text{if}\quad i=1,3,\\ 0 & \text{otherwise},\end{cases}$$
$$\dim\operatorname{Hom}(\mathcal{T}_2,\mathcal{R}_i)=\dim\operatorname{Hom}(E_i\mathcal{T}_2,\hat{\mathcal{U}}_i)=\begin{cases}1 & \text{if}\quad i=2,\\ 0 & \text{otherwise}.\end{cases}$$

Thus, $\mathcal{R}_2$ is the projective cover of $\mathcal{T}_2$. Since both of $\mathcal{R}_1$ and $\mathcal{R}_3$ are indecomposable projective modules which surjects to $\mathcal{T}_1$, $\mathcal{R}_1\simeq\mathcal{R}_3$ is the projective cover of $\mathcal{T}_1$.

In the crystal of the Fock space $\mathcal{F}$, we have $\varepsilon_1(\mathcal{U}_2)=\varepsilon_2(\mathcal{U}_1)=1$. Thus, Lemma 3.2 implies that $E_1(\mathcal{U}_2)$ and $E_2(\mathcal{U}_1)$ are irreducible $R^{\Lambda_0}(\beta_0-\alpha_1-\alpha_2)$-modules, and the uniqueness of the irreducible $R^{\Lambda_0}(\beta_0-\alpha_1-\alpha_2)$-modules implies $E_1(\mathcal{U}_2)\simeq E_2(\mathcal{U}_1)$. We have the exact sequence

$$0 \to E_1\mathcal{U}_2 \to E_1\hat{\mathcal{U}}_2 \to E_1\mathcal{U}_2 \to 0. \tag{4.3}$$

Since $E_1(\mathcal{U}_2)$ is not projective, it does not split, and $E_1\hat{\mathcal{U}}_2$ is indecomposable projective. The same argument shows that $E_2\hat{\mathcal{U}}_1$ is indecomposable projective. Hence, the indecomposable projective $R^{\Lambda_0}(\beta_0-\alpha_1-\alpha_2)$-module is given by

$$E_1(\hat{\mathcal{U}}_2) \simeq E_2(\hat{\mathcal{U}}_1) \simeq \begin{matrix} E_1(\mathcal{U}_2) \\ E_1(\mathcal{U}_2) \end{matrix}.$$

It follows that, for $i, j = 1, 2$ with $i \neq j$, we have

$$\dim \operatorname{Hom}(\mathcal{R}_i, \mathcal{R}_j) = \dim \operatorname{Hom}(E_j\hat{\mathcal{U}}_i, E_i\hat{\mathcal{U}}_j) = 2,$$
$$\dim \operatorname{Hom}(\mathcal{R}_i, \mathcal{R}_i) = \dim \operatorname{Hom}(\hat{\mathcal{U}}_i, E_iF_i\hat{\mathcal{U}}_i) = \dim \operatorname{Hom}(\hat{\mathcal{U}}_i, \hat{\mathcal{U}}_i^{\bigoplus\langle h_i, \Lambda_0-\beta_0+\alpha_i\rangle}) = 4.$$

Therefore, $\mathcal{R}_1$ and $\mathcal{R}_2$ are self-dual modules whose composition multiplicities are given by

$$[\mathcal{R}_1] = 4[\mathcal{T}_1] + 2[\mathcal{T}_2], \quad [\mathcal{R}_2] = 2[\mathcal{T}_1] + 4[\mathcal{T}_2].$$

Let $\mathcal{V}_i := F_i\mathcal{U}_i$, for $i = 1, 2$. By the same argument as above, we have

$$\dim \operatorname{Hom}(\mathcal{V}_i, \mathcal{T}_j) = \dim \operatorname{Hom}(\mathcal{T}_i, \mathcal{V}_j) = \delta_{i,j}.$$

We have the exact sequence

$$0 \to \mathcal{V}_i \to \mathcal{R}_i \to \mathcal{V}_i \to 0, \tag{4.4}$$

which does not split because $\mathcal{R}_i$ are indecomposable. As $\operatorname{Top}(\mathcal{V}_i) \simeq \mathcal{T}_i \simeq \operatorname{Soc}(\mathcal{V}_i)$, we have

$$\mathcal{V}_1 \simeq \begin{matrix} \mathcal{T}_1 \\ \mathcal{T}_2 \\ \mathcal{T}_1 \end{matrix}, \quad \mathcal{V}_2 \simeq \begin{matrix} \mathcal{T}_2 \\ \mathcal{T}_1 \\ \mathcal{T}_2 \end{matrix}. \tag{4.5}$$

Proposition 4.1. *The radical series of $\mathcal{R}_1$ and $\mathcal{R}_2$ are given as follows*:

$$\mathcal{R}_1 \simeq \begin{matrix} \mathcal{T}_1 \\ \mathcal{T}_1 \oplus \mathcal{T}_2 \\ \mathcal{T}_2 \oplus \mathcal{T}_1 \\ \mathcal{T}_1 \end{matrix}, \quad \mathcal{R}_2 \simeq \begin{matrix} \mathcal{T}_2 \\ \mathcal{T}_2 \oplus \mathcal{T}_1 \\ \mathcal{T}_1 \oplus \mathcal{T}_2 \\ \mathcal{T}_2 \end{matrix}.$$

Proof. As the argument is symmetric in $i = 1$ and $i = 2$, we only consider $\mathcal{R}_1$. It is clear from (4.5) that $\mathcal{T}_2$ appears in $\operatorname{Rad}(\mathcal{R}_1)/\operatorname{Rad}^2(\mathcal{R}_1)$. If $\operatorname{Rad}(\mathcal{R}_1)/\operatorname{Rad}^2(\mathcal{R}_1)$ is irreducible, then $\mathrm{Ext}^1(\mathcal{T}_1, \mathcal{T}_1) = 0$. Since $\mathrm{Rad}^2(\mathcal{R}_1)/\mathrm{Rad}^3(\mathcal{R}_1)$ contains $\mathcal{T}_1$ by (4.4) and (4.5), it implies that $\mathcal{R}_1$ has the radical series of the following form.

$$\mathcal{R}_1 \simeq \begin{array}{c} \mathcal{T}_1 \\ \mathcal{T}_2 \\ \mathcal{T}_1 \oplus \mathcal{T}_1 \\ \mathcal{T}_2 \\ \mathcal{T}_1 \end{array}.$$

But if we look at $\mathrm{Rad}(\mathcal{R}_1)/\mathrm{Rad}^3(\mathcal{R}_1)$, we have $\dim \mathrm{Ext}^1(\mathcal{T}_2, \mathcal{T}_1) \geq 2$, and the self-duality of irreducible modules implies that $\dim \mathrm{Ext}^1(\mathcal{T}_1, \mathcal{T}_2) \geq 2$. It contradicts $\dim \mathrm{Ext}^1(\mathcal{T}_1, \mathcal{T}_2) = 1$. Thus, $\mathrm{Rad}(\mathcal{R}_1)/\mathrm{Rad}^2(\mathcal{R}_1)$ is not irreducible, and we have the desired shape of the radical series. □

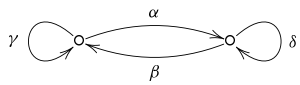
Theorem 4.2. *The algebra* $R^{\Lambda_0}(2\delta - \varpi_4)$ *is wild.*

Proof. By (4.4), $\mathcal{R}_1$ has a submodule $\mathcal{V}$ which is isomorphic to $\mathcal{V}_1$. Let $\gamma\colon \mathcal{R}_1 \twoheadrightarrow \mathcal{V} \hookrightarrow \mathcal{R}_1$ be the homomorphism induced by (4.4), which is a lift of $\mathcal{R}_1 \twoheadrightarrow \mathcal{T}_1 \hookrightarrow \mathrm{Rad}(R_1)/\mathrm{Rad}^2(R_1)$. We have $\gamma^2 = 0$. Similarly, we take a lift δ of $\mathcal{R}_2 \twoheadrightarrow \mathcal{T}_2 \hookrightarrow \mathrm{Rad}(R_2)/\mathrm{Rad}^2(R_2)$ such that $\delta^2 = 0$. We now choose

$$\alpha = \text{a lift of } \mathcal{R}_1 \twoheadrightarrow \mathcal{T}_1 \hookrightarrow \mathrm{Rad}(\mathcal{R}_2)/\mathrm{Rad}^2(\mathcal{R}_2),$$
$$\beta = \text{a lift of } \mathcal{R}_2 \twoheadrightarrow \mathcal{T}_2 \hookrightarrow \mathrm{Rad}(\mathcal{R}_1)/\mathrm{Rad}^2(\mathcal{R}_1).$$

Then, the quiver of the basic algebra of $R^{\Lambda_0}(2\delta - \varpi_4)$ is given as follows:

(4.6)



Considering the configuration of the radical series from Proposition 4.1, we must have

$$\mathrm{Im}(\alpha\beta) \simeq \begin{array}{c} \mathcal{T}_1 \\ \mathcal{T}_1 \end{array}, \quad \mathrm{Im}(\beta\alpha) \simeq \begin{array}{c} \mathcal{T}_2 \\ \mathcal{T}_2 \end{array}, \quad \mathrm{Im}(\alpha) \simeq \begin{array}{c} \mathcal{T}_1 \\ \mathcal{T}_1 \oplus \mathcal{T}_2 \\ \mathcal{T}_2 \end{array}, \quad \mathrm{Im}(\beta) \simeq \begin{array}{c} \mathcal{T}_2 \\ \mathcal{T}_2 \oplus \mathcal{T}_1 \\ \mathcal{T}_1 \end{array}$$

and it follows that

$$\alpha\beta\alpha = \beta\alpha\beta = 0, \quad \mathrm{Im}(\gamma\alpha) = \mathrm{Im}(\alpha\delta) \simeq \begin{array}{c} \mathcal{T}_1 \\ \mathcal{T}_2 \end{array}, \quad \mathrm{Im}(\delta\beta) = \mathrm{Im}(\beta\gamma) \simeq \begin{array}{c} \mathcal{T}_2 \\ \mathcal{T}_1 \end{array}.$$

By adjusting γ and δ by nonzero scalar multiples, we may assume $\gamma\alpha = \alpha\delta$. Thus, we have the defining relations for the basic algebra as follows, where $c \in \mathbf{k}$ is a nonzero scalar:

$$\gamma^2 = \delta^2 = \alpha\beta\alpha = \beta\alpha\beta = 0, \quad \gamma\alpha = \alpha\delta, \quad \delta\beta = c\beta\gamma.$$

Since the algebra of the quiver (4.6) with the defining relations

$$\gamma^2 = \delta^2 = \alpha\beta\alpha = \beta\alpha\beta = 0, \quad \gamma\alpha = \alpha\delta, \quad \delta\beta = \beta\gamma = 0$$

is of wild representation type by [11, Theorem 1, Table W (32)], so is $R^{\Lambda_0}(2\delta - \varpi_4)$. □

5. Representations type of $R^{\Lambda_0}(\beta)$

By the categorification theorem, $R^{\Lambda}(\beta) \neq 0$ if and only if $\Lambda - \beta$ is a weight of $V(\Lambda)$. A weight μ of $V(\Lambda)$ is *maximal* if $\mu + \delta$ is not a weight of $V(\Lambda)$. Let $\max(\Lambda)$ be the set of all maximal weights of $V(\Lambda)$.

Proposition 5.1. *For the weight system of the $\mathfrak{g}(A)$-module $V(\Lambda_0)$ in type $C_l^{(1)}$, we have*
(1) $\max(\Lambda_0) \cap \mathsf{P}^+ = \{\Lambda_0 + \varpi_i - (i/2)\delta \mid i \in I,\ i$ *is even*$\}$,
(2) μ *is a weight of $V(\Lambda_0)$ if and only if $\mu = w\eta - k\delta$ for some $w \in \mathsf{W}$, $\eta \in \max(\Lambda_0) \cap \mathsf{P}^+$ and $k \in \mathbb{Z}_{\geq 0}$.*

Proof. (1) Let $\mu \in \max(\Lambda_0) \cap \mathsf{P}^+$. Since $\mu \in \mathsf{P}^+$ and $\varpi_1, \dots, \varpi_l$ form a basis of $\sum_{i \in I \setminus \{0\}} \mathbb{Q}\alpha_i$, μ can be written as

$$\mu = \Lambda_0 + \sum_{i \in I \setminus \{0\}} p_i \varpi_i + t\delta$$

for some $p_i = \mu(h_i) \in \mathbb{Z}_{\geq 0}$ and $t \in \mathbb{Z}$. Then, the computations

$$0 \leq \mu(h_0) = 1 - p_1 - \cdots - p_n,$$
$$0 \leq \mu(h_1 + \cdots + h_n) = p_1 + \cdots + p_n$$

imply that $\mu = \Lambda_0 + \varpi_i + t\delta$ for some $i \in I \setminus \{0\}$, or $\mu = \Lambda_0 + t\delta$. In the latter case, $\mu \in \max(\Lambda_0)$ implies that $\mu = \Lambda_0$, which is equal to $\Lambda_0 + \varpi_0$. In the former case, $\Lambda_0 - \mu \in \mathsf{Q}^+$ implies that i is even by the definition (1.2). We show that $t = -i/2$. We consider the Young diagram

$$\lambda(i) = (\underbrace{i, i, \dots, i}_{i/2})$$

in the Fock space $\mathcal{F}$. Considering the residue pattern, we have

$$\mathrm{wt}(\lambda(i)) = \Lambda_0 - \left(\frac{i}{2}\alpha_0 + (i-1)\alpha_1 + (i-2)\alpha_2 + \cdots + \alpha_{i-1}\right) = \Lambda_0 + \varpi_i - \frac{i}{2}\delta.$$

Thus, Theorem 2.6 implies

$$\dim R^{\Lambda_0}\left(\frac{i}{2}\delta - \varpi_i\right) \neq 0,$$

and $\Lambda_0 + \varpi_i - (i/2)\delta$ is a weight of $V(\Lambda_0)$. It follows from

$$\left(-\varpi_i + \frac{i}{2}\delta\right) - \delta \notin \mathsf{Q}^+$$

that $\Lambda_0 + \varpi_i - (i/2)\delta$ is maximal.

(2) $\max(\Lambda_0)$ is W-invariant by [13, Proposition 10.1] and we have

$$\max(\Lambda_0) = \mathsf{W}(\max(\Lambda_0) \cap \mathsf{P}^+)$$

by [13, Corollary 10.1]. Then, for any weight μ of $V(\Lambda_0)$, there exist a unique $\zeta \in \max(\Lambda_0)$ and a unique $k \in \mathbb{Z}_{\geq 0}$ such that $\mu = \zeta - k\delta$ [13, (12.6.1)]. □

Lemma 5.2 ([9, Proposition 2.3], [3, Remark 5.10])**.** *Let A and B be finite dimensional* **k***-algebras and suppose that there exists a constant $C > 0$ and functors*

$$F\colon A\text{-mod} \to B\text{-mod}, \quad G\colon B\text{-mod} \to A\text{-mod}$$

such that, for any A-module M,
(1) *M is a direct summand of $GF(M)$ as an A-module,*
(2) $\dim F(M) \leq C \dim M$.
Then, if A is wild, so is B.

Lemma 5.3. (1) *If $R^{\Lambda_0}(\beta - \alpha_j)$ is wild and $\langle h_j, \Lambda_0 - \beta + \alpha_j\rangle \geq 1$, then $R^{\Lambda_0}(\beta)$ is wild.*
(2) *Suppose that $R^{\Lambda_0}(k\delta - \varpi_i)$ is wild. Then, we have*
 (a) *$R^{\Lambda_0}((k+1)\delta - \varpi_i)$ is wild,*
 (b) *if $i + 2 \in I$, then $R^{\Lambda_0}((k+1)\delta - \varpi_{i+2})$ is wild.*

Proof. (1) Considering the functors

$$F_j\colon R^{\Lambda_0}(\beta - \alpha_j)\text{-mod} \to R^{\Lambda_0}(\beta)\text{-mod}, \quad E_j\colon R^{\Lambda_0}(\beta)\text{-mod} \to R^{\Lambda_0}(\beta - \alpha_j)\text{-mod},$$

the assertion follows from Lemma 5.2 and Theorem 2.3.

(2) For $0 \leq i \leq l-1$ and $k \in \mathbb{Z}_{\geq 0}$, direct computation shows

$$\Lambda_0 + \varpi_{i+2} - (k+1)\delta + \alpha_{i+1} = r_i r_{i-1} \cdots r_1 r_0 r_1 \cdots r_i(\Lambda_0 + \varpi_i - k\delta),$$
$$\Lambda_0 + \varpi_i - (k+1)\delta + \alpha_l = r_{l-1} r_{l-2} \cdots r_1 r_0 r_1 \cdots r_i(\Lambda_0 + \varpi_i - k\delta).$$

Thus, (2) (a), for $i \neq l$, and (2) (b) follow from Proposition 2.2 and (1) because

$$\langle h_{i+1}, \Lambda_0 + \varpi_{i+2} - (k+1)\delta + \alpha_{i+1}\rangle = 2,$$
$$\langle h_l, \Lambda_0 + \varpi_i - (k+1)\delta + \alpha_l\rangle = 2.$$

Similarly, we consider

$$\Lambda_0 + \varpi_l - (k+1)\delta + \alpha_0 = r_1 r_2 \cdots r_l(\Lambda_0 + \varpi_l - k\delta).$$

Then (2) (a), for $i = l$, follows from Proposition 2.2 and (1) because

$$\langle h_0, \Lambda_0 + \varpi_l - (k+1)\delta + \alpha_0\rangle = 2.$$

We have proved the lemma. □

Lemma 5.4. *The algebras $R^{\Lambda_0}(2\delta - \varpi_2)$ and $R^{\Lambda_0}(2\delta)$ are wild.*

Proof. Note that $2\delta - \varpi_2 = \delta + \alpha_0 + \alpha_1$. If $l \geq 3$, Lemma 5.3 (1) and Theorem 3.7 imply that $R^{\Lambda_0}(2\delta - \varpi_2)$ is wild, because we have

$$\langle h_0, \Lambda_0 - \delta\rangle = 1, \quad \langle h_1, \Lambda_0 - \delta - \alpha_0\rangle = 2.$$

Applying Lemma 5.3 (2) (a), Theorem 3.7 also implies that $R^{\Lambda_0}(2\delta)$ is wild.

In the following, we suppose that $l = 2$. We set

$$e_0 = \sum_{\nu \in I^\delta} e(\nu, 0), \quad e_1 = \sum_{\nu' \in I^{\delta+\alpha_0}} e(\nu', 1), \quad e = \sum_{\nu \in I^\delta} e(\nu, 0, 1).$$

Considering the residue pattern and Theorem 2.6, we have

$$E_0 R^{\Lambda_0}(\delta) = 0.$$

Since $\langle h_0, \Lambda_0 - \delta\rangle = 1$, Theorem 2.3 gives an algebra isomorphism

$$R^{\Lambda_0}(\delta) \simeq E_0 F_0 R^{\Lambda_0}(\delta) = e_0 R^{\Lambda_0}(\delta + \alpha_0) e_0.$$

We also have $E_1 R^{\Lambda_0}(\delta + \alpha_0) = 0$ by Theorem 2.6. It follows from

$$\langle h_1, \Lambda_0 - \delta - \alpha_0\rangle = 2$$

and Theorem 2.3 that there is a bimodule isomorphism

$$\mathbf{k}[t]/(t^2) \otimes_{\mathbf{k}} R^{\Lambda_0}(\delta + \alpha_0) \simeq E_1 F_1 R^{\Lambda_0}(\delta + \alpha_0) = e_1 R^{\Lambda_0}(2\delta - \varpi_2) e_1. \tag{5.1}$$

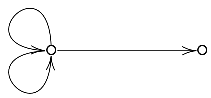
Thus, multiplying $e = ee_1 = e_1e$ on the both sides and factoring out the square of the radicals, (5.1) gives the isomorphism of algebras

$$\begin{aligned}&eR^{\Lambda_0}(2\delta - \varpi_2)e/\mathrm{Rad}^2(eR^{\Lambda_0}(2\delta - \varpi_2)e)\\&\simeq \mathbf{k}[t]/(t^2) \otimes_{\mathbf{k}} R^{\Lambda_0}(\delta)/(t^2, t\,\mathrm{Rad}(R^{\Lambda_0}(\delta)), \mathrm{Rad}^2(R^{\Lambda_0}(\delta))).\end{aligned}$$

We denote the algebra by B and let $\mathcal{O}$ be the irreducible $\mathbf{k}[t]/(t^2)$-module. Then B has irreducible modules $\mathcal{O} \otimes \mathcal{S}_1$ and $\mathcal{O} \otimes \mathcal{S}_2$, where $\mathcal{S}_1$ and $\mathcal{S}_2$ are the irreducible $R^{\Lambda_0}(\delta)$-modules in Proposition 3.1. By Proposition 3.5, the projective cover of $\mathcal{O} \otimes \mathcal{S}_1$ has the radical series

$$\begin{array}{ccc} & \mathcal{O} \otimes \mathcal{S}_1 & \\ \mathcal{O} \otimes \mathcal{S}_1 & \mathcal{O} \otimes \mathcal{S}_1 & \mathcal{O} \otimes \mathcal{S}_2, \end{array}$$

which implies that the quiver of $eR^{\Lambda_0}(2\delta - \varpi_2)e$ contains



as a subquiver. By [8, I.10.8 (i)], $eR^{\Lambda_0}(2\delta - \varpi_2)e$ is wild, and so is $R^{\Lambda_0}(2\delta - \varpi_2)$. Then, $R^{\Lambda_0}(2\delta) = R^{\Lambda_0}(2\delta - \varpi_2 + \alpha_1 + \alpha_2)$ is wild by Lemma 5.3 (1) because we have

$$\langle h_2, \Lambda_0 - 2\delta + \alpha_1 + \alpha_2\rangle = 1, \quad \langle h_1, \Lambda_0 - 2\delta + \alpha_1\rangle = 2.$$

We have proved the lemma. □

We summarize the results which are obtained so far. Suppose that $i \geq 4$ is even. Then, Theorem 4.2 and Lemma 5.3 (2) (a) (b) imply that $R^{\Lambda_0}(k\delta - \varpi_i)$, for $k \geq i/2$, are all wild. If $i = 2$, then $R^{\Lambda_0}(\delta - \varpi_2) = R^{\Lambda_0}(\alpha_0 + \alpha_1)$ is of finite type by Lemma 3.3 (1), and $R^{\Lambda_0}(k\delta - \varpi_2)$, for $k \geq 2$, are wild by Lemma 5.4 and Lemma 5.3 (2) (a). If $i = 0$, $R^{\Lambda_0}(0)$ is a simple algebra, and $R^{\Lambda_0}(\delta)$ is tame if $l = 2$ and wild if $l > 2$ by Theorem 3.7. As $R^{\Lambda_0}(2\delta)$ is wild by Lemma 5.4, Lemma 5.3 (2) (a) implies that $R^{\Lambda_0}(k\delta)$, for $k \geq 2$, are wild. Thus, we have the following theorem.

Theorem 5.5. *Let $i \in I$ be an even index. For $\kappa \in \mathsf{W}(\Lambda_0 - \varpi_i)$ and $k \geq i/2$, the finite quiver Hecke algebra $R^{\Lambda_0}(\Lambda_0 - \kappa + k\delta)$ of type $C_l^{(1)}$ is*
(1) *a simple algebra if $i = k = 0$,*
(2) *of finite representation type if $i = 2$ and $k = 1$,*
(3) *of tame representation type if $i = 0$, $k = 1$ and $l = 2$,*
(4) *of wild representation type otherwise.*

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