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RUSSELL'S HYPERSURFACE FROM A GEOMETRIC POINT OF VIEW

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Abstract

The famous Russell hypersurface is a smooth complex affine threefold which is diffeomorphic to a euclidean space but not algebraically isomorphic to the three dimensional affine space. This fact was first established by Makar-Limanov, using algebraic minded techniques. In this article, we give an elementary argument which adds a greater insight to the geometry behind the original proof and which also may be applicable in other situations.

1. Introduction

Russell's hypersurface

$$X := \{(x, y, z, t) \in \mathbb{C}^4 \mid x + x^2y + z^3 + t^2 = 0\} \hookrightarrow \mathbb{C}^4,$$

is one of the most prominent examples of an exotic variety, i.e. a variety which is diffeomorphic to an affine space ([1], [6, Lemma 5.1]), but not isomorphic to it. The latter is an immediate consequence of Theorem 1 below, which states that there are not sufficiently many actions of the additive group \mathbb{G}_a on X , and the aim of this paper is to give an elementary argument for this theorem. It includes some important elements of the original proof, but gives a greater geometrical insight to the situation.

The study of exotic varieties goes back to a paper of Ramanujam [16], where a nontrivial example of a topologically contractible smooth affine algebraic surface S over \mathbb{C} is constructed. Ramanujam observed that $S \times \mathbb{C}$ is diffeomorphic to \mathbb{C}^3 , and asked whether this product is also isomorphic to \mathbb{C}^3 . This was later proven not to be the case, and thus the algebraic structure on \mathbb{C}^3 coming from $S \times \mathbb{C}^2$ is exotic [17]. Later on, many other exotic structures on \mathbb{C}^3 have been constructed, see e.g. the introduction of [18] for a list. Note also that there are no exotic structures on affine space in dimension ≤ 2 [16].

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The motivation for studying Russell's hypersurface originally came from the linearization conjecture for \mathbb{C}^3 , which claims that each \mathbb{G}_m -action on \mathbb{C}^3 is linearizable. In the proof of this result by Koras and Russell, they described a list of smooth affine threefolds diffeomorphic to \mathbb{C}^3 which contains all the potential counterexamples to the conjecture, and thus it was reduced to determining whether all of these so called Koras–Russell threefolds are exotic [13]. Kaliman and Makar-Limanov established exoticity for some of them [10], and Russell's hypersurface is the “most simple” among the remaining ones. The difficulty with Russell's hypersurface was that all the usual algebraic and geometric invariants failed to distinguish it from \mathbb{C}^3 . Makar-Limanov finally established exoticity of Russell's hypersurface (Theorem 1), and later on Kaliman and Makar-Limanov were able to prove exoticity of the remaining Koras–Russell threefolds [8, 9] as well, elaborating on Makar-Limanov's methods. This confirmed the linearization conjecture [7].

From now on, we will focus on Makar-Limanov's result, stated in the following theorem.

Theorem 1 (Makar-Limanov, [14]). *The projection $\text{pr}_1: X \rightarrow \mathbb{C}$, $(x, y, z, t) \mapsto x$ is invariant with respect to any \mathbb{G}_a -action on X .*

Some years after Makar-Limanov proved Theorem 1, Kaliman proved, using non-elementary birational geometry, that morphisms $\mathbb{C}^3 \rightarrow \mathbb{C}$ with generic fiber \mathbb{C}^2 cannot have any other fibers [5]. Since all the fibers of $\text{pr}_1: X \rightarrow \mathbb{C}$ are \mathbb{C}^2 except the zero fiber $\text{pr}_1^{-1}(0)$, it follows also from Kaliman's result that $X \not\cong \mathbb{C}^3$. In 2005, Makar-Limanov gave another proof of the exoticity of Russell's hypersurface [15]; yet another proof was given by Derksen [3], and Crachiola also proved the exoticity in the positive characteristic case [2]. The original proof of Theorem 1 used algebraic techniques, while we rather focus on a geometric approach using fibrations and quotient maps.

An outline of our proof. In order to prove Theorem 1, we make use of an isomorphism $X \cong U \subset M$ with an open subset U of a blowup $\pi: M \rightarrow \mathbb{C}^3$, such that $D := M \setminus U$ is the strict transform of $\{0\} \times \mathbb{C}^2 \hookrightarrow \mathbb{C}^3$ and

$$X \cong U \subset M \rightarrow \mathbb{C}^3$$

is the map $(x, y, z, t) \mapsto (x, z, t)$. That is, X is isomorphic to an affine modification U of \mathbb{C}^3 . The key-result is then that $\mathcal{O}(M) \subset \mathcal{O}(X)$ is invariant for any \mathbb{G}_a -action on X . Since $\mathcal{O}(M) \cong \mathcal{O}(\mathbb{C}^3)$, this allows us to conclude that for any given \mathbb{G}_a -action on X , there is an induced \mathbb{G}_a -action on \mathbb{C}^3 which makes $\pi|_X: X \rightarrow \mathbb{C}^3$ equivariant. Then $\pi(U)$ is obviously invariant, and it follows that its interior $\mathbb{C}^* \times \mathbb{C}^2$ is invariant as well. Theorem 1 is obtained from this by observing that any \mathbb{G}_a -action on $\mathbb{C}^* \times \mathbb{C}^2$ leaves the first coordinate invariant: a nontrivial \mathbb{G}_a -orbit is isomorphic to \mathbb{C} , but there are no non-constant morphisms from \mathbb{C} to \mathbb{C}^* .

2. Russell's hypersurface in a blowup of \mathbb{C}^3

We recall the realization of Russell's hypersurface as an affine modification of \mathbb{C}^3 , see also [11, Example 1.5]. Let $N \hookrightarrow \mathbb{C}^2 = \text{Spec}(\mathbb{C}[z, t])$ denote the affine cuspidal cubic curve given by

$$N := \{(z, t) \in \mathbb{C}^2 \mid z^3 + t^2 = 0\},$$

and let $I := (g, h) \subset \mathbb{C}[x, z, t]$ denote the ideal which is generated by the two relatively prime polynomials $g(x, z, t) = x^2$ and $h(x, z, t) = x + z^3 + t^2$. The zero set of I is $\{0\} \times N$, and the blowup

$$M := \text{Bl}_I(\mathbb{C}^3) \cong \{((x, z, t), [u : v]) \in \mathbb{C}^3 \times \mathbb{P}^1 \mid h(x, z, t)u + g(x, z, t)v = 0\}$$

of \mathbb{C}^3 along I is a hypersurface in $\mathbb{C}^3 \times \mathbb{P}^1$ with singular locus of codimension two: $\text{Sing}(M) = \{0\} \times N \times \{[0 : 1]\}$. In particular, M is a normal variety.

REMARK 2.1. Russell's hypersurface is isomorphic to the open subset U of M given by $u \neq 0$, via the embedding $X \hookrightarrow M$, $(x, y, z, t) \mapsto ((x, z, t), [1 : y])$.

We denote the complement of U in M by D . Note that D is then given by $u = 0$, and that the image of U under the blowup morphism is $\pi(U) = \mathbb{C}^* \times \mathbb{C}^2 \cup (\{0\} \times N)$.

3. Additive group actions on Russell's hypersurface

In order to see that $\mathcal{O}(M) \subset A := \mathcal{O}(X)$ is invariant for every \mathbb{G}_a -action on X , we show the equivalent fact that $\mathcal{O}(M) \subset A$ is stable under every locally nilpotent derivation $\partial: A \rightarrow A$. This obviously holds for the trivial \mathbb{G}_a -action on X , so we may assume that $\partial \neq 0$. The first step is to characterize $\mathcal{O}(M)$ in terms of a filtration on A .

REMARK 3.1. With the filtration

$$A_{\leq n} := \mathcal{O}_{nD}(M) = \{f \in \mathcal{O}(M)^* \mid \text{div}(f) \geq -nD\} \cup \{0\}$$

we have $A_{\leq 0} = \mathcal{O}(M) = \pi^*(\mathcal{O}(\mathbb{C}^3))$, so $\pi^*(\mathcal{O}(\mathbb{C}^3))$ is stable with respect to a locally nilpotent derivation $\partial: A \rightarrow A$ if and only if $\partial(A_{\leq 0}) \subseteq A_{\leq 0}$.

In order to understand the above filtration, we treat A as a subset of $\mathbb{C}(x, z, t)$ and note that multiplicities along D are simply multiplicities along $\{0\} \times \mathbb{C}^2$; so x, y, z and t have multiplicities 1, $-2, 0$ and 0 , respectively. Now every element $f \in A \setminus \{0\}$ can be written in the form

$$f = \sum_{i=0}^k y^i p_i(x, z, t),$$

where each p_i is at most linear in x for $i \geq 1$ (and $p_k \neq 0$). Thus

$$A = \bigoplus_{k=-\infty}^{\infty} A_k$$

is a direct sum of free $\mathbb{C}[z, t]$ -modules A_k of rank 1 defined as

$$(1) \quad A_k := \begin{cases} \mathbb{C}[z, t]x^{|k|}, & \text{if } k \leq 0, \\ \mathbb{C}[z, t]y^l, & \text{if } k = 2l > 0, \\ \mathbb{C}[z, t]xy^l, & \text{if } k = 2l - 1 > 0, \end{cases}$$

and one can check that

$$A_{\leq n} = \bigoplus_{k \leq n} A_k.$$

We thus obtain an explicit description of the associated graded algebra

$$B := \text{Gr}(A) = \bigoplus_{n \in \mathbb{Z}} B_n \quad \text{with} \quad B_n := A_{\leq n} / A_{\leq n-1}.$$

It is generated by the elements $\text{gr}(x) \in B_{-1}$, $\text{gr}(y) \in B_2$, $\text{gr}(z)$, $\text{gr}(t) \in B_0$ and

$$W := \text{Spec}(B) \cong \{(x, y, z, t) \in \mathbb{C}^4 \mid x^2y + z^3 + t^2 = 0\}.$$

In particular $B_0 = \mathbb{C}[\text{gr}(z), \text{gr}(t)] \simeq \mathbb{C}[z, t]$.

REMARK 3.2. This grading was also used by M. Zaidenberg, see [18, Lemma 7.4].

Let $l = l(\partial) \in \mathbb{Z}$ be minimal with the property that $\partial(A_{\leq n}) \subset A_{\leq n+l}$ for all $n \in \mathbb{Z}$; the existence of such an l follows from the fact that both $A_{\leq 0}$ and B are finitely generated graded algebras, and $l \neq -\infty$ since we consider a nontrivial \mathbb{G}_a -action. It follows that $\partial: A \rightarrow A$ induces a nontrivial homogeneous locally nilpotent derivation on B of degree l ; we will denote it by δ . With this notation it is enough to show that $l \leq 0$ in order to obtain $\partial(A_{\geq 0}) \subseteq A_{\geq 0}$. In fact, more is true:

Proposition 3.3. *With B as above, any nontrivial locally nilpotent homogeneous derivation $\delta: B \rightarrow B$ has degree $l < 0$.*

Before going into the proof, let us start with a discussion of the geometry of $W \hookrightarrow \mathbb{C}^4$ and prove Lemma 3.6 below. As a hypersurface in \mathbb{C}^4 , W is a normal variety since its singular set $\text{Sing}(W) = \{0\} \times \mathbb{C} \times \{0\} \times \{0\}$ has codimension two. It admits two different group actions: the \mathbb{G}_a -action $(\tau, w) \mapsto \tau \cdot w$ corresponding to the locally

nilpotent derivation $\delta: B \rightarrow B$; and the \mathbb{G}_m -action corresponding to the grading of B . The latter is given by

$$\mathbb{G}_m \times W \rightarrow W, \quad (\lambda, (x, y, z, t)) \mapsto (\lambda^{-1}x, \lambda^2y, z, t),$$

and since $B_0 = \mathbb{C}[z, t]$, the \mathbb{G}_m -quotient morphism is given by

$$p: W \rightarrow \mathbb{C}^2 \cong \text{Spec}(\mathbb{C}[z, t]), \quad (x, y, z, t) \mapsto (z, t).$$

It is trivial above $\mathbb{C}^2 \setminus N$: the map

$$(\mathbb{C}^2 \setminus N) \times \mathbb{G}_m \xrightarrow{\sim} p^{-1}(\mathbb{C}^2 \setminus N), \quad ((z, t), \lambda) \mapsto (\lambda^{-1}, -(z^3 + t^2)\lambda^2, z, t),$$

is a \mathbb{G}_m -equivariant isomorphism with inverse

$$p^{-1}(\mathbb{C}^2 \setminus N) \xrightarrow{\sim} (\mathbb{C}^2 \setminus N) \times \mathbb{G}_m, \quad (x, y, z, t) \mapsto ((z, t), x^{-1}).$$

As for N , we have $p^{-1}(N) = F_- \cup F_+$, where F_- and F_+ are the subsets of $p^{-1}(N)$ given by $y = 0$ and $x = 0$ respectively.

REMARK 3.4. The set F_- consists exactly of the points $w \in W$ for which $\lim_{\lambda \rightarrow \infty} \lambda w$ exists, and F_+ consists exactly of the points $w \in W$ for which $\lim_{\lambda \rightarrow 0} \lambda w$ exists.

REMARK 3.5. The above trivialization extends to a trivialization $\mathbb{C}^2 \times \mathbb{G}_m \xrightarrow{\sim} W \setminus F_+$, but for $W \setminus F_-$ there is no such trivialization since the \mathbb{G}_m -isotropy group of a point in $F_+ \setminus F_-$ has order 2.

Now let us turn to the \mathbb{G}_a -action $\mathbb{G}_a \times W \rightarrow W, (\tau, w) \mapsto \tau \cdot w$, corresponding to $\delta: B \rightarrow B$. Since δ is homogeneous of degree l , it is normalized by the \mathbb{G}_m -action, i.e. for $w \in W$, $\tau \in \mathbb{G}_a$ and $\lambda \in \mathbb{G}_m$, we have

$$(\lambda^{-l}\tau) \cdot (\lambda w) = \lambda(\tau \cdot w).$$

In particular this implies that λO is a \mathbb{G}_a -orbit for any \mathbb{G}_a -orbit O .

Lemma 3.6. *Let $\delta: B \rightarrow B$ be a nontrivial locally nilpotent derivation, homogeneous of degree l . Then either*

- (1) $l < 0$ and F_+ is invariant, or
- (2) $l > 0$ and F_- is invariant.

Proof. Since the locally nilpotent derivation $\delta: B \rightarrow B$ is homogeneous, its kernel

$$B^\delta := \{f \in B \mid \delta(f) = 0\} = \{f \in B \mid f(\tau \cdot w) = f(w), \forall \tau \in \mathbb{G}_a\}$$

is a graded subalgebra, i.e.:

$$B^\delta = \bigoplus_{n \in \mathbb{Z}} B_n^\delta.$$

Given $f \in B_k \setminus \{0\}$, we have $\delta^\nu f \in B_{k+\nu l}^\delta \setminus \{0\}$ for a suitable $\nu \in \mathbb{N}$. It follows that

- (1) if $l = 0$, we have $B_n^\delta \neq \{0\}$ for all $n \neq 0$,
- (2) if $l > 0$ we have $B_n^\delta \neq \{0\}$ for some $n > 0$,
- (3) if $l < 0$ we have $B_n^\delta \neq \{0\}$ for some $n < 0$.

First assume that $l > 0$, so that $B_n^\delta \neq 0$ for some n , and let $f \in B_n^\delta \setminus \{0\}$. Then f vanishes on F_- since $f(\lambda x) = \lambda^n f(x)$ and since $\lim_{\lambda \rightarrow \infty} \lambda x$ exists in W for $x \in F_-$. It follows that F_- is invariant since it is an irreducible component of the invariant set $V(f) \subset W$ of dimension two. If $l < 0$, it follows analogously that $F_+ \subset W$ is invariant. It remains to show that l cannot be zero.

If $l = 0$, both F_- and F_+ are invariant. So $p^{-1}(N)$ is invariant and $W \setminus p^{-1}(N)$ as well. Then for any nontrivial \mathbb{G}_a -orbit $O \subset W \setminus p^{-1}(N)$ the map $(z^3 + t^2) \circ p|_O$ has no zeros, and thus must be constant, say with value $a \in \mathbb{C}^*$, since $O \cong \mathbb{C}$. However, any morphism $p|_O: O \rightarrow V(\mathbb{C}^2; z^3 + t^2 - a)$ from the complex line to the smooth affine elliptic curve $V(\mathbb{C}^2; z^3 + t^2 - a)$ is constant, so O is contained in a p -fiber. Since $p(O) \in \mathbb{C}^2 \setminus N$, this p -fiber is isomorphic to \mathbb{G}_m , as p is a \mathbb{G}_m -principal bundle over $\mathbb{C}^2 \setminus N$. This gives a contradiction since \mathbb{C} cannot be embedded into \mathbb{G}_m . \square

Proof of Proposition 3.3. By Lemma 3.6 it is enough to show that F_- , given by $y = 0$, is not invariant. Suppose to the contrary that F_- is invariant; then its complement in W , given by $y \neq 0$, is invariant as well. Since there is no non-constant invertible function on \mathbb{G}_a -orbits, all \mathbb{G}_a -orbits in $W \setminus F_-$ are contained in level hypersurfaces of y . In particular the hypersurface $V \subset W$ which is given by $y = 1$ is invariant and we have $V \simeq \{(x, y, z) \in \mathbb{C}^3 \mid x^2 + z^3 + t^2 = 0\}$. The restriction of the \mathbb{G}_m -quotient projection

$$\psi := p|_V: V \rightarrow \mathbb{C}^2, \quad (x, z, t) \mapsto (z, t)$$

is a two sheeted branched covering of \mathbb{C}^2 with branch locus $\psi^{-1}(N) \cong N$ and deck transformation

$$\sigma: V \rightarrow V, \quad (x, z, t) \mapsto (-x, z, t),$$

which is simply the action of $-1 \in \mathbb{G}_m$. In particular $\sigma(O)$ is a \mathbb{G}_a -orbit for any \mathbb{G}_a -orbit O . Assume for the moment that every nontrivial \mathbb{G}_a -orbit intersects $\psi^{-1}(N)$ exactly once. Since the hypersurface V is a normal surface, there is a quotient map

$$\chi: V \rightarrow V//\mathbb{G}_a := \text{Spec}(\mathcal{O}(V)^{\mathbb{G}_a}),$$

the generic fiber of which is a \mathbb{G}_a -orbit [4, Lemma 1.1]. Thus the restriction

$$\chi|_{\psi^{-1}(N)}: \psi^{-1}(N) \rightarrow V//\mathbb{G}_a$$

is injective on a nonempty open subset of $\psi^{-1}(N)$. Hence, V/\mathbb{G}_a being a smooth curve, it follows from Zariski's main theorem that this restriction is an open embedding. However, this is a contradiction since the affine cuspidal cubic curve $\psi^{-1}(N)$ has a singular point.

Finally, any nontrivial \mathbb{G}_a -orbit O intersects $\psi^{-1}(N)$: otherwise $\psi|_O$ would be a non-constant morphism to an affine elliptic curve $z^3 + t^2 = a$ for some $a \in \mathbb{C}^*$, which is impossible. Note that a point in $O \cap \psi^{-1}(N)$ is a common point of the two \mathbb{G}_a -orbits $\sigma(O)$ and O , so $\sigma(O) = O$. Choose an equivariant isomorphism $\mathbb{C} \cong O$ such that $0 \in \mathbb{C}$ corresponds to a point in $\psi^{-1}(N)$. Then the involution $\sigma: O \rightarrow O$ corresponds to $\mathbb{C} \rightarrow \mathbb{C}$, $\zeta \mapsto -\zeta$, and as a consequence every nontrivial \mathbb{G}_a -orbit $O \hookrightarrow V$ intersects the branch locus $\psi^{-1}(N) = W^\sigma$ (the fixed point set of σ) in exactly one point. \square

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