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## ON GENERA OF LEFSCHETZ FIBRATIONS AND FINITELY PRESENTED GROUPS

RYOMA KOBAYASHI

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### Abstract

It is known that every finitely presented group is the fundamental group of the total space of a Lefschetz fibration. In this paper, we give another proof which improves the result of Korkmaz. In addition, Korkmaz defined the genus of a finitely presented group. We also evaluate upper bounds for genera of some finitely presented groups.

### 1. Introduction

Gompf [5] proved that every finitely presented group is the fundamental group of a closed symplectic 4-manifold. Donaldson [4] proved that every closed symplectic 4-manifold admits a Lefschetz pencil. By blowing up the base locus of a Lefschetz pencil, we obtain a Lefschetz fibration over  $S^2$ . In addition, blowing up does not change the fundamental group of a 4-manifold. Therefore, it immediately follows that every finitely presented group is the fundamental group of the total space of a Lefschetz fibration.

Amoros–Bogomolov–Katzarkov–Pantev [1] and Korkmaz [8] also constructed Lefschetz fibrations whose fundamental groups are a given finitely presented group. In particular, Korkmaz [8] provided explicitly a genus and a monodromy of such a Lefschetz fibration.

Let  $F_n = \langle g_1, \dots, g_n \rangle$  be the free group of rank  $n$ . For  $x \in F_n$ , the *syllable length*  $l(x)$  of  $x$  is defined by

$$l(x) = \min \{s \mid x = g_{i(1)}^{m(1)} \cdots g_{i(s)}^{m(s)}, 1 \leq i(j) \leq n, m(j) \in \mathbb{Z}\}.$$

For a finitely presented group  $\Gamma$  with a presentation  $\Gamma = \langle g_1, \dots, g_n \mid r_1, \dots, r_k \rangle$ , Korkmaz [8] proved that for any  $g \geq 2(n + \sum_{1 \leq i \leq k} l(r_i) - k)$  there exists a genus- $g$  Lefschetz fibration  $f: X \rightarrow S^2$  such that the fundamental group  $\pi_1(X)$  is isomorphic to  $\Gamma$ , providing explicitly a monodromy.

In this paper, we improve this result.

**Theorem 1.1.** *Let  $\Gamma$  be a finitely presented group with a presentation  $\Gamma = \langle g_1, \dots, g_n \mid r_1, \dots, r_k \rangle$ , and let  $l = \max_{1 \leq i \leq k} \{l(r_i)\}$ . Then for any  $g \geq 2n + l - 1$ , there*

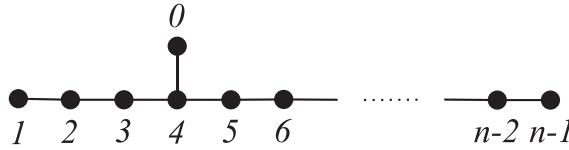


Fig. 1. The Dynkin diagram.

exists a genus- $g$  Lefschetz fibration  $f: X \rightarrow S^2$  such that the fundamental group  $\pi_1(X)$  is isomorphic to  $\Gamma$ .

In this theorem, if  $k = 0$ , we suppose  $l = 1$ . We will prove the theorem by providing an explicit monodromy.

In addition, Korkmaz [8] defined the *genus*  $g(\Gamma)$  of a finitely presented group  $\Gamma$  to be the minimal genus of a Lefschetz fibration with sections whose fundamental group is isomorphic to  $\Gamma$ . The Lefschetz fibrations constructed in Theorem 1.1 have sections. Hence the definition of the genus of a finitely presented group is well-defined.

We will also prove the following theorem.

**Theorem 1.2.** (1) Let  $B_n$  denote the  $n$ -strands braid group. Then for  $n \geq 3$ , we have  $2 \leq g(B_n) \leq 4$ .  
 (2) Let  $\mathcal{H}_g$  be the hyperelliptic mapping class group of a closed connected orientable surface of genus  $g \geq 1$ . Then we have  $2 \leq g(\mathcal{H}_g) \leq 4$ .  
 (3) Let  $\mathcal{M}_{0,n}$  denote the mapping class group of a sphere with  $n$  punctures. Then for  $n \geq 3$ , we have  $2 \leq g(\mathcal{M}_{0,n}) \leq 4$ .  
 (4) Let  $S_n$  denote the  $n$ -symmetric group. Then for  $n \geq 3$ , we have  $2 \leq g(S_n) \leq 4$ .  
 (5) Let  $\mathcal{A}_n$  denote the  $n$ -Artin group associated to the Dynkin diagram shown in Fig. 1. Then for  $n \geq 6$ , we have  $2 \leq g(\mathcal{A}_n) \leq 5$ .  
 (6) Let  $n, k \geq 0$  be integers with  $n + k \geq 3$ , and let  $m_1, \dots, m_k \geq 2$  be integers. Then we have  $(n + k + 1)/2 \leq g(\mathbb{Z}^n \oplus \mathbb{Z}_{m_1} \oplus \dots \oplus \mathbb{Z}_{m_k}) \leq n + k + 1$ .

## 2. A Lefschetz fibration and preliminaries

**2.1. A Lefschetz fibration and its monodromy.** Here, we review briefly the theory of Lefschetz fibrations.

Let  $X$  be a closed connected orientable smooth 4-manifold. A smooth map  $f: X \rightarrow S^2$  is a genus- $g$  *Lefschetz fibration* over  $S^2$  if it satisfies following properties:

- All regular fibers are diffeomorphic to a closed connected oriented surface of genus  $g$ .
- Each critical point of  $f$  has an orientation-preserving chart on which  $f(z_1, z_2) = z_1^2 + z_2^2$  relative to a suitable smooth chart on  $S^2$ .
- Each singular fiber contains only one critical point.

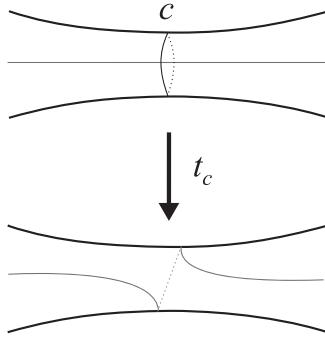
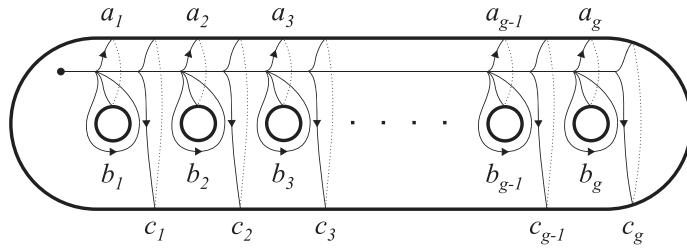
Fig. 2. The right Dehn twist about  $c$ .

Fig. 3.

- $f$  is *relatively minimal*, that is, no fiber contains an embedded sphere with the self-intersection number  $-1$ .

Let  $\mathcal{M}_g$  be the mapping class group of a closed connected oriented surface  $\Sigma_g$  of genus  $g$ , that is, the group of isotopy classes of orientation-preserving diffeomorphisms  $\Sigma_g \rightarrow \Sigma_g$ . In this paper, for elements  $x$  and  $y$  of a group, the composition  $xy$  means that we first apply  $x$  and then  $y$ . So for  $f, g \in \mathcal{M}_g$ , the composition  $fg$  means that we first apply  $f$  and then  $g$ . For a simple closed curve  $c$  on  $\Sigma_g$ , let  $t_c$  be the isotopy class of the right Dehn twist about  $c$  (see Fig. 2). For a genus- $g$  Lefschetz fibration which has  $n$  singular fibers, there are simple closed curves  $c_1, \dots, c_n$  on  $\Sigma_g$ , each of which is called the *vanishing cycle*, such that each singular fiber  $F_i$  is obtained by collapsing  $c_i$  to a point to create a transverse self-intersection, and  $t_{c_1} \cdots t_{c_n} = 1$ . This equation is called the *monodromy* of a Lefschetz fibration. Conversely, if there are simple closed curves  $c_1, \dots, c_n$  on  $\Sigma_g$  such that  $t_{c_1} \cdots t_{c_n} = 1$ , then we can construct a genus- $g$  Lefschetz fibration with the monodromy  $t_{c_1} \cdots t_{c_n} = 1$ .

For a Lefschetz fibration  $f: X \rightarrow S^2$ , a smooth map  $s: S^2 \rightarrow X$  is a section of  $f$  if  $f \circ s: S^2 \rightarrow S^2$  is the identity map.

For a closed connected orientable surface  $\Sigma_g$  of genus  $g$ , let  $a_1, \dots, a_g, b_1, \dots, b_g$  and  $c_1, \dots, c_g$  be loops on  $\Sigma_g$  as shown in Fig. 3. Then the fundamental group  $\pi_1(\Sigma_g)$

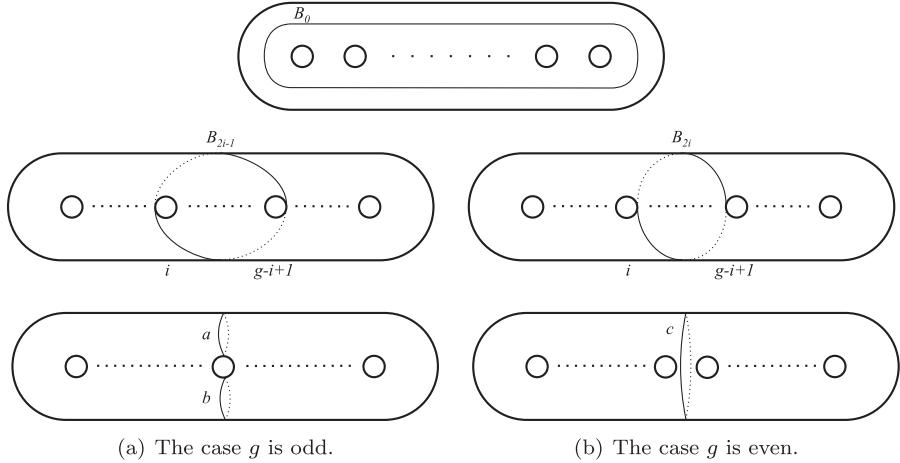


Fig. 4.

of  $\Sigma_g$  has a following presentation

$$\pi_1(\Sigma_g) = \langle a_1, b_1, \dots, a_g, b_g \mid r \rangle,$$

where  $r = b_g^{-1} \cdots b_1^{-1} (a_1 b_1 a_1^{-1}) \cdots (a_g b_g a_g^{-1})$ .

Let  $B_0, \dots, B_g$  and  $a, b, c$  be simple closed curves on  $\Sigma_g$  as shown in Fig. 4. In this paper, let  $W$  denote the following

$$W = \begin{cases} (t_c t_{B_g} \cdots t_{B_0})^2 & \text{when } g \text{ is even,} \\ (t_a^2 t_b^2 t_{B_g} \cdots t_{B_0})^2 & \text{when } g \text{ is odd.} \end{cases}$$

It was shown in [7] that  $W = 1$  in the mapping class group  $\mathcal{M}_g$  of  $\Sigma_g$ . In addition, the Lefschetz fibration  $f_W: X_W \rightarrow S^2$  with the monodromy  $W = 1$  has a section (see [7] and [8]).

**2.2. Preliminaries.** We now state the way to obtain the presentation of the fundamental group of a Lefschetz fibration with a section. For a group  $\Gamma$  and  $\{x_1, \dots, x_n\} \subset \Gamma$ , let  $\langle x_1, \dots, x_n \rangle$  denote the normal closure of  $\{x_1, \dots, x_n\}$  in  $\Gamma$ .

**Proposition 2.1** (cf. [6]). *Let  $f: X \rightarrow S^2$  be a genus- $g$  Lefschetz fibration with the monodromy  $t_{c_1} \cdots t_{c_n} = 1$ . Suppose that  $f$  has a section. Then we have*

$$\pi_1(X) \cong \pi_1(\Sigma_g) / \langle c_1, \dots, c_n \rangle,$$

where we regard  $c_1, \dots, c_n$  as elements in  $\pi_1(\Sigma_g)$ .

For  $x, y \in \mathcal{M}_g$ , let  $x^y = y^{-1}xy$ . For example, for simple closed curves  $c_1, \dots, c_n$  on  $\Sigma_g$  and  $h \in \mathcal{M}_g$ , we have  $(t_{c_1} \cdots t_{c_n})^h = (h^{-1}t_{c_1}h) \cdots (h^{-1}t_{c_n}h) = t_{(c_1)h} \cdots t_{(c_n)h}$ , where  $(c_i)h$  means the image of  $c_i$  by  $h$ .

**Proposition 2.2** ([8]). *Let  $f: X \rightarrow S^2$  be a genus- $g$  Lefschetz fibration with the monodromy  $V = t_{c_1} \cdots t_{c_n} = 1$ . Suppose that  $f$  has a section. Let  $d$  be a simple closed curve on  $\Sigma_g$  which intersects some  $c_i$  transversely at only one point. Let  $f': X' \rightarrow S^2$  be the genus- $g$  Lefschetz fibration with the monodromy  $VV^{t_d} = 1$ . Then we have*

$$\pi_1(X') \cong \pi_1(\Sigma_g)/\langle c_1, \dots, c_n, d \rangle,$$

where we regard  $c_1, \dots, c_n$  and  $d$  as elements in  $\pi_1(\Sigma_g)$ .

In this paper, we denote the Lefschetz fibration with the monodromy  $V = 1$  by  $f_V: X_V \rightarrow S^2$ . For example, in the above proposition,  $f = f_V$ ,  $X = X_V$  and  $f' = f_{VV^{t_d}}$ ,  $X' = X_{VV^{t_d}}$ .

We next state results of Korkmaz [8].

**Theorem 2.3** ([8]). (1) *Let  $\Sigma_g$  be a closed connected orientable surface of genus  $g \geq 0$ . Then we have  $g(\pi_1(\Sigma_g)) = g$ .*  
 (2) *Let  $m(\Gamma)$  denote the minimal number of generators for  $\Gamma$ . Then we have  $m(\Gamma)/2 \leq g(\Gamma)$ , with the equality if and only if  $\Gamma$  is isomorphic to  $\pi_1(\Sigma_g)$ .*  
 (3) *For the mapping class group  $\mathcal{M}_1$  of  $\Sigma_1$ , we have  $2 \leq g(\mathcal{M}_1) \leq 4$ .*  
 (4) *Let  $B_n$  denote the  $n$ -strands braid group. Then for  $n \geq 3$ , we have  $2 \leq g(B_n) \leq 5$ .*  
 (5) *Let  $n, k \geq 0$  be integers with  $n + k \geq 3$ , and let  $m_1, \dots, m_k \geq 2$  be integers. Then we have  $(n + k + 1)/2 \leq g(\mathbb{Z}^n \oplus \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_k}) \leq 2(n + k) + 1$ .*

Theorem 1.2 improves Theorem 2.3 (4) and (5).

### 3. Proof of Theorem 1.1

First of all, we show a proposition used in proofs of Theorem 1.1 and 1.2. For elements  $x$  and  $y$  in a group, let  $[x, y] = xyx^{-1}y^{-1}$ . For a real number  $a$ ,  $[a]$  is the maximal integer less than or equal to  $a$ .

**Proposition 3.1.** *Let  $f_W: X_W \rightarrow S^2$  be the genus- $g$  Lefschetz fibration with the monodromy  $W = 1$ , where  $W$  is as above, and let  $a_1, b_1, \dots, a_g, b_g$  be the generators of  $\pi_1(\Sigma_g)$  as shown in Fig. 3. Then we have followings:*

(1) (See [8].) Let  $U = WW^{t_{b_1}} \cdots W^{t_{b_g}}$ , then the fundamental group  $\pi_1(X_U)$  of the Lefschetz fibration  $X_U$  has the following presentation

$$\pi_1(X_U) = \begin{cases} \left\langle a_1, b_1, \dots, a_g, b_g \mid \begin{array}{l} b_1, \dots, b_g, \\ a_1 a_g, \dots, a_{g/2} a_{(g+2)/2} \end{array} \right\rangle & \text{when } g \text{ is even,} \\ \left\langle a_1, b_1, \dots, a_g, b_g \mid \begin{array}{l} b_1, \dots, b_g, \\ a_1 a_g, \dots, a_{(g-1)/2} a_{(g+3)/2}, \\ a_{(g+1)/2} \end{array} \right\rangle & \text{when } g \text{ is odd,} \end{cases}$$

and, the group  $\pi_1(X_U)$  is isomorphic to the free group of rank  $[g/2]$ .

(2) Let  $U' = WW^{t_{b_2}} \cdots W^{t_{b_{g-1}}}$ , then the fundamental group  $\pi_1(X_{U'})$  of the Lefschetz fibration  $X_{U'}$  has the following presentation

$$\pi_1(X_{U'}) = \begin{cases} \left\langle a_1, b_1, \dots, a_g, b_g \mid \begin{array}{l} [a_1, b_1], \\ b_2, \dots, b_{g-1}, \\ b_1 b_g, \\ a_1 a_g, \dots, a_{g/2} a_{(g+2)/2} \end{array} \right\rangle & \text{when } g \text{ is even,} \\ \left\langle a_1, b_1, \dots, a_g, b_g \mid \begin{array}{l} [a_1, b_1], \\ b_2, \dots, b_{g-1}, \\ b_1 b_g, \\ a_1 a_g, \dots, a_{(g-1)/2} a_{(g+3)/2}, \\ a_{(g+1)/2} \end{array} \right\rangle & \text{when } g \text{ is odd,} \end{cases}$$

and, the group  $\pi_1(X_{U'})$  is isomorphic to the free product of the free group of rank  $([g/2] - 1)$  with  $\mathbb{Z} \oplus \mathbb{Z}$ .

Proof. Simple closed curves  $B_0, \dots, B_g$  and  $a, b, c$  as shown in Fig. 4 can be described in  $\pi_1(\Sigma_g)$ , up to conjugation, as follows

- $B_{2k} = a_k b_{k+1} b_{k+2} \cdots b_{g-k-1} b_{g-k} c_{g-k} a_{g-k+1}$ , where  $0 \leq k \leq g/2$ ,
- $B_{2k+1} = a_{k+1} b_{k+1} b_{k+2} \cdots b_{g-k-1} b_{g-k} c_{g-k} a_{g-k}$ , where  $0 \leq k \leq g/2$ ,
- $a = a_{(g+1)/2}$ ,  $b = c_{(g-1)/2} a_{(g+1)/2}$  and  $c = c_{g/2}$ ,

where let  $a_0 = a_{g+1} = 1$ . In addition, note that  $c_i = b_i^{-1} \cdots b_1^{-1} (a_1 b_1 a_1^{-1}) \cdots (a_i b_i a_i^{-1})$  up to conjugation, for  $1 \leq i \leq g$ . Since  $X_W$  has a section, by Proposition 2.1, we first

obtain a presentation of  $\pi_1(X_W)$  as follows.

$$\pi_1(X_W) = \begin{cases} \left\langle a_1, b_1, \dots, a_g, b_g \mid \begin{array}{l} c_g, c_{g/2}, \\ a_1 a_g, \dots, a_{g/2} a_{(g+2)/2}, \\ b_1 a_g b_g a_g^{-1}, \dots, b_{g/2} a_{(g+2)/2} b_{(g+2)/2} a_{(g+2)/2}^{-1} \end{array} \right\rangle & \text{when } g \text{ is even,} \\ \left\langle a_1, b_1, \dots, a_g, b_g \mid \begin{array}{l} c_g, a_{(g+1)/2}, b_{(g+1)/2}, c_{(g-1)/2}, \\ a_1 a_g, \dots, a_{(g-1)/2} a_{(g+3)/2}, \\ b_1 a_g b_g a_g^{-1}, \dots, b_{(g-1)/2} a_{(g+3)/2} b_{(g+3)/2} a_{(g+3)/2}^{-1} \end{array} \right\rangle & \text{when } g \text{ is odd.} \end{cases}$$

(We have that  $\pi_1(X_W)$  is isomorphic to  $\pi_1(\Sigma_{[g/2]})$ .) Since each  $b_i$  intersects some  $B_j$  transversely at only one point, by Proposition 2.2, we obtain the claim.  $\square$

REMARK. From Proposition 3.1, we have followings.

- For  $n \geq 1$ , there are genus- $2n$  and  $(2n+1)$  Lefschetz fibrations whose fundamental groups are isomorphic to the free group of rank  $n$ .
- For  $n \geq 2$ , there are genus- $(2n-2)$  and  $(2n-1)$  Lefschetz fibrations whose fundamental groups are isomorphic to the free product of the free group of rank  $(n-2)$  with  $\mathbb{Z} \oplus \mathbb{Z}$ .

Let  $\Gamma$  be a finitely presented group with a presentation  $\Gamma = \langle g_1, \dots, g_n \mid r_1, \dots, r_k \rangle$  and let  $l = \max_{1 \leq i \leq k} \{l(r_i)\}$ . For  $g \geq n + l - 1$  and  $r_i$ , we construct a simple closed curve  $R_i$  on  $\Sigma_g$  as below.

At first, we construct a simple closed curve  $R$  in the case  $n = 4$  and  $r = g_2 g_1 g_2^2 g_4^{-1} g_3^{-2}$  as an example. Note that  $l(r) = 5$ . Let  $x_1, x_2, x_3, x_4, x_5$  be loops on  $\Sigma_g$  which are homotopic to  $a_2, a_1, a_2, a_4$  and  $a_3$ , respectively, as shown in Fig. 5 (a). Let  $y_1, y_2, y_3, y_4$  be loops on  $\Sigma_g$  which are homotopic to  $a_5, a_6, a_7, a_8$ , respectively, and let  $z_1, z_2, z_3, z_4$  be loops on  $\Sigma_g$  which are homotopic to  $a_5, a_6, a_7, a_8$ , respectively, as shown in Fig. 5 (a). First we deform  $\Sigma_g$  around  $y_1, z_1, \dots, y_4, z_4$  as shown in Fig. 5 (b). Then let  $D$  be a subsurface containing  $y_t$  and  $z_t$  which is surrounded by a simple closed curve on  $\Sigma_g$  as shown in Fig. 5 (b). Next, for  $1 \leq t \leq 4$ , we move  $y_t$  to the right side of  $x_t$  in  $D$ , and  $z_t$  to the left side of  $x_{t+1}$  in  $D$ , as shown in Fig. 5 (c). Let  $\bar{R}$  be the loop as shown in Fig. 6 (a), and let  $R = (\bar{R})t_{x_1}^{-1}t_{x_2}^{-1}t_{x_3}^{-2}t_{x_4}t_{x_5}^2$ , as shown in Fig. 6 (b). Finally, we deform the surface so that  $y_1, \dots, y_4$  and  $z_1, \dots, z_4$  go back to their original position as shown in Fig. 6 (c).

In general, a loop  $R_i$  is constructed as follows. Let  $r_i = g_{j(1)}^{m(1)} \cdots g_{j(l(r_i))}^{m(l(r_i))}$ . For  $1 \leq t \leq l(r_i)$ , let  $x_t$  be a loop on  $\Sigma_g$  which is homotopic to  $a_{j(t)}$ . If  $j(s) = j(s')$  for some  $s < s'$ , we put  $x_{s'}$  to the right side of  $x_s$ . For  $1 \leq t \leq l(r_i) - 1$ , let  $y_t$  and  $z_t$  be loops on  $\Sigma_g$  which are homotopic to  $a_{n+t}$ , such that  $z_t$  is in the right side of  $y_t$ .

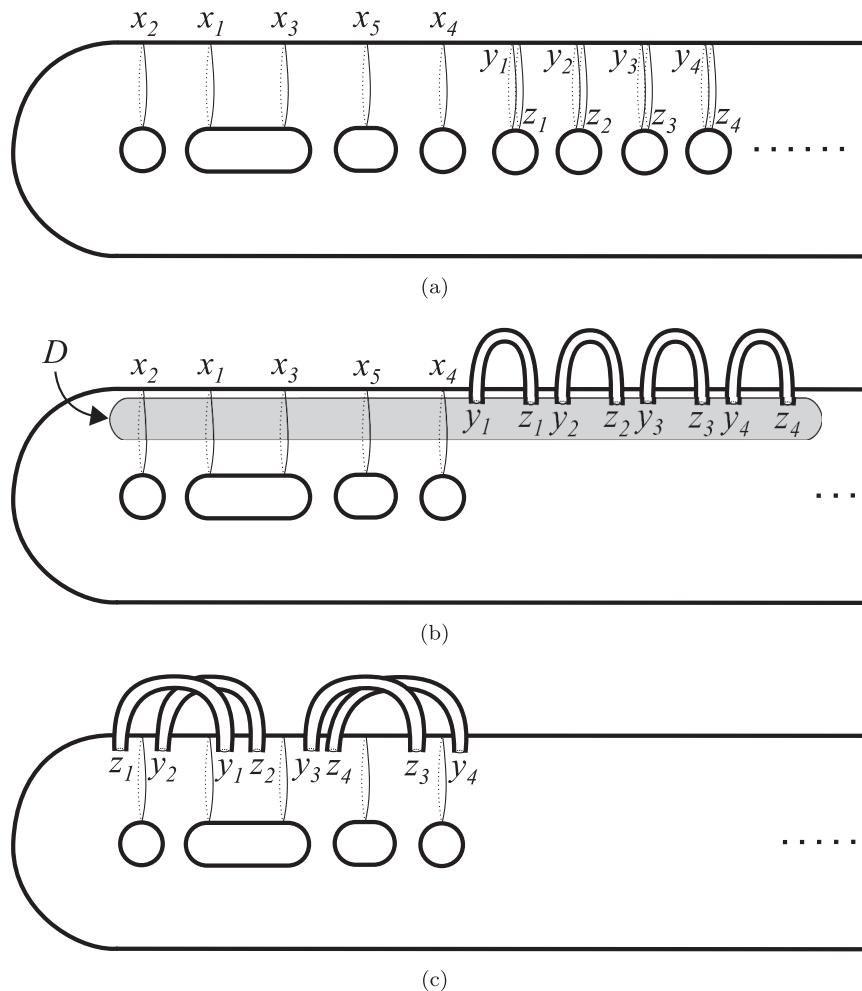


Fig. 5. The loop  $R$  in the case  $n = 4$ ,  $r = g_2g_1g_2^2g_4^{-1}g_3^{-2}$ .

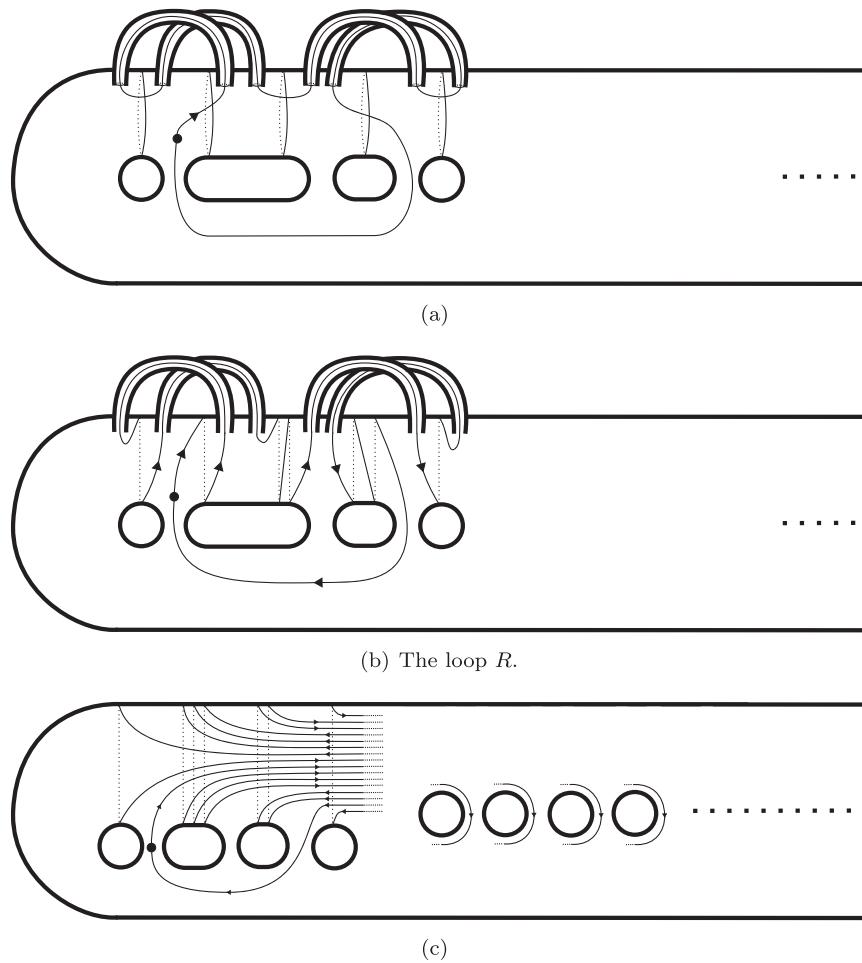
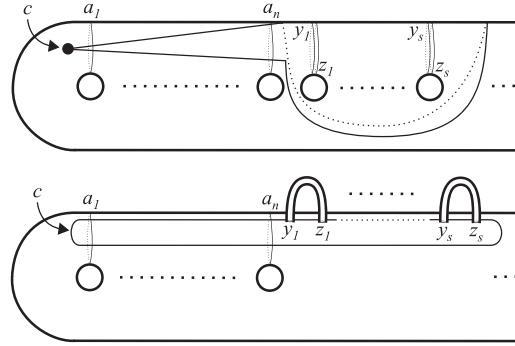


Fig. 6. The loop  $R$  in the case  $n = 4$ ,  $r = g_2g_1g_2^2g_4^{-1}g_3^{-2}$ .

Fig. 7. The loop  $c$  where  $s = l(r_i) - 1$ .

First we deform  $\Sigma_g$  around  $y_1, z_1, \dots, y_{l(r_i)-1}, z_{l(r_i)-1}$ , similarly to the above example. Let  $c$  be a simple closed curve which is described in  $\pi_1(\Sigma_g)$  as follows

$$c = (a_{n+1}b_{n+1}a_{n+1}^{-1}) \cdots (a_{n+l(r_i)-1}b_{n+l(r_i)-1}a_{n+l(r_i)-1}^{-1})b_{n+l(r_i)-1}^{-1} \cdots b_{n+1}^{-1},$$

and intersects each of  $a_1, \dots, a_n$  at two points, as shown in Fig. 7. Then let  $D$  be a subsurface whose boundary is  $c$ , and which contains  $y_t$  and  $z_t$ .

Next we deform  $D$  as follows. For  $1 \leq t \leq l(r_i) - 1$ , we move  $y_t$  to just right side of  $x_t$  in  $D$ , and  $z_t$  to just left side of  $x_{t+1}$  in  $D$  as shown in Fig. 5 (c). We regard that this motion does not affect on loops  $a_i, b_i$  and  $c_i$ . Hence  $x_1, \dots, x_{l(r_i)}$  also do not deform, as shown in Fig. 5 (c).

After that, we define a simple closed curve as shown in Fig. 6 (a). More precisely, we construct arcs  $L_i$  and  $L'_i$  as follows. The arc  $L_i$  is in  $D$ .  $L_i$  begins from the point at the left side of  $x_1$  on the loop  $c$ , crosses  $x_1, y_1, z_1, x_2, y_2, z_2, \dots$ , in this order, finally crosses  $x_{l(r_i)}$ , and stops at the right side of  $x_{l(r_i)}$  on the loop  $c$ . Let  $L'_i$  be an arc whose base point is the end point of  $L_i$ , end point is the base point of  $L_i$ , and which does not intersect the interior of  $D$  and loops  $a_1, b_1, \dots, a_n, b_n$  and  $c_n$ . Note that the surface which is obtained by removing loops  $c, a_1, b_1, \dots, a_n, b_n$  and  $c_n$  from  $\Sigma_g$ , and which contains  $L'_i$  is a disk. Hence the arc  $L'_i$  is unique up to homotopy relative to the base point and the end point. Let  $L_i \cdot L'_i$  denote the composition of  $L_i$  and  $L'_i$ .

We now define  $R_i = (L_i \cdot L'_i)t_{x_1}^{-m(1)} \cdots t_{x_{l(r_i)}}^{-m(l(r_i))}$ . Finally, we deform the surface so that  $y_1, z_1, \dots, y_{l(r_i)-1}, z_{l(r_i)-1}$  go back to their original position.

Note that the loop  $R_i$  is described in  $\pi_1(\Sigma_g)$ , up to conjugation, as follows:

$$(*) \quad R_i = \left( \prod_{1 \leq t \leq m(1)} x_{i,1,t} a_{j(1)} \right) \cdots \left( \prod_{1 \leq t \leq m(l(r_i))} x_{i,l(r_i),t} a_{j(l(r_i))} \right) \tilde{L}_i,$$

where  $x_{i,s,t}$  is a loop which is some products of  $a_{n+1}, b_{n+1}, \dots, a_{l(r_i)-1}, b_{l(r_i)-1}$  and  $c_{n+1}$ , and  $\tilde{L}_i$  is a loop which is described in  $\pi_1(\Sigma_g)$  as follows:

$$\tilde{L}_i = \begin{cases} b_{j(l(r_i))}^{-1} b_{j(l(r_i))-1}^{-1} \cdots b_{j(1)+1}^{-1} b_{j(1)}^{-1} & \text{when } j(1) \leq j(l(r_i)), \\ b_{j(l(r_i))} b_{j(l(r_i))-1} \cdots b_{j(1)} b_{j(1)-1} & \text{when } j(1) > j(l(r_i)). \end{cases}$$

We now prove Theorem 1.1.

Proof of Theorem 1.1. For  $g \geq 2n + l - 1$ , let  $V$  be the following

$$V = UW^{t_{a_{n+1}}} \cdots W^{t_{a_{[g/2]}}},$$

where  $U = WW^{t_{b_1}} \cdots W^{t_{b_g}}$ . In addition, let  $V'$  be the following

$$V' = VV^{t_{R_1}} \cdots V^{t_{R_k}},$$

where  $R_i$  is the loop constructed previously. We show that the fundamental group  $\pi_1(X_{V'})$  is isomorphic to  $\Gamma$ .

Since each of  $b_1, \dots, b_g$  and  $a_{n+1}, \dots, a_{[g/2]}$  intersects some  $B_i$  transversely at only one point, by Proposition 2.2, we have

$$\begin{aligned} \pi_1(X_V) &= \pi_1(\Sigma_g) / \langle b_1, \dots, b_g, a_{n+1}, \dots, a_{[g/2]} \rangle \\ &= \pi_1(X_U) / \langle a_{n+1}, \dots, a_{[g/2]} \rangle. \end{aligned}$$

In addition, by the presentation of (1) of Proposition 3.1, we have

$$\pi_1(X_U) = \langle a_1, \dots, a_{[g/2]} \rangle.$$

Therefore we have

$$\begin{aligned} \pi_1(X_V) &= \langle a_1, \dots, a_{[g/2]} \mid a_{n+1}, \dots, a_{[g/2]} \rangle \\ &= \langle a_1, \dots, a_n \rangle, \end{aligned}$$

Because of the presentation of  $\pi_1(X_U)$  in (1) of Proposition 3.1, we assume  $g \geq 2n + l - 1$  in place of  $g \geq n + l - 1$ .

For any  $1 \leq i \leq k$ , consider the vanishing cycle  $((B_0)t_{a_{n+1}})t_{R_i}$  of  $X_{V'}$ . Note that  $(B_0)t_{a_{n+1}}$  and  $(a_{n+1})t_{R_i}$  are described in  $\pi_1(\Sigma_g)$ , up to conjugation, as follows:

- $(B_0)t_{a_{n+1}} = a_{n+1}(b_1 \cdots b_g)$ ,
- $(a_{n+1})t_{R_i} = a_{n+1}(zR_iz^{-1})$  for some  $z \in \pi_1(\Sigma_g)$ .

Then, we have that  $((B_0)t_{a_{n+1}})t_{R_i}$  is described in  $\pi_1(\Sigma_g)$  as follows:

$$\begin{aligned} ((B_0)t_{a_{n+1}})t_{R_i} &= (x \cdot a_{n+1}(b_1 \cdots b_n) \cdot x^{-1})t_{R_i} \\ &= (x)t_{R_i}(a_{n+1})t_{R_i}(b_1 \cdots b_n)t_{R_i}(x^{-1})t_{R_i} \\ &= (x)t_{R_i}(y \cdot a_{n+1}(zR_iz^{-1}) \cdot y^{-1})(w \cdot (B_0)t_{R_i} \cdot w^{-1})((x)t_{R_i})^{-1}, \end{aligned}$$

for some elements  $x, y$  and  $w$  in  $\pi_1(\Sigma_g)$ . Since  $a_{n+1} = (B_0)t_{R_i} = 1$  in  $\pi_1(X_V)$ , we have  $R_i = 1$  from  $((B_0)t_{a_{n+1}})t_{R_i} = 1$ , in  $\pi_1(X_V)$ . For a vanishing cycle  $c$  of  $X_V$ , if  $R_i$  intersects  $c$  transversely at  $s$  points, then the vanishing cycle  $(c)t_{R_i}$  of  $X_V$  is described in  $\pi_1(\Sigma_g)$ , up to conjugation, as follows:

$$(c)t_{R_i} = x_1 R_i^{\varepsilon_1} \cdots x_s R_i^{\varepsilon_s} x_{s+1},$$

where  $\varepsilon_j = \pm 1$  and  $x_1, \dots, x_{s+1}$  are elements in  $\pi_1(\Sigma_g)$  such that  $c = x_1 \cdots x_{s+1}$ . Since  $R_i = 1$  and  $c = 1$  in  $\pi_1(X_V)$ , we can delete the relation  $(c)t_{R_i} = 1$  of  $\pi_1(X_V)$ . We now define  $\hat{r}_i = a_{j(1)}^{m(1)} \cdots a_{j(l(r_i))}^{m(l(r_i))}$  for  $r_i = g_{j(1)}^{m(1)} \cdots g_{j(l(r_i))}^{m(l(r_i))}$ . Since  $x_{i,s,t}$  and  $\tilde{L}_i$  in the description  $(*)$  of  $R_i$  are 1 in  $\pi_1(X_V)$ , the natural epimorphism  $\pi_1(\Sigma_g) \twoheadrightarrow \pi_1(X_V)$  sends  $R_i$  to  $\hat{r}_i$ . Note that the vanishing cycles of  $X_V$  consist of  $c$  and  $(c)t_{R_i}$  for all vanishing cycles  $c$  of  $X_V$  and  $1 \leq i \leq k$ . Therefore, we have

$$\begin{aligned} \pi_1(X_V) &= \langle a_1, \dots, a_n \mid \hat{r}_1, \dots, \hat{r}_k \rangle \\ &\cong \Gamma. \end{aligned}$$

Thus, the proof of Theorem 1.1 is completed.  $\square$

#### 4. Proof of Theorem 1.2

In this section, we prove Theorem 1.2.

**4.1. Proof of (1) of Theorem 1.2.** For  $n \geq 2$ , let  $B_n$  denote the  $n$ -strands braid group. The group  $B_n$  has a presentation with generators  $\sigma_1, \dots, \sigma_{n-1}$  and with relations

- $\sigma_i \sigma_j \sigma_i^{-1} \sigma_j^{-1} = 1$ , where  $1 \leq i < j - 1 \leq n - 2$ ,
- $\sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1} = 1$ , where  $1 \leq i \leq n - 2$ .

Let  $x = \sigma_1$  and  $y = \sigma_1 \cdots \sigma_{n-1}$ . Then  $B_n$  can be presented with generators  $x, y$  and with relations

- $xy^k xy^{-k} x^{-1} y^k x^{-1} y^{-k} = 1$ , where  $2 \leq k \leq n - 2$ ,
- $xyxy^{-1} xyx^{-1} y^{-1} x^{-1} yx^{-1} y^{-1} = 1$ ,
- $(xy)^{n-1} y^{-n} = 1$ .

A correspondence between the first presentation and the second presentation is given by  $\sigma_i = y^{i-1} xy^{1-i}$  for  $1 \leq i \leq n - 1$ . See [8] for this presentation.

We now prove (1) of Theorem 1.2.

**Proof of (1) of Theorem 1.2.** For  $n \geq 3$ , since  $B_n$  is generated by two generators  $x, y$ , we have  $g(B_n) \geq 2$  from (2) of Theorem 2.3 (cf. [8]). Therefore, we prove  $g(B_n) \leq 4$  for  $n \geq 3$ .

Let  $R_{1,k}, R_2$  and  $R_{3,n}$  be simple closed curves on  $\Sigma_4$  as shown in Fig. 8, where  $2 \leq k \leq n - 2$ . Note that  $R_{1,k}, R_2$  and  $R_{3,n}$  intersect  $B_4$  transversely at only one point, for  $2 \leq k \leq n - 2$ . Loops  $R_{1,k}, R_2$  and  $R_{3,n}$  can be described in  $\pi_1(\Sigma_4)$ , up to conjugation, as follows

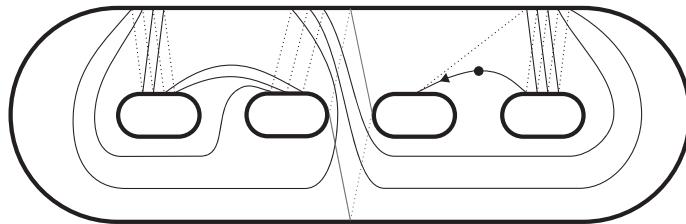
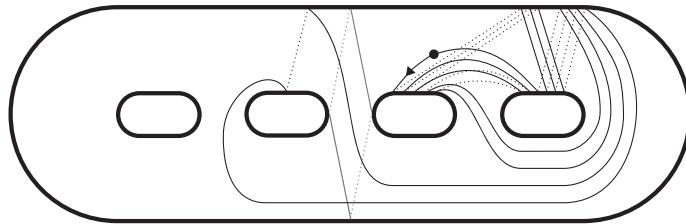
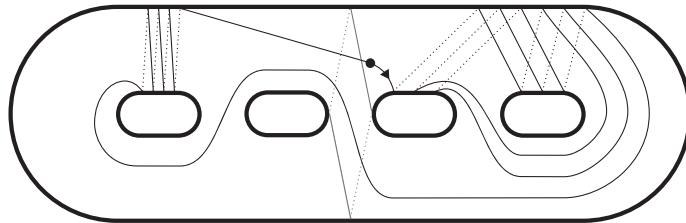
(a) The loop  $R_{1,k}$  with  $k = 2$ .(b) The loop  $R_2$ .(c) The loop  $R_{3,n}$  with  $n = 4$ .

Fig. 8.

- $R_{1,k} = a_3^{-1}a_4^{-k}(b_3b_4)^{-1}a_2a_1^{-k}(b_1)a_2^{-1}(b_1b_2)^{-1}a_1^ka_2^{-1}(b_3b_4)a_4^k$ , where  $2 \leq k \leq n-2$ ,
- $R_2 = a_3^{-1}a_4^{-1}(b_4^{-1})a_3^{-1}a_4a_3^{-1}a_4^{-1}(b_2b_3b_4)^{-1}a_2^{-1}(b_3b_4)a_4a_3a_4^{-1}a_3(b_4)a_4$ ,
- $R_{3,n} = (a_3^{-1}a_4^{-1}(b_4^{-1}))^{n-1}(b_1b_3)^{-1}a_1^{-n}$ .

Let  $V_1$  be the following:

$$V_1 = WW^{t_{b_1}}W^{t_{b_2}}W^{t_{b_3}}W^{t_{b_4}}\left(\prod_{2 \leq k \leq n-2} W^{t_{R_{1,k}}}\right)W^{t_{R_2}}W^{t_{R_{3,n}}}.$$

Then, from Proposition 2.2 and (1) of Proposition 3.1, the fundamental group  $\pi_1(X_{V_1})$  can be presented with generators  $a_2, a_1$  and with relations

- $a_2a_1^ka_2a_1^{-k}a_2^{-1}a_1^ka_2^{-1}a_1^{-k} = 1$ , where  $2 \leq k \leq n-2$ ,
- $a_2a_1a_2a_1^{-1}a_2a_1a_2^{-1}a_1^{-1}a_2^{-1}a_1a_2^{-1}a_1^{-1} = 1$ ,
- $(a_2a_1)^{n-1}a_1^{-n} = 1$ .

Let  $a_2 = x$  and  $a_1 = y$ . Then it follows that  $\pi_1(X_{V_1})$  is isomorphic to  $B_n$ . Therefore, for  $n \geq 3$  we have  $g(B_n) \leq 4$ .

Thus, the proof of (1) of Theorem 1.2 is completed.  $\square$

**4.2. Proof of (2) of Theorem 1.2.** For  $g \geq 1$ , let  $\mathcal{H}_g$  be the hyperelliptic mapping class group of  $\Sigma_g$ , that is, a subgroup of the mapping class group  $\mathcal{M}_g$  which consists of elements commutative with a hyperelliptic involution. It is well known that there is the natural epimorphism  $B_{2g+2} \rightarrow \mathcal{H}_g$ . For  $g \geq 2$ , Birman and Hilden [2] gave a presentation of the group  $\mathcal{H}_g$  with generators  $\sigma_1, \dots, \sigma_{2g+1}$  and with relations

- $\sigma_i\sigma_j\sigma_i^{-1}\sigma_j^{-1} = 1$ , where  $1 \leq i < j-1 \leq 2g$ ,
- $\sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1} = 1$ , where  $1 \leq i \leq 2g$ ,
- $(\sigma_1 \cdots \sigma_{2g+1})^{2g+2} = 1$ ,
- $(\sigma_1 \cdots \sigma_{2g+1}\sigma_{2g+1} \cdots \sigma_1)^2 = 1$ ,
- $[\sigma_1 \cdots \sigma_{2g+1}\sigma_{2g+1} \cdots \sigma_1, \sigma_1] = 1$ .

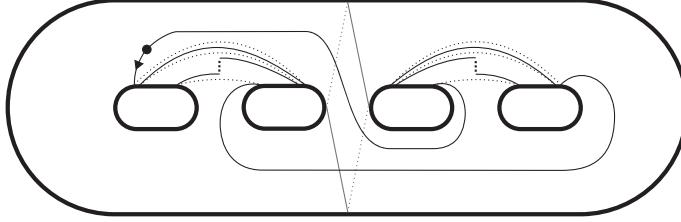
Similarly to Subsection 4.1, let  $x = \sigma_1$  and  $y = \sigma_1 \cdots \sigma_{2g+1}$ . Then, note that  $y^{2g+2} = 1$ . We calculate

$$\begin{aligned} \sigma_1 \cdots \sigma_{2g+1}\sigma_{2g+1} \cdots \sigma_1 &= y(y^{2g}xy^{-2g}) \cdots (yxy^{-1})x \\ &= y^{2g+1}(xy^{-1})^{2g}x \\ &= y^{-1}(xy^{-1})^{2g}x \\ &= (y^{-1}x)^{2g+1}. \end{aligned}$$

Then we have  $(\sigma_1 \cdots \sigma_{2g+1}\sigma_{2g+1} \cdots \sigma_1)^2 = (y^{-1}x)^{4g+2}$ . In addition, we have

$$\begin{aligned} [\sigma_1 \cdots \sigma_{2g+1}\sigma_{2g+1} \cdots \sigma_1, \sigma_1] &= (y^{-1}x)^{2g+1}x(x^{-1}y)^{2g+1}x^{-1} \\ &= (y^{-1}x)^{2g+1}(yx^{-1})^{2g+1}. \end{aligned}$$

Therefore,  $\mathcal{H}_g$  can be presented with generators  $x, y$  and with relations

Fig. 9. The loop  $R_6$ .

- $xy^kxy^{-k}x^{-1}y^kx^{-1}y^{-k} = 1$ , where  $2 \leq k \leq 2g$ ,
- $xyxy^{-1}xyx^{-1}y^{-1}x^{-1}yx^{-1}y^{-1} = 1$ ,
- $(xy)^{2g+1}y^{-2g-2} = 1$ ,
- $y^{2g+2} = 1$ ,
- $(y^{-1}x)^{4g+2} = 1$ ,
- $(y^{-1}x)^{2g+1}(yx^{-1})^{2g+1} = 1$ .

We now prove (2) of Theorem 1.2.

Proof of (2) of Theorem 1.2. For  $g \geq 2$ , since  $\mathcal{H}_g$  is generated by two generators  $x, y$ , we have  $g(\mathcal{H}_g) \geq 2$  from (2) of Theorem 2.3 (cf. [8]). Therefore, we prove  $g(\mathcal{H}_g) \leq 4$  for  $g \geq 2$ .

Let  $R_4$ ,  $R_5$  and  $R_6$  be simple closed curves on  $\Sigma_4$  described in  $\pi_1(\Sigma_4)$ , up to conjugation, as follows

- $R_4 = a_1^{2g+2}(b_1^{-1})$ ,
- $R_5 = (a_1^{-1}a_2)^{4g+2}(b_1^{-1})$ ,
- $R_6 = (a_1^{-1}a_2)^{2g+1}(b_2b_3b_4)(a_4^{-1}a_3)^{2g+1}(b_3^{-1})$ .

For the loop  $R_6$ , see Fig. 9. Note that  $R_4$ ,  $R_5$  and  $R_6$  intersect  $B_2$ ,  $B_1$  and  $B_4$  transversely at only one point, respectively. Let  $V_2$  be the following:

$$V_2 = WW^{t_{b_1}}W^{t_{b_2}}W^{t_{b_3}}W^{t_{b_4}} \left( \prod_{2 \leq k \leq 2g} W^{t_{R_{1,k}}} \right) W^{t_{R_2}}W^{t_{R_{3,2g+2}}}W^{t_{R_4}}W^{t_{R_5}}W^{t_{R_6}}.$$

Then, from Proposition 2.2 and (1) of Proposition 3.1, the fundamental group  $\pi_1(X_{V_2})$  can be presented with generators  $a_2, a_1$  and with relations

- $a_2a_1^k a_2 a_1^{-k} a_2^{-1} a_1^k a_2^{-1} a_1^{-k} = 1$ , where  $2 \leq k \leq 2g$ ,
- $a_2a_1a_2a_1^{-1}a_2a_1a_2^{-1}a_1^{-1}a_2^{-1}a_1a_2^{-1}a_1^{-1} = 1$ ,
- $(a_2a_1)^{2g+1}a_1^{-2g-2} = 1$ ,
- $a_1^{2g+2} = 1$ ,
- $(a_1^{-1}a_2)^{4g+2} = 1$ ,
- $(a_1^{-1}a_2)^{2g+1}(a_1a_2^{-1})^{2g+1} = 1$ .

Let  $a_2 = x$  and  $a_1 = y$ . Then it follows that  $\pi_1(X_{V_2})$  is isomorphic to  $\mathcal{H}_g$ . Therefore, for  $g \geq 2$  we have  $g(\mathcal{H}_g) \leq 4$ . In particular, since the group  $\mathcal{H}_1$  is isomorphic to  $\mathcal{M}_1$ ,

we have  $2 \leq g(\mathcal{H}_1) \leq 4$  from (3) of Theorem 2.3 (cf. [8]).

Thus, the proof of (2) of Theorem 1.2 is completed.  $\square$

**4.3. Proof of (3) of Theorem 1.2.** For  $n \geq 3$ , let  $\mathcal{M}_{0,n}$  denote the mapping class group of an  $n$ -punctured sphere, that is, the group of isotopy classes of orientation-preserving diffeomorphisms  $S^2 \setminus \{p_1, \dots, p_n\} \rightarrow S^2 \setminus \{p_1, \dots, p_n\}$ . Magnus [9] gave a presentation of the group  $\mathcal{M}_{0,n}$  with generators  $\sigma_1, \dots, \sigma_{n-1}$  and with relations

- $\sigma_i \sigma_j \sigma_i^{-1} \sigma_j^{-1} = 1$ , where  $1 \leq i < j - 1 \leq n - 2$ ,
- $\sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1} = 1$ , where  $1 \leq i \leq n - 2$ ,
- $(\sigma_1 \cdots \sigma_{n-1})^n = 1$ ,
- $\sigma_1 \cdots \sigma_{n-1} \sigma_{n-1} \cdots \sigma_1 = 1$ .

Similarly to Subsection 4.1 and 4.2, let  $x = \sigma_1$  and  $y = \sigma_1 \cdots \sigma_{n-1}$ . Then  $\mathcal{M}_{0,n}$  can be presented with generators  $x, y$  and with relations

- $xy^k xy^{-k} x^{-1} y^k x^{-1} y^{-k} = 1$ , where  $2 \leq k \leq n - 2$ ,
- $xyxy^{-1} xyx^{-1} y^{-1} x^{-1} yx^{-1} y^{-1} = 1$ ,
- $(xy)^{n-1} y^{-n} = 1$ ,
- $y^n = 1$ ,
- $(y^{-1} x)^{n-1} = 1$ .

We now prove (3) of Theorem 1.2.

**Proof of (3) of Theorem 1.2.** For  $n \geq 3$ , since  $\mathcal{M}_{0,n}$  is generated by two generators  $x, y$ , we have  $g(\mathcal{M}_{0,n}) \geq 2$  from (2) of Theorem 2.3 (cf. [8]). Therefore, we prove  $g(\mathcal{M}_{0,n}) \leq 4$  for  $n \geq 3$ .

Let  $R_7$  and  $R_8$  be simple closed curves on  $\Sigma_4$  described in  $\pi_1(\Sigma_4)$ , up to conjugation, as follows

- $R_7 = a_1^n (b_1^{-1})$ ,
- $R_8 = (a_1^{-1} a_2)^{n-1} (b_1^{-1})$ .

Note that  $R_7$  and  $R_8$  intersect  $B_2$  and  $B_1$  transversely at only one point, respectively. Let  $V_3$  be the following:

$$V_3 = V_1 W^{t_{R_7}} W^{t_{R_8}}.$$

Then, from Proposition 2.2 and (1) of Proposition 3.1, the fundamental group  $\pi_1(X_{V_3})$  can be presented with generators  $a_2, a_1$  and with relations

- $a_2 a_1^k a_2 a_1^{-k} a_2^{-1} a_1^k a_2^{-1} a_1^{-k} = 1$ , where  $2 \leq k \leq n - 2$ ,
- $a_2 a_1 a_2 a_1^{-1} a_2 a_1 a_2^{-1} a_1^{-1} a_2^{-1} a_1 a_2^{-1} a_1^{-1} = 1$ ,
- $(a_2 a_1)^{n-1} a_1^{-n} = 1$ ,
- $a_1^n = 1$ ,
- $(a_1^{-1} a_2)^{n-1} = 1$ .

Let  $a_2 = x$  and  $a_1 = y$ . Then it follows that  $\pi_1(X_{V_3})$  is isomorphic to  $\mathcal{M}_{0,n}$ . Therefore, for  $n \geq 3$  we have  $g(\mathcal{M}_{0,n}) \leq 4$ .

Thus, the proof of (3) of Theorem 1.2 is completed.  $\square$

**4.4. Proof of (4) of Theorem 1.2.** For  $n \geq 3$ , let  $S_n$  denote the  $n$ -symmetric group. It is well known that the group  $S_n$  has a presentation with generators  $\sigma_1, \dots, \sigma_{n-1}$  and with relations

- $\sigma_i \sigma_j \sigma_i^{-1} \sigma_j^{-1} = 1$ , where  $1 \leq i < j - 1 \leq n - 2$ ,
- $\sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1} = 1$ , where  $1 \leq i \leq n - 2$ ,
- $\sigma_i^2 = 1$ , where  $1 \leq i \leq n - 1$ .

Similarly to Subsection 4.1, let  $x = \sigma_1$  and  $y = \sigma_1 \cdots \sigma_{n-1}$ . Since  $\sigma_i = y^{i-1} x y^{1-i}$ ,  $\sigma_i^2 = 1$  if and only if  $x^2 = 1$ . Therefore  $S_n$  can be presented with generators  $x, y$  and with relations

- $xy^k xy^{-k} x^{-1} y^k x^{-1} y^{-k} = 1$ , where  $2 \leq k \leq n - 2$ ,
- $xyxy^{-1} yxyx^{-1} y^{-1} x^{-1} yx^{-1} y^{-1} = 1$ ,
- $(xy)^{n-1} y^{-n} = 1$ ,
- $x^2 = 1$ .

We now prove (4) of Theorem 1.2.

**Proof of (4) of Theorem 1.2.** For  $n \geq 3$ , since  $S_n$  is generated by two generators  $x, y$ , we have  $g(S_n) \geq 2$  from (2) of Theorem 2.3 (cf. [8]). Therefore, we prove  $g(S_n) \leq 4$  for  $n \geq 3$ .

Let  $R_9$  be the simple closed curve on  $\Sigma_4$  described in  $\pi_1(\Sigma_4)$ , up to conjugation, as follows

- $R_9 = a_2^2(b_2^{-1})$ .

Note that  $R_9$  intersects  $B_4$  transversely at only one point. Let  $V_4$  be the following:

$$V_4 = V_1 W^{t_{R_9}}.$$

Then, from Proposition 2.2 and (1) of Proposition 3.1, the fundamental group  $\pi_1(X_{V_4})$  can be presented with generators  $a_2, a_1$  and with relations

- $a_2 a_1^k a_2 a_1^{-k} a_2^{-1} a_1^k a_2^{-1} a_1^{-k} = 1$ , where  $2 \leq k \leq n - 2$ ,
- $a_2 a_1 a_2 a_1^{-1} a_2 a_1 a_2^{-1} a_1^{-1} a_2^{-1} a_1 a_2^{-1} a_1^{-1} = 1$ ,
- $(a_2 a_1)^{n-1} a_1^{-n} = 1$ ,
- $a_2^2 = 1$ .

Let  $a_2 = x$  and  $a_1 = y$ . Then it follows that  $\pi_1(X_{V_4})$  is isomorphic to  $S_n$ . Therefore, for  $n \geq 3$  we have  $g(S_n) \leq 4$ .

Thus, the proof of (4) of Theorem 1.2 is completed. □

**4.5. Proof of (5) of Theorem 1.2.** The Artin group is introduced by [3]. For  $n \geq 6$ , the  $n$ -Artin group  $\mathcal{A}_n$  associated to the Dynkin diagram shown in Fig. 1 is defined by a presentation with generators  $\sigma_1, \dots, \sigma_{n-1}, \tau$  and with relations

- $\sigma_i \sigma_j \sigma_i^{-1} \sigma_j^{-1} = 1$ , where  $1 \leq i < j - 1 \leq n - 2$ ,
- $\sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1} = 1$ , where  $1 \leq i \leq n - 2$ ,
- $\sigma_4 \tau \sigma_4 \tau^{-1} \sigma_4^{-1} \tau^{-1} = 1$ ,
- $\tau \sigma_i \tau^{-1} \sigma_i^{-1} = 1$ , where  $1 \leq i \leq n - 1$  with  $i \neq 4$ .

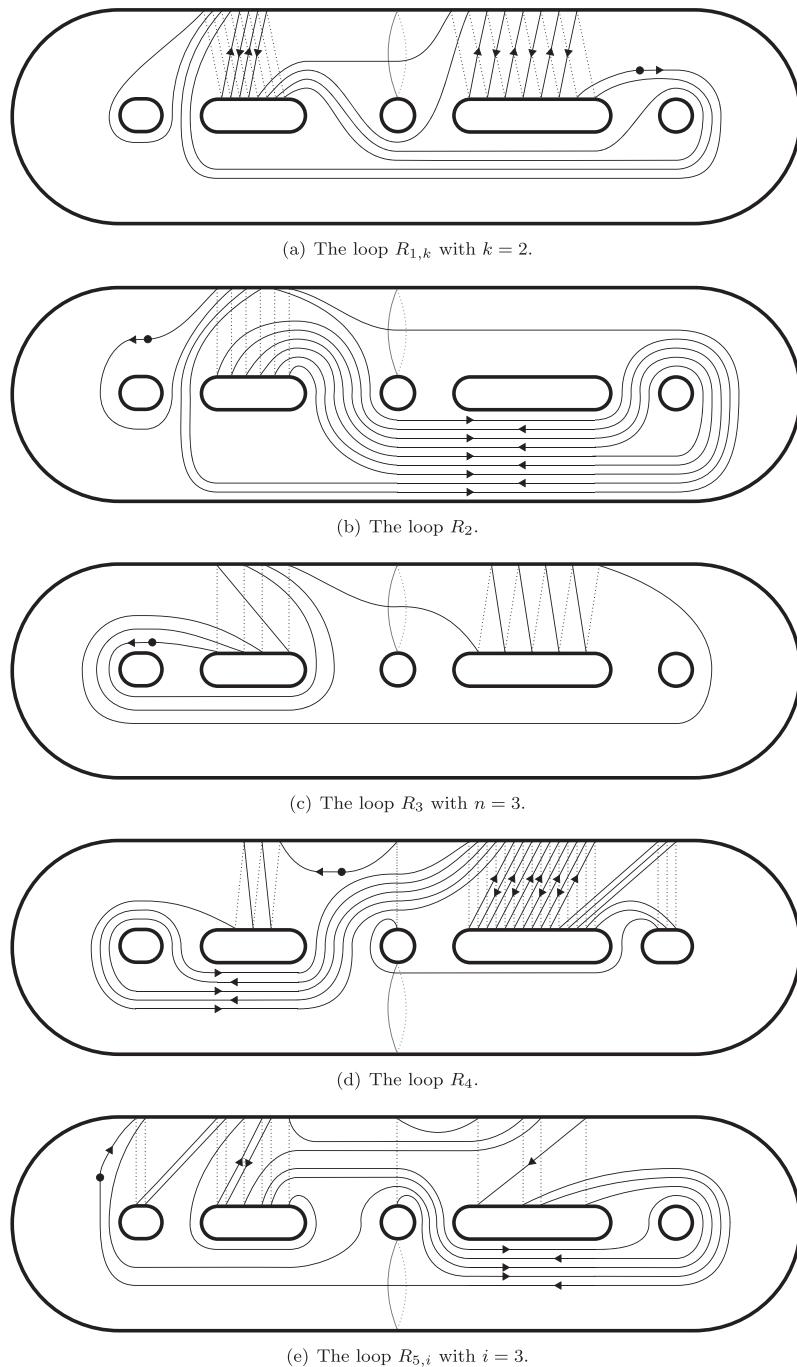


Fig. 10.

It is known that there is the natural epimorphism  $\mathcal{A}_{2g+1} \twoheadrightarrow \mathcal{M}_g$ . Similarly to Subsection 4.1, let  $x = \sigma_1$  and  $y = \sigma_1 \cdots \sigma_{n-1}$ . In addition, let  $z = \tau$ . Then the group  $\mathcal{A}_n$  can be presented with generators  $x, y, z$  and with relations

- $xy^kxy^{-k}x^{-1}y^kx^{-1}y^{-k} = 1$ , where  $2 \leq k \leq n-2$ ,
- $xyxy^{-1}xyx^{-1}y^{-1}x^{-1}yx^{-1}y^{-1} = 1$ ,
- $(xy)^{n-1}y^{-n} = 1$ ,
- $(y^3xy^{-3})z(y^3xy^{-3})z^{-1}(y^3x^{-1}y^{-3})z^{-1} = 1$ ,
- $z(y^{i-1}xy^{1-i})z^{-1}(y^{i-1}x^{-1}y^{1-i}) = 1$ , where  $1 \leq i \leq n-1$  with  $i \neq 4$ .

We now prove (5) of Theorem 1.2.

Proof of (5) of Theorem 1.2. Since  $\mathcal{A}_n$  is generated by three generators  $x, y$  and  $z$ , we have  $g(\mathcal{A}_n) \geq 2$  from (2) of Theorem 2.3 (cf. [8]). Therefore, we prove  $g(\mathcal{A}_n) \leq 5$ .

Let  $R_{1,k}, R_2, R_3, R_4$  and  $R_{5,i}$  be simple closed curves on  $\Sigma_5$  as shown in Fig. 10, where  $2 \leq k \leq n-2$  and  $2 \leq i \leq n-1$  with  $i \neq 4$ . Note that we can not consider the loop  $R_{5,1}$ . Note that  $R_{1,k}, R_2$  and  $R_3$  intersect  $a$  transversely at only one point, for  $2 \leq k \leq n-2$ , and that  $R_4$  and  $R_{5,i}$  intersect  $b$  transversely at only one point, for  $2 \leq i \leq n-1$  with  $i \neq 4$ . Loops  $R_{1,k}, R_2, R_3, R_4$  and  $R_{5,i}$  can be described in  $\pi_1(\Sigma_5)$ , up to conjugation, as follows

- $R_{1,k} = b_5^{-1}(b_2b_3b_4)^{-1}a_2^k(b_3b_4)b_5^{-1}(b_3b_4)^{-1}a_2^{-k}(b_2b_3b_4)b_5a_4^{-2k}(b_3^{-1})a_2^{-k}b_1^{-1}a_2^k a_4^{2k}$ , where  $2 \leq k \leq n-2$ ,
- $R_2 = b_1a_2(b_3b_4)b_5^{-1}(b_3b_4)^{-1}a_2^{-1}(b_3b_4)b_5^{-1}(b_2b_3b_4)^{-1}a_2(b_3b_4)b_5(b_3b_4)^{-1}a_2^{-1}(b_2b_3b_4) \times b_5a_2(b_3b_4)b_5(b_3b_4)^{-1}a_2^{-1}$ ,
- $R_3 = (b_1(b_2)a_2)^{n-1}(b_1(b_2b_3b_4)b_5)a_4^{n+2}a_2^2$ ,
- $R_4 = a_2^3b_1(b_2)a_3^3a_5^{-1}a_4^{-3}(b_2^{-1})b_1(b_2)a_3^3a_5a_4^{-3}(b_2^{-1})b_1^{-1}(b_2)a_4^3a_5(a_3b_3b_4)^{-1}$ ,
- $R_{5,i} = a_1a_2^{i-1}(b_4)b_5^{-1}(b_4)a_2^{1-i}a_1^{-1}(b_1(b_2b_4)b_5)a_4^{1-i}(a_3b_4)b_5(a_4^{2-i}a_2^{2-i}(b_2))a_2^{-1}a_4^{i-2} \times (b_1(b_2b_3b_4)b_5)^{-1}$ , where  $2 \leq i \leq n-1$  with  $i \neq 4$ .

Let  $V_5$  be the following:

$$V_5 = WW^{t_{b_2}}W^{t_{b_3}}W^{t_{b_4}}\left(\prod_{2 \leq k \leq n-2} W^{t_{R_{1,k}}}\right)W^{t_{R_2}}W^{t_{R_3}}W^{t_{R_4}}\left(\prod_{2 \leq i \leq n-1, i \neq 4} W^{t_{R_{5,i}}}\right).$$

Then, from Proposition 2.2 and (2) of Proposition 3.1, the fundamental group  $\pi_1(X_{V_5})$  can be presented with generators  $b_1, a_2, a_1$  and with relations

- $b_1a_2^k b_1 a_2^{-k} b_1^{-1} a_2^k b_1^{-1} a_2^{-k} = 1$ , where  $2 \leq k \leq n-2$ ,
- $b_1a_2b_1a_2^{-1}b_1a_2b_1^{-1}a_2^{-1}b_1^{-1}a_2b_1^{-1}a_2^{-1} = 1$ ,
- $(b_1a_2)^{n-1}a_2^{-n} = 1$ ,
- $(a_2^3b_1a_2^{-3})a_1(a_2^3b_1a_2^{-3})a_1^{-1}(a_2^3b_1^{-1}a_2^{-3})a_1^{-1} = 1$ ,
- $a_1(a_2^{i-1}b_1a_2^{1-i})a_1^{-1}(a_2^{i-1}b_1^{-1}a_2^{1-i}) = 1$ , where  $2 \leq i \leq n-1$  with  $i \neq 4$ ,
- $a_1b_1a_1^{-1}b_1^{-1}$ .

Let  $b_1 = x, a_2 = y$  and  $a_1 = z$ . Then  $\pi_1(X_{V_5})$  is isomorphic to  $\mathcal{A}_n$ . Therefore, for  $n \geq 6$  we have  $g(\mathcal{A}_n) \leq 5$ .

Thus, the proof of (5) of Theorem 1.2 is completed.  $\square$

#### 4.6. Proof of (6) of Theorem 1.2.

Proof of (6) of Theorem 1.2. Let  $n, k \geq 0$  be integers with  $n + k \geq 3$ .

At first, we consider the case  $n + k$  is even. We put  $n + k = 2r$ . Let  $A_{i,j}$  and  $B_{i,j}$  be simple closed curves on  $\Sigma_{n+k+1}$  as shown in (a) and (b) of Fig. 11, respectively, where  $1 \leq i < j \leq r$ , and let  $C_{i,j}$  be the simple closed curve on  $\Sigma_{n+k+1}$  as shown in (c), (d) and (e) of Fig. 11, where  $1 \leq i, j \leq r$ . Note that each of  $A_{i,j}$ ,  $B_{i,j}$  and  $C_{i,j}$  intersects  $a_{r+1}$  transversely at only one point. Loops  $A_{i,j}$ ,  $B_{i,j}$  and  $C_{i,j}$  can be described in  $\pi_1(\Sigma_{n+k+1})$ , up to conjugation, as follows

- $A_{i,j} = a_i a_j^{-1} a_{2r-i+2} a_{2r-j+2}^{-1} (c_{r+1}^{-1} b_{r+1}^{-1})$ , where  $1 \leq i < j \leq r$ ,
- $B_{i,j} = b_i b_j b_i^{-1} a_{2r-j+2} b_{2r-j+2} a_{2r-j+2}^{-1} (b_{r+1}^{-1} c_r)$ , where  $1 \leq i < j \leq r$ ,
- $C_{i,j} = a_i b_j^{-1} a_i^{-1} a_{2r-j+2} b_{2r-j+2}^{-1} a_{2r-j+2}^{-1} (a_{r+1} b_{r+1}^{-1})$ , where  $1 \leq i, j \leq r$  and  $i \neq j$ ,
- $C_{i,i} = b_i^{-1} a_i b_i a_i^{-1} (b_{r+1}^{-1})$ , where  $1 \leq i \leq r$ .

Let  $V_6$  be the following:

$$V_6 = W \left( \prod_{1 \leq i < j \leq r} W^{t_{A_{i,j}}} \right) \left( \prod_{1 \leq i < j \leq r} W^{t_{B_{i,j}}} \right) \left( \prod_{1 \leq i, j \leq r} W^{t_{C_{i,j}}} \right).$$

Note that we have relations  $a_{r+1} = 1$ ,  $b_{r+1} = 1$ ,  $c_r = 1$  and  $c_{r+1} = 1$  in  $\pi_1(X_W)$ . In addition, we have the relation  $a_{2r-j+2} b_{2r-j+2} a_{2r-j+2}^{-1} = b_j^{-1}$  in  $\pi_1(X_W)$  (see the presentation of  $\pi_1(X_W)$  in the proof of Proposition 3.1). Then, from Proposition 2.2, the fundamental group  $\pi_1(X_{V_6})$  can be presented with generators  $a_1, b_1, \dots, a_r, b_r$  and with relations

- $a_i a_j^{-1} a_i^{-1} a_j$ , where  $1 \leq i < j \leq r$ ,
- $b_i b_j b_i^{-1} b_j^{-1}$ , where  $1 \leq i < j \leq r$ ,
- $a_i b_j^{-1} a_i^{-1} b_j$ , where  $1 \leq i, j \leq r$  and  $i \neq j$ ,
- $b_i^{-1} a_i b_i a_i^{-1}$ , where  $1 \leq i \leq r$ .

Namely,  $\pi_1(X_{V_6})$  is isomorphic to  $\mathbb{Z}^{2r}$ . We next consider the simple closed curve  $R_i^{m_i}$  on  $\Sigma_{n+k+1}$  as shown in Fig. 12, where  $1 \leq i \leq 2r$  and  $m_i \geq 2$ . Note that  $R_i^{m_i}$  intersects  $a_{r+1}$  transversely at only one point. Loops  $R_i^{m_i}$  can be described in  $\pi_1(\Sigma_{n+k+1})$ , up to conjugation, as follows

- $R_i^{m_i} = a_i^{m_i} (a_{2r-i+2} b_{2r-i+2}^{-1} a_{2r-i+2}^{-1} a_{r+1} b_{r+1}^{-1} b_i^{-1})$ , where  $1 \leq i \leq r$ ,
- $R_{r+i}^{m_{r+i}} = b_i^{m_{r+i}} (a_i^{-1} a_{2r-i+2}^{-1} a_{r+1} b_{r+1}^{-1})$ , where  $1 \leq i \leq r$ .

Let  $V_7$  be the following:

$$V_7 = V_6 \left( \prod_{1 \leq i \leq k} W^{t_{R_i^{m_i}}} \right).$$

Then, from Proposition 2.2, the fundamental group  $\pi_1(X_{V_7})$  is isomorphic to  $\mathbb{Z}^n \oplus \mathbb{Z}_{m_1} \oplus \dots \oplus \mathbb{Z}_{m_k}$ . Therefore, if  $n + k$  is even, we have  $g(\mathbb{Z}^n \oplus \mathbb{Z}_{m_1} \oplus \dots \oplus \mathbb{Z}_{m_k}) \leq n + k + 1$ .

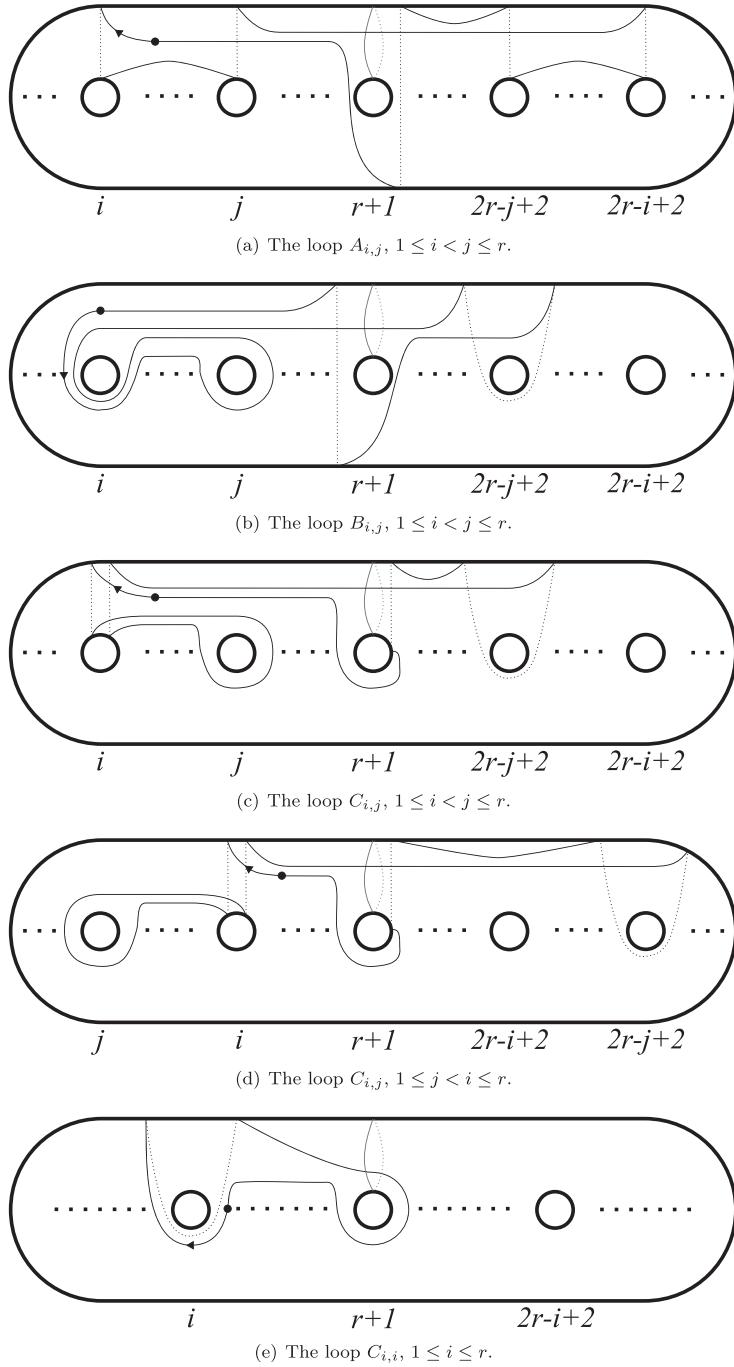


Fig. 11.

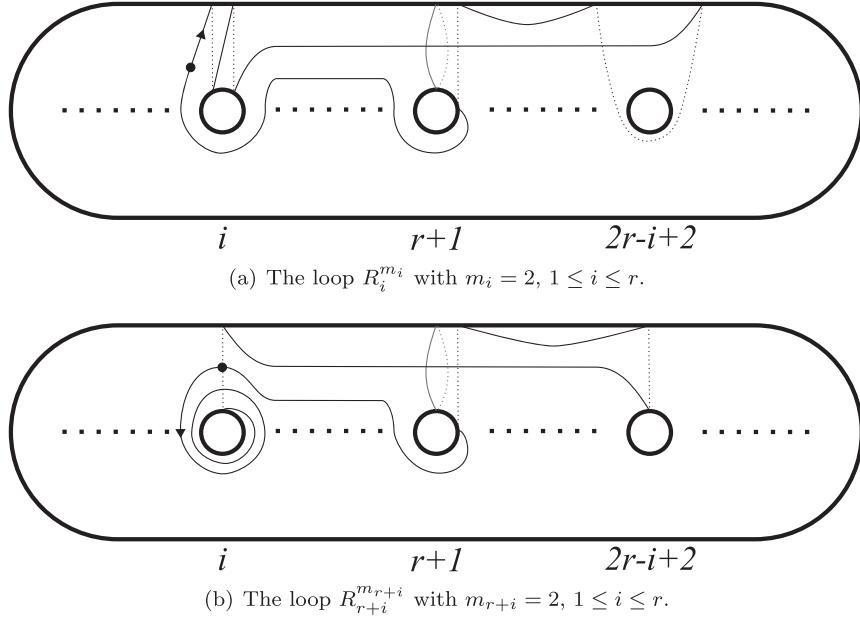


Fig. 12.

Next, we consider the case  $n+k$  is odd. We put  $n+k = 2r+1$ . Let  $A_{i,j}$  and  $B_{i,j}$  be simple closed curves on  $\Sigma_{n+k+1}$  as shown in (a) and (b) of Fig. 13, respectively, where  $1 \leq i < j \leq r$ , and let  $C_{i,j}$  be the simple closed curve on  $\Sigma_{n+k+1}$  as shown in (c), (d) and (e) of Fig. 13, where  $1 \leq i, j \leq r$ . In addition, let  $A_{i,r+1}$  and  $C_{r+1,i}$  be simple closed curves on  $\Sigma_{n+k+1}$  as shown in (a) and (b) of Fig. 14, where  $1 \leq i \leq r$ . Note that each of  $A_{i,j}$ ,  $B_{i,j}$  and  $C_{i,j}$  intersects  $B_{2r+2}$  transversely at only one point. Loops  $A_{i,j}$ ,  $B_{i,j}$  and  $C_{i,j}$  can be described in  $\pi_1(\Sigma_{n+k+1})$ , up to conjugation, as follows

- $A_{i,j} = a_i a_j^{-1} a_{2r-i+3} a_{2r-j+3}^{-1} (c_{r+1}^{-1} b_{r+1}^{-1})$ , where  $1 \leq i < j \leq r$ ,
- $A_{i,r+1} = a_i a_{r+1}^{-1} (b_{r+2}) a_{2r-i+3} (c_{r+2}) a_{r+1}$ , where  $1 \leq i \leq r$ ,
- $B_{i,j} = b_i b_j b_i^{-1} (b_{r+2}) a_{2r-j+3} b_{2r-j+3} a_{2r-j+3}^{-1} (b_{r+2}^{-1} b_{r+1} c_{r+1})$ , where  $1 \leq i < j \leq r$ ,
- $C_{i,j} = a_i b_j a_i^{-1} (b_{r+2}) a_{2r-j+3} b_{2r-j+3} a_{2r-j+3}^{-1} (b_{r+2}^{-1} b_{r+1} c_{r+1})$ , where  $1 \leq i, j \leq r$  and  $i \neq j$ ,
- $C_{i,i} = b_i^{-1} a_i b_i a_i^{-1} (b_{r+1}^{-1})$ , where  $1 \leq i \leq r$ ,
- $C_{r+1,i} = a_{r+1} b_i a_{r+1}^{-1} (b_{r+2}) a_{2r-i+3} b_{2r-i+3} a_{2r-i+3}^{-1} (c_{r+2})$ , where  $1 \leq i \leq r$ .

Let  $V_8$  be the following:

$$V_8 = WW^{t_{b_{r+1}}}\left(\prod_{1 \leq i < j \leq r+1} W^{t_{A_{i,j}}}\right)\left(\prod_{1 \leq i < j \leq r} W^{t_{B_{i,j}}}\right)\left(\prod_{1 \leq i \leq r+1, 1 \leq j \leq r} W^{t_{C_{i,j}}}\right).$$

Since  $b_{r+1}$  intersects  $B_{2r+2}$  transversely at only one point, we have the relation  $b_{r+1} = 1$

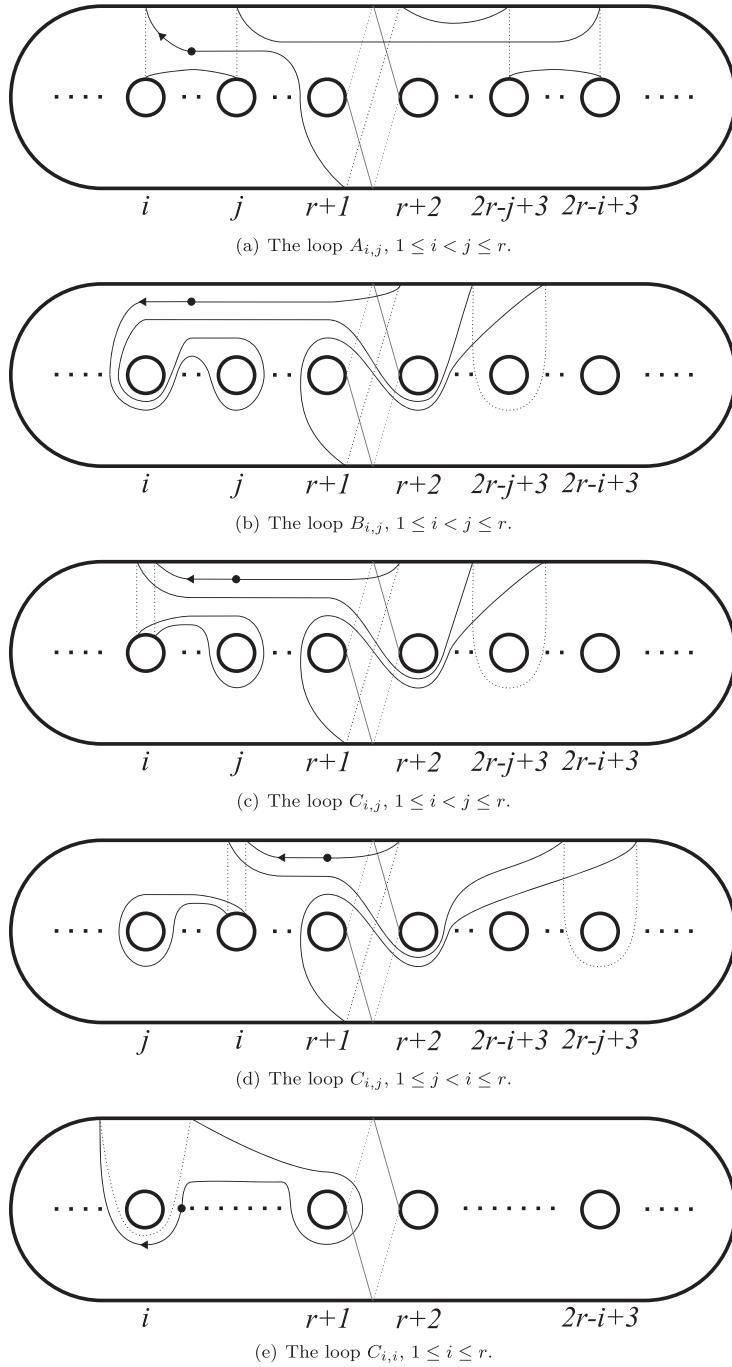


Fig. 13.

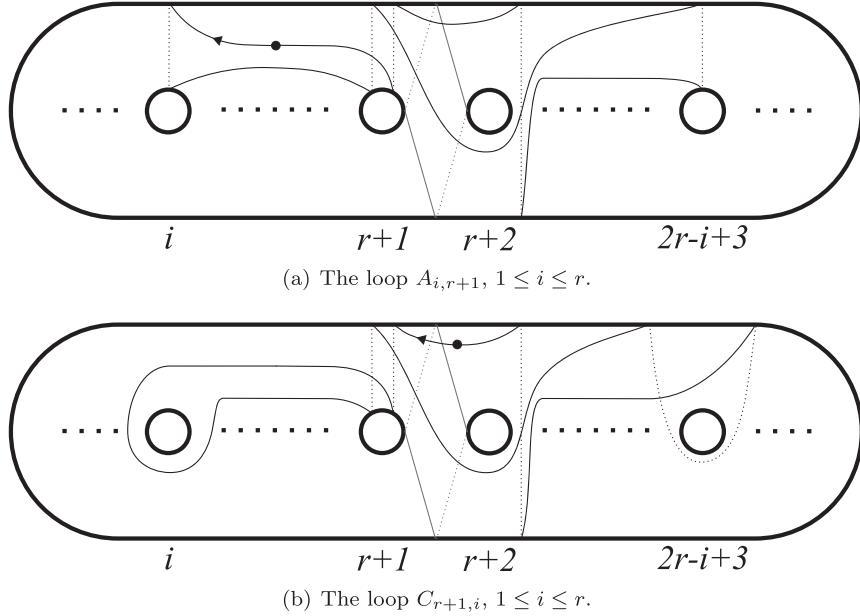


Fig. 14.

in  $\pi_1(X_{WW^{b_{r+1}}})$  from Proposition 2.2. Hence we have relations  $b_{r+2} = 1$  and  $c_{r+2} = 1$  in  $\pi_1(X_{WW^{b_{r+1}}})$ . Then, from Proposition 2.2 and the presentation of  $\pi_1(X_W)$  in the proof of Proposition 3.1, the fundamental group  $\pi_1(X_{V_8})$  is isomorphic to an abelian generated by  $a_1, b_1, \dots, a_r, b_r$  and  $a_{r+1}$ . We next consider the simple closed curve  $R_i^{m_i}$  on  $\Sigma_{n+k+1}$  as shown in Fig. 15, where  $1 \leq i \leq 2r+1$  and  $m_i \geq 2$ . Note that  $R_i^{m_i}$  intersects  $B_{2r+2}$  transversely at only one point. Loops  $R_i^{m_i}$  can be described in  $\pi_1(\Sigma_{n+k+1})$ , up to conjugation, as follows

- $R_i^{m_i} = a_i^{m_i} (a_{2r-i+3} b_{2r-i+3}^{-1} a_{2r-i+3}^{-1} c_{r+1}^{-1} b_{r+1}^{-1} b_i^{-1})$ , where  $1 \leq i \leq r$ ,
- $R_{r+i}^{m_{r+i}} = b_i^{m_{r+i}} (a_i^{-1} a_{2r-i+3}^{-1} c_{r+1}^{-1} b_{r+1}^{-1})$ , where  $1 \leq i \leq r$ ,
- $R_{2r+1}^{m_{2r+1}} = a_{r+1}^{m_{2r+1}} (b_{r+1}^{-1})$ .

Let  $V_9$  be the following:

$$V_9 = V_8 \left( \prod_{1 \leq i \leq k} W^{t_{R_i^{m_i}}} \right).$$

Then, from Proposition 2.2, the fundamental group  $\pi_1(X_{V_9})$  is isomorphic to  $\mathbb{Z}^n \oplus \mathbb{Z}_{m_1} \oplus \dots \oplus \mathbb{Z}_{m_k}$ . Therefore, if  $n+k$  is odd, we have  $g(\mathbb{Z}^n \oplus \mathbb{Z}_{m_1} \oplus \dots \oplus \mathbb{Z}_{m_k}) \leq n+k+1$ .

Moreover, it is immediately follows from Theorem 2.3 (2) or (5) (cf. [8]) that  $g(\mathbb{Z}^n \oplus \mathbb{Z}_{m_1} \oplus \dots \oplus \mathbb{Z}_{m_k}) \geq (n+k+1)/2$ . Thus, the proof of (6) of Theorem 1.2 is completed.  $\square$

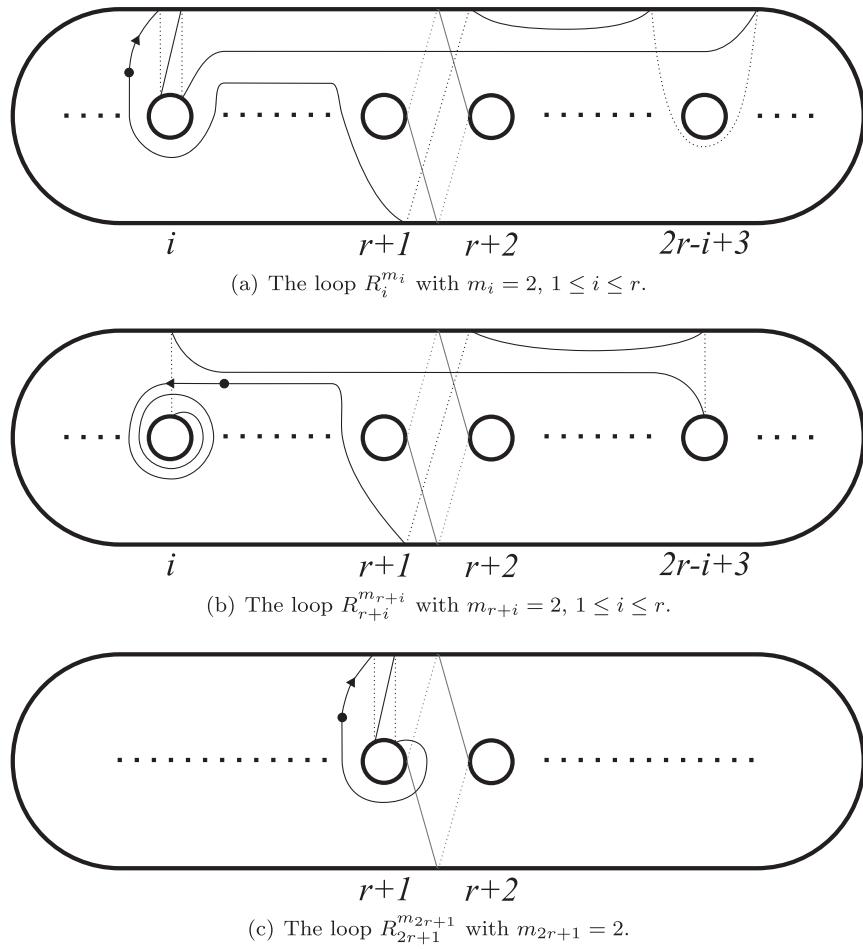


Fig. 15.

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Department of Mathematics  
 Faculty of Science and Technology  
 Tokyo University of Science  
 2641 Yamazaki, Noda  
 Chiba 278-8510  
 Japan

Current address:  
 General Education  
 Ishikawa National College of Technology  
 Tsubata, Ishikawa, 929-0392  
 Japan  
 e-mail: kobayashi\_ryoma@ishikawa-nct.ac.jp