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E-POLYNOMIALS OF
SL(2, ℂ)-CHARACTER VARIETIES OF
COMPLEX CURVES OF GENUS 3

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Abstract

We compute the E-polynomials of the moduli spaces of representations of the fundamental group of a complex curve of genus $g = 3$ into SL(2, ℂ), and also of the moduli space of twisted representations. The case of genus $g = 1, 2$ has already been done in [12]. We follow the geometric technique introduced in [12], based on stratifying the space of representations, and on the analysis of the behaviour of the E-polynomial under fibrations.

1. Introduction

Let X be a smooth complex projective curve of genus $g \geq 1$, and let G be a complex reductive group. The G-character variety of X is defined as the moduli space of semisimple representations of $\pi_1(X)$ into G, that is,

$$\mathcal{M}(G) = \left\{ (A_1, B_1, \ldots, A_g, B_g) \in G^{2g} \,\middle|\, \prod_{i=1}^{g} [A_i, B_i] = \mathrm{Id} \right\} /\!\!/ G.$$

For complex linear groups $G = \mathrm{GL}(n, \mathbb{C})$, $\mathrm{SL}(n, \mathbb{C})$, the representations of $\pi_1(X)$ into G can be understood as G-local systems $E \to X$, hence defining a flat bundle E which has $\deg E = 0$. A natural generalization consists of allowing bundles E of non-zero degree d. The G-local systems on X correspond to representations $\rho: \pi_1(X - \{p_0\}) \to G$, where $p_0 \in X$ is a fixed point, and $\rho(\gamma) = \zeta_n \mathrm{Id}$, γ a loop around p_0 and ζ_n a primitive n-th root of unity, giving rise to the moduli space of twisted representations

$$\mathcal{M}^d(G) = \left\{ (A_1, B_1, \ldots, A_g, B_g) \in G^{2g} \,\middle|\, \prod_{i=1}^{g} [A_i, B_i] = e^{2\pi \sqrt{-1} d/n} \mathrm{Id} \right\} /\!\!/ G.$$

The space $\mathcal{M}^d(G)$ is known in the literature as the Betti moduli space. This space is closely related to two other spaces: the de Rham and Dolbeault moduli spaces. The de Rham moduli space $\mathcal{D}^d(G)$ is the moduli space parameterizing flat bundles, i.e.,

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(E, ∇), where $E \to X$ is an algebraic bundle of degree d and rank n (and fixed determinant in the case $G = \mathrm{SL}(n, \mathbb{C})$), and ∇ is an algebraic connection on $X - \{p_0\}$ with a logarithmic pole at p_0 with residue $-(d/n)\mathrm{Id}$. The Riemann–Hilbert correspondence [3, 20] gives an analytic isomorphism (but not an algebraic isomorphism) $\mathcal{D}^d(G) \cong \mathcal{M}^d(G)$. The Dolbeault moduli space $\mathcal{H}^d(G)$ is the moduli space of G-Higgs bundles (E, Φ), consisting of an algebraic bundle $E \to X$ of degree d and rank n (also with fixed determinant in the case $G = \mathrm{SL}(n, \mathbb{C})$), and a homomorphism $\Phi \colon E \to E \otimes K_X$, known as the Higgs field (in the case $G = \mathrm{SL}(n, \mathbb{C})$, the Higgs field has trace 0). In this situation the theory of harmonic bundles [2, 19] gives a homeomorphism $\mathcal{M}^d(G) \cong \mathcal{H}^d(G)$.

When $\gcd(n, d) = 1$ these moduli spaces are smooth and the underlying differentiable manifold is hyperkähler, but the complex structures do not correspond under the previous isomorphisms. The cohomology of these moduli spaces has been computed in several particular cases, mostly from the point of view of the Dolbeault moduli space $\mathcal{H}^d(G)$. Poincaré polynomials for $G = \mathrm{SL}(2, \mathbb{C})$ were computed by Hitchin in his seminal paper on Higgs bundles [11] and for $G = \mathrm{SL}(3, \mathbb{C})$ by Gothen in [8]. More recently the case of $G = \mathrm{GL}(4, \mathbb{C})$ has been solved in [6]. A recursive formula for the motive of the moduli space of Higgs bundles of arbitrary rank and degree coprime to the rank has been given in [7] and formulas for the number of points of these spaces over finite fields of sufficiently large characteristic have been computed in [17, 18] for arbitrary rank and degree. In particular, these results give the Betti numbers of the character variety.

Hausel and Rodriguez-Villegas started the computation of the E-polynomials of G-character varieties focusing on $G = \mathrm{GL}(n, \mathbb{C})$, $\mathrm{SL}(n, \mathbb{C})$ and $\mathrm{PGL}(n, \mathbb{C})$ using arithmetic methods inspired by the Weil conjectures. In [9] they obtained the E-polynomials of $\mathcal{M}^d(G)$ for $G = \mathrm{GL}(n, \mathbb{C})$ in terms of a simple generating function. Following these methods, Mereb [16] studied the case of $\mathrm{SL}(n, \mathbb{C})$ giving an explicit formula for the E-polynomial in the case $G = \mathrm{SL}(2, \mathbb{C})$, while for $\mathrm{SL}(n, \mathbb{C})$ these polynomials are given in terms of a generating function. He proved also that the E-polynomials for $\mathrm{SL}(n, \mathbb{C})$ are palindromic and monic.

Logares, Newstead and the second author [12] introduced a geometric technique to compute the E-polynomial of character varieties by using stratifications and also handling fibrations which are locally trivial in the analytic topology but not in the Zariski topology. The main results of [12] are explicit formulae for the E-polynomials of character varieties for $G = \mathrm{SL}(2, \mathbb{C})$ and $g = 1, 2$. They also deal with "twisted" character varieties in which the holonomy around the puncture is not diagonalizable (in this case there is no correspondence with a de Rham or Dolbeault moduli space as mentioned above).

The purpose in this paper is to give another step in the geometric technique of [12] to the case of $g = 3$. We prove the following

Theorem 1. *Let X be a complex curve of genus $g = 3$. Then the E-polynomials of the character varieties are*:

$$e(\mathcal{M}(\mathrm{SL}(2, \mathbb{C}))) = q^{12} - 4q^{10} + 74q^8 + 375q^6 + 16q^4 + q^2 + 1,$$
$$e(\mathcal{M}^1(\mathrm{SL}(2, \mathbb{C}))) = q^{12} - 4q^{10} + 6q^8 - 252q^7 - 14q^6 - 252q^5 + 6q^4 - 4q^2 + 1,$$

where $q = uv$.

The new input consists of using the map $\mathcal{M}(\mathrm{SL}(2, \mathbb{C})) \to \mathbb{C}^3$ given by $(A_1, B_1, A_2, B_2, A_3, B_3) \mapsto (t_1, t_2, t_3)$, $t_i = \mathrm{tr}([A_i, B_i])$, to stratify conveniently the space of representations in $\mathrm{SL}(2, \mathbb{C})^6$. This leads us to analyse a fibration over a two-dimensional basis (the surface C in Fig. 1), for which we have to compute the E-polynomial. Therefore the arguments of [12] about Hodge monodromy fibrations over one-dimensional bases have to be extended to general bases (see Theorem 2). In the case of $\mathcal{M}(\mathrm{SL}(2, \mathbb{C}))$ we also have to deal with reducible representations. The arguments used in [12, Section 8.2.2] to compute the E-polynomial of the irreducible locus give an increasingly large number of strata as the genus increases. So here we use an alternative route (suggested by the ideas in [14]), consisting of computing the E-polynomial of the irreducible locus by subtracting the E-polynomial of the subset of reducible representations.

As a byproduct of our analysis, we obtain the behaviour of the E-polynomial of the parabolic character variety ($G = \mathrm{SL}(2, \mathbb{C})$)

$$\mathcal{M}^\lambda(G) = \left\{ (A_1, B_1, \ldots, A_g, B_g) \in G^{2g} \,\middle|\, \prod_{i=1}^{g}[A_i, B_i] = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \right\} /\!\!/ G,$$

when λ varies in $\mathbb{C} - \{0, \pm 1\}$, for the case $g = 2$. This is given by the following formula

(1)
$$\begin{aligned} R(\mathcal{M}^\lambda) &= (q^8 + q^7 - 2q^6 - 2q^5 + 4q^4 - 2q^3 - 2q^2 + q + 1)T \\ &\quad + 15(q^5 - 2q^4 + q^3)N, \end{aligned}$$

which means that the E-polynomial of the invariant part of the cohomology is the polynomial accompanying T, and the E-polynomial of the non-invariant part is the polynomial accompanying N. This monodromy is of interest from the point of view of mirror symmetry [10] and answers a question raised to the authors by Tamás Hausel.

After finishing and submitting a first version of this paper, the authors have managed to complete the computation of E-polynomials of character varieties for surfaces of general genus $g \geq 3$ in [15]. The case provided here cannot however be subsumed in [15], since it serves as the starting point of induction on the genus carried there. Moreover, this specific case of $g = 3$ has special features and has to be treated by an *ad hoc* method (it is the only case that needs the analysis of Hodge monodromy representations over a 2-dimensional basis).

2. E-polynomials

Our main goal is to compute the Hodge–Deligne polynomial of $SL(2, \mathbb{C})$-character varieties. We will follow the methods in [12], so we collect some basic results from [12] in this section.

2.1. E-polynomials.

We start by reviewing the definition of E-polynomials. A pure Hodge structure of weight k consists of a finite dimensional complex vector space H with a real structure, and a decomposition $H = \bigoplus_{k=p+q} H^{p,q}$ such that $H^{q,p} = \overline{H^{p,q}}$, the bar meaning complex conjugation on H. A Hodge structure of weight k gives rise to the so-called Hodge filtration, which is a descending filtration $F^p = \bigoplus_{s \geq p} H^{s,k-s}$. We define $\mathrm{Gr}_F^p(H) := F^p/F^{p+1} = H^{p,k-p}$.

A mixed Hodge structure consists of a finite dimensional complex vector space H with a real structure, an ascending (weight) filtration $\cdots \subset W_{k-1} \subset W_k \subset \cdots \subset H$ (defined over $\mathbb{R}$) and a descending (Hodge) filtration F such that F induces a pure Hodge structure of weight k on each $\mathrm{Gr}_k^W(H) = W_k/W_{k-1}$. We define

$$H^{p,q} := \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W(H)$$

and write $h^{p,q}$ for the *Hodge number* $h^{p,q} := \dim H^{p,q}$.

Let Z be any quasi-projective algebraic variety (maybe non-smooth or non-compact). The cohomology groups $H^k(Z)$ and the cohomology groups with compact support $H_c^k(Z)$ are endowed with mixed Hodge structures [4, 5]. We define the *Hodge numbers* of Z by

$$h_c^{k,p,q}(Z) = h^{p,q}(H_c^k(Z)) = \dim \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W H_c^k(Z).$$

The Hodge–Deligne polynomial, or E-polynomial is defined as

$$e(Z) = e(Z)(u, v) := \sum_{p,q,k} (-1)^k h_c^{k,p,q}(Z) u^p v^q.$$

When $h_c^{k,p,q} = 0$ for $p \neq q$, the polynomial $e(Z)$ depends only on the product uv. This will happen in all the cases that we shall investigate here. In this situation, it is conventional to use the variable $q = uv$. If this happens, we say that the variety is *of balanced type*. For instance, $e(\mathbb{C}^n) = q^n$.

The key property of Hodge–Deligne polynomials that permits their calculation is that they are additive for stratifications of Z. If Z is a complex algebraic variety and $Z = \bigsqcup_{i=1}^n Z_i$, where all Z_i are locally closed in Z, then $e(Z) = \sum_{i=1}^n e(Z_i)$.

There is another useful property that we shall use often: if there is an action of $\mathbb{Z}_2$ on X, then we have polynomials $e(X)^+$, $e(X)^-$, which are the E-polynomials of the invariant and anti-invariant parts of the cohomology of X, respectively. More concretely, $e(X)^+ = e(X/\mathbb{Z}_2)$ and $e(X)^- = e(X) - e(X)^+$. Then if $\mathbb{Z}_2$ acts on X and on

Y, we have the equality (see [12, Proposition 2.6])

$$(2) \qquad e((X \times Y)/\mathbb{Z}_2) = e(X)^+ e(Y)^+ + e(X)^- e(Y)^-.$$

2.2. Fibrations. We have to deal with fibrations

$$(3) \qquad F \to Z \xrightarrow{\pi} B$$

that are locally trivial in the analytic topology. We want to compute the E-polynomial of the total space in terms of the polynomials of the base and the fiber. The fibration defines a local system $\mathcal{H}_c^k$, whose fibers are the cohomology groups $H_c^k(F_b)$, where $b \in B$, $F_b = \pi^{-1}(b)$. The fibers possess mixed Hodge structures, and the subspaces $W_t(H_c^k(F_b))$ are preserved by the holonomy. We will assume from now on that F is of balanced type, so $\mathrm{Gr}_{2p}^W H_c^k(F_b) = H_c^{k,p,p}(F_b)$. Associated to the fibration, there is a monodromy representation:

$$(4) \qquad \rho \colon \pi_1(B) \to \mathrm{GL}(H_c^{k,p,p}(F)).$$

Suppose that the monodromy group $\Gamma = \mathrm{im}(\rho)$ is an abelian and finite group. Then $H_c^{k,p,p}(F)$ are modules over the representation ring $R(\Gamma)$. So there is a well defined element, the *Hodge monodromy representation*,

$$(5) \qquad R(Z) := \sum (-1)^k H_c^{k,p,p}(F) q^p \in R(\Gamma)[q].$$

As the monodromy representation (4) has finite image, there is a finite covering $B_\rho \to B$ such that the pull-back fibration

$$(6) \qquad \begin{array}{ccc} Z' & \longrightarrow & Z \\ \pi' \downarrow & & \pi \downarrow \\ B_\rho & \longrightarrow & B \end{array}$$

has trivial monodromy.

Let $S_1, \ldots, S_N$ be the irreducible representations of Γ (there are $N = \#\Gamma$ of them, and all of them are one-dimensional). These are generators of $R(\Gamma)$ as a free abelian group. We write the Hodge monodromy representation of (5) as

$$R(Z) = a_1(q)S_1 + \cdots + a_N(q)S_N.$$

Theorem 2. *Suppose that B_ρ is of balanced type. Then Z is of balanced type. Moreover, there are polynomials $s_1(q), \ldots, s_N(q) \in \mathbb{Z}[q]$ (only dependent on B, B_ρ and Γ, but not on the fibration or the fiber) such that*

$$e(Z) = a_1(q)s_1(q) + \cdots + a_N(q)s_N(q),$$

for $R(Z) = a_1(q)S_1 + \cdots + a_N(q)S_N$.

Proof.   The Leray spectral sequence of the fibration (3) has E_2-term

(7) $$E_2^{l,m}(Z) = H_c^l(B, H_c^m(F))$$

and abuts to $H_c^k(Z)$. By [1], $E_j^{l,m}(Z)$ has a mixed Hodge structure for $j \geq 2$, and the differentials d_j are compatible with the mixed Hodge structure. Therefore $e(Z) = e(H_c^*(Z)) = e(E_2^{*,*}(Z))$.

The mixed Hodge structure associated to π' in (6) is the product mixed Hodge structure $E_2^{l,m}(Z') = H_c^l(B_\rho) \otimes H_c^m(F)$. By our assumption, B_ρ and F are of balanced type, so Z' is of balanced type. There is a map $E_2^{l,m}(Z') \to E_2^{l,m}(Z)$, which preserves the mixed Hodge structures. This map is surjective, so Z is of balanced type.

By definition,

$$R(Z) = \sum (-1)^m H_c^{m,p,p}(F)q^p = a_1(q)S_1 + \cdots + a_N(q)S_N.$$

The local systems $S_i \to B$ are 1-dimensional and have a mixed Hodge structure. When we pull-back $S_i \to B$ to B_ρ, we get trivial local systems. Hence there is a surjection $H_c^l(B_\rho) \twoheadrightarrow H_c^l(B, S_i)$ and it has the induced mixed Hodge structure. So $e(H_c^*(B, H_c^*(F))) = e(H_c^*(B, R(Z))) = \sum a_i(q)e(H_c^*(B, S_i))$. We define $s_i(q) = e(H_c^*(B, S_i))$ to have the statement. $\square$

Write $e(S_i) = s_i(q)$, $1 \leq i \leq N$. So there is a $\mathbb{Z}[q]$-linear map

$$e \colon R(\Gamma)[q] \to \mathbb{Z}[q]$$

satisfying the property that $e(R(Z)) = e(Z)$.

A trivial application of Theorem 2 happens when $\pi \colon Z \to B$ is a fibre bundle with fibre F such that the action of $\pi_1(B)$ on $H_c^*(F)$ is trivial. Then $R(Z) = e(F)T$, where T is the trivial local system and $e(Z) = e(F)e(B)$ (this result appears in [12, Proposition 2.4]).

The hypothesis that the action of $\pi_1(B)$ on $H_c^*(F)$ is trivial holds in particular in the following cases:

• B is irreducible and π is locally trivial in the Zariski topology.

• π is a principal G-bundle with G a connected algebraic group.

• Z is a G-space with isotropy $H < G$ such that $G/H \to Z \to B$ is a fiber bundle, and G is a connected algebraic group.

We can also recover easily the result in [12, Proposition 2.10], which is the main tool used in [12].

Corollary 3. *Let $B = \mathbb{C} - \{q_1, \ldots, q_l\}$. Suppose that B_ρ is a rational curve. Then $e(Z) = (q - 1)e(F)^{\mathrm{inv}} - (l - 1)e(F)$, where $e(F)^{\mathrm{inv}}$ denotes the E-polynomial of the invariant part of the cohomology of F.*

Proof. Let Γ be the monodromy, $N = \#\Gamma$, and $S_1 = T, S_2, \ldots, S_N$ the irreducible representations, where T denotes the trivial representation. We only need to see that $e(T) = q - l$ and $e(S_i) = -(l - 1)$ for $i \geq 2$. Given that, if $R(Z) = aT + \sum_{i \geq 2} a_i S_i$, then $e(F)^{\mathrm{inv}} = a$, $e(F) = a + \sum a_i$, and $e(Z) = (q - l)a - (l - 1) \sum a_i = (q - 1)a - (l - 1)(a + \sum a_i)$, as required.

For the trivial representation, it is clear that $e(T) = e(B) = q - l$.

Now let S be a (one-dimensional) irreducible representation of Γ. Let $\tilde{\Gamma}$ be the image of $S \colon \Gamma \to \mathbb{C}^*$, and let $e = \#\tilde{\Gamma}$. Take the e-cover associated to this group, $\tilde{B} \to B$. Then $\tilde{\Gamma}$ acts on $\tilde{B}$ with quotient B. Clearly, $\tilde{B}$ is a rational curve (the quotient $\Gamma \to \tilde{\Gamma}$ produces a covering map $B_\rho \to \tilde{B}$, and B_ρ is a rational curve by assumption). Then we have a fibration $Y \to \tilde{B} \to B$, where Y is a finite set of e points. Clearly, $R(\tilde{B}) = T + \sum_{p=2}^{e} S_{i_p}$, for some $i_p \in \{2, \ldots, N\}$, where $S_{i_2} = S$.

The covering $\tilde{B} \to B$ can be extended to a ramified covering $\varphi \colon \mathbb{P}^1 \to \mathbb{P}^1$. The Hurwitz formula then says that $-2 = e(-2) + r$, where r is the degree of the ramification divisor. Then $\varphi^{-1}(q_1, \ldots, q_l, \infty)$ has $e(l + 1) - r$ points. Therefore $\tilde{B} = \mathbb{P}^1 - \varphi^{-1}(q_1, \ldots, q_l, \infty)$ has

$$e(\tilde{B}) = q + 1 - e(l + 1) + r$$
$$= q + 1 - e(l + 1) + (2e - 2) = (q - l) - (e - 1)(l - 1).$$

The formula in Theorem 2 says that $e(\tilde{B}) = (q - l) + \sum_{p=2}^{e} s_{i_p}$. Therefore $\sum_{p=2}^{e} s_{i_p} = -(e - 1)(l - 1)$.

This happens for all choices of coverings associated to all representations $S_2, \ldots, S_N$. Hence $s_i = -(l - 1)$, for all $i = 2, \ldots, N$. $\qquad\square$

Another case that we are going to use later is the following.

Corollary 4. *Let (3) be a fibration over $B = \mathbb{C}^* \times \mathbb{C}^*$ with finite monodromy, such that F is of balanced type. Then Z is also of balanced type and its E-polynomial is given by*

$$e(Z) = (q - 1)^2 e(F)^{\mathrm{inv}}.$$

Proof. First note that $\pi_1(B) = \mathbb{Z} \times \mathbb{Z}$, so the monodromy Γ, being a quotient $\mathbb{Z} \times \mathbb{Z} \twoheadrightarrow \Gamma$, must be abelian. Moreover, there is some $n > 0$ such that $\mathbb{Z}_n \times \mathbb{Z}_n \twoheadrightarrow \Gamma$, and the covering associated to $\mathbb{Z}_n \times \mathbb{Z}_n$ is of balanced type (is $\mathbb{C}^* \times \mathbb{C}^*$ again). Hence B_ρ is of balanced type, since there are coverings $\mathbb{C}^* \times \mathbb{C}^* \to B_\rho \to \mathbb{C}^* \times \mathbb{C}^*$. Now let

S be an irreducible representation of $R(\Gamma)$. If $S = T$, the trivial representation, then $e(T) = e(B) = (q - 1)^2$.

If S is a non-trivial representation, then take the covering $\tilde{B} \to B$ associated to S. Again there are coverings $\mathbb{C}^* \times \mathbb{C}^* \to B_\rho \to \tilde{B} \to B = \mathbb{C}^* \times \mathbb{C}^*$. The Hodge monodromy representation associated to $\tilde{B}$ is $T + \sum_{p=2}^e S_{i_p}$, where $S_{i_2} = S$. In cohomology $H_c^*(\mathbb{C}^* \times \mathbb{C}^*) \to H_c^*(\tilde{B}) \to H_c^*(B)$ are surjections, but the composition is an isomorphism (multiplication by n). Therefore $e(R(\tilde{B})) = e(\tilde{B}) = (q - 1)^2$, and $e\left(\sum_{p=2}^e S_{i_p}\right) = 0$. This happens for all choices of S, so it must be $e(S) = 0$, for any irreducible non-trivial S. $\square$

2.3. Building blocks. We need to recall some E-polynomials from [12] which serve as building blocks for the E-polynomials of character varieties for any genus $g \geq 2$. They are basically associated to the case of $g = 1$. First, we have that $e(\mathrm{SL}(2, \mathbb{C})) = q^3 - q$ and $e(\mathrm{PGL}(2, \mathbb{C})) = q^3 - q$. Consider the following subsets of $\mathrm{SL}(2, \mathbb{C})$:

- $E_0 :=$ conjugacy class of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. It has $e(E_0) = 1$.

- $E_1 :=$ conjugacy class of $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. It has $e(E_1) = 1$.

- $E_2 :=$ conjugacy class of $J_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. It is $E_2 \cong \mathrm{PGL}(2, \mathbb{C})/U$, with $U = \left\{ \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \mid y \in \mathbb{C} \right\}$. It has $e(E_2) = q^2 - 1$.

- $E_3 :=$ conjugacy class of $J_- = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$. It is $E_3 \cong \mathrm{PGL}(2, \mathbb{C})/U$ and $e(E_3) = q^2 - 1$.

- $E_{4,\lambda} :=$ conjugacy class of $\xi_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, where $\lambda \in \mathbb{C} - \{0, \pm 1\}$. Note that $E_{4,\lambda} = E_{4,\lambda^{-1}}$, since the matrices ξ_λ and $\xi_{\lambda^{-1}}$ are conjugate. We have $E_{4,\lambda} \cong \mathrm{PGL}(2, \mathbb{C})/D$, where $D = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \mid x \in \mathbb{C}^* \right\}$. So $e(E_{4,\lambda}) = q^2 + q$.

- We also need the set $E_4 := \{A \in \mathrm{SL}(2, \mathbb{C}) \mid \mathrm{Tr}(A) \neq \pm 2\}$, which is the union of the conjugacy classes $E_{4,\lambda}$, $\lambda \in \mathbb{C} - \{0, \pm 1\}$. This has $e(E_4) = q^3 - 2q^2 - q$.

Now consider the map

$$f \colon \mathrm{SL}(2, \mathbb{C})^2 \to \mathrm{SL}(2, \mathbb{C}),$$

$$(A, B) \mapsto [A, B] = ABA^{-1}B^{-1}.$$

Note that f is equivariant under the action of $\mathrm{SL}(2, \mathbb{C})$ by conjugation on both spaces. We stratify $X = \mathrm{SL}(2, \mathbb{C})^2$ as follows:

- $X_0 := f^{-1}(E_0)$,
- $X_1 := f^{-1}(E_1)$,
- $X_2 := f^{-1}(E_2)$,
- $X_3 := f^{-1}(E_3)$,
- $X_4 := f^{-1}(E_4)$.

We also introduce the varieties $f^{-1}(C)$ for fixed $C \in \mathrm{SL}(2, \mathbb{C})$ and define accordingly

- $\bar{X}_2 := f^{-1}(J_+)$. If we take $(A, B) \in X_2$, there exists $P \in \mathrm{PGL}(2, \mathbb{C})$ such that $P[A,B]P^{-1} = J_+$, so there is a fibration $U \to \mathrm{PGL}(2,\mathbb{C}) \times \bar{X}_2 \to X_2$ given by $(A, B) \mapsto$

(PAP^{-1}, PBP^{-1}). We get $e(X_2) = (q^2 - 1)e(\bar{X}_2)$.

- $\bar{X}_3 := f^{-1}(J_-)$. Similarly, there is a fibration $U \to \mathrm{PGL}(2, \mathbb{C}) \times \bar{X}_3 \to X_3$, and $e(X_3) = (q^2 - 1)e(\bar{X}_3)$.

- $\bar{X}_{4,\lambda} := f^{-1}(\xi_\lambda)$, for $\lambda \neq 0, \pm 1$. We define also $X_{4,\lambda} = f^{-1}(W_{4,\lambda})$. There is a fibration $D \to \mathrm{PGL}(2, \mathbb{C}) \times \bar{X}_{4,\lambda} \to X_{4,\lambda}$, and $e(X_{4,\lambda}) = (q^2 + q)e(\bar{X}_{4,\lambda})$.

  It will also be convenient to define

- $\bar{X}_4 := \left\{ (A, B, \lambda) \mid [A, B] = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \text{ for some } \lambda \neq 0, \pm 1, A, B \in \mathrm{SL}(2, \mathbb{C}) \right\}$.

There is an action of $\mathbb{Z}_2$ on $\bar{X}_4$ given by interchanging $(A, B, \lambda) \mapsto (P_0^{-1}AP_0, P_0^{-1}BP_0, \lambda^{-1})$, with $P_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The set $\bar{X}_4/\mathbb{Z}_2$ coincides with the union of all $\bar{X}_{4,\lambda}$.

The Hodge–Deligne polynomials computed in [12] are as follows:

$$e(X_0) = q^4 + 4q^3 - q^2 - 4q,$$
$$e(X_1) = q^3 - q,$$
$$e(\bar{X}_2) = q^3 - 2q^2 - 3q,$$
$$e(\bar{X}_3) = q^3 + 3q^2,$$
$$e(\bar{X}_{4,\lambda}) = q^3 + 3q^2 - 3q - 1.$$

There is a fibration

$$\bar{X}_4 \to B = \mathbb{C} - \{0, \pm 1\}.$$

Let γ_0, γ_{-1}, γ_1 be loops around the points 0, -1, 1, respectively. The monodromies around γ_1, γ_{-1} are trivial, and the monodromy around γ_0 is of order 2. So the monodromy group is $\Gamma = \mathbb{Z}_2$ and the Hodge monodromy representation is computed in [12, Theorem 6.1],

$$(8) \qquad R(\bar{X}_4) = (q^3 - 1)T + (3q^2 - 3q)N,$$

where T is the trivial representation and N the non-trivial one. Then Corollary 3 gives

$$(9) \qquad e(\bar{X}_4) = (q - 3)a - 2b = q^4 - 3q^3 - 6q^2 + 5q + 3.$$

(where we have written $R(\bar{X}_4) = aT + bN$, $a = q^3 - 1$, $b = 3q^2 - 3q$).

Taking the quotient by $\mathbb{Z}_2$, we have the fibration

$$\bar{X}_4/\mathbb{Z}_2 \to B = (\mathbb{C} - \{0, \pm 1\})/\mathbb{Z}_2 \cong \mathbb{C} - \{\pm 2\},$$

and the Hodge monodromy representation is a representation of the group $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$, generated by the loops γ_{-2}, γ_2 around the points -2, 2, respectively. The Hodge monodromy representation is computed in [12, Theorem 7.1],

$$(10) \qquad R(\bar{X}_4/\mathbb{Z}_2) = q^3 T - 3q S_2 + 3q^2 S_{-2} - S_0,$$

where T is the trivial representation, $S_{\pm 2}$ is the representation that is non-trivial around the loop $\gamma_{\pm 2}$ and $S_0 = S_{-2} \otimes S_2$. Now writing $R(\bar{X}_4/\mathbb{Z}_2) = aT + bS_2 + cS_{-2} + dS_0$, Corollary 3 says that

$$(11) \qquad e(\bar{X}_4/\mathbb{Z}_2) = (q-1)a - (a+b+c+d) = q^4 - 2q^3 - 3q^2 + 3q + 1.$$

Note that if $R(\bar{X}_4/\mathbb{Z}_2) = aT + bS_2 + cS_{-2} + dS_0$, then $R(\bar{X}_4) = (a+d)T + (b+c)N$, which agrees with (8).

To recover $e(X_4)$, we note that

$$X_4 \cong (\bar{X}_4 \times \mathrm{PGL}(2,\mathbb{C})/D)/\mathbb{Z}_2.$$

By [12, Proposition 3.2], we have that $e(\mathrm{PGL}(2,\mathbb{C})/D)^+ = q^2$ and $e(\mathrm{PGL}(2,\mathbb{C})/D)^- = q$. Using (2),

$$
\begin{aligned}
(12) \qquad e(X_4) &= q^2 e(\bar{X}_4)^+ + q e(\bar{X}_4)^- \\
&= q^2 e(\bar{X}_4/\mathbb{Z}_2) + q(e(\bar{X}_4) - e(\bar{X}_4/\mathbb{Z}_2)) \\
&= (q^2 - q)e(\bar{X}_4/\mathbb{Z}_2) + q e(\bar{X}_4).
\end{aligned}
$$

Now using (11) and (9) we get

$$e(X_4) = q^6 - 2q^5 - 4q^4 + 3q^2 + 2q.$$

3. E-polynomial of the twisted character variety

We start by computing the E-polynomial of the twisted character variety $\mathcal{M}^1 = \mathcal{M}^1(\mathrm{SL}(2,\mathbb{C}))$ for a curve X of genus 3. This space can be described as the quotient

$$(13) \qquad\qquad\qquad \mathcal{M}^1 = W/\mathrm{PGL}(2,\mathbb{C}),$$

where

$$W = \{(A_1, B_1, A_2, B_2, A_3, B_3) \in \mathrm{SL}(2,\mathbb{C})^6 \mid [A_1, B_1][A_2, B_2][A_3, B_3] = -\mathrm{Id}\}.$$

Notice that we only need to consider a geometric quotient, since all representations are irreducible: if there was a common eigenvector v for $(A_1, B_1, A_2, B_2, A_3, B_3)$, we would obtain that $[A_1, B_1](v) = [A_2, B_2](v) = [A_3, B_3](v) = v$ and therefore $[A_1, B_1][A_2, B_2][A_3, B_3](v) = v \neq -\mathrm{Id}(v) = -v$.

To compute the E-polynomial of the moduli space, we will stratify the space into locally closed subvarieties and compute the E-polynomial of each stratum, since E-polynomials are additive with respect to stratifications. We stratify the space W according to the possible values of the traces of the commutators. To simplify the notation,
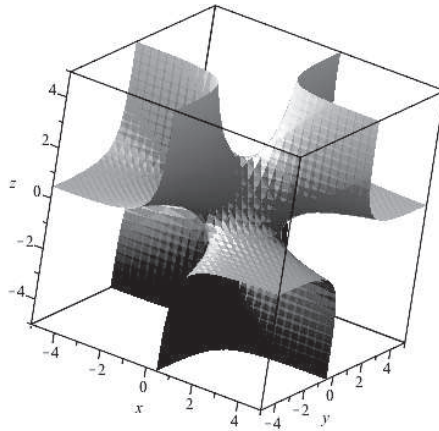
Fig. 1. The cubic surface C.

we write $[A_1, B_1] = \xi_1$, $[A_2, B_2] = \xi_2$, $[A_3, B_3] = \xi_3$. Consider the map

$$F \colon W \to \mathbb{C}^3,$$

$$(A_1, B_1, A_2, B_2, A_3, B_3) \mapsto (t_1, t_2, t_3) = (\operatorname{tr} \xi_1, \operatorname{tr} \xi_2, \operatorname{tr} \xi_3).$$

We are interested in the following condition: if ξ_1, ξ_2, ξ_3 share an eigenvector v with eigenvalue λ_i in each case, then in a suitable basis

$$[A_1, B_1] = \begin{pmatrix} \lambda_1 & * \\ 0 & \lambda_1^{-1} \end{pmatrix}, \quad [A_2, B_2] = \begin{pmatrix} \lambda_2 & * \\ 0 & \lambda_2^{-1} \end{pmatrix}, \quad [A_3, B_3] = \begin{pmatrix} \lambda_3 & * \\ 0 & \lambda_3^{-1} \end{pmatrix},$$

and therefore, as $\xi_1 \xi_2 \xi_3 = -\operatorname{Id}$, we obtain that $\lambda_1 \lambda_2 \lambda_3 = -1$. Other possibility is that v has eigenvalue λ_1 for ξ_1, λ_2 for ξ_2 and λ_3^{-1} for ξ_3, yielding $\lambda_1 \lambda_2 \lambda_3^{-1} = -1$, etc. Working out all possibilities, we have that ξ_1, ξ_2, ξ_3 can share an eigenvector when

$$(14) \qquad \lambda_3 = -\lambda_1 \lambda_2, \quad \lambda_3 = -\lambda_1^{-1} \lambda_2, \quad \lambda_3 = -\lambda_1 \lambda_2^{-1}, \quad \text{or} \quad \lambda_3 = -\lambda_1^{-1} \lambda_2^{-1}.$$

Equivalently, in terms of the traces, when

$$(15) \qquad t_1^2 + t_2^2 + t_3^2 + t_1 t_2 t_3 = 4.$$

This defines a (smooth) cubic surface $C \subset \mathbb{C}^3$, depicted in Fig. 1.

Lemma 5. *ξ_1, ξ_2, ξ_3 share an eigenvector if and only if $(t_1, t_2, t_3) \in C$.*

Proof. We have already seen the if part. For the only if part, note that if (t_1, t_2, t_3) satisfies (15) then the eigenvalues λ_1, λ_2, λ_3 satisfy one of the four relations in (14). Changing some λ_i by λ_i^{-1} if necessary, we can suppose that $\lambda_3 = -\lambda_1 \lambda_2$.

Suppose that some ξ_i is diagonalizable. Without loss of generality, we suppose it is ξ_1, and choose a basis with $\xi_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}$. Write $\xi_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then we have the equation

$$\xi_3^{-1} = -\xi_1\xi_2 = -\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = -\begin{pmatrix} \lambda_1 a & \lambda_1 b \\ \lambda_1^{-1}c & \lambda_1^{-1}d \end{pmatrix}.$$

We get the equations $t_2 = a + d$ and $t_3 = \operatorname{tr}\xi_3 = \operatorname{tr}\xi_3^{-1} = -\lambda_1 a - \lambda_1^{-1}d$. Hence $\lambda_2 + \lambda_2^{-1} = a + d$ and $\lambda_1\lambda_2 + \lambda_1^{-1}\lambda_2^{-1} = \lambda_1 a + \lambda_1^{-1}d$. If $\lambda_1 \neq \pm1$, it must be $a = \lambda_2$, $d = \lambda_2^{-1}$; thus $ad = 1$ and hence $bc = 0$, which implies that the matrices share an eigenvector. If $\lambda_1 = \pm1$, then $\xi_1 = \pm\operatorname{Id}$, and $\xi_2 = \pm\xi_3$, so the matrices share their eigenvectors.

Now suppose that none of the ξ_i are diagonalizable, so they are of Jordan type. Let v_i be the only eigenvector (up to scalar multiples) of ξ_i. If ξ_i do not share an eigenvector, then v_1, v_2 is a basis, on which $\xi_1 = \lambda_1\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, $\xi_2 = \lambda_2\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$, where $\lambda_i \in \{\pm1\}$. Then $\xi_3^{-1} = -\lambda_1\lambda_2\begin{pmatrix} 1+bc & b \\ c & 1 \end{pmatrix}$ hence $t_3 = 2\lambda_3 = -2\lambda_1\lambda_2 = -\lambda_1\lambda_2(2+bc)$. So $bc = 0$, which is a contradiction. $\qquad\square$

We stratify the space W according to the traces of ξ_1, ξ_2, ξ_3. Consider the planes $t_i = \pm2$ and the cubic surface C above. Then consider as well the intersections of these seven subvarieties. This gives the required stratification. We shall compute the E-polynomials of the chunk of W lying above each of these strata, starting by the lower-dimensional ones (points) and going up in dimension.

We shall use some of the polynomials computed in Sections 8–12 in [12] which correspond to some of the strata of the spaces of representations for the case of a curve of genus $g = 2$. We shall point out which stratum we use each time.

3.1. Special points. The intersections of three planes, or of the cubic and two planes, is the collection of eight points given by $(t_1, t_2, t_3) = (\pm2, \pm2, \pm2)$. Let W_1 be the subset of $(A_1, B_1, A_2, B_2, A_3, B_3) \in W$ with traces given by these possibilities. Note that if $t_i = 2$ then $\xi_i = \operatorname{Id}$ or ξ_i is of Jordan type J_+; analogously, if $t_i = -2$ then $\xi_i = -\operatorname{Id}$ or ξ_i is of Jordan type J_-.

- $W_{11} = \{(t_1, t_2, t_3) = (2, 2, 2)\} = F^{-1}((2, 2, 2))$. Then ξ_i are all of Jordan type (if $\xi_1 = \operatorname{Id}$ then $\xi_2\xi_3 = -\operatorname{Id}$, so $\xi_2 = -\xi_3^{-1}$ and $t_2 = -t_3$). Choosing an adequate basis, we can assume that $[A_1, B_1] = J_+$, so $[A_2, B_2][A_3, B_3] = -(J_+)^{-1} = J_-$. This set has been determined in [12, Stratum Z_3, Section 12]. It has polynomial $qe(\bar{X}_2)^2$. Therefore

$$\begin{aligned} e(W_{11}) &= qe(\bar{X}_2)^3 e(\operatorname{PGL}(2, \mathbb{C})/U) \\ &= q^{12} - 6q^{11} + 2q^{10} + 34q^9 - 12q^8 - 82q^7 - 18q^6 + 54q^5 + 27q^4. \end{aligned}$$

- $W_{12} = \{(t_1, t_2, t_3) = (2, 2, -2)\} = F^{-1}((2, 2, -2))$ and the cyclic permutations (accounting for three cases). If $\xi_1 = \operatorname{Id}$, then $[A_2, B_2][A_3, B_3] = -\operatorname{Id}$, with $t_2 = 2$ and

$t_3 = -2$. If $\xi_2 = \mathrm{Id}$ then $\xi_3 = -\mathrm{Id}$; and if $\xi_2 \sim J_+$ then $\xi_3 \sim J_-$, so the E-polynomial is $e(X_0)(e(X_0)e(X_1) + e(\mathrm{PGL}(2, \mathbb{C})/U)e(\bar{X}_2)e(\bar{X}_3))$. If $\xi_1 \sim J_+$, then choosing an adequate basis, we can assume that $[A_1, B_1] = J_+$, so that $[A_2, B_2][A_3, B_3] = -(J_+)^{-1} = J_-$. This set has been determined in [12, Stratum Z_1, Section 12] to be $(q-2)e(\bar{X}_2)e(\bar{X}_3) + e(\bar{X}_2)e(X_1) + e(\bar{X}_3)e(X_0)$. Therefore, multiplying by 3 because of the three cyclic permutations,

$$e(W_{12}) = 3e(X_0)(e(X_0)e(X_1) + e(\mathrm{PGL}(2, \mathbb{C})/U)e(\bar{X}_2)e(\bar{X}_3))$$
$$+ 3e(\bar{X}_2)((q-2)e(\bar{X}_2)e(\bar{X}_3) + e(\bar{X}_2)e(X_1) + e(\bar{X}_3)e(X_0))e(\mathrm{PGL}(2, \mathbb{C})/U)$$
$$= 9q^{12} + 27q^{11} - 69q^{10} - 192q^9 - 27q^8 + 186q^7 + 129q^6 - 42q^4 - 21q^3.$$

- $W_{13} = \{(t_1, t_2, t_3) = (-2, -2, 2)\} = F^{-1}((-2, -2, 2))$ and cyclic permutations (which account for three cases). As in the case of W_{11}, ξ_1 is of Jordan type, and in a suitable basis we can write $\xi_1 = J_-$, so $[A_1, B_1][A_2, B_2] = -J_-^{-1} = J_+$. This set has been determined in [12, Stratum Z_3, Section 11] and it has polynomial $qe(\bar{X}_2)e(\bar{X}_3)$. Therefore

$$e(W_{13}) = 3qe(\bar{X}_3)^2 e(\bar{X}_2)e(\mathrm{PGL}(2, \mathbb{C})/U)$$
$$= 3q^{12} + 12q^{11} - 21q^{10} - 120q^9 - 63q^8 + 108q^7 + 81q^6.$$

- $W_{14} = \{(t_1, t_2, t_3) = (-2, -2, -2)\} = F^{-1}((-2, -2, -2))$. There are two possibilities for ξ_1. If $\xi_1 \sim J_-$, then choosing an adequate basis, we can assume that $[A_1, B_1] = J_-$, so $[A_2, B_2][A_3, B_3] = -(J_-)^{-1} = J_+$. This set has been determined in [12, Stratum Z_2, Section 11] to be $(q-2)e(\bar{X}_3)^2 + 2e(\bar{X}_3)e(X_1)$. If $\xi_1 = -\mathrm{Id}$, then $[A_2, B_2][A_3, B_3] = \mathrm{Id}$, with $t_2 = -2$, and $t_3 = -2$. This case produces the contribution $e(X_1)(e(X_1)^2 + e(\mathrm{PGL}(2, \mathbb{C})/U)e(\bar{X}_3)^2)$. Therefore

$$e(W_{14}) = e(\bar{X}_3)((q-2)e(\bar{X}_3)^2 + 2e(\bar{X}_3)e(X_1))e(\mathrm{PGL}(2, \mathbb{C})/U)$$
$$+ e(X_1)(e(X_1)^2 + e(\mathrm{PGL}(2, \mathbb{C})/U)e(\bar{X}_3)^2)$$
$$= q^{12} + 10q^{11} + 26q^{10} - 12q^9 - 99q^8 - 27q^7 + 72q^6 + 30q^5 - q^3.$$

Adding all up, we obtain

$$e(W_1) = 14q^{12} + 43q^{11} - 62q^{10} - 290q^9$$
$$- 201q^8 + 185q^7 + 264q^6 + 84q^5 - 15q^4 - 22q^3.$$

3.2. Special lines. The intersection of two of the planes are the lines $\{(t_1, t_2, t_3) = (\pm 2, \pm 2, t_3), t_3 \in \mathbb{C} - \{\pm 2\}\}$, and the cyclic permutations of these. The intersection of one of the planes and the cubic surface is given by the lines $\{t_1 = \pm 2, t_2 = \mp t_3 \in \mathbb{C} - \{\pm 2\}\}$ and the cyclic permutations. We denote by W_2 the portion of W lying over these lines, and we stratify in the following sets:

- W_{21} given by $t_1 = 2$, $t_2 = \pm 2$ and $t_3 \in \mathbb{C} - \{\pm 2\}$. Note that if $\xi_1 = \mathrm{Id}$ then $\xi_2 \xi_3 = -\mathrm{Id}$, with $\mathrm{tr}\, \xi_2 = \pm 2$ and $\mathrm{tr}\, \xi_3 \neq \pm 2$, a contradiction. Then ξ_1 is of Jordan type. Choosing an adequate basis, we can assume that $\xi_1 = J_+$. Then $[A_2, B_2][A_3, B_3] = -(J_+)^{-1} = J_-$, where $\mathrm{tr}\, \xi_2 = \pm 2$ and $\mathrm{tr}\, \xi_3 \neq \pm 2$. This set has been computed in [12, Stratum Z_4, Section 12] and it has E-polynomial $q(e(\bar{X}_2) + e(\bar{X}_3))e(\bar{X}_4/\mathbb{Z}_2)$. So

$$
\begin{aligned}
e(W_{21}) &= e(\bar{X}_2)e(\mathrm{PGL}(2, \mathbb{C})/U)q(e(\bar{X}_2) + e(\bar{X}_3))e(\bar{X}_4/\mathbb{Z}_2) \\
&= 2q^{13} - 7q^{12} - 13q^{11} + 47q^{10} + 40q^9 \\
&\quad - 103q^8 - 58q^7 + 93q^6 + 38q^5 - 30q^4 - 9q^3.
\end{aligned}
$$

- W_{22} given by $t_1 = -2$, $t_2 = \pm 2$ and $t_3 \in \mathbb{C} - \{\pm 2\}$. Analogous computations give

$$
\begin{aligned}
e(W_{22}) &= e(\bar{X}_3)e(\mathrm{PGL}(2, \mathbb{C})/U)q(e(\bar{X}_2) + e(\bar{X}_3))e(\bar{X}_4/\mathbb{Z}_2) \\
&= 2q^{13} + 3q^{12} - 22q^{11} - 27q^{10} + 61q^9 \\
&\quad + 58q^8 - 68q^7 - 43q^6 + 27q^5 + 9q^4.
\end{aligned}
$$

- W_{23} given by $t_1 = 2$ and $t_2 = -t_3 \in \mathbb{C} - \{\pm 2\}$. If $\xi_1 = \mathrm{Id}$, $\xi_2 \xi_3 = -\mathrm{Id}$, $t_2 = -t_3 \neq \pm 2$. This is computed in [12, Stratum W_4, Section 9], $e(W_4) = q^9 - 2q^8 - 7q^7 - 18q^6 + 24q^5 + 28q^4 - 17q^3 - 8q^2 - q$. On the other hand, if $\xi_1 \sim J_+$, using an adequate basis, we have $\xi_1 = J_+$, $\xi_2 \xi_3 = -J_+^{-1} = J_-$ and $t_2 = -t_3 \neq \pm 2$. This is computed in [12, Stratum Z_5, Section 12]

$$
\begin{aligned}
e(W_{23}) &= e(X_0)e(W_4) + e(\bar{X}_2)e(\mathrm{PGL}(2, \mathbb{C})/U)e(Z_5) \\
&= 2q^{13} - 3q^{12} - 17q^{11} - 63q^{10} + 115q^9 \\
&\quad + 142q^8 - 193q^7 - 88q^6 + 99q^5 + 17q^4 - 6q^3 - 5q^2.
\end{aligned}
$$

- W_{24} given by $t_1 = -2$, $t_2 = t_3 \neq \pm 2$. If $\xi_1 = -\mathrm{Id}$, then $\xi_2 \xi_3 = \mathrm{Id}$, with $t_2 = t_3 \neq \pm 2$. This is computed in [12, Stratum Y_4, Section 8], however it has a misprint and the correct polynomial is $e(Y_4) = q^9 - 2q^8 + 2q^7 - 18q^6 + 6q^5 + 28q^4 - 8q^3 - 8q^2 - q$. If $\xi_1 \sim J_-$, then in a suitable basis $\xi_2 \xi_3 = J_+$ with $t_2 = t_3 \neq \pm 2$ again, which is computed in [12, Stratum Z_5, Section 11]. So:

$$
\begin{aligned}
e(W_{24}) &= e(X_1)e(Y_4) + e(\bar{X}_3)e(\mathrm{PGL}(2, \mathbb{C})/U)e(Z_5) \\
&= q^{13} + q^{12} - 15q^{11} - 43q^{10} - 40q^9 + 240q^8 \\
&\quad + 26q^7 - 242q^6 + 11q^5 + 43q^4 + 17q^3 + q^2.
\end{aligned}
$$

Considering the possible permutations, and adding the E-polynomials of the strata, we get

$$
\begin{aligned}
e(W_2) &= 3e(W_{21}) + 3e(W_{22}) + 3e(W_{23}) + 3e(W_{24}) \\
&= 21q^{13} - 18q^{12} - 201q^{11} - 258q^{10} + 528q^9 \\
&\quad + 1011q^8 - 879q^7 - 840q^6 + 525q^5 + 117q^4 + 6q^3 - 12q^2.
\end{aligned}
$$

3.3. Special planes. Now consider the planes given by the equations $\{t_i = \pm 2\}$, from which we remove the previous strata. We do the case $t_1 = \pm 2$ and multiply by three the result to account for the three cases $i = 1, 2, 3$. The planes are given by $t_1 = \pm 2$, where $t_2, t_3 \neq \pm 2$ and $t_2 \neq \mp t_3$. Note that it cannot be $\xi_1 = \pm \mathrm{Id}$, since this would imply $t_2 = \mp t_3$. We have the following cases:

- W_{31} given by $\xi_1 \sim J_+$. This implies that, in a suitable basis, $\xi_2 \xi_3 = J_-$, together with the previous restrictions for the traces. This is the set given in [12, Stratum Z_6, Section 12], so

$$
e(W_{31}) = e(\bar{X}_2)e(\mathrm{PGL}(2, \mathbb{C})/U)e(Z_6)
$$
$$
= q^{14} - 7q^{13} + 6q^{12} + 46q^{11} - 35q^{10} - 211q^9 + 94q^8
$$
$$
+ 351q^7 - 103q^6 - 218q^5 + 28q^4 + 39q^3 + 9q^2.
$$

- W_{32} given by $\xi_1 \sim J_-$. This implies that, in a suitable basis, $\xi_2 \xi_3 = J_+$, together with the restrictions $t_2, t_3 \neq \pm 2$, $t_2 \neq t_3$. This is the set given in [12, Stratum Z_6, Section 11]. Therefore

$$
e(W_{32}) = e(\bar{X}_3)e(\mathrm{PGL}(2, \mathbb{C})/U)e(Z_6)
$$
$$
= q^{14} - 2q^{13} - 16q^{12} + 17q^{11} + 71q^{10} - 33q^9 - 194q^8
$$
$$
+ 74q^7 + 174q^6 - 47q^5 - 36q^4 - 9q^3.
$$

The total contribution of the planes is

$$
e(W_3) = 3(e(W_{31}) + e(W_{32}))
$$
$$
= 6q^{14} - 27q^{13} - 30q^{12} + 189q^{11} + 108q^{10} - 732q^9 - 300q^8
$$
$$
+ 1275q^7 + 213q^6 - 795q^5 - 24q^4 + 90q^3 + 27q^2.
$$

3.4. The cubic C. One eigenvector. We now deal with the part of W lying over the cubic surface C, with $t_1, t_2, t_3 \neq \pm 2$. As we saw in Lemma 5, this corresponds to the case that ξ_1, ξ_2, ξ_3 share (at least) one eigenvector.

We deal now with the case where ξ_1, ξ_2 and ξ_3 share just one eigenvector (up to scalar multiples), which we denote by v. Using it as the first vector of a basis that diagonalizes ξ_1, we can arrange that

$$
(16) \qquad \xi_1 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} \mu & 1 \\ 0 & \mu^{-1} \end{pmatrix}, \quad \xi_3 = \begin{pmatrix} -\lambda^{-1}\mu^{-1} & \lambda \\ 0 & -\lambda\mu \end{pmatrix}
$$

where $\lambda = \lambda_1$, $\mu = \lambda_2$ are the eigenvalues of ξ_1, ξ_2 associated to the eigenvector v. Our choice of basis for the above expressions of ξ_1, ξ_2, ξ_3 gives us a slice for the $\mathrm{PGL}(2, \mathbb{C})$-action, since their stabilizer is trivial. So we shall only need to multiply by $e(\mathrm{PGL}(2, \mathbb{C}))$ after computing the E-polynomial of the space

$$
\{(A_1, B_1, A_2, B_2, A_3, B_3 \in \mathrm{SL}(2, \mathbb{C})^6 \mid [A_i, B_i] = \xi_i, \ i = 1, \ldots, 3\}
$$

where ξ_1, ξ_2, ξ_3 satisfy (16). This set can be regarded as a fibration

$$Z \to B = \{(\lambda, \mu) \in (\mathbb{C}^*)^2 \mid \lambda, \mu, \lambda\mu \neq \pm 1\},$$

with fiber

$$\bar{X}_{4,\lambda} \times \bar{X}_{4,\mu} \times \bar{X}_{4,-\lambda^{-1}\mu^{-1}}.$$

To compute its E-polynomial, we would like to extend the fibration to the six curves λ, μ, $\lambda\mu = \pm 1$ and apply Corollary 4. We cannot extend the fibration; however we can extend the local system defining the Hodge monodromy fibration. By (8), the Hodge monodromy fibration $R(\bar{X}_4)$ is trivial over $\lambda = \pm 1$, and it is of order 2 over $\lambda = 0$. Consider the projections:

$$\pi_1 \colon B \to \mathbb{C}^* - \{\pm 1\},$$

$$(\lambda, \mu) \mapsto \lambda,$$

$$\pi_2 \colon B \to \mathbb{C}^* - \{\pm 1\},$$

$$(\lambda, \mu) \mapsto \mu,$$

$$\pi_3 \colon B \to \mathbb{C}^* - \{\pm 1\},$$

$$(\lambda, \mu) \mapsto -\lambda^{-1}\mu^{-1}.$$

Then

$$\bar{Z} = \pi_1^*(\bar{X}_4) \times \pi_2^*(\bar{X}_4) \times \pi_3^*(\bar{X}_4).$$

The Hodge monodromy fibration $R(\bar{Z})$ can be extended (locally trivially) over the lines $\lambda = \pm 1$, $\mu = \pm 1$ and $\lambda\mu = \pm 1$, to a Hodge monodromy fibration $\tilde{R}(\bar{Z})$ over $\tilde{B} = \mathbb{C}^* \times \mathbb{C}^*$. Moreover, the monodromy around $\lambda = 0$ and $\mu = 0$ is of order two. The corresponding group is $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$ and the representation ring is generated by representations T, N_1, N_2, $N_{12} = N_1 \otimes N_2$, where T is the trivial representation, N_1 is the representation with non-trivial monodromy around the origin of the first copy of $\mathbb{C}^*$, and N_2 is the representation with non-trivial monodromy around the origin of the second copy of $\mathbb{C}^*$.

Pulling back the Hodge monodromy representation of $\bar{X}_4$ given in (8), we have that $R(\pi_1^*(\bar{X}_4)) = aT + bN_1$, $R(\pi_2^*(\bar{X}_4)) = aT + bN_2$ and $R(\pi_3^*(\bar{X}_4)) = aT + bN_{12}$, where $a = q^3 - 1$, $b = 3q^2 - 3q$. So the Hodge monodromy representation of $\bar{Z}$ is

$$R(\bar{Z}) = (aT + bN_1) \otimes (aT + bN_2) \otimes (aT + bN_{12})$$
$$= (a^3 + b^3)T + (a^2b + ab^2)N_1 + (a^2b + ab^2)N_2 + (a^2b + ab^2)N_{12}.$$

We extend $R(\bar{Z})$ to a Hodge monodromy fibration $\tilde{R}(\bar{Z})$ over $\tilde{B} = \mathbb{C}^* \times \mathbb{C}^*$ with the same formula, and compute its E-polynomial, applying Corollary 4,

$$
\begin{aligned}
e(\tilde{R}(\bar{Z})) &= (q-1)^2 e(F)^{\mathrm{inv}} = (q-1)^2(a^3 + b^3) \\
&= (q-1)^2((q^3-1)^3 + (3q^2-3q)^3) \\
&= q^{11} - 2q^{10} + q^9 + 24q^8 \\
&\quad - 129q^7 + 267q^6 - 267q^5 + 129q^4 - 24q^3 - q^2 + 2q - 1.
\end{aligned}
$$

Now we substract the contribution of $\tilde{R}(\bar{Z})$ over the lines $\lambda = \pm 1$, $\mu = \pm 1$ and $\lambda\mu = \pm 1$.

• Consider the curve defined by $\lambda = 1$, $\mu \neq \pm 1$. The fibration has the Hodge monodromy of a fibration over $\mathbb{C}^* - \{\pm 1\}$ with fiber

$$
\bar{X}_{4,\lambda_0} \times \bar{X}_{4,\mu} \times \bar{X}_{4,-\mu^{-1}}
$$

This has Hodge monodromy representation equal to $e(\bar{X}_{4,\lambda_0})R(\bar{X}_4) \otimes \tau^* R(\bar{X}_4)$, where $\tau(\mu) = -\mu^{-1}$. This is equal to

$$
\begin{aligned}
e(\bar{X}_{4,\lambda_0})R(\bar{X}_4) \otimes \tau^* R(\bar{X}_4) &= e(\bar{X}_{4,\lambda_0})R(\bar{X}_4) \otimes R(\bar{X}_4) \\
&= e(\bar{X}_{4,\lambda_0})(aT + bN) \otimes (aT + bN) \\
&= e(\bar{X}_{4,\lambda_0})((a^2 + b^2)T + (2ab)N).
\end{aligned}
$$

Using Corollary 3 and (9), we get that the contribution equals

$$
(17) \quad
\begin{aligned}
&e(\bar{X}_{4,\lambda_0})((q-3)(a^2 + b^2) - 2(2ab)) \\
&= q^{10} - 15q^8 - 36q^7 - 24q^6 + 300q^5 - 238q^4 - 60q^3 + 39q^2 + 20q + 3.
\end{aligned}
$$

• The computation for $\lambda = -1$, $\mu \neq \pm 1$ is analogous and gives the same quantity.
• By symmetry, the contribution for $\mu = \pm 1$, and for $\lambda\mu = \pm 1$ is the same as for $\lambda = \pm 1$. So we have to multiply (17) by 6.
• The contribution of the four points $(\pm 1, \pm 1)$ is $4e(X_{4,\lambda_0})^3 = 4q^9 + 36q^8 + 72q^7 - 120q^6 - 288q^5 + 288q^4 + 120q^3 - 72q^2 - 36q - 4$.

Therefore, the E-polynomial of the original fibration is:

$$
\begin{aligned}
e(\bar{Z}) &= e(\tilde{R}(\bar{Z})) \\
&\quad - 6(q^{10} - 15q^8 - 36q^7 - 24q^6 + 300q^5 - 238q^4 - 60q^3 + 39q^2 + 20q + 3) \\
&\quad - 4e(X_{4,\lambda_0})^3 \\
&= q^{11} - 8q^{10} - 3q^9 + 78q^8 \\
&\quad + 15q^7 + 531q^6 - 1779q^5 + 1209q^4 + 216q^3 - 163q^2 - 82q - 15
\end{aligned}
$$

and

$$e(W_4) = e(\mathrm{PGL}(2, \mathbb{C}))e(\bar{Z})$$
$$= q^{14} - 8q^{13} - 4q^{12} + 86q^{11} + 18q^{10} + 453q^9 - 1794q^8$$
$$+ 678q^7 + 1995q^6 - 1372q^5 - 298q^4 + 148q^3 + 82q^2 + 15q.$$

3.5. The cubic C. Two eigenvectors. Suppose now that $(t_1, t_2, t_3) \in C$ and ξ_1, ξ_2, ξ_3 share two eigenvectors. Then they can all be simultaneously diagonalized. With respect to a suitable basis, we have that:

$$[A_1, B_1] = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad [A_2, B_2] = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix},$$

$$[A_3, B_3] = \begin{pmatrix} e - \lambda^{-1}\mu^{-1} & 0 \\ 0 & -\lambda\mu \end{pmatrix}.$$

This defines a fibration $\bar{Z} \to B := \{(\lambda, \mu) \in (\mathbb{C}^*)^2 \mid \lambda, \mu, \lambda\mu \neq \pm 1\}$, with fiber

$$\bar{X}_{4,\lambda} \times \bar{X}_{4,\mu} \times \bar{X}_{4,-\lambda^{-1}\mu^{-1}}.$$

The stabilizer of ξ_1, ξ_2, ξ_3 is $D \times \mathbb{Z}_2$, where D are the diagonal matrices and the $\mathbb{Z}_2$-action is given by the simultaneous permutation of the eigenvalues, i.e., by conjugation by $P_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Therefore the stratum we are analysing is

$$W_5 \cong Z = (\bar{Z} \times \mathrm{PGL}(2, \mathbb{C})/D)/\mathbb{Z}_2.$$

The action on the basis of $\bar{Z} \to B$ takes (λ, μ) to (λ^{-1}, μ^{-1}), producing a fibration

$$(18) \qquad \begin{array}{ccc} \bar{Z} & \longrightarrow & \bar{Z}' := \bar{Z}/\mathbb{Z}_2 \\ \downarrow & & \downarrow \\ B & \longrightarrow & B' = B/\mathbb{Z}_2. \end{array}$$

If we write

$$(19) \qquad \begin{aligned} \pi_1 &: B' \to \mathbb{C}^*/\mathbb{Z}_2, \\ (\lambda, \mu) &\mapsto \lambda, \\ \pi_2 &: B' \to \mathbb{C}^*/\mathbb{Z}_2, \\ (\lambda, \mu) &\mapsto \mu, \\ \pi_3 &: B' \to \mathbb{C}^*/\mathbb{Z}_2, \\ (\lambda, \mu) &\mapsto \lambda^{-1}\mu^{-1} \end{aligned}$$

($\mathbb{Z}_2$ acts on $\mathbb{C}^*$ by $x \sim x^{-1}$), we can obtain three pullback bundles

$$
\begin{array}{ccc}
\bar{Z}'_i & \longrightarrow & \bar{X}_4/\mathbb{Z}_2 \\
\downarrow & & \downarrow \\
B' & \xrightarrow{\pi_i} & (\mathbb{C}^* - \{\pm 1\})/\mathbb{Z}_2
\end{array}
$$

$i = 1, 2, 3$, such that $\bar{Z}' \cong \bar{Z}'_1 \times \bar{Z}'_2 \times \bar{Z}'_3 \cong \pi_1^*(\bar{X}_4/\mathbb{Z}_2) \times \pi_2^*(\bar{X}_4/\mathbb{Z}_2) \times f^*\pi_3^*(\bar{X}_4/\mathbb{Z}_2)$, with $f(x) = -x$.

As a consequence, if we write $R(\bar{X}_4/\mathbb{Z}_2) = aT + bS_2 + cS_{-2} + dS_0$, as in (10), the Hodge monodromy representation is

$$
\begin{aligned}
R(\bar{Z}') = {} & \pi_1^*(R(\bar{X}_4/\mathbb{Z}_2)) \otimes \pi_2^*(R(\bar{X}_4/\mathbb{Z}_2)) \otimes f^*\pi_3^*(R(\bar{X}_4/\mathbb{Z}_2)) \\
= {} & (aT + bS_2^\lambda + cS_{-2}^\lambda + dS_0^\lambda) \\
& \otimes (aT + bS_2^\mu + cS_{-2}^\mu + dS_0^\mu) \otimes (aT + cS_2^{\lambda\mu} + bS_{-2}^{\lambda\mu} + dS_0^{\lambda\mu}) \\
= {} & a^3 T + a^2 b S_2^\mu + a^2 c S_{-2}^\mu + a^2 d S_0^\mu + a^2 b S_2^\lambda + ab^2 S_2^\lambda \otimes S_2^\mu \\
& + abc S_2^\lambda \otimes S_{-2}^\mu + abd S_2^\lambda \otimes S_0^\mu + a^2 c S_{-2}^\lambda + abc S_{-2}^\lambda \otimes S_2^\mu \\
& + ac^2 S_{-2}^\lambda \otimes S_{-2}^\mu + acd S_{-2}^\lambda \otimes S_0^\mu + a^2 d S_0^\lambda + abd S_0^\lambda \otimes S_2^\mu \\
& + acd S_0^\lambda \otimes S_{-2}^\mu + ad^2 S_0^\lambda \otimes S_0^\mu + a^2 c S_2^{\lambda\mu} + abc S_2^\mu \otimes S_2^{\lambda\mu} \\
& + ac^2 S_{-2}^\mu \otimes S_2^{\lambda\mu} + acd S_0^\mu \otimes S_2^{\lambda\mu} + abc S_2^\lambda \otimes S_2^{\lambda\mu} + b^2 c S_2^\lambda \otimes S_2^\mu \otimes S_2^{\lambda\mu} \\
& + bc^2 S_2^\lambda \otimes S_{-2}^\mu \otimes S_2^{\lambda\mu} + bcd S_2^\lambda \otimes S_0^\mu \otimes S_2^{\lambda\mu} + ac^2 S_{-2}^\lambda \otimes S_2^{\lambda\mu} \\
& + bc^2 S_{-2}^\lambda \otimes S_2^\mu \otimes S_2^{\lambda\mu} + c^3 S_{-2}^\lambda \otimes S_{-2}^\mu \otimes S_2^{\lambda\mu} + c^2 d S_{-2}^\lambda \otimes S_0^\mu \otimes S_2^{\lambda\mu} \\
& + acd S_0^\lambda \otimes S_2^{\lambda\mu} + bcd S_0^\lambda \otimes S_2^\mu \otimes S_2^{\lambda\mu} + c^2 d S_0^\lambda \otimes S_{-2}^\mu \otimes S_2^{\lambda\mu} \\
& + cd^2 S_0^\lambda \otimes S_0^\mu \otimes S_2^{\lambda\mu} + a^2 b S_{-2}^{\lambda\mu} + ab^2 S_2^\mu \otimes S_{-2}^{\lambda\mu} + abc S_{-2}^\mu \otimes S_{-2}^{\lambda\mu} \\
& + abd S_0^\mu \otimes S_{-2}^{\lambda\mu} + ab^2 S_2^\lambda \otimes S_{-2}^{\lambda\mu} + b^3 S_2^\lambda \otimes S_2^\mu \otimes S_{-2}^{\lambda\mu} \\
& + b^2 c S_2^\lambda \otimes S_{-2}^\mu \otimes S_{-2}^{\lambda\mu} + b^2 d S_2^\lambda \otimes S_0^\mu \otimes S_{-2}^{\lambda\mu} \\
& + abc S_{-2}^\lambda \otimes S_{-2}^{\lambda\mu} + b^2 c S_{-2}^\lambda \otimes S_2^\mu \otimes S_{-2}^{\lambda\mu} + bc^2 S_{-2}^\lambda \otimes S_{-2}^\mu \otimes S_{-2}^{\lambda\mu} \\
& + bcd S_{-2}^\lambda \otimes S_0^\mu \otimes S_{-2}^{\lambda\mu} + abd S_0^\lambda \otimes S_{-2}^{\lambda\mu} + b^2 d S_0^\lambda \otimes S_2^\mu \otimes S_{-2}^{\lambda\mu} \\
& + bcd S_0^\lambda \otimes S_{-2}^\mu \otimes S_{-2}^{\lambda\mu} + bd^2 S_0^\lambda \otimes S_0^\mu \otimes S_{-2}^{\lambda\mu} + a^2 d S_0^{\lambda\mu} + abd S_2^\mu \otimes S_0^{\lambda\mu} \\
& + acd S_{-2}^\mu \otimes S_0^{\lambda\mu} + ad^2 S_0^\mu \otimes S_0^{\lambda\mu} + abd S_2^\lambda \otimes S_0^{\lambda\mu} + b^2 d S_2^\lambda \otimes S_2^\mu \otimes S_0^{\lambda\mu} \\
& + bcd S_2^\lambda \otimes S_{-2}^\mu \otimes S_0^{\lambda\mu} + bd^2 S_2^\lambda \otimes S_0^\mu \otimes S_0^{\lambda\mu} + acd S_{-2}^\lambda \otimes S_0^{\lambda\mu} \\
& + bcd S_{-2}^\lambda \otimes S_2^\mu \otimes S_0^{\lambda\mu} + c^2 d S_{-2}^\lambda \otimes S_{-2}^\mu \otimes S_0^{\lambda\mu} + cd^2 S_{-2}^\lambda \otimes S_0^\mu \otimes S_0^{\lambda\mu} \\
& + ad^2 S_0^\lambda \otimes S_0^{\lambda\mu} + bd^2 S_0^\lambda \otimes S_2^\mu \otimes S_0^{\lambda\mu} + cd^2 S_0^\lambda \otimes S_{-2}^\mu \otimes S_0^{\lambda\mu} \\
& + d^3 S_0^\lambda \otimes S_0^\mu \otimes S_0^{\lambda\mu},
\end{aligned}
$$

(20)

where $S_\bullet^\lambda = \pi_1^*(S_\bullet)$, $S_\bullet^\mu = \pi_2^*(S_\bullet)$ and $S_\bullet^{\lambda\mu} = \pi_3^*(S_\bullet)$. To obtain the E-polynomial of the total space, we need to substitute each representation by its associated E-polynomial, by Theorem 2.

Proposition 6. *We have*

- $e(T) = e(S_2^\lambda \otimes S_2^\mu \otimes S_2^{\lambda\mu}) = e(S_2^\lambda \otimes S_{-2}^\mu \otimes S_{-2}^{\lambda\mu}) = q^2 - 6q + 9$,
- $e(S_0^a) = e(S_{-2}^\lambda \otimes S_{-2}^\mu \otimes S_{-2}^{\lambda\mu}) = e(S_{-2}^\lambda \otimes S_2^\mu \otimes S_2^{\lambda\mu}) = -2q + 6$,
- $e(S_{\pm2}^a) = e(S_{\pm2}^b \otimes S_{\pm2}^c) = -q + 5$,

for a, b, $c = \lambda$, μ, $\lambda\mu$.

Proof. Recall that the basis is $B' = \{(\lambda, \mu) \in (\mathbb{C}^*)^2 \mid \lambda, \mu, \lambda\mu \neq \pm1\}/\mathbb{Z}_2$. We compute the E-polynomial of each representation case by case:

- $e(T)$. Using (2), we compute $e((\mathbb{C} - \{0, \pm1\})^2/\mathbb{Z}_2) = (q - 2)^2 + 1$, since $e(\mathbb{C} - \{0, \pm1\})^+ = q - 2$ and $e(\mathbb{C} - \{0, \pm1\})^- = -1$. Also $e(\{(\lambda, \mu) \in (\mathbb{C} - \{0, \pm1\})^2 \mid \lambda\mu = \pm1\}/\mathbb{Z}_2) = 2e((\mathbb{C} - \{0, \pm1\})/\mathbb{Z}_2) = 2(q - 2)$. So $e(T) = e(B') = q^2 - 4q + 5 - (2q - 4) = q^2 - 6q + 9$.

- $e(S_0^\lambda)$. Since $S_0^\lambda = \pi_1^*(S_0)$, we need to compute the E-polynomial of the pullback bundle of the fibration $\mathbb{C} - \{0, \pm1\} \to \mathbb{C} - \{\pm2\}$ that maps $\lambda \mapsto \lambda + \lambda^{-1}$, and which has Hodge monodromy representation equal to $T + S_0$. The pullback bundle is

$$
\begin{array}{ccc}
E_{S_0} \subset \bar{E}_{S_0} = (\mathbb{C} - \{0, \pm1\}) \times (\mathbb{C} - \{0, \pm1\}) & \longrightarrow & \mathbb{C} - \{0, \pm1\} \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle g} \\
B' \subset \bar{B}' = (\mathbb{C} - \{0, \pm1\}) \times (\mathbb{C} - \{0, \pm1\})/\mathbb{Z}_2 \xrightarrow{\ \pi_1\ } & \mathbb{C} - \{\pm2\} \cong (\mathbb{C} - \{0, \pm1\})/\mathbb{Z}_2
\end{array}
$$

where $g(\lambda) = \lambda + \lambda^{-1}$ and p is the quotient map. $E_{S_0} = p^{-1}(B')$, and $B' = \bar{B}' - \{(\lambda, \mu) \mid \lambda\mu = \pm1\}$. Then $e(\bar{E}_{S_0}) = (q - 3)^2$, $e(p^{-1}(\{(\lambda, \mu) \mid \lambda\mu = \pm1\})) = e(\{(\lambda, \mu) \in (\mathbb{C} - \{0, \pm1\})^2 \mid \lambda\mu = \pm1\}) = 2(q - 3)$. So $e(E_{S_0}) = e(T + S_0^\lambda) = (q - 3)^2 - 2(q - 3) = q^2 - 8q + 15$. Finally,

$$
e(S_0^\lambda) = e(T + S_0^\lambda) - e(T) = -2q + 6.
$$

The diagram still commutes if we change π_1 by π_2 or π_3, so we infer that $S_0^\lambda \cong S_0^\mu \cong S_0^{\lambda\mu}$ as local systems. This implies that certain reductions can be made: since $S_2^a \otimes S_{-2}^a \cong S_0^a$ ($a = \lambda, \mu, \lambda\mu$), whenever S_0^λ, S_0^μ or $S_0^{\lambda\mu}$ appear in a tensor product we can switch one by another in order to simplify the expression. For example:

$$
S_2^\lambda \otimes S_2^\mu \otimes S_0^{\lambda\mu} \cong S_2^\lambda \otimes S_2^\mu \otimes S_0^\mu \cong S_2^\lambda \otimes S_{-2}^\mu,
$$
$$
S_0^\lambda \otimes S_{-2}^\mu \otimes S_0^{\lambda\mu} \cong S_0^\mu \otimes S_{-2}^\mu \otimes S_0^\mu \cong S_{-2}^\mu,
$$

and so on. In fact, we deduce from this that, for all representations R that appear in (20), either $e(R) = e(S_0^\lambda)$, $e(R) = e(S_{\pm2}^\lambda)$, $e(R) = e(S_{\pm2}^\lambda \otimes S_{\pm2}^\mu)$ or $e(R) = e(S_{\pm2}^\lambda \otimes$

$S^\mu_{\pm 2} \otimes S^{\lambda\mu}_{\pm 2}$), as any representation where S^a_0 appears can be simplified to one of these expressions.

- $e(S^\lambda_{-2})$. To obtain $e(S^\lambda_{-2})$, we take the pullback under the map $\mathbb{C} - \{\pm 2, 0\} \to \mathbb{C} - \{\pm 2\}$, $x \mapsto x^2 - 2$, which ramifies at -2. The pullback fibration is

$$
\begin{array}{ccc}
E_{S_{-2}} \subset \bar{E}_{S_{-2}} = \left\{ \dfrac{(y, \mu)}{(y, \mu) \sim (y^{-1}, \mu^{-1})} \right\} & \longrightarrow & x = y + y^{-1} \in \mathbb{C} - \{\pm 2, 0\} \\
\Big\downarrow {\scriptstyle p} & & \Big\downarrow {\scriptstyle 2:1} \\
B' \subset \bar{B}' = \{[(\lambda = y^2, \mu)]\} & \longrightarrow & x^2 - 2 = y^2 + y^{-2} = \lambda + \lambda^{-1} \in \mathbb{C} - \{\pm 2\}
\end{array}
$$

where $p(y, \mu) = (y^2, \mu) \in B'$, $y \in \mathbb{C} - \{0, \pm 1, \pm\sqrt{-1}\}$. Therefore $e(\bar{E}_{S_{-2}}) = (q-3)(q-2) + 2$. Substracting the contribution corresponding to the two hyperbolas $\{y^2\mu = \pm 1\}$, which has E-polynomial $2(q-3)$, we get $e(E_{S_{-2}}) = e(T + S^\lambda_{-2}) = (q-3)(q-2) + 2 - 2(q-3) = q^2 - 7q + 14$, and

$$
e(S^\lambda_{-2}) = e(T + S^\lambda_{-2}) - e(T) = -q + 5.
$$

The diagram for $e(S^\lambda_2)$ is similar:

$$
\begin{array}{ccc}
E_{S_2} \subset \bar{E}_{S_2} = \left\{ \dfrac{(y, \mu)}{(y, \mu) \sim (y^{-1}, \mu^{-1})} \right\} & \longrightarrow & x = y + y^{-1} \in \mathbb{C} - \{\pm 2, 0\} \\
\Big\downarrow {\scriptstyle p} & & \Big\downarrow {\scriptstyle 2:1} \\
B' \subset \bar{B}' = \{[(\lambda = -y^2, \mu)]\} & \longrightarrow & 2 - x^2 = y^2 + y^{-2} = \lambda + \lambda^{-1} \in \mathbb{C} - \{\pm 2\}
\end{array}
$$

but now $p(y, \mu) = (-y^2, \mu)$. Since $E_{S_2} \cong E_{S_{-2}}$, we obtain that $e(S^\lambda_2) = e(S^\lambda_{-2})$ and analogous computations yield that $e(S^\mu_{\pm 2}) = e(S^\mu_{\pm 2}) = e(S^{\lambda\mu}_{\pm 2})$.

- $e(S^\lambda_{-2} \otimes S^\mu_{-2})$. To compute the E-polynomial of this representation, we can use the fibration with Hodge monodromy representation $(T + S^\lambda_{-2}) \otimes (T + S^\mu_{-2})$, which corresponds to the fibered product of two copies of the pullback fibration $E_{S_{-2}}$, one with $\lambda = y^2$, the other with $\mu = z^2$. The total space is

$$
\begin{array}{c}
E_{S_{-2,-2}} \subset \bar{E}_{S_{-2,-2}} = \left\{ \left(\dfrac{(y, \mu)}{(y, \mu) \sim (y^{-1}, \mu^{-1})}, \dfrac{(\lambda, z)}{(\lambda, z) \sim (\lambda^{-1}, z^{-1})} \right) \right\} \\
\to B' \subset \bar{B}' = \{[\lambda, \mu]\},
\end{array}
$$

where $\lambda = y^2$, $\mu = z^2$ and $(y, z) \sim (y^{-1}, z^{-1})$, $y, z \in \mathbb{C} - \{0, \pm 1, \pm\sqrt{-1}\}$. The E-polynomial is $e(\bar{E}_{S_{-2,-2}}) = (q-3)^2 + 4$. Recall that we need to substract the contribution of the (now four) hyperbolas given by $\{yz = \pm 1\}$ and $\{yz = \pm\sqrt{-1}\}$. The action $(y, z) \sim (y^{-1}, z^{-1})$ acts on each of the first two hyperbolas giving a contribution

of $2(q - 3)$, whereas it interchanges the last two, which gives $q - 5$. We get $e((T + S_{-2}^{\lambda}) \otimes (T + S_{-2}^{\mu})) = e(E_{S_{-2,-2}}) = (q - 3)^2 + 4 - 2(q - 3) - (q - 5) = q^2 - 9q + 24$ and

$$e(S_{-2}^{\lambda} \otimes S_{-2}^{\mu}) = e((T + S_{-2}^{\lambda}) \otimes (T + S_{-2}^{\mu})) - e(T) - e(S_{-2}^{\lambda}) - e(S_{-2}^{\mu})$$
$$= q^2 - 9q + 24 - (q^2 - 6q + 9) - 2(-q + 5) = -q + 5.$$

Computing the different fibered products of E_{S_2} and $E_{S_{-2}}$ gives us

$$e(S_2^{\lambda} \otimes S_{-2}^{\mu}) = e(S_{-2}^{\lambda} \otimes S_2^{\mu}) = e(S_2^{\lambda} \otimes S_2^{\mu}) = -q + 5.$$

Changing the projection does not alter the computations, so $e(S_{\pm 2}^{a} \otimes S_{\pm 2}^{b}) = -q + 5$, for a, $b = \lambda$, μ, $\lambda\mu$.

• $e(S_{-2}^{\lambda} \otimes S_{-2}^{\mu} \otimes S_{-2}^{\lambda\mu})$. To compute this E-polynomial, we can compute the E-polynomial of the representation $(T + S_2^{\lambda}) \otimes (T + S_2^{\mu}) \otimes (T + S_2^{\lambda\mu})$, which corresponds to the fibered product of three copies of $E_{S_{-2}}$. The total space of this fibered product is given by

$$\left(\frac{(y, \mu)}{(y, \mu) \sim (y^{-1}, \mu^{-1})}, \frac{(\lambda, z)}{(\lambda, z) \sim (\lambda^{-1}, z^{-1})}, \frac{(w, \lambda^{-1})}{(w, \lambda^{-1}) \sim (w^{-1}, \lambda)} \right),$$

where $\lambda = y^2$, $\mu = z^2$, $\lambda\mu = w^2$ and $yz = \pm w$, $y, z, w \in \mathbb{C} - \{0, \pm 1, \pm\sqrt{-1}\}$ and $(y, z) \sim (y^{-1}, z^{-1})$. Taking into account the two possible signs for w, and substracting the contribution from the hyperbolas in $\bar{B}'$, we obtain that the E-polynomial is $2(q^2 - 9q + 24)$. This implies that

$$e(S_{-2}^{\lambda} \otimes S_{-2}^{\mu} \otimes S_{-2}^{\lambda\mu})$$
$$= e((T + S_{-2}^{\lambda}) \otimes (T + S_{-2}^{\mu}) \otimes (T + S_{-2}^{\lambda\mu})) - e(T) - 3e(S_{-2}^{\lambda}) - 3e(S_{-2}^{\lambda} \otimes S_{-2}^{\mu})$$
$$= 2(q^2 - 9q + 24) - (q^2 - 6q + 9) - 6(-q + 5) = q^2 - 6q + 9 = e(T).$$

An analogous computation gives the same polynomial for $e(S_2^{\lambda} \otimes S_2^{\mu} \otimes S_{-2}^{\lambda\mu})$ and cyclic permutations of signs.

• $e(S_{-2}^{\lambda} \otimes S_{-2}^{\lambda} \otimes S_2^{\lambda})$. As we did in the previous case, to compute the E-polynomial it suffices to take the fibered product of two copies of $E_{S_{-2}}$ and E_{S_2}. The total space is again parametrized by

$$\left(\frac{(y, \mu)}{(y, \mu) \sim (y^{-1}, \mu^{-1})}, \frac{(\lambda, z)}{(\lambda, z) \sim (\lambda^{-1}, z^{-1})}, \frac{(w, \lambda^{-1})}{(w, \lambda^{-1}) \sim (w^{-1}, \lambda)} \right),$$

where now $\lambda = y^2$, $\mu = z^2$ and $\lambda\mu = -w^2$, $y, z, w \in \mathbb{C} - \{0, \pm 1, \pm\sqrt{-1}\}$. In particular, this implies that $yz = \sqrt{-1}w$ and $yz = -\sqrt{-1}w$, which gives two components that get identified under the $\mathbb{Z}_2$-action given by $(y, z, w) \sim (y^{-1}, z^{-1}, w^{-1})$. Therefore, the quotient is parametrized by $(y, z) \in (\mathbb{C} - \{0, \pm 1, \pm\sqrt{-1}\})^2$, which produces $(q - 5)^2$.

Substracting the four hyperbolas, we get that the E-polynomial of the fibered product is $(q-5)^2 - 4(q-5) = q^2 - 14q + 45$. So

$$e(S^{\lambda}_{-2} \otimes S^{\mu}_{-2} \otimes S^{\lambda\mu}_2)$$

$$= e((T + S^{\lambda}_{-2}) \otimes (T + S^{\mu}_{-2}) \otimes (T + S^{\lambda\mu}_2)) - e(T) - 3e(S^{\lambda}_{-2}) - 3e(S^{\lambda}_{-2} \otimes S^{\mu}_{-2})$$

$$= (q^2 - 14q + 45) - (q^2 - 6q + 9) - 6(-q + 5) = -2(q-3) = e(S^a_0). \qquad \square$$

Substituting the values just obtained for the E-polynomial of every irreducible representation in (20), and the values $a = q^3$, $b = -3q$, $c = 3q^2$, $d = -1$, we obtain the E-polynomial of the total fibration

$$e(\bar{Z}/\mathbb{Z}_2) = e(\bar{Z}') = e(R(\bar{Z}'))$$

(21)
$$= q^{11} - 6q^{10} + 54q^8 - 12q^7 + 189q^6$$
$$- 915q^5 + 666q^4 + 153q^3 - 81q^2 - 43q - 6.$$

Using the formula (12), we get that

$$e(W_5) = e(Z) = (q^2 - q)e(\bar{Z}') + qe(\bar{Z})$$
$$= q^{13} - 6q^{12} - 2q^{11} + 51q^{10} + 12q^9 + 216q^8$$
$$- 573q^7 - 198q^6 + 696q^5 - 18q^4 - 125q^3 - 45q^2 - 9q.$$

3.6. Generic case. Let us finally deal with the case $(t_1, t_2, t_3) \notin C$, $t_i \neq \pm 2$, which corresponds to the open subset U in $(\mathbb{C}^*)^3$ defined by those representations whose ξ_1, ξ_2, ξ_3 are diagonalizable and do not share an eigenvector. Choosing a basis that diagonalizes ξ_1, note that, if we write $\xi_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \xi_3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

so that

$$\xi_3 = \begin{pmatrix} -\lambda_1^{-1}d & \lambda_1 b \\ \lambda_1^{-1}c & -\lambda_1 a \end{pmatrix}$$

and $bc \neq 0$ (see Lemma 5). Conjugating by a diagonal matrix, we can assume that $b = 1$. As $t_2 = a + d$ and $t_3 = -\lambda_1 a - \lambda_1^{-1}d$, we have that a, d are determined by the values of (t_2, t_3); and c is determined by the equation $\det \xi_2 = ad - bc = 1$. We see that for fixed (λ_1, t_2, t_3), a, b, c, d are fully determined, and so are ξ_2, ξ_3.

Consider the $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$-cover given by $(\lambda_1, \lambda_2, \lambda_3) \mapsto (t_1, t_2, t_3)$ over $(\mathbb{C} - \{0, \pm 1\})^3$. Let $\bar{E}$ be the pull-back fibration. The fiber over $(\lambda_1, \lambda_2, \lambda_3)$ is isomorphic to

$$\bar{X}_{4, \lambda_1} \times \bar{X}_{4, \lambda_2} \times \bar{X}_{4, \lambda_3}.$$

Let $(A_1, B_1, A_2, B_2, A_3, B_3) \in \bar{E}$. Then $[A_1, B_1] = \xi_1$, $[A_2, B_2] = \xi_2$, $[A_3, B_3] = \xi_3$, where $a = a(\lambda_1, \lambda_2, \lambda_3)$, $b = 1$, $c = c(\lambda_1, \lambda_2, \lambda_3)$, $d = d(\lambda_1, \lambda_2, \lambda_3)$. Take $Q = Q(\lambda_1, \lambda_2, \lambda_3)$, $S = S(\lambda_1, \lambda_2, \lambda_3)$ matrices such that $Q^{-1}\xi_2 Q = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix}$ and $S^{-1}\xi_3 S = \begin{pmatrix} \lambda_3 & 0 \\ 0 & \lambda_3^{-1} \end{pmatrix}$. Then $\Theta(A_1, B_1, A_2, B_2, A_3, B_3) = (A_1, B_1, Q^{-1}A_2 Q, Q^{-1}B_2 Q, S^{-1}A_3 S, S^{-1}B_3 S)$ identifies $\bar{E}$ with the subset of $\bar{X}_4 \times \bar{X}_4 \times \bar{X}_4$ where $(t_1, t_2, t_3) \notin C$. The second and third copies of $\mathbb{Z}_2$ act as the standard $\mathbb{Z}_2$-action on the second and third copies of $\bar{X}_4$, respectively. The action of the first copy of $\mathbb{Z}_2$ is more delicate: it acts as conjugation by $P_0 = \begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix}$, sending $\xi_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix} \mapsto \begin{pmatrix} \lambda_1^{-1} & 0 \\ 0 & \lambda_1 \end{pmatrix}$, $\xi_2 = \begin{pmatrix} a & 1 \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & 1 \\ c & a \end{pmatrix}$. Then under the isomorphism Θ, it acts by conjugation by $Q^{-1}P_0 Q$ (resp. $S^{-1}P_0 S$) on the second (resp. third) factor of $\bar{X}_4$. This matrix is easily computed to be diagonal, so the action is (homologically) trivial.

The conclusion is that $\bar{E}/\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is isomorphic to the open set of $\bar{X}_4/\mathbb{Z}_2 \times \bar{X}_4/\mathbb{Z}_2 \times \bar{X}_4/\mathbb{Z}_2$, where $(t_1, t_2, t_3) \notin C$.

Now we need to compute the E-polynomial of $T = (\bar{X}_4/\mathbb{Z}_2 \times \bar{X}_4/\mathbb{Z}_2 \times \bar{X}_4/\mathbb{Z}_2) \cap \{(t_1, t_2, t_3) \in C\}$. Here we can parametrize $C \cong \{(\lambda, \mu) \in (\mathbb{C}^*)^3 \mid \lambda, \mu, \lambda\mu \neq \pm 1\}/\mathbb{Z}_2$. The fiber over (λ, μ) is isomorphic to $\bar{X}_{4,\lambda} \times \bar{X}_{4,\mu} \times \bar{X}_{4,-\lambda^{-1}\mu^{-1}}$, hence this space is isomorphic to $\bar{Z}'$, studied in (18). Using (21), we get

$$e(W_6) = e(\mathrm{PGL}(2, \mathbb{C}))(e(\bar{X}_4/\mathbb{Z}_2)^3 - e(\bar{Z}'))$$
$$= q^{15} - 7q^{14} + 8q^{13} + 44q^{12} - 105q^{11} - 109q^{10} + 9q^9$$
$$+ 1068q^8 - 666q^7 - 1182q^6 + 852q^5 + 238q^4 - 92q^3 - 52q^2 - 7q.$$

3.7. Final result. If we add all the E-polynomials of the different strata, we get

$$e(W) = e(W_1) + e(W_2) + e(W_3) + e(W_4) + e(W_5) + e(W_6)$$
$$= q^{15} - 5q^{13} + 10q^{11} - 252q^{10} - 20q^9 + 20q^7 + 252q^6 - 10q^5 + 5q^3 - q.$$

Then

$$e(\mathcal{M}^1) = e(W)/e(\mathrm{PGL}(2, \mathbb{C}))$$
$$= q^{12} - 4q^{10} + 6q^8 - 252q^7 - 14q^6 - 252q^5 + 6q^4 - 4q^2 + 1.$$

This agrees with the result in [16], obtained by arithmetic methods.

REMARK 7. As the referee kindly pointed out, the arithmetic methods developed by Hausel and Rodriguez-Villegas in [9] may be used for the computation of the E-polynomials of some of the strata, even in those cases where the monodromy around the puncture is not semisimple. It would require the use of character tables for $\mathrm{SL}(2, \mathbb{C})$ and [9, Proposition 2.3.2]; this was the approach taken by Mereb to obtain in [16] the E-polynomial of the moduli space $e(\mathcal{M}^1(\mathrm{SL}(n, \mathbb{C})))$. However, these non-semisimple

cases do not appear in the literature and a detailed analysis is needed when the GIT quotient is not smooth, as it is the case for $\mathcal{M}(\mathrm{SL}(2, \mathbb{C}))$ in the next section.

4. Hodge monodromy representation for the genus 2 character variety

We introduce the following sets associated to the representations of a genus 2 complex curve, and give the E-polynomials computed in [12]:

- $Y_0 := \{(A_1, B_1, A_2, B_2) \in \mathrm{SL}(2, \mathbb{C})^4 \mid [A_1, B_1][A_2, B_2] = \mathrm{Id}\}$. Then $e(Y_0) = q^9 + q^8 + 12q^7 + 2q^6 - 3q^4 - 12q^3 - q$, by [12, Section 8.1].
- $Y_1 := \{(A_1, B_1, A_2, B_2) \in \mathrm{SL}(2, \mathbb{C})^4 \mid [A_1, B_1][A_2, B_2] = -\mathrm{Id}\}$. Then $e(Y_1) = q^9 - 3q^7 - 30q^6 + 30q^4 + 3q^3 - q$, by [12, Section 9].
- $\bar{Y}_2 := \left\{(A_1, B_1, A_2, B_2) \in \mathrm{SL}(2, \mathbb{C})^4 \;\middle|\; [A_1, B_1][A_2, B_2] = J_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right\}$, $e(\bar{Y}_2) = q^9 - 3q^7 - 4q^6 - 39q^5 - 4q^4 - 15q^3$, by [12, Section 11].
- $\bar{Y}_3 := \left\{(A_1, B_1, A_2, B_2) \in \mathrm{SL}(2, \mathbb{C})^4 \;\middle|\; [A_1, B_1][A_2, B_2] = J_- = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}\right\}$. Then $e(\bar{Y}_3) = q^9 - 3q^7 + 15q^6 + 6q^5 + 45q^4$, by [12, Section 12].
- $\bar{Y}_{4,\lambda} := \left\{(A_1, B_1, A_2, B_2) \;\middle|\; [A_1, B_1][A_2, B_2] = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\right\}$, for $\lambda \neq 0, \pm1$. Then $e(\bar{Y}_{4,\lambda}) = q^9 - 3q^7 + 15q^6 - 39q^5 + 39q^4 - 15q^3 + 3q^2 - 1$, by [12, Section 10].

Let $\bar{Y}_4 := \left\{(A_1, B_1, A_2, B_2, \lambda) \in \mathrm{SL}(2, \mathbb{C})^4 \times \mathbb{C}^* \mid [A_1, B_1][A_2, B_2] = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \lambda \neq 0, \pm1\right\}$. We have a fibration

$$\bar{Y}_4 \to \mathbb{C} - \{0, \pm1\}.$$

If we take the quotient by the $\mathbb{Z}_2$-action there is another fibration

$$\bar{Y}_4/\mathbb{Z}_2 \to \mathbb{C} - \{\pm2\}.$$

We are interested in the Hodge monodromy representations $R(\bar{Y}_4)$ and $R(\bar{Y}_4/\mathbb{Z}_2)$.

Proposition 8. $R(\bar{Y}_4) = (q^9 - 3q^7 + 6q^5 - 6q^4 + 3q^2 - 1)T + (15q^6 - 45q^5 + 45q^4 - 15q^3)N$.

Proof. We follow the stratification $\bar{Y}_{4,\lambda_0} = \bigsqcup_{i=1}^{7} Z_i$ given in [12, Section 10], and study the behaviour of each stratum when λ varies in $\mathbb{C} - \{0, \pm1\}$ to obtain the Hodge monodromy representation of $\bar{Y}_4$. Let $\xi = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$. As in [12, Section 10], we write

$$\nu = [B_2, A_2] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \delta = [A_1, B_1] = \xi\nu = \begin{pmatrix} \lambda a & \lambda b \\ \lambda^{-1}c & \lambda^{-1}d \end{pmatrix},$$

and $t_1 = \mathrm{tr}\,\nu$, $t_2 = \mathrm{tr}\,\delta$. Note that every (t_1, t_2, λ) determines a, d by

$$a = \frac{t_2 - \lambda^{-1}t_1}{\lambda - \lambda^{-1}}, \quad d = \frac{\lambda t_1 - t_2}{\lambda - \lambda^{-1}}.$$

Then $bc = ad - 1$.

We look at the strata:

• Z_1, corresponding to $t_1 = \pm 2$, $t_2 = \pm 2$. In this case, both ν, δ are of Jordan type. If we take the basis given by $\{u_1, u_2\}$, where u_1 is an eigenvector for ν and u_2 an eigenvector for δ, then

$$\nu = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad \delta = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$$

for certain $x, y \in \mathbb{C}^*$. Now, since $\delta\nu^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, we obtain that $\lambda + \lambda^{-1} = 2 - xy$. We can fix $x = 1$ by rescaling the basis, so y is fixed and there is no monodromy around the origin. Therefore

$$R(Z_1) = e(Z_1)T = (q-1)(e(\bar{X}_2) + e(\bar{X}_3))^2 T$$
$$= (4q^7 - 15q^5 + 5q^4 + 15q^3 - 9q^2)T,$$

where T is the trivial representation.

• Z_2, corresponding to $t_1 = 2$, $t_2 = \lambda + \lambda^{-1}$ and $t_2 = 2$, $t_1 = \lambda + \lambda^{-1}$. We focus on the first case. In this situation, $bc = 0$, so there are three possibilities: either $b = c = 0$ (in which case $\nu = \mathrm{Id}$) or $b = 0$, $c \neq 0$ or $b \neq 0$, $c = 0$ (in either case there is a parameter in $\mathbb{C}^*$ and ν is of Jordan type). In every situation, ν has trivial monodromy, whereas $\delta \sim \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$. This contributes $R(\bar{X}_4)$. Therefore

$$R(Z_2) = 2(e(X_0) + 2(q-1)e(\bar{X}_2))R(\bar{X}_4)$$
$$= (6q^7 - 4q^6 - 6q^5 - 2q^4 + 4q^3 + 6q^2 - 4q)T$$
$$+ (18q^6 - 30q^5 - 6q^4 + 30q^3 - 12q^2)N.$$

• Z_3, given by $t_1 = -2$, $t_2 = -\lambda - \lambda^{-1}$ and $t_2 = -2$, $t_1 = -\lambda - \lambda^{-1}$. This is analogous to the previous case, so

$$R(Z_3) = 2(e(X_1) + 2(q-1)e(\bar{X}_3))R(\bar{X}_4)$$
$$= (4q^7 + 10q^6 - 12q^5 - 6q^4 - 10q^3 + 12q^2 + 2q)T$$
$$+ (12q^6 + 18q^5 - 66q^4 + 30q^3 + 6q^2)N.$$

• Z_4, defined by $t_1 = 2$, $t_2 \neq \pm 2$, $\lambda + \lambda^{-1}$ and $t_2 = 2$, $t_1 \neq \pm 2$, $\lambda + \lambda^{-1}$. Both cases are similar, so we do the first case. For each λ, (t_1, t_2) move in a punctured line $\{(t_1, t_2) \mid t_1 = 2, t_2 \neq \pm 2, \lambda + \lambda^{-1}\}$, where ν is of Jordan form and δ is of diagonal type, with trace t_2. Both families can be trivialized, giving a contribution of $e(\bar{X}_2)$ and $e(\bar{X}_4/\mathbb{Z}_2)$. The missing fiber $\bar{X}_{4,\lambda}$ over $\lambda + \lambda^{-1}$, which needs to be removed, has

monodromy representation $R(\bar{X}_4)$ as λ varies. Therefore

$$R(Z_4) = 2(q-1)e(\bar{X}_2)(e(\bar{X}_4/\mathbb{Z}_2)T - R(\bar{X}_4))$$
$$= (2q^8 - 12q^7 + 10q^6 + 36q^5 - 26q^4 - 36q^3 + 14q^2 + 12q)T$$
$$+ (-6q^6 + 24q^5 - 12q^4 - 24q^3 + 18q^2)N.$$

The factor $(q-1)$ corresponds to the fact that now $bc \neq 0$, so there is the extra freedom given by $\mathbb{C}^*$.

- Z_5, defined by $t_1 = -2$, $t_2 \neq \pm 2$, $-\lambda - \lambda^{-1}$ and $t_2 = -2$, $t_1 \neq \pm 2$, $-\lambda - \lambda^{-1}$. Similarly to Z_4, we obtain

$$R(Z_5) = 2(q-1)e(\bar{X}_3)(e(\bar{X}_4/\mathbb{Z}_2)T - R(\bar{X}_4))$$
$$= (2q^8 - 2q^7 - 24q^6 + 12q^5 + 34q^4 - 10q^3 - 12q^2)T$$
$$+ (-6q^6 - 6q^5 + 30q^4 - 18q^3)N.$$

- Z_6. This stratum corresponds to the set $\{(t_1, t_2) \mid t_1, t_2 \neq \pm 2, ad = 1\}$, which is a hyperbola H_λ for every λ. Since $bc = 0$, we get a contribution of $2q - 1$, arising from the disjoint cases $b = c = 0$; $b = 0$, $c \neq 0$; and $b \neq 0$, $c = 0$. Parametrizing H_λ by a parameter $\mu \in \mathbb{C}^* - \{\pm 1, \pm \lambda^{-1}\}$ as in [12, Section 10], we obtain a fibration over $\mathbb{C}^* - \{\pm 1, \pm \lambda^{-1}\}$ whose fiber over μ is $\bar{X}_{4,\mu} \times \bar{X}_{4,\lambda\mu}$, for fixed λ. When λ varies over $\mathbb{C}^* - \{\pm 1\}$, note that we can extend the local system trivially to the cases $\lambda, \mu = \pm 1$. This extension can be regarded as a local system over the set of $(\lambda, \mu) \in \mathbb{C}^* \times \mathbb{C}^*$

$$\bar{Z}_6 = \bar{X}_4 \times m^* \bar{X}_4 \to \mathbb{C}^* \times \mathbb{C}^*,$$

where $m: \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^*$ maps $(\lambda, \mu) \mapsto \lambda\mu$. The Hodge monodromy representation of $\bar{Z}_6$ belongs to $R(\mathbb{Z}_2 \times \mathbb{Z}_2)[q]$ (with generators N_1, N_2 denoting the representation which is not trivial over the generator of the fundamental group of the first and second copy of $\mathbb{C}^*$ respectively, and $N_{12} = N_1 \otimes N_2$). Since $R(\bar{X}_4) = (q^3 - 1)T + (3q^2 - 3q)N$, we get

$$R_{\mathbb{C}^* \times \mathbb{C}^*}(\bar{Z}_6) = ((q^3-1)T + (3q^2-3q)N_2) \otimes ((q^3-1)T + (3q^2-3q)N_{12})$$
$$= (q^3-1)^2 T + (3q^2-3q)^2 N_1$$
$$+ (3q^2-3q)(q^3-1)N_2 + (3q^2-3q)(q^3-1)N_{12}.$$

We write this as $R_{\mathbb{C}^* \times \mathbb{C}^*}(\bar{Z}_6) = aT + bN_1 + cN_2 + dN_{12}$. To obtain the Hodge monodromy representation over $\lambda \in \mathbb{C}^*$, we use the projection $\pi_1: \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^*$, $(\lambda, \mu) \mapsto \lambda$. Then $T \mapsto e(T)T$, $N_2 \mapsto e(N_2)T$, $N_1 \mapsto e(T)N$, $N_{12} \mapsto e(N_2)N$ for the representations. Using that $e(T) = q - 1$ and $e(N_2) = 0$ and substracting the contribution from the sets

$\mu = \pm 1, \pm \lambda^{-1}$, which yield $4e(\bar{X}_{4,\lambda})R(\bar{X}_4)$, we get

$$R(\bar{Z}_6) = ae(T)T + be(T)N + ce(N_2)T + de(N_2)N - 4e(\bar{X}_{4,\lambda})R(\bar{X}_4)$$
$$= (q^7 - 5q^6 - 12q^5 + 10q^4 + 10q^3 + 12q^2 - 11q - 5)T$$
$$+ (-3q^5 - 51q^4 + 99q^3 - 33q^2 - 12q)N$$

and

$$R(Z_6) = (2q - 1)R(\bar{Z}_6)$$
$$= (2q^8 - 11q^7 - 19q^6 + 32q^5 + 10q^4 + 14q^3 - 34q^2 + q + 5)T$$
$$+ (-6q^6 - 99q^5 + 249q^4 - 165q^3 + 9q^2 + 12q)N.$$

- Z_7, corresponding to the open stratum given by the set of (t_1, t_2) such that $t_i \neq \pm 2$, $i = 1, 2$ and $(t_1, t_2) \notin H_\lambda$. If we forget about the condition $(t_1, t_2) \in H_\lambda$, Z_7 is a fibration over (t_1, t_2) with fiber isomorphic to $\bar{X}_{4,\mu_1} \times \bar{X}_{4,\mu_2}$, $t_i = \mu_i + \mu_i^{-1}$, $i = 1, 2$. Its monodromy is trivial, as the local system is trivial when λ varies. The contribution over H_λ, already computed in the previous stratum, is $R(\bar{Z}_6)$. So we get

$$R(Z_7) = (q - 1)(e(\bar{X}_4/\mathbb{Z}_2)^2 T - R(\bar{Z}_6))$$
$$= (q^9 - 6q^8 + 8q^7 + 27q^6 - 41q^5 - 21q^4 + 23q^3 + 26q^2 - 11q - 6)T$$
$$+ (3q^6 + 48q^5 - 150q^4 + 132q^3 - 21q^2 - 12q)N.$$

Adding all pieces, we get

$$R(\bar{Y}_4) = (q^9 - 3q^7 + 6q^5 - 6q^4 + 3q^2 - 1)T$$
$$+ (15q^6 - 45q^5 + 45q^4 - 15q^3)N. \qquad \square$$

Dividing by $q - 1$, we get the formula (1).

We want to compute the Hodge monodromy representation of $\bar{Y}_4/\mathbb{Z}_2$. We have the following.

Lemma 9. *The Hodge monodromy representation $R(\bar{Y}_4/\mathbb{Z}_2)$ is of the form $R(\bar{Y}_4/\mathbb{Z}_2) = aT + bS_2 + cS_{-2} + dS_0$, for some polynomials $a, b, c, d \in \mathbb{Z}[q]$.*

Proof. The Hodge monodromy representation $R(\bar{Y}_4/\mathbb{Z}_2)$ lies in the representation ring of the fundamental group of $\mathbb{C} - \{\pm 2\}$. Under the double cover $\mathbb{C} - \{0, \pm 1\} \to \mathbb{C} - \{\pm 2\}$, it reduces to $R(\bar{Y}_4)$. By Proposition 8, $R(\bar{Y}_4)$ is of order 2. Hence $R(\bar{Y}_4/\mathbb{Z}_2)$ has only monodromy of order 2 over the loops $\gamma_{\pm 2}$ around the points ± 2. This is the statement of the lemma. $\qquad \square$

To compute $a, b, c, d \in \mathbb{Z}[q]$, we compute the E-polynomial of the twisted SL(2,ℂ)-character variety (13) in another way. Stratify

$$W := \{(A_1, B_1, A_2, B_2, A_3, B_3) \in \mathrm{SL}(2, \mathbb{C})^6 \mid [A_1, B_1][A_2, B_2] = -[A_3, B_3]\}$$

as follows:

- $W_0' = \{(A_1, B_1, A_2, B_2, A_3, B_3) \mid [A_1, B_1][A_2, B_2] = -[B_3, A_3] = \mathrm{Id}\}$. Then

$$\begin{aligned} e(W_0') &= e(Y_0)e(X_1) \\ &= q^{12} + q^{11} + 11q^{10} + q^9 - 12q^8 \\ &\quad - 5q^7 - 12q^6 + 3q^5 + 11q^4 + q^2. \end{aligned}$$

- $W_1' = \{(A_1, B_1, A_2, B_2, A_3, B_3) \mid [A_1, B_1][A_2, B_2] = -[B_3, A_3] = -\mathrm{Id}\}$. Then

$$\begin{aligned} e(W_1') &= e(Y_1)e(X_0) \\ &= q^{13} + 4q^{12} - 4q^{11} - 46q^{10} - 117q^9 + 72q^8 \\ &\quad + 243q^7 - 18q^6 - 124q^5 - 16q^4 + q^3 + 4q^2. \end{aligned}$$

- $W_2' = \{(A_1, B_1, A_2, B_2, A_3, B_3) \mid [A_1, B_1][A_2, B_2] = -[B_3, A_3] \sim J_+\}$. Then

$$\begin{aligned} e(W_2') &= e(\mathrm{PGL}(2, \mathbb{C})/U)e(\bar{Y}_2)e(\bar{X}_3) \\ &= q^{14} + 3q^{13} - 4q^{12} - 16q^{11} - 48q^{10} - 108q^9 + 24q^8 \\ &\quad + 76q^7 + 27q^6 + 45q^5. \end{aligned}$$

- $W_3' = \{(A_1, B_1, A_2, B_2, A_3, B_3) \mid [A_1, B_1][A_2, B_2] = -[B_3, A_3] \sim J_-\}$. Then

$$\begin{aligned} e(W_3') &= e(\mathrm{PGL}(2, \mathbb{C})/U)e(\bar{Y}_3)e(\bar{X}_2) \\ &= q^{14} - 2q^{13} - 7q^{12} + 23q^{11} - 9q^{10} - 33q^9 - 93q^8 \\ &\quad - 123q^7 + 108q^6 + 135q^5. \end{aligned}$$

- $W_4' = \left\{(A_1, B_1, A_2, B_2, A_3, B_3) \mid [A_1, B_1][A_2, B_2] = -[B_3, A_3] \sim \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \right.$ $\left. \lambda \neq 0, \pm 1 \right\}$.

- $\bar{W}_4' = \left\{(A_1, B_1, A_2, B_2, A_3, B_3, \lambda) \mid [A_1, B_1][A_2, B_2] = -[B_3, A_3] = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \right.$ $\left. \lambda \neq 0, \pm 1 \right\}$.

- $\bar{W}_{4,\lambda}' = \left\{(A_1, B_1, A_2, B_2, A_3, B_3) \mid [A_1, B_1][A_2, B_2] = -[B_3, A_3] = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \right\},$ where $\lambda \neq 0, \pm 1$.

Using the formula in Section 3.7,

$$
\begin{aligned}
e(W_4') &= e(W) - e(W_0') - e(W_1') - e(W_2') - e(W_3') \\
&= q^{15} - 2q^{14} - 7q^{13} + 6q^{12} + 6q^{11} - 160q^{10} + 237q^9 + 9q^8 \\
&\quad - 171q^7 + 147q^6 - 69q^5 + 5q^4 + 4q^3 - 5q^2 - q.
\end{aligned}
$$

For the last stratum, note that:

$$
\bar{W}_{4,\lambda}' = \bar{Y}_{4,\lambda} \times \bar{X}_{4,-\lambda}.
$$

So $R(\bar{W}_4'/\mathbb{Z}_2) = R(\bar{Y}_4/\mathbb{Z}_2) \otimes R(\tau^* \bar{X}_4/\mathbb{Z}_2)$, where $\tau\colon \mathbb{C} - \{\pm 2\} \to \mathbb{C} - \{\pm 2\}$, $\tau(x) = -x$. Then

$$
\begin{aligned}
R(\bar{W}_4'/\mathbb{Z}_2) &= R(\bar{Y}_4/\mathbb{Z}_2) \otimes \tau^* R(\bar{X}_4/\mathbb{Z}_2) \\
&= (aT + bS_2 + cS_{-2} + dS_0) \otimes (q^3 T + 3q^2 S_2 - 3q S_{-2} - S_0) \\
&= (q^3 a + 3q^2 b - 3qc - d)T + (3q^2 a + q^3 b - c - 3qd)S_2 \\
&\quad + (-3qa - b + q^3 c + 3q^2 d)S_{-2} + (-a - 3qb + 3q^2 c + q^3 d)S_0 \\
&= a'T + b'S_2 + c'S_{-2} + d'S_0.
\end{aligned}
$$

Using (12), we get

$$
\begin{aligned}
e(W_4') &= (q^2 - q)e(\bar{W}_4'/\mathbb{Z}_2) + q e(\bar{W}_4') \\
&= (q^2 - q)((q - 2)a' - (b' + c' + d')) + q((q - 3)(a' + d') - 2(b' + c')),
\end{aligned}
$$

which gives us the equation

$$
\begin{aligned}
(22) \qquad e(W_4') &= a(q^6 - 2q^5 - 4q^4 + 3q^2 + 2q) + b(2q^5 - 7q^4 - 3q^3 + 7q^2 + q)S \\
&\quad + c(-q^5 - 4q^4 + 4q^2 + q) + d(-5q^4 - q^3 + 5q^2 + q).
\end{aligned}
$$

We can obtain another equation if we recall that

$$
Y_4 := \left\{ (A_1, B_1, A_2, B_2) \ \middle| \ [A_1, B_1][A_2, B_2] \sim \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \lambda \neq 0, \pm 1 \right\}.
$$

Using that $\mathrm{SL}(2, \mathbb{C})^4 = \bigsqcup_{i=0}^4 Y_i$, we obtain that:

$$
\begin{aligned}
e(Y_4) &= e(\mathrm{SL}(2, \mathbb{C})^4) - e(Y_0) - e(Y_1) - e(Y_2) - e(Y_3) \\
&= q^{12} - 2q^{11} - 4q^{10} + 6q^9 - 6q^8 \\
&\quad + 18q^7 - 6q^6 - 18q^5 + 15q^4 - 6q^3 + 2q.
\end{aligned}
$$

But the E-polynomial of Y_4 can again be obtained using the Hodge monodromy representation $R(\bar{Y}_4/\mathbb{Z}_2)$, using (12),

$$e(Y_4) = (q^2 - q)e(\bar{Y}_4/\mathbb{Z}_2) + qe(\bar{Y}_4)$$

(23)
$$= (q^2 - q)((q - 2)a - (b + c + d)) + q((q - 3)(a + d) - 2(b + c))$$

$$= (q^3 - 2q^2 - q)a - (q^2 + q)(b + c) - 2qd.$$

Finally, two more equations arise from the E-polynomial of the fiber of $\bar{Y}_4/\mathbb{Z}_2 \to \mathbb{C} - \{\pm 2\}$,

(24)
$$e(Y_{4,\lambda}) = a + b + c + d,$$

and from the Hodge monodromy representation $R(\bar{Y}_4) = (q^9 - 3q^7 + 6q^5 - 6q^4 + 3q^2 - 1)T + (15q^6 - 45q^5 + 45q^4 - 15q^3)N$ given in Proposition 8. Since $R(\bar{Y}_4) = (a + d)T + (b + c)N$, we get the equation

(25)
$$a + d = q^9 - 3q^7 + 6q^5 - 6q^4 + 3q^2 - 1.$$

From equations (22), (23), (24) and (25), we find

$$a = q^9 - 3q^7 + 6q^5,$$
$$b = -45q^5 - 15q^3,$$
$$c = 15q^6 + 45q^4,$$
$$d = -6q^4 + 3q^2 - 1.$$

We have proved:

Proposition 10. $R(\bar{Y}_4/\mathbb{Z}_2) = (q^9 - 3q^7 + 6q^5)T - (45q^5 + 15q^3)S_2 + (15q^6 + 45q^4)S_{-2} + (-6q^4 + 3q^2 - 1)S_0.$

5. E-polynomial of the character variety of genus 3

Let $\mathcal{M} = \mathcal{M}(SL(2, \mathbb{C}))$ be the character variety of a genus 3 complex curve X, i.e., the moduli space of semisimple representations of its fundamental group into $SL(2, \mathbb{C})$. It can be defined as the space

$$\mathcal{M} = V//PGL(2, \mathbb{C}),$$

where

$$V = \{(A_1, B_1, A_2, B_2, A_3, B_3) \in SL(2, \mathbb{C})^6 \mid [A_1, B_1][A_2, B_2][A_3, B_3] = \mathrm{Id}\}.$$

We stratify V as follows:

- $V_0 = \{(A_1, B_1, A_2, B_2, A_3, B_3) \mid [A_1, B_1][A_2, B_2] = [B_3, A_3] = \mathrm{Id}\}$. Then

$$
\begin{aligned}
e(V_0) &= e(Y_0)e(X_0) \\
&= q^{13} + 5q^{12} + 15q^{11} + 45q^{10} - 8q^9 - 53q^8 \\
&\quad - 32q^7 - 45q^6 + 23q^5 + 44q^4 + q^3 + 4q^2.
\end{aligned}
$$

- $V_1 = \{(A_1, B_1, A_2, B_2, A_3, B_3) \mid [A_1, B_1][A_2, B_2] = [B_3, A_3] = -\mathrm{Id}\}$. Then

$$
\begin{aligned}
e(V_1) &= e(Y_1)e(X_1) \\
&= q^{12} - 4q^{10} - 30q^9 + 3q^8 \\
&\quad + 60q^7 + 3q^6 - 30q^5 - 4q^4 + q^2.
\end{aligned}
$$

- $V_2 = \{(A_1, B_1, A_2, B_2, A_3, B_3) \mid [A_1, B_1][A_2, B_2] = [B_3, A_3] \sim J_+\}$.

$$
\begin{aligned}
e(V_2) &= e(\mathrm{PGL}(2, \mathbb{C})/U)e(\bar{Y}_2)e(\bar{X}_2) \\
&= q^{14} - 2q^{13} - 7q^{12} + 4q^{11} - 16q^{10} + 84q^9 + 132q^8 \\
&\quad - 44q^7 - 65q^6 - 42q^5 - 45q^4.
\end{aligned}
$$

- $V_3 = \{(A_1, B_1, A_2, B_2, A_3, B_3) \mid [A_1, B_1][A_2, B_2] = [B_3, A_3] \sim J_-\}$.

$$
\begin{aligned}
e(V_3) &= e(\mathrm{PGL}(2, \mathbb{C})/U)e(\bar{Y}_3)e(\bar{X}_3) \\
&= q^{14} + 3q^{13} - 4q^{12} + 3q^{11} + 54q^{10} + 57q^9 + 84q^8 \\
&\quad - 63q^7 - 135q^6.
\end{aligned}
$$

- $V_4 = \left\{(A_1, B_1, A_2, B_2, A_3, B_3) \mid [A_1, B_1][A_2, B_2] = [B_3, A_3] \sim \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \lambda \neq 0, \pm 1\right\}$.
  For computing $e(Y_4)$, we define

$$
\begin{aligned}
\bar{V}_{4,\lambda} := \Big\{ (A_1, B_1, A_2, B_2, A_3, B_3) \mid \\
[A_1, B_1][A_2, B_2] = [B_3, A_3] = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \Big\},
\end{aligned}
$$

for $\lambda \neq 0, \pm 1$. There is a fibration $\bar{V}_4 \to \mathbb{C} - \{0, \pm 1\}$ with fiber $\bar{V}_{4,\lambda}$. Note that $\bar{V}_{4,\lambda} \cong$

$\bar{Y}_{4,\lambda} \times \bar{X}_{4,\lambda}$, so

$$
\begin{aligned}
R(\bar{V}_4/\mathbb{Z}_2) &= R(\bar{Y}_4/\mathbb{Z}_2) \otimes R(\bar{X}_4/\mathbb{Z}_2) \\
&= ((q^9 - 3q^7 + 6q^5)T - (45q^5 + 15q^3)S_2 + (15q^6 + 45q^4)S_{-2} \\
&\quad + (-6q^4 + 3q^2 - 1)S_0) \otimes (q^3 T - 3q S_2 + 3q^2 S_{-2} - S_0) \\
&= (q^{12} - 3q^{10} + 51q^8 + 270q^6 + 51q^4 - 3q^2 + 1)T \\
&\quad + (-3q^{10} - 36q^8 - 66q^6 - 36q^4 - 3q^2)S_2 \\
&\quad + (3q^{11} + 6q^9 + 63q^7 + 63q^5 + 6q^3 + 3q)S_{-2} \\
&\quad + (-q^9 - 183q^7 - 183q^5 - q^3)S_0,
\end{aligned}
$$

which we write as $R(\bar{V}_4/\mathbb{Z}_2) = \tilde{a}T + \tilde{b}S_2 + \tilde{c}S_{-2} + \tilde{d}S_0$. If we apply (12),

$$
\begin{aligned}
e(V_4) &= q(q^2 - 2q - 1)\tilde{a} - q(q + 1)(\tilde{b} + \tilde{c}) - 2q\tilde{d} \\
&= q^{15} - 2q^{14} - 7q^{13} + 6q^{12} + 51q^{11} - 70q^{10} + 192q^9 - 171q^8 \\
&\quad - 216q^7 + 237q^6 - 24q^5 + 5q^4 + 4q^3 - 5q^2 - q.
\end{aligned}
$$

From this

(26)
$$
\begin{aligned}
e(V) &= e(V_0) + e(V_1) + e(V_2) + e(V_3) + e(V_4) \\
&= q^{15} - 5q^{13} + q^{12} + 73q^{11} + 9q^{10} + 295q^9 - 5q^8 \\
&\quad - 295q^7 - 5q^6 - 73q^5 + 5q^3 - q.
\end{aligned}
$$

5.1. Contribution from reducibles. To compute the E-polynomial of $\mathcal{M} = \mathcal{M}(\mathrm{SL}(2, \mathbb{C}))$, we need to take a GIT quotient and differentiate between reducible and irreducible orbits.

In [12, Section 8], this analysis is done in the case of $g = 2$, by stratifying the set of irreducible orbits, and computing the E-polynomial of each stratum. The number of strata increases rapidly with the genus. Therefore, for $g = 3$ we are going to follow the method in [14] which consists on computing the E-polynomial of the reducible locus (which has fewer strata) and substracting it from the total.

A reducible representation given by $(A_1, B_1, A_2, B_2, A_3, B_3)$ is S-equivalent to

(27)
$$
\left(\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}, \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix}, \begin{pmatrix} \lambda_3 & 0 \\ 0 & \lambda_3^{-1} \end{pmatrix}, \begin{pmatrix} \lambda_4 & 0 \\ 0 & \lambda_4^{-1} \end{pmatrix}, \begin{pmatrix} \lambda_5 & 0 \\ 0 & \lambda_5^{-1} \end{pmatrix}, \begin{pmatrix} \lambda_6 & 0 \\ 0 & \lambda_6^{-1} \end{pmatrix} \right),
$$

under the equivalence relation $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) \sim (\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1}, \lambda_4^{-1}, \lambda_5^{-1}, \lambda_6^{-1})$ given by the permutation of the eigenvectors. Under the action $\lambda \mapsto \lambda^{-1}$ we have that

$e(\mathbb{C}^*)^+ = q$, $e(\mathbb{C}^*)^- = -1$, so we obtain

$$
\begin{aligned}
e(\mathcal{M}^{\mathrm{red}}) &= e((\mathbb{C}^*)^6/\mathbb{Z}_2) \\
&= (e(\mathbb{C}^*)^+)^6 + \binom{6}{2}(e(\mathbb{C}^*)^+)^4(e(\mathbb{C}^*)^-)^2 \\
&\quad + \binom{6}{4}(e(\mathbb{C}^*)^+)^2(e(\mathbb{C}^*)^-)^4 + (e(\mathbb{C}^*)^-)^6 \\
&= q^6 + 15q^4 + 15q^2 + 1.
\end{aligned}
$$

A reducible representation happens when there is a common eigenvector. So in a suitable basis, it is

(28)
$$
\left(\begin{pmatrix} \lambda_1 & a_1 \\ 0 & \lambda_1^{-1} \end{pmatrix}, \begin{pmatrix} \lambda_2 & a_2 \\ 0 & \lambda_2^{-1} \end{pmatrix}, \begin{pmatrix} \lambda_3 & a_3 \\ 0 & \lambda_3^{-1} \end{pmatrix}, \begin{pmatrix} \lambda_4 & a_4 \\ 0 & \lambda_4^{-1} \end{pmatrix}, \begin{pmatrix} \lambda_5 & a_5 \\ 0 & \lambda_5^{-1} \end{pmatrix}, \begin{pmatrix} \lambda_6 & a_6 \\ 0 & \lambda_6^{-1} \end{pmatrix} \right).
$$

This is parametrized by $(\mathbb{C}^* \times \mathbb{C})^6$. The condition $[A_1, B_1][A_2, B_2][A_3, B_3] = \mathrm{Id}$ is rewritten as

(29)
$$
\begin{aligned}
&\lambda_2(\lambda_1^2 - 1)a_2 - \lambda_1(\lambda_2^2 - 1)a_1 + \lambda_4(\lambda_3^2 - 1)a_4 \\
&\quad - \lambda_3(\lambda_4^2 - 1)a_3 + \lambda_6(\lambda_5^2 - 1)a_6 - \lambda_5(\lambda_6^2 - 1)a_5 = 0.
\end{aligned}
$$

There are four cases:

• R_1 given by $(a_1, a_2, a_3, a_4, a_5, a_6) \in \langle (\lambda_1 - \lambda_1^{-1}, \lambda_2 - \lambda_2^{-1}, \lambda_3 - \lambda_3^{-1}, \lambda_4 - \lambda_4^{-1}, \lambda_5 - \lambda_5^{-1}, \lambda_6 - \lambda_6^{-1}) \rangle$ and $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) \neq (\pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1)$. Then we can conjugate the representation (28) to the diagonal form (27). In this case we can suppose all $a_i = 0$, and the stabilizer are the diagonal matrices $D \subset \mathrm{PGL}(2, \mathbb{C})$. There is an action of $\mathbb{Z}_2$ given by interchanging of the two basis vectors, and if we write $A := (\mathbb{C}^*)^6 - \{(\pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1)\}$, the stratum is $(A \times \mathrm{PGL}(2, \mathbb{C})/D)/\mathbb{Z}_2$. Note that $e(A)^+ = q^6 + 15q^4 + 15q^2 - 63$, $e(A)^- = e(A) - e(A)^+ = -(6q^5 + 20q^3 + 6q)$, and $e(\mathrm{PGL}(2, \mathbb{C})/D)^+ = q^2$, $e(\mathrm{PGL}(2, \mathbb{C})/D)^- = q$. So

$$
e(R_1) = q^8 + 9q^6 - 5q^4 - 69q^2.
$$

• R_2 given by $(a_1, a_2, a_3, a_4, a_5, a_6) \notin \langle (\lambda_1 - \lambda_1^{-1}, \lambda_2 - \lambda_2^{-1}, \lambda_3 - \lambda_3^{-1}, \lambda_4 - \lambda_4^{-1}, \lambda_5 - \lambda_5^{-1}, \lambda_6 - \lambda_6^{-1}) \rangle$ and $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) \neq (\pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1)$. Then (29) determines a hyperplane $H \subset \mathbb{C}^6$, and the condition for $(a_1, a_2, a_3, a_4, a_5, a_6)$ defines a line $l \subset H$. If $U' \cong D \times U$ denotes the upper triangular matrices, we have a surjective map $A \times (H - l) \times \mathrm{PGL}(2, \mathbb{C}) \to R_2$ and the fiber is isomorphic to U'. So

$$
\begin{aligned}
e(R_2) &= ((q - 1)^6 - 64)(q^5 - q)(q^3 - q)/(q^2 - q) \\
&= q^{12} - 5q^{11} + 9q^{10} - 5q^9 - 6q^8 \\
&\quad + 14q^7 - 78q^6 - 58q^5 + 5q^4 - 9q^3 + 69q^2 + 63q.
\end{aligned}
$$

- R_3, given by $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) = (\pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1)$, $(a_1, a_2, a_3, a_4, a_5, a_6) = (0, 0, 0, 0, 0, 0)$. This is the case where $A_i = B_i = \pm \mathrm{Id}$, $i = 1, 2, 3$, which gives 64 points. Therefore

$$e(R_3) = 64.$$

- R_4, given by $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) = (\pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1)$, $(a_1, a_2, a_3, a_4, a_5, a_6) \neq (0, 0, 0, 0, 0, 0)$. In this case, there is at least a matrix of Jordan type. The diagonal matrices act projectivizing the set $(a_1, a_2, a_3, a_4, a_5, a_6) \in \mathbb{C} - \{(0, 0, 0, 0, 0, 0)\}$ and the stabilizer is isomorphic to U. So

$$e(R_4) = 64 e(\mathbb{P}^5) e(\mathrm{PGL}(2, \mathbb{C})/U)$$
$$= 64q^7 + 64q^6 - 64q - 64.$$

Adding all up, we have

$$e(V^{\mathrm{red}}) = e(R_1) + e(R_2) + e(R_3) + e(R_4)$$
$$= q^{12} - 5q^{11} + 9q^{10} - 5q^9 - 5q^8 + 78q^7 - 5q^6 - 58q^5 - 9q^3 - q,$$

and hence

$$e(V^{\mathrm{irr}}) = e(V) - e(V^{\mathrm{red}})$$
$$= q^{15} - 5q^{13} + 78q^{11} + 300q^9 - 373q^7 - 15q^5 + 14q^3,$$

and thus

$$e(\mathcal{M}^{\mathrm{irr}}) = e(V^{\mathrm{irr}})/e(\mathrm{PGL}(2, \mathbb{C}))$$
$$= q^{12} - 4q^{10} + 74q^8 + 374q^6 + q^4 - 14q^2.$$

Finally, we have

$$e(\mathcal{M}) = e(\mathcal{M}^{\mathrm{red}}) + e(\mathcal{M}^{\mathrm{irr}})$$
$$= q^{12} - 4q^{10} + 74q^8 + 375q^6 + 16q^4 + q^2 + 1.$$

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