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<td>Iiyori, Nobuo; Sawabe, Masato</td>
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Osaka University
PARTIALLY ORDERED SETS OF NON-TRIVIAL NILPOTENT $\pi$-SUBGROUPS

Nobuo Iiyori and Masato Sawabe

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Abstract

In this paper, we introduce a subposet $\mathcal{L}_\pi(G)$ of a poset $\mathcal{N}_\pi(G)$ of all non-trivial nilpotent $\pi$-subgroups of a finite group $G$. We examine basic properties of subgroups in $\mathcal{L}_\pi(G)$ which contain the notion of both radical $p$-subgroups and centric $p$-subgroups of $G$. It is shown that $\mathcal{L}_\pi(G)$ is homotopy equivalent to $\mathcal{N}_\pi(G)$. As examples, we investigate in detail the case where symmetric groups.

1. Introduction

Let $G$ be a finite group, and $\text{Sgp}(G)$ the totality of subgroups of $G$. We regard $\text{Sgp}(G)$ as a partially ordered set (poset for short) with respect to the inclusion-relation $\subseteq$. Then any subset $\mathcal{X} \subseteq \text{Sgp}(G)$ can be thought of a subposet of $(\text{Sgp}(G), \subseteq)$ which is identified with the associated order complex. Let $p \in \pi(G)$. Denote by $\mathcal{S}_p(G)$ the totality of non-trivial $p$-subgroups of $G$. A $p$-subgroup complex $\mathcal{X} \subseteq \mathcal{S}_p(G)$ itself is studied well by many authors (see [9] and various references in it). On the other hand, for distinct $p, q \in \pi(G)$, it is also quite important to investigate $\mathcal{X} \subseteq \mathcal{S}_p(G)$ and $\mathcal{Y} \subseteq \mathcal{S}_q(G)$ simultaneously. In order to do so, we focus on nilpotent subgroups, and actually deal with a poset $\mathcal{N}_\pi(G)$ of all non-trivial nilpotent $\pi$-subgroups of $G$ where $\pi \subseteq \pi(G)$. In particular, we introduce a subposet $\mathcal{L}_\pi(G)$ of $\mathcal{N}_\pi(G)$, and show that they are homotopy equivalent each other. It is worth mentioning that a subgroup in $\mathcal{L}_\pi(G)$ contains the notion of both radical $p$-subgroups and centric $p$-subgroups of $G$.

The paper is organized as follows: In Section 2, we establish some notations, and prepare a number of standard posets of subgroups like $\mathcal{N}_\pi(G)$. In Section 3, we introduce a new poset $\mathcal{L}_\pi(G)$ consisting of certain nilpotent $\pi$-subgroups of $G$. We give another description of $\mathcal{L}_\pi(G)$ which is different from the form of the definition. Furthermore some tools for determining $\mathcal{L}_\pi(G)$ are developed. Then by using those results, we classify subgroups in $\mathcal{L}_\pi(G)$ for some groups $G$ as examples. In Section 4, we provide homotopy equivalences among $\mathcal{L}_\pi(G)$ and the other standard posets of subgroups. Relations with known $p$-subgroup posets are examined. In Section 5, we investigate in detail the case where the symmetric group $\mathcal{S}_n$ of degree $n$. In particular, we give a strategy to determine $\mathcal{L}_\pi(\mathcal{S}_n)$ which is focused on irreducible subgroups (see Definition 5.5). Then, as

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examples, we classify subgroups in \( \mathcal{L}_\pi(\mathcal{G}_n) \) for \( n \leq 6 \) by using our method.

Finally, this work is derived from a series of our papers [5, 6, 7].

2. Preliminaries

In this section, we establish some notations which will be used in this paper. Let \( G \) be a finite group with the identity element \( e \). Denote by \( \pi(G) \) the set of all prime divisors of the order of \( G \). Let \( \pi \) be a subset of \( \pi(G) \). A subgroup \( H \) of \( G \) is called a \( \pi \)-subgroup if \( \pi(H) \subseteq \pi \). The notation \( \text{Sgp}(G) \) stands for the totality of subgroups of \( G \). Note that \( \text{Sgp}(G) \) is regarded as a poset together with the usual inclusion-relation \( \leq \). We define the following subposets of \( \text{Sgp}(G), \leq \):

\[
\mathcal{N}_\pi(G) := \{ U \in \text{Sgp}(G) \mid U \text{ is a non-trivial nilpotent } \pi \text{-subgroup of } G \},
\]

\[
\text{Ab}_\pi(G) := \{ U \in \text{Sgp}(G) \mid U \text{ is a non-trivial abelian } \pi \text{-subgroup of } G \}.
\]

Furthermore let \( \mathcal{A}_\pi(G) \) be a subposet consisting of all non-trivial direct products of elementary abelian \( p \)-subgroups of \( G \) where \( p \) runs over primes in \( \pi \). Then we have three posets \( \mathcal{A}_\pi(G) \subseteq \text{Ab}_\pi(G) \subseteq \mathcal{N}_\pi(G) \) on which the group \( G \) acts by conjugation. The set of all maximal elements in \( \mathcal{N}_\pi(G), \leq \) is denoted by \( \mathcal{N}_\pi(G)^\text{max} \). For \( \pi = \{ p_1, \ldots, p_k \} \subseteq \pi(G) \), we sometimes write \( \mathcal{N}_{p_1,\ldots,p_k}(G) \) in place of \( \mathcal{N}_\pi(G) \). The ways of writing \( \mathcal{N}_\pi(G)^\text{max} \) and \( \mathcal{N}_{p_1,\ldots,p_k}(G) \) are applied to the other posets. Let \( p \in \pi(G) \). Denote by \( \mathcal{S}_p(G) \) the totality of non-trivial \( p \)-subgroups of \( G \). Then we note that \( \mathcal{N}_\pi(G) = \mathcal{S}_p(G) \).

Denote by \( Z(G) \) and \( O_{\pi}(G) \) respectively the center of \( G \), and the largest normal \( \pi \)-subgroup of \( G \). For \( A \in \text{Ab}_\pi(G) \), suppose that \( A = A_1 \times \cdots \times A_k \) is the direct product of \( p \)-subgroups \( A_i \) (\( 1 \leq i \leq k \)) of \( A \). Then denote by \( \Omega_i(A) := \Omega_i(A_1) \times \cdots \times \Omega_i(A_k) \in \mathcal{A}_\pi(G) \) where \( \Omega_i(A_i) \in \mathcal{A}_p(G) \) is a subgroup generated by all elements in \( A_i \) of order \( p_i \). For a subgroup \( H \leq G \), if \( O_{\pi}(Z(H)) \neq \{ e \} \) then \( O_{\pi}(Z(H)) \in \text{Ab}_\pi(G) \) and \( \Omega_i(O_{\pi}(Z(H))) \in \mathcal{A}_\pi(G) \). We express these subgroups as \( O_{\pi}(Z(H)) \) and \( \Omega_i(O_{\pi}(Z(H))) \) for short. In this way, we frequently omit parentheses of the composition of group operators throughout this paper.

Let \( (\mathcal{P}, \leq) \) be a poset. For \( z \in \mathcal{P} \), put \( \mathcal{P}_{<z} := \{ x \in \mathcal{P} \mid x \leq z \} \). Similarly, we define \( \mathcal{P}_{\leq z}, \mathcal{P}_{\geq z}, \) and \( \mathcal{P}_{>z} \).

3. Subposets of \( \mathcal{N}_\pi(G) \)

Let \( G \) be a finite group, and \( \pi \subseteq \pi(G) \). We introduce subposets of \( (\mathcal{N}_\pi(G), \leq) \) as follows:

\[
\mathcal{L}_\pi(G) := \{ U \in \mathcal{N}_\pi(G) \mid U \geq O_{\pi}(Z\mathcal{N}_G(U)) \},
\]

\[
\mathcal{L}_\pi^*(G) := \{ U \in \mathcal{N}_\pi(G) \mid U \geq \Omega_0(O_{\pi}(Z\mathcal{N}_G(U))) \}.
\]

Both families are closed under \( G \)-conjugation. In this section, we study basic properties of \( \mathcal{L}_\pi(G) \subseteq \mathcal{L}_\pi^*(G) \), and provide some examples. Note that, for a subgroup \( U \) of \( G \),
\[ U \geq O_{\pi}ZN_G(U) \text{ if and only if } Z(U) \geq O_{\pi}ZN_G(U). \]

**Remark 3.1** (\(p\)-radicals and \(p\)-centrics). Let \( p \in \pi(G) \).

1. Denote by \( B_p(G) \) the totality of non-trivial \( p\)-subgroups \( U \) of \( G \) satisfying \( O_pN_G(U) = U \). A subgroup in \( B_p(G) \) is called a radical \( p\)-subgroup (or just \( p\)-radical) of \( G \). The poset \( B_p(G) \) is a generalized object of the Tits building, and it plays an important role in the area of group geometry. For a \( p\)-radical \( U \in B_p(G) \), we have that \( U \geq Z(U) = ZO_pN_G(U) \geq O_pZN_G(U) \). It follows that \( B_p(G) \subseteq L_p(G) \), and thus, a subgroup in \( L_\pi(G) \) contains the notion of \( p\)-radicals. Furthermore, we see later in Remark 4.9 that \( B_p(G) \) is homotopy equivalent to \( L_p(G) \).

2. A centric \( p\)-subgroup (or just \( p\)-centric) \( U \) of \( G \) is defined as a subgroup in \( S_p(G) \) such that any \( p\)-element in \( C_G(U) \) is contained in \( U \). This is also important in the area of group geometry or representation theory. Then it is now easy to check that a condition \( U \geq O_pZN_G(U) \) holds for a \( p\)-centric \( U \). Thus \( L_p(G) \) includes all \( p\)-centrics.

**Lemma 3.2.** Suppose that \( p \in \pi \). Then \( L_\pi(G) \cap N_{\pi}(G) \subseteq L_p(G) \), and \( L_{\pi}^*(G) \cap N_{\pi}(G) \subseteq L_{\pi}^p(G) \).

**Proof.** For any \( U \in L_\pi(G) \cap N_{\pi}(G) \), we have that \( U \geq O_{\pi}ZN_G(U) \). But \( U \) is a \( p\)-subgroup, so that, \( O_{\pi}ZN_G(U) = O_pZN_G(U) \). Thus \( U \in L_p(G) \). The second assertion similarly holds.

**Lemma 3.3.** For \( U \in N_{\pi}(G) \), put \( K_U := O_{\pi}ZN_G(U) \). Then the product \( UK_U \) is a member of \( L_\pi(G) \).

**Proof.** Since \( U \) and \( K_U \) are nilpotent \( \pi\)-subgroups such that \([U, K_U] = \{e\}\), so is the product \( UK_U \). Set \( H := ZN_G(UK_U) \). Since \( U \leq N_G(U) \leq N_G(UK_U) \), we have that \( H \leq C_G(U) \leq N_G(U) \). It follows that \( H \) is contained in \( ZN_G(U) \). Thus \( O_{\pi}(H) \leq O_{\pi}ZN_G(U) = K_U \leq UK_U \). This shows that \( UK_U \in L_\pi(G) \).

Below is a description of \( L_\pi(G) \) by using \( UK_U \).

**Proposition 3.4.** Under the notation in Lemma 3.3, \( L_\pi(G) = \{UK_U \mid U \in N_{\pi}(G)\} \).

**Proof.** By Lemma 3.3, it is enough to show that a map \( f : N_{\pi}(G) \rightarrow L_\pi(G) \) defined by \( f(U) := UK_U \) is surjective. Indeed, for any \( X \in L_\pi(G) \subseteq N_{\pi}(G) \), we have that \( X \geq O_{\pi}ZN_G(X) =: K_X \) by the definition of \( X \). Thus \( X = XK_X = f(X) \) as desired.

From here, we want to develop some tools for determining \( L_\pi(G) \).

**Lemma 3.5.** The followings hold.

1. \( N_{\pi}(G)^{\text{max}} \subseteq L_\pi(G) \) and \( A_{\pi}(G)^{\text{max}} \subseteq L_{\pi}^*(G) \).
(2) For $U \in \text{Ab}_\pi(G)^{\text{max}}$, $N^\pi_U(G) \subseteq \mathcal{L}^\pi(G)$. In particular, $\text{Ab}_\pi(G)^{\text{max}} \subseteq \text{Ab}_\pi(G) \cap \mathcal{L}^\pi(G)$.

(3) $\text{Ab}_\pi(G)^{\text{max}} = (\text{Ab}_\pi(G) \cap \mathcal{L}^\pi(G))^{\text{max}}$.

Proof. (1) For $U \in N^\pi_U(G)^{\text{max}}$, put $K_U := O^\pi_Z N_G(U)$. Since $U \leq U K_U \in N^\pi_U(G)$ and the maximality of $U$, we have that $U K_U = U$ and $U \geq K_U$. Thus $U \in \mathcal{L}^\pi(G)$. On the other hand, for $V \in \mathcal{A}_\pi(G)^{\text{max}}$, put $K_V^V := \Omega_1 \pi Z N_G(V) \in \mathcal{A}_\pi(G)$. Since $V \leq V K_V^V \in \mathcal{A}_\pi(G)$, we have the second assertion by the same way.

(2) For $U \in \text{Ab}_\pi(G)^{\text{max}}$, take $V \in N^\pi_U(G)^{\geq U}$. Since $U \leq V \leq N_G(V)$, any element $t \in K_V := O^\pi_Z N_G(V)$ commutes with $U$. Thus $U \leq (t) U \in \text{Ab}_\pi(G)$. By the maximality of $U$, we have that $t \in U \leq V$, and so $K_V \leq V$ as desired.

(3) Set $\mathcal{L}^{\text{ab}}(G) := \text{Ab}_\pi(G) \cap \mathcal{L}^\pi(G)$. For $U \in \text{Ab}_\pi(G)^{\text{max}} \subseteq \mathcal{L}^{\text{ab}}(G)$, there exists $R \in \mathcal{L}^{\text{ab}}(G)^{\text{max}} \subseteq \text{Ab}_\pi(G)$ such that $U \leq R$. Then by the maximality of $U$, $U = R \in \mathcal{L}^{\text{ab}}(G)^{\text{max}}$. The converse inclusion similarly holds.

\textbf{Proposition 3.6.} For $V \leq U \in \mathcal{L}^\pi(G)$, suppose that $Z(U) \leq V \leq U$ and $N_G(U) \leq N_G(V)$. Then $V \in \mathcal{L}^\pi(G)$.

Proof. Take any $x \in ZN_G(V)$. Since $N_G(U) \leq N_G(V)$, we have that $[x, N_G(U)] = \{e\}$. This yields that $x \in ZN_G(U)$ and $ZN_G(V) \leq ZN_G(U)$. Thus $O^\pi_Z N_G(V) \leq O^\pi_Z N_G(U) \leq Z(U) \leq V$ as wanted.

\textbf{Definition 3.7.} For subgroups $A \leq B \leq G$, $A$ is said to be weakly closed in $B$ with respect to $G$ if $A^g \leq B$ for some $g \in G$ implies $A^g = A$. In particular, $N_G(B) \leq N_G(A)$ holds.

The next result is an immediate consequence of Proposition 3.6.

\textbf{Proposition 3.8.} For $V \leq U \in \mathcal{L}^\pi(G)$, suppose that $Z(U) \leq V \leq U$.

(1) If $V$ is weakly closed in $U$ with respect to $G$ then $V \in \mathcal{L}^\pi(G)$.

(2) If $V$ is a characteristic subgroup of $U$ then $V \in \mathcal{L}^\pi(G)$. In particular, $Z(U) \in \mathcal{L}^\pi(G)$, and that $O^\pi_Z N_G Z(U) \leq Z(U)$ holds.

Before giving examples, we recall some notations. For a subgroup $H \leq G$, we set $H^G := \{g^{-1} H g \mid g \in G\}$. For an integer $n \geq 2$, the symmetric and alternating group of degree $n$ are denoted by $S_n$ and $A_n$. The notation $C_n$ means the cyclic group of order $n$.

\textbf{Example 3.9 (Solvable group $S_4$).} Let $G = S_4$ of order $2^3 \cdot 3$, and $\pi := \pi(G) = \{2, 3\}$. We determine $\mathcal{L}_\pi(G)$. By Lemma 3.5 (1), $D_8 \simeq U \in \text{Syl}_2(G) \subseteq N^\pi_U(G)^{\text{max}} \subseteq \mathcal{L}^\pi(G)$. Since any subgroup $V$ of $U$ containing $Z(U)$ is weakly closed in $U$ with respect to $G$, we have that $V \in \mathcal{L}^\pi(G)$ by Proposition 3.8 (1). Let $W := \langle (12) \rangle$ be a remaining
2-subgroup of \( G \). Since \( N_G(W) = \langle (12), (34) \rangle \), we have that \( O_\pi ZN_G(W) = \langle (12), (34) \rangle \not\subseteq W \), so that, \( W \notin \mathcal{L}_\pi(G) \). Finally, by Lemma 3.5 (1), \( \text{Syl}_3(G) \subseteq N_\pi(G)^{\text{max}} \subseteq \mathcal{L}_\pi(G) \). Therefore, we get

\[
\mathcal{L}_{2,3}^\ast(G) = \mathcal{L}_{2,3}(G) = N_{2,3}(G) \setminus \langle (12) \rangle^G = (S_2(G) \setminus \langle (12) \rangle^G) \cup \text{Syl}_3(G).
\]

**Example 3.10** (Non-solvable group \( S_3 \)). Let \( G = S_3 \) of order \( 2^3 \cdot 3 \cdot 5 \), and \( \pi := \{2, 3\} \subseteq \pi(G) \). We determine \( \mathcal{L}_\pi(G) \). By the same way as in Example 3.9, we have that \( S_2(G) \setminus \langle (12) \rangle^G \subseteq \mathcal{L}_\pi(G) \). Let \( W := \langle (12) \rangle \) be a remaining 2-subgroup of \( G \). Since \( N_G(W) = \langle (12) \rangle \), we have that \( O_\pi ZN_G(W) = \langle (12) \rangle \), so that, \( W \in \mathcal{L}_\pi(G) \). Let \( X := \langle (123) \rangle \in \text{Syl}_3(G) \subseteq N_\pi(G) \). Since \( N_G(X) = \langle (123), (12), (45) \rangle \), we have that \( O_\pi ZN_G(X) = \langle (45) \rangle \not\subseteq X \). Thus \( X \notin \mathcal{L}_\pi(G) \). Finally, by Lemma 3.5 (2), \( C_6 \equiv \langle (123)(45) \rangle \in \text{Ab}_2(G)^{\text{max}} \subseteq \mathcal{L}_\pi(G) \). Therefore, we get

\[
\mathcal{L}_{2,3}^\ast(G) = \mathcal{L}_{2,3}(G) = N_{2,3}(G) \setminus \langle (123) \rangle^G = S_2(G) \cup \langle (123)(45) \rangle^G.
\]

**Example 3.11** (Simple group \( J_1 \)). Let \( G = J_1 \) be the Janko simple group of order \( 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \), and \( \pi := \{2, 3, 5\} \subseteq \pi(G) \). We determine \( \mathcal{L}_\pi(G) \) referring [2, p.36]. There is a unique class of involutions with a representative \( z \). Set \( U = \langle z \rangle \). Since \( N_G(U) \cong \langle U \times A_4 \rangle \), we have that \( O_\pi ZN_G(U) = U \), so that, \( U \in \mathcal{L}_\pi(G) \). By Lemma 3.5 (1), \( C_2 \times C_2 \times C_2 \cong V \in \text{Syl}_2(G) \subseteq N_\pi(G)^{\text{max}} \subseteq \mathcal{L}_\pi(G) \). Since \( N_G(V) \cong V \times (C_7 \times C_3) \), all subgroups of order \( 2^2 \) are \( G \)-conjugate each other. Take the four group \( C_2 \times C_2 \cong W < A_4 < A_5 < U \times A_5 \cong N_G(U) \). Then \( N_G(W) \cong U \times A_4 \) and \( O_\pi ZN_G(W) = U \not\subseteq W \). Thus \( W \notin \mathcal{L}_\pi(G) \). By looking at the normalizers, we see that \( \text{Syl}_3(G) \cup \text{Syl}_5(G) \subseteq \mathcal{L}_\pi(G) \). Finally, by Lemma 3.5 (2), subgroups isomorphic to \( C_6 \) or \( C_{10} \) are in \( \text{Ab}_2(G)^{\text{max}} \subseteq \mathcal{L}_\pi(G) \). Therefore, we get

\[
\mathcal{L}_{2,3,5}^\ast(G) = \mathcal{L}_{2,3,5}(G) = N_{2,3,5}(G) \setminus W^G
\]

\[= (S_2(G) \setminus W^G) \cup \text{Syl}_3(G) \cup \text{Syl}_5(G) \cup (C_6)^G \cup (C_{10})^G.
\]

4. Homotopy equivalences

Let \( (\mathcal{P}, \leq) \) be a poset. Denote by \( O(\mathcal{P}) = O(\mathcal{P}, \leq) \) the order complex of \( \mathcal{P} \), which is a simplicial complex defined by all inclusion-chains \( (x_0 < \cdots < x_i) \), where \( x_i \in \mathcal{P} \), as simplices. We identify a poset \( \mathcal{P} \) with the associated order complex \( O(\mathcal{P}) \). We write \( \mathcal{P} \simeq \mathcal{Q} \) when posets \( \mathcal{P} \) and \( \mathcal{Q} \) (namely, complexes \( O(\mathcal{P}) \) and \( O(\mathcal{Q}) \)) are homotopy equivalent. Now any subset \( X \subseteq \text{Sgp}(G) \) is thought of a subposet of \( (\text{Sgp}(G), \leq) \). Thus we can consider homotopy properties of \( X \). In this section, we give homotopy equivalences among \( \mathcal{L}_\pi(G) \) and the other standard posets of subgroups. Relations with known \( p \)-subgroup posets are also investigated. The next lemma is fundamental in the theory of subgroup complexes.
Lemma 4.1. Let $\mathcal{P}$ and $\mathcal{Q}$ be posets. Let $\varphi: \mathcal{P} \rightarrow \mathcal{P}$ and $\psi: \mathcal{P} \rightarrow \mathcal{Q}$ be poset maps.

1. (cf. Lemma 3.3.3 in [9]) If there exists $x_0 \in \mathcal{P}$ such that $\varphi(x) \geq x$ and $\varphi(x) \geq x_0$ for any $x \in \mathcal{P}$ (that is, $\mathcal{P}$ is conically contractible) then $\mathcal{P}$ is contractible.

2. (cf. Proposition 3.1.12 (2) in [9]) Suppose that $\varphi(x) \leq x$ for any $x \in \mathcal{P}$. Then for any subset $\text{Im} \varphi \subseteq R \subseteq \mathcal{P}$, we have that $\mathcal{P} \simeq R$. (And dually for $\varphi(x) \geq x$.)

3. (Quillen's fiber theorem; cf. Theorem 4.2.1 in [9]) Suppose that $\psi^{-1}(\mathcal{Q}_{\geq z})$ is contractible for any $z \in \mathcal{Q}$. Then $\mathcal{P} \simeq \mathcal{Q}$. (And dually for $\mathcal{Q}_{\geq z}$).

4. (cf. Theorem 4.3.2 in [9]) Suppose that $\mathcal{P}$ is finite. Let

$$\mathcal{P}^< := \{z \in \mathcal{P} \mid \mathcal{P}_{<z} \text{ is not contractible}\},$$

$$\mathcal{P}^> := \{z \in \mathcal{P} \mid \mathcal{P}_{>z} \text{ is not contractible}\}.$$ 

Then for any subset $\mathcal{P}^< \subseteq R \subseteq \mathcal{P}$, we have that $\mathcal{P} \simeq R$. (And dually for $\mathcal{P}^>$.)

**Proposition 4.2.** The inclusions $\mathcal{A}_\pi(G) \hookrightarrow \mathcal{N}_\pi(G)$ and $\mathcal{A}_{\nu}(G) \hookrightarrow \mathcal{N}_\pi(G)$ induce homotopy equivalences.

Proof. Let $f: \mathcal{A}_\pi(G) \hookrightarrow \mathcal{N}_\pi(G)$ be the inclusion map. Then by Lemma 4.1 (3), it is enough to show that $f^{-1}(\mathcal{N}_\pi(G)_{\leq U}) = \{E \in \mathcal{A}_\pi(G) \mid E \leq U\} = \mathcal{A}_\pi(U)$ is contractible for any $U \in \mathcal{N}_\pi(G)$. Express $U = U_1 \times \cdots \times U_m$ as the direct product of Sylow subgroups $U_i$ $(1 \leq i \leq m)$ of $U$. Then $A := \Omega_1 Z(U) = \Omega_1 Z(U_1) \times \cdots \times \Omega_1 Z(U_m) \neq \{e\}$ is a member of $\mathcal{A}_\pi(U)$. Let $\varphi: \mathcal{A}_\pi(U) \rightarrow \mathcal{A}_\pi(U)$ be a poset map defined by $\varphi(E) := AE$ for $E \in \mathcal{A}_\pi(U)$, which satisfies $\varphi(E) \geq E$ and $\varphi(E) \geq A$. This yields that $\mathcal{A}_\pi(U)$ is contractible by Lemma 4.1 (1).

By the same way, we obtain $\mathcal{A}_{\nu}(G) \simeq \mathcal{N}_\pi(G)$ although we may replace $A := \Omega_1 Z(U)$ with just $Z(U)$ in the above discussion. \(\square\)

**Proposition 4.3.** $\mathcal{N}_\pi^c(G)^c \subseteq \mathcal{L}_\pi(G) \subseteq \mathcal{L}_\pi^c(G) \subseteq \mathcal{N}_\pi(G)$ holds. In particular, $\mathcal{N}_\pi(G)$, $\mathcal{L}_\pi(G)$, and $\mathcal{L}_\pi^c(G)$ are homotopy equivalent each other by Lemma 4.1 (4).

Proof. It is enough to show that $\mathcal{N}_\pi(G)^c \subseteq \mathcal{L}_\pi(G)$.

For $U \in \mathcal{N}_\pi(G)$, we have that $\mathcal{N}_\pi(G)_{>U} \simeq \mathcal{N}_\pi(N_G(U))_{>U}$. Indeed, for any $V \in \mathcal{N}_\pi(G)_{>U}$, $N_V(U) > U$ as $V$ is nilpotent. Then a poset map

$$f: \mathcal{N}_\pi(G)_{>U} \rightarrow \mathcal{N}_\pi(G)_{>U}$$

defined by $V \mapsto N_V(U) \leq V$ provides us $\mathcal{N}_\pi(G)_{>U} \simeq \text{Im} f = \mathcal{N}_\pi(N_G(U))_{>U}$ by Lemma 4.1 (2).

Set $K_U := O_\pi Z N_G(U)$. Since $U$ and $K_U$ are normal nilpotent $\pi$-subgroups of $N_G(U)$, we have that $UK_U \in \mathcal{N}_\pi(N_G(U))$. Suppose that $U \not\supseteq K_U$, that is, $U \not\in \mathcal{L}_\pi(G)$. 


Then $U K_U \in \mathcal{N}_\pi(N_G(U))_U$. Furthermore, for $X \in \mathcal{N}_\pi(N_G(U))_U$, we have that $[X, K_U] = \{e\}$. This yields that $\mathcal{N}_\pi(N_G(U))_U \ni X K_U = X(U K_U)$, and that a poset map

$$\varphi : \mathcal{N}_\pi(N_G(U))_U \to \mathcal{N}_\pi(N_G(U))_U$$

defined by $X \mapsto X(U K_U)$ induces contractibility of $\mathcal{N}_\pi(N_G(U))_U$ by Lemma 4.1 (1). It follows that $\mathcal{N}_\pi(G)^\triangleright \subseteq \mathcal{L}_\pi(G)$.

**Remark 4.4.** The converse inclusion $\mathcal{N}_\pi(G)^\triangleright \supseteq \mathcal{L}_\pi(G)$ is not necessarily established. For example, let $G = M_{12}$ be the Mathieu group of degree 12 of order $2^6 \cdot 3^3 \cdot 5 \cdot 11$, and $\pi := \{2\} \subseteq \pi(G)$. Referring to [2, p. 33], there exists a subgroup $U \cong C_4 \times C_4$ of $G$ with $N_G(U) \cong U \rtimes D_{12}$ and $O_2 Z N_G(U) = \{e\} \leq U$. Thus $U \in \mathcal{L}_2(G)$. However, $\mathcal{N}_2(N_G(U))_U \ni \mathcal{N}_2(D_{12}) = S_2(D_{12})$ is contractible since $O_2(D_{12}) \cong C_2$. This shows that $U \notin \mathcal{N}_2(G)^\triangleright$.

**Proposition 4.5.** The followings hold.

1. $\mathcal{A}_\pi(G)^\triangleright \subseteq \mathcal{A}_\pi(G) \cap \mathcal{L}_\pi(G) \subseteq \mathcal{A}_\pi(G)$.
2. $\mathcal{A}_\pi(G)^\triangleright \subseteq \mathcal{A}_\pi(G) \cap \mathcal{L}_\pi^G(G) \subseteq \mathcal{A}_\pi(G)$.

In particular, we have homotopy equivalences $\mathcal{A}_\pi(G) \simeq \mathcal{A}_\pi(G) \cap \mathcal{L}_\pi(G)$ and $\mathcal{A}_\pi(G) \simeq \mathcal{A}_\pi(G) \cap \mathcal{L}_\pi^G(G)$ by Lemma 4.1 (4).

Proof. For $U \in \mathcal{A}_\pi(G)$, set $K_U := O_\pi Z N_G(U)$. Since $[U, K_U] = \{e\}$, we have that $U K_U \in \mathcal{A}_\pi(G)$. Suppose that $U \notin K_U$, that is, $U \notin \mathcal{A}_\pi(G) \cap \mathcal{L}_\pi(G)$. Then $U K_U \in \mathcal{A}_\pi(G)^\triangleright$. Furthermore, for $X \in \mathcal{A}_\pi(G)^\triangleright$, we have that $X \leq C_G(U) \leq N_G(U)$, and thus $[X, K_U] = \{e\}$. This yields that $\mathcal{A}_\pi(G)^\triangleright \ni X K_U = X(U K_U)$, and that a poset map

$$\varphi : \mathcal{A}_\pi(G)^\triangleright \to \mathcal{A}_\pi(G)^\triangleright$$

defined by $X \mapsto X(U K_U)$ induces contractibility of $\mathcal{A}_\pi(G)^\triangleright$ by Lemma 4.1 (1). It follows that $\mathcal{A}_\pi(G)^\triangleright \subseteq \mathcal{A}_\pi(G) \cap \mathcal{L}_\pi(G)$.

By the same way, we obtain $\mathcal{A}_\pi(G)^\triangleright \subseteq \mathcal{A}_\pi(G) \cap \mathcal{L}_\pi^G(G) \subseteq \mathcal{A}_\pi(G)$ by using $K_U^\triangleright := \Omega_1 O_\pi Z N_G(U)$ in place of $K_U := O_\pi Z N_G(U)$ in the above discussion.

Summarizing Propositions 4.2, 4.3, and 4.5, we obtain the next.

**Proposition 4.6.** The following homotopy equivalences hold.

1. $\mathcal{N}_\pi(G) \simeq \mathcal{L}_\pi(G) \simeq \mathcal{L}_\pi^G(G) \simeq \mathcal{A}_\pi(G) \simeq \mathcal{A}_\pi(G)$.
2. $\mathcal{A}_\pi(G) \simeq \mathcal{A}_\pi(G) \cap \mathcal{L}_\pi(G)$.
3. $\mathcal{A}_\pi(G) \simeq \mathcal{A}_\pi(G) \cap \mathcal{L}_\pi^G(G)$.

Note that equivalences in Proposition 4.6 can be extended to $G$-homotopy equivalences (see [9, Section 3.5] or [11]).
Remark 4.7 (The whole $\pi(G)$ case). In the case of $\pi = \pi(G)$, our equivalence ($\alpha$) in Proposition 4.6 gives $N(G) \simeq \text{Ab}(G) \simeq A(G)$ where these three posets are respectively the totality of non-trivial nilpotent subgroups, abelian subgroups, and direct products of elementary abelian subgroups of $G$. This result coincides with a part of [8, Proposition 1.2].

Like Lemma 4.1, posets $S_p(G)$, $A_p(G)$, and $B_p(G)$ (see Remark 3.1) are also fundamental in the theory of subgroup complexes. In particular, those three posets are homotopy equivalent each other (cf. [9, p. 165]). Below is an immediate consequence of Proposition 4.6 with $\pi = \{p\}$. In particular, equivalences related to $L_p(G)$ should be new.

**Corollary 4.8.** The following homotopy equivalences hold.

$$S_p(G) = N_p(G) \simeq \text{Ab}_p(G) \simeq A_p(G) \simeq L_p(G) \simeq L_p^*(G),$$

$$\text{Ab}_p(G) \simeq \{U \in \text{Ab}_p(G) \mid U \geq O_pZN_G(U)\},$$

$$A_p(G) \simeq \{U \in A_p(G) \mid U \geq \Omega_1O_pZN_G(U)\}.$$

**Remark 4.9.** (1) Recall that a poset $Z_p(G) := \{U \in A_p(G) \mid \Omega_1O_pZN_G(U) = U\}$ is introduced by Benson (see [1, p. 226]). It is known that $A_p(G)^* \subseteq Z_p(G)$ (cf. [9, Remark 4.3.5]), so that, $A_p(G) \simeq Z_p(G)$. But this equivalence of $A_p(G)$ is different from $A_p(G) \simeq A_p(G) \cap L_p(G)$ in Corollary 4.8.

(2) As mentioned in Remark 3.1, $B_p(G)$ is included in $L_p(G)$. Thus a relation $B_p(G) = B_p(G) \cap L_p(G)$ holds. Furthermore, we have that $B_p(G) \simeq S_p(G) \simeq L_p(G)$ by Corollary 4.8.

**Remark 4.10.** We investigated $N_p(G)^*$ in Proposition 4.3, and also $\text{Ab}_p(G)^*$ and $A_p(G)^*$ in Proposition 4.5. On the other hand, it is known (cf. [9, p. 152]) that $S_p(G)^* = A_p(G)$ and $S_p(G)^* \subseteq B_p(G)$ in general. Furthermore the equality $S_p(G)^* = B_p(G)$ holds assuming Quillen conjecture which is saying that if $S_p(G)$ is contractible then $O_p(G)$ is non-trivial. From this viewpoint, a subgroup in $N_p(G)^* \subseteq L_p(G)$ might be a candidate of “$\pi$-radicals”. In addition, we already saw in Remark 3.1 that a subgroup in $L_p(G)$ contains the notion of $p$-radicals.

**Remark 4.11.** Suppose that $O_p(G) \neq \{e\}$. Then a relation $U \leq U O_p(G) \geq O_p(G)$ for any $U \in S_p(G)$ gives us (conical) contractibility of $S_p(G)$. The converse is Quillen conjecture. How about $N_p(G)$? Let $G$ be the symmetric group $S_4$ of degree 4, and $\pi := \pi(G) = \{2, 3\}$. Then $N_\pi(G) = S_2(G) \cup S_3(G)$ is disconnected (i.e. non-contractible) even if $O_\pi(G) = G \neq \{e\}$ or $O_\pi F(G) = F(G) \cong C_2 \times C_2 \neq \{e\}$ where $F(G)$ is the Fitting subgroup of $G$. 
5. Investigations on $L_\pi(\mathfrak{S}_n)$

For a positive integer $n$, denote by $\mathfrak{S}(\Omega) = \mathfrak{S}_n$ the symmetric group on a set $\Omega := \{1, 2, \ldots, n\}$. In this section, we investigate subgroups in $L_\pi(\mathfrak{S}(\Omega))$. It is shown that the determination of $H \in L_\pi(\mathfrak{S}(\Omega))$ can be reduced to the case where $H$ is irreducible (see Definition 5.5) such that there is no fixed point of $H$ on $\Omega$. Then focusing on the irreducibility of subgroups, we provide a strategy to determine $L_\pi(\mathfrak{S}_n)$. As examples, we classify subgroups in $L_\pi(\mathfrak{S}_n)$ for $n \leq 6$ by using our method.

For a family $\mathcal{H} \subseteq \text{Sgp}(\mathfrak{S}_n)$ of subgroups closed under $\mathfrak{S}_n$-conjugation, denote by $\mathcal{H}/\sim_{\mathfrak{S}_n}$ a set of $\mathfrak{S}_n$-conjugate representatives of $\mathcal{H}$.

5.1. The symmetric group. We establish some notations on $\mathfrak{S}(\Omega)$. For $x, y \in \mathfrak{S}(\Omega)$, the composition $xy \in \mathfrak{S}(\Omega)$ is read from left to right, and denote by $\alpha^x \in \Omega$ the image of $\alpha \in \Omega$ under $x$. Let $e \in \mathfrak{S}(\Omega)$ be the identity element. The notation $E := \{e\}$ stands for the trivial subgroup of $\mathfrak{S}(\Omega)$. For a subgroup $H \leq \mathfrak{S}(\Omega)$, as in [3, p. 19], the set of fixed points and support of $H$ are defined by

$$\text{fix}(H) := \{\alpha \in \Omega \mid \alpha^h = \alpha \text{ for all } h \in H\},$$

$$\text{supp}(H) := \Omega \setminus \text{fix}(H) = \{\alpha \in \Omega \mid \alpha^h \neq \alpha \text{ for some } h \in H\}.$$  

It is clear that $H = E$ if and only if $\text{supp}(H) = \emptyset$.

**Notation 5.1.** For an $H$-invariant subset $\Gamma \subseteq \Omega$, denote by $H|_\Gamma \leq \mathfrak{S}(\Omega)$ the group of permutations which agree with an element of $H$ on $\Gamma$ and are the identity on $\Omega \setminus \Gamma$. In other words, for an element $h \in H$, we identify a bijective restriction map $h|_{\Gamma} : \Gamma \to \Gamma$ with a permutation on $\Omega$ which is the identity on $\Omega \setminus \Gamma$. Then the group $H|_{\Gamma}$ is defined by $\{h|_{\Gamma} \mid h \in H\} \leq \mathfrak{S}(\Gamma) \leftrightarrow \mathfrak{S}(\Omega)$.

A subset $\text{supp}(H) \subseteq \Omega$ is $N_{\mathfrak{S}(\Omega)}(H)$-invariant, and $H$ is identified with $H|_{\text{supp}(H)} \leq \mathfrak{S}(\text{supp}(H))$. For any $H$-invariant subset $\Gamma \subseteq \Omega$, it is clear that $\text{supp}(H|_{\Gamma}) = \text{supp}(H) \cap \Gamma$.

5.2. Reduction to the fixed point free case. In this section, we show that the determination of $H \in L_\pi(\mathfrak{S}(\Omega))$ can be reduced to the case where $H$ has no fixed points in $\Omega$. Put

$$L_\pi(\mathfrak{S}(\Omega))^0 := \{H \in L_\pi(\mathfrak{S}(\Omega)) \mid \text{fix}(H) = \emptyset\}.$$  

**Lemma 5.2.** Let $H \leq \mathfrak{S}(\Omega)$ be a non-trivial subgroup.

1. Suppose $2 \notin \pi$. Then $H \in L_\pi(\mathfrak{S}(\Omega))$ if and only if $H \in L_\pi(\mathfrak{S}(\Omega / \text{fix}(H)))^0$.
2. Suppose $2 \in \pi$. Then $H \in L_\pi(\mathfrak{S}(\Omega))$ if and only if $H \in L_\pi(\mathfrak{S}(\Omega / \text{fix}(H)))^0$ and $|\text{fix}(H)| \neq 2$. 

Proof. Set $G := \mathcal{G}(\Omega)$, $\Omega_+ := \text{supp}(H)$, and $\Omega_0 := \text{fix}(H)$. Recall that $H$ is identified with $H_+ := H|_{\text{supp}(H)}$. In order to prove this lemma, it is enough to show that $H \in L_\pi(\mathcal{G}(\Omega))$ if and only if $H_+ \in L_\pi(\mathcal{G}(\Omega_+))^0$, and $|\Omega_0| \neq 2$ or $2 \not\in \pi$. Now since $N_G(H)$ acts on both $\Omega_0$ and $\Omega_+$, we have that $N_G(H) \leq \mathcal{G}(\Omega_0) \times \mathcal{G}(\Omega_+)$. Hence

\[ N_G(H) = N_{\mathcal{G}(\Omega_0) \times \mathcal{G}(\Omega_+)}(H_+) = \mathcal{G}(\Omega_0) \times N_{\mathcal{G}(\Omega_+)}(H_+), \]

\[ O_\pi Z N_G(H) = O_\pi Z(\mathcal{G}(\Omega_0)) \times O_\pi Z(N_{\mathcal{G}(\Omega_+)}(H_+)). \]

Suppose that $H \in L_\pi(G)$, that is, $H_+ = H \geq O_\pi Z N_G(H)$. Then $O_\pi Z(\mathcal{G}(\Omega_0)) = E$ and $H_+ \geq O_\pi Z(N_{\mathcal{G}(\Omega_+)}(H_+))$. Thus $H_+ \in L_\pi(\mathcal{G}(\Omega_+))^0$. Furthermore $Z(\mathcal{G}(\Omega_0))$ is non-trivial if and only if $|\Omega_0| = 2$. This yields that $O_\pi Z(\mathcal{G}(\Omega_0)) = E$ if and only if $|\Omega_0| \neq 2$ or $2 \not\in \pi$. The converse is now clear. The proof is complete. \qed

The following result is a consequence of Lemma 5.2.

**Proposition 5.3.** For positive integers $n \geq 3$ and $2 \leq k \leq n - 1$, set $[k] := \{1, \ldots, k\} \subseteq \Omega$. Then we have that

\[ L_\pi(\mathcal{G}(\Omega))/\sim_{\mathcal{G}(\Omega)} = \begin{cases} \bigcup_{k=2}^{n-1} L_\pi(\mathcal{G}([k])/\sim_{\mathcal{G}([k])}) \cup L_\pi(\mathcal{G}(\Omega))/\sim_{\mathcal{G}(\Omega)} & \text{if } 2 \not\in \pi, \\ \bigcup_{k=2}^{n-1} L_\pi(\mathcal{G}([k])/\sim_{\mathcal{G}([k])}) \cup L_\pi(\mathcal{G}(\Omega))/\sim_{\mathcal{G}(\Omega)} & \text{if } 2 \in \pi. \end{cases} \]

By Proposition 5.3 together with the inductive argument, the determination of $L_\pi(\mathcal{G}(\Omega))$ can be reduced to that of $L_\pi(\mathcal{G}(\Omega))^0$.

**5.3. Reduction to components.** In this section, we introduce the irreducibility of a subgroup of $\mathcal{G}(\Omega)$, and show that any non-trivial subgroup $H$ of $\mathcal{G}(\Omega)$ can be uniquely decomposed into irreducible subgroups of $H$. Using such a decomposition of $H$, the notion of components of $H$ comes out. Then we show that the determination of $H \in L_\pi(\mathcal{G}(\Omega))^0$ can be reduced to the case where $H$ itself is a component of $H$.

**Notation 5.4.** If a direct product subgroup $H = H_1 \times H_2 \leq \mathcal{G}(\Omega)$ satisfies $\text{supp}(H_1) \cap \text{supp}(H_2) = \emptyset$, then we denote it by $H = H_1 \perp H_2$. In this case, we have a disjoint union $\text{supp}(H) = \text{supp}(H_1) \uplus \text{supp}(H_2)$. Furthermore, we recursively define $H_1 \perp H_2 \perp \cdots \perp H_l$ for any finite number of subgroups $H_l \leq \mathcal{G}(\Omega)$ by $(H_1 \perp \cdots \perp H_{l-1}) \perp H_l$.

**Definition 5.5.** Let $H \leq \mathcal{G}(\Omega)$ be a subgroup. $H$ is said to be reducible if there exist non-trivial subgroups $H_1, H_2 \leq H$ such that $H = H_1 \perp H_2$. On the other
hand, we call $H$ irreducible if $H \neq E$ and $H$ is not reducible, that is, whenever $H = K \perp L$ for subgroups $K, L \leq H$ then $K = E$ or $L = E$.

**Lemma 5.6.** (1) For a subgroup $H = H_1 \perp H_2 \leq \mathcal{G}(\Omega)$ and an $H$-invariant subset $\Gamma \subseteq \Omega$, we have that $H|_{\Gamma} = H_1|_{\Gamma} \perp H_2|_{\Gamma}$.

(2) Suppose that $A \perp B = A \perp C \leq \mathcal{G}(\Omega)$. Then $B = C$.

Proof. (1) Straightforward.

(2) Set $D := A \perp B$. Then $\Gamma_B := \text{supp}(B) = \text{supp}(D) \setminus \text{supp}(A) = \text{supp}(C) = : \Gamma_C$.

For a $D$-invariant subset $\Gamma_B = \Gamma_C$, we have by (1) that

\[ D|_{\Gamma_a} = (A \perp B)|_{\Gamma_a} = A|_{\Gamma_a} \perp B|_{\Gamma_a} = E \perp B = B, \]

\[ D|_{\Gamma_c} = (A \perp C)|_{\Gamma_c} = A|_{\Gamma_c} \perp C|_{\Gamma_c} = E \perp C = C. \]

Thus $B = C$ as wanted. \(\square\)

**Proposition 5.7.** Let $H \leq \mathcal{G}(\Omega)$ be a non-trivial subgroup. Then $H$ is decomposed as

\[ H = H_1 \perp \cdots \perp H_l \]

where the $H_i \leq H$ are irreducible and unique up to order.

Proof. We proceed by induction on $|\text{supp}(H)| > 0$. For the existence, we may assume that $H$ is reducible. Then there exist non-trivial subgroups $H_1, H_2 \leq H$ such that $H = H_1 \perp H_2$. Since the supports of $H_1$ and $H_2$ are strictly contained in $\text{supp}(H)$, we have that each $H_i$ can be decomposed into irreducible subgroups by induction. This shows the existence of the decomposition.

Suppose next that $H = H_1 \perp \cdots \perp H_l = K_1 \perp \cdots \perp K_m$ for some irreducible subgroups $H_i, K_j \leq \mathcal{G}(\Omega)$. Since $\Gamma := \supp(H_1) \subseteq \supp(H) = \bigcup_{j=1}^{l} \supp(K_j)$, we may assume that $\Gamma \cap \Lambda \neq \emptyset$ for $\Lambda := \supp(K_1)$. Then $\supp(K_1|_{\Gamma}) \subseteq \supp(K_1) \cap \Gamma = \Lambda \cap \Gamma \neq \emptyset$ and $K_1|_{\Gamma} \neq E$. Now

\[ H_1 = H|_{\Gamma} = (K_1 \perp \cdots \perp K_m)|_{\Gamma} = K_1|_{\Gamma} \perp \cdots \perp K_m|_{\Gamma}. \]

By the irreducibility of $H_1$, $H_1 = K_1|_{\Gamma}$ and $\Gamma = \supp(H_1) = \supp(K_1|_{\Gamma}) \subseteq \Lambda$. Exchanging roles of $\Gamma$ and $\Lambda$, we can obtain that $\Lambda \subseteq \Gamma$, so that, $\Gamma = \Lambda$. This yields that $H_1 = K_1|_{\Gamma} = K_1|_{\Lambda} = K_1$. Then by Lemma 5.6, $H' := H_2 \perp \cdots \perp H_l = K_2 \perp \cdots \perp K_m$. Since the support of $H'$ is strictly contained in $\supp(H)$, the uniqueness also holds by induction. \(\square\)

**Corollary 5.8.** Let $H \leq \mathcal{G}(\Omega)$ be a non-trivial subgroup, and let $H = H_1 \perp \cdots \perp H_l$ be a decomposition of $H$ as in Proposition 5.7. Set $\Gamma_i := \supp(H_i)$ for $1 \leq i \leq l$. 


Suppose that $\text{supp}(H) = \Omega$. Then we have that if $H_i \in \mathcal{L}_n(\mathfrak{S}(\Gamma_i))^0$ for all $1 \leq i \leq l$ then $H \in \mathcal{L}_n(\mathfrak{S}(\Omega))^0$.

Proof. Any element $g \in O_\pi \mathcal{Z} N_{\mathfrak{S}(\Omega)}(H)$ commutes with $H_i$ for all $1 \leq i \leq l$. So $\Gamma_i$ is $\langle g \rangle$-invariant. Since $\text{supp}(H) = \Omega$, we have that $g = \prod_{i=1}^l g|_{\Gamma_i}$ which is contained in $\prod_{i=1}^l O_\pi \mathcal{Z} N_{\mathfrak{S}(\Gamma_i)}(H_i)$. Thus

$$O_\pi \mathcal{Z} N_{\mathfrak{S}(\Omega)}(H) \leq \prod_{i=1}^l O_\pi \mathcal{Z} N_{\mathfrak{S}(\Gamma_i)}(H_i),$$

and this completes the proof. $\square$

We establish the situation once more here. Set $G := \mathfrak{S}(\Omega)$, and let $H \leq \mathfrak{S}(\Omega)$ be a non-trivial subgroup. Suppose that $H = H_1 \perp \cdots \perp H_t$ be a decomposition of $H$ into irreducible subgroups $H_i$ $(1 \leq i \leq t)$ as in Proposition 5.7. Then a set $\mathcal{X}_H := \{H_1, \ldots, H_t\}$ is uniquely determined by $H$. Let $\{K_1, \ldots, K_t\} \subseteq \mathcal{X}_H$ be a set of representatives of $G$-conjugate classes in $\mathcal{X}_H$. For each $K_i$, denote by $[K_i] := \{H_j \in \mathcal{X}_H \mid H_j \sim_G K_i\}$ the class containing $K_i$. We set $[K_i] = \{K_i^{(1)}, K_i^{(2)}, \ldots, K_i^{(m_i)}\}$, and define a subgroup

$$M(K_i) := \langle K \mid K \in [K_i]\rangle = K_i^{(1)} \perp K_i^{(2)} \perp \cdots \perp K_i^{(m_i)} \leq H.$$ 

Then $H = M(K_1) \perp M(K_2) \perp \cdots \perp M(K_t)$. We call each subgroup $M(K_i)$ a “component” of $H$. Put

$$X_i := \text{supp}(M(K_i)) = \bigcup_{j=1}^{m_i} \text{supp}(K_i^{(j)}), \quad G_i := \mathfrak{S}(X_i) \leq G.$$

**Proposition 5.9.** With the above notations, suppose that $\text{supp}(H) = \Omega$. Then we have that

1. $N_G(H) = N_{G_1}(M(K_1)) \perp N_{G_2}(M(K_2)) \perp \cdots \perp N_{G_t}(M(K_t))$.
2. $H \in \mathcal{L}_n(G)^0$ if and only if $M(K_i) \in \mathcal{L}_n(G_i)^0$ for all $1 \leq i \leq t$.

Proof. (1) For any $g \in N_G(H)$, $H = H^g = H_1^g \perp \cdots \perp H_t^g$. Since $\mathcal{X}_H$ is uniquely determined by $H$ by Proposition 5.7, we have that $\langle g \rangle$ acts on $\mathcal{X}_H$ and $[K_i]$ for any $1 \leq i \leq t$. This yields that $X_i$ is $\langle g \rangle$-invariant, and thus $g|_{X_i} \in N_{G_i}(M(K_i))$. Since $\text{supp}(H) = \Omega$, we have that $g = \prod_{i=1}^l g|_{X_i}$ which is contained in $N_{G_1}(M(K_1)) \perp \cdots \perp N_{G_t}(M(K_t))$. The converse inclusion is trivial.

(2) Straightforward from (1). $\square$

By Proposition 5.9 (2), the determination of $H \in \mathcal{L}_n(\mathfrak{S}(\Omega))^0$ can be reduced to the case where $H$ itself is a component of $H$, that is, all subgroups in $\mathcal{X}_H$ are $\mathfrak{S}(\Omega)$-conjugate each other.
5.4. Reduction to irreducible subgroups. In this section, we show that the determination of $H \in \mathcal{L}_\pi(\mathcal{S}(\Omega))^0$ can be reduced to the case where $H$ is irreducible.

Set $G := \mathcal{S}(\Omega)$. By reason of Proposition 5.9 (2), we assume the following Hypothesis 5.10.

**Hypothesis 5.10.** Let $H \leq \mathcal{S}(\Omega)$ be a non-trivial subgroup. Suppose that $H = H_1 \cdot \cdots \cdot H_l$ be a decomposition of $H$ into irreducible subgroups $H_i$ ($1 \leq i \leq l$) as in Proposition 5.7. Then $H_i \sim_G H_j$ for any $1 \leq i, j \leq l$.

We examine the structure of $N_G(H)$. Set $\Gamma_i := \text{supp}(H_i)$ and $G_i := \mathcal{S}(\Gamma_i)$ for $1 \leq i \leq l$. By Hypothesis 5.10, for each $2 \leq i \leq l$, there exists $g_i \in G$ such that $H_i = H_1^{g_i} := g_i^{-1}H_1g_i$ which induces a permutation equivalence $(H_1, \Gamma_1) \simeq (H_i, \Gamma_i)$. In other words, there exist bijections $f_i : H_1 \to H_i$ defined by $x \mapsto x^{g_i} := g_i^{-1}xg_i$ for $x \in H_1$, and $\phi_i : \Gamma_1 \to \Gamma_i$ defined by $\alpha \mapsto \alpha^{\phi_i}$ for $\alpha \in \Gamma_1$ satisfying $(\alpha^{\phi_i})^{f_i} = (\alpha^x)^{\phi_i}$ for any $x \in H_1$ and $\alpha \in \Gamma_1$. Now we define an involution

$$\sigma_i := \prod_{\alpha \in \Gamma_1} (\alpha, \alpha^{\phi_i}) \in \mathcal{S}(\Gamma_1 \cup \Gamma_i) \leq \mathcal{S}(\Omega) \quad (2 \leq i \leq l)$$

which acts on $\mathcal{X}_H = \{H_1, \ldots, H_l\}$ as a transposition $(H_1, H_i)$. Then $S := \langle \sigma_2, \ldots, \sigma_l \rangle \cong \mathcal{S}_l$ acts on both $\mathcal{X}_H$ and $\{N_G(H_1, \ldots, N_G(H_l)\}$ as $\mathcal{S}_l$ respectively, and a subgroup $N_G(H_1) \cdot S \cong B \rtimes S \leq N_G(H)$ is defined where $B := N_G(H_1) \times \cdots \times N_G(H_l)$.

**Proposition 5.11.** Assume Hypothesis 5.10. With the above notations, suppose that $\text{supp}(H) = \Omega$. Then we have that

1. $N_G(H) = B \rtimes S$.
2. $H \in \mathcal{L}_\pi(G)^0$ if and only if $H_i \in \mathcal{L}_\pi(G_i)^0$.

Proof. (1) For any element $g \in N_G(H)$, $\langle g \rangle$ acts on $\mathcal{X}_H$ as in the proof of Proposition 5.9. Then there exists $\sigma \in S$ such that $\sigma$ is equal to $g$ as elements of $\mathcal{S}(\mathcal{X}_H)$. Thus $g\sigma^{-1}$ fixes $H_i$ for all $1 \leq i \leq l$, so that, $(g^{\sigma^{-1}})|_{\Gamma_i} \in N_G(H_i)$. Since $\text{supp}(H) = \Omega$, we have that $g\sigma^{-1} = \prod_{i=1}^l (g^{\sigma^{-1}})|_{\Gamma_i}$, which is contained in $B$. So $g \in B\sigma \subseteq B \rtimes S$.

(2) Suppose that $H_i \notin \mathcal{L}_\pi(G_i)^0$, and then we will show that $H \notin \mathcal{L}_\pi(G)^0$. We may assume that $l \geq 2$. Now there exists $z_1 \in O_\pi ZN_G(H_1) \setminus H_1$. For $2 \leq i \leq l$, put

$$z_i := \sigma_i^{-1}z_1\sigma_i \in O_\pi ZN_G(H_i) \setminus H_i, \quad z_0 := \prod_{i=1}^l z_i \in N_G(H) \setminus H.$$

Then $[z_0, B] = E$. Furthermore, for each $\sigma_j \in S$ ($2 \leq j \leq l$), we have that

$$z_0^{\sigma_j} = z_1^{\sigma_j} \cdot \prod_{i=1}^l z_i^{\sigma_j} \neq z_1^{\sigma_j} \cdot \prod_{i=1}^l z_i^{\sigma_j} \times z_1 = z_0.$$
This implies that \([z_0, S] = E\) and \(z_0 \in ZN_G(H)\) by Proposition 5.11 (1). Thus \(z_0\) is in \(O_2 ZN_G(H) \setminus H\), and \(H \notin \mathcal{L}_\pi(G)^0\) as desired. The converse follows from Corollary 5.8.

Summarizing Propositions 5.9 and 5.11, we have the following.

**Theorem 5.12.** Let \(H \leq \mathfrak{S}(\Omega)\) be a non-trivial subgroup, and let

\[
H = (H_1^{(1)} \perp \cdots \perp H_1^{(m_1)}) \perp (H_2^{(1)} \perp \cdots \perp H_2^{(m_2)}) \perp \cdots \perp (H_t^{(1)} \perp \cdots \perp H_t^{(m_t)})
\]

be a decomposition of \(H\) as in Proposition 5.7 where each \(H_i^{(1)} \perp \cdots \perp H_i^{(m_i)}\) is a component of \(H\). Set \(\Gamma_i := \text{supp}(H_i^{(1)})\) for \(1 \leq i \leq t\). Suppose that \(\text{supp}(H) = \Omega\). Then we have that \(H \in \mathcal{L}_\pi(\mathfrak{S}(\Omega))^0\) if and only if \(H_i^{(1)} \in \mathcal{L}_\pi(\mathfrak{S}(\Gamma_i))^0\) for all \(1 \leq i \leq t\).

By Theorem 5.12, the determination of \(H \in \mathcal{L}_\pi(\mathfrak{S}(\Omega))^0\) can be reduced to the case where \(H\) is irreducible.

**5.5. On intransitive subgroups.** In this section, we show that intransitive subgroups of \(\mathfrak{S}(\Omega)\) can be described inductively in terms of smaller irreducible subgroups. This idea will be used in Section 5.6. First we recall pullbacks.

**Remark 5.13.** (1) Let \(G\) and \(H\) be groups, and let \(\theta : G/N \to H/K\) be a group isomorphism between quotient groups. Then the pullback \(G \times^\theta H\) of \(G\) and \(H\) via \(\theta\) is a subgroup \(\{(g, h) \in G \times H \mid (gN)^\theta = hK\}\) of \(G \times H\) (cf. [4, Definition 13.11]). Note that if \(\theta\) is trivial, that is, \(G/N\) is the trivial group, then \(G \times^\theta H = G \times H\).

(2) Let \(G = K \times L\) be a direct product. Then any subgroup \(H\) of \(G\) can be realized as the pullback of certain subgroups in \(K\) and \(L\). More precisely, there exist subgroups \(K \supseteq K_1 \supseteq K_2\) and \(L \supseteq L_1 \supseteq L_2\), and also a group isomorphism \(\theta : K_1/K_2 \to L_1/L_2\) such that \(H = K_1 \times^\theta L_1\) (cf. [10, (4.19)]).

Let \(H \leq \mathfrak{S}(\Omega)\) be a non-trivial subgroup. Suppose that \(\text{supp}(H) = \Omega\), and that \(H\) acts intransitively on \(\Omega\). Let

\[
\Omega = \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_{m-1} \cup \mathcal{O}_m \quad (m \geq 2)
\]

be a decomposition of \(\Omega\) into \(H\)-orbits. Set \(\Lambda_1 := \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_{m-1}\) and \(\Lambda_2 := \mathcal{O}_m\). Then a subgroup \(B := H|_{\Lambda_1} \leq \mathfrak{S}(\Lambda_2)\) is transitive on \(\Lambda_2\), that is, irreducible. On the other hand, a subgroup \(H|_{\Lambda_1} \leq \mathfrak{S}(\Lambda_1)\) is decomposed as \(H|_{\Lambda_1} = A_1 \perp \cdots \perp A_l\) into irreducible subgroups \(A_i\) (\(1 \leq i \leq l\)) by Proposition 5.7. It follows that

\[
H \leq H|_{\Lambda_1} \times H|_{\Lambda_2} = (A_1 \perp \cdots \perp A_l) \perp B.
\]
Since the supports of $A_i$ and $B$ are strictly contained in $\text{supp}(H) = \Omega$, we may assume that a list of irreducible subgroups $A_i$ and $B$ is already known by induction. Thus $H$ can be concretely described as the pullback $H_1 \times^\theta H_2$ of certain subgroups $H_1 \leq A_1 \perp \cdots \perp A_l$ and $H_2 \leq B$ where $\theta$ is a group isomorphism between quotients (see Remark 5.13). Note that, if $H$ is irreducible then $\theta$ must not be trivial. In the next, we give a result on irreducible pullbacks under the above situation.

**Proposition 5.14.** Let $B \leq \mathcal{G}(\Omega)$ be an irreducible subgroup, and let $A := A_1 \perp \cdots \perp A_l \leq \mathcal{G}(\Omega)$ where $A_i$ is irreducible for all $1 \leq i \leq l$. Suppose that $\text{supp}(A) \cap \text{supp}(B) = \emptyset$ and $\text{supp}(A \perp B) = \Omega$. Suppose further that there exists a group isomorphism $\theta : A/N_1 \to B/N_2$ (for some $N_1 \leq A$ and $N_2 \leq B$ such that $A_i \not\leq N_1$ for all $1 \leq i \leq l$). Then the pullback $P := A \times^\theta B = \{(a, b) \in A \times B \mid (aN_1)\theta = bN_2\}$ is irreducible.

**Proof.** Set $\Gamma_i := \text{supp}(A_i)$ $(1 \leq i \leq l)$ and $\Gamma := \text{supp}(B)$. Suppose that $P$ is reducible. Then there exist non-trivial subgroups $K, L \leq P$ such that $P = K \perp L$. Let $\pi_A : P \to A$ and $\pi_B : P \to B$ be the projections of $P$ on $A$ and $B$ respectively. Both $\pi_A$ and $\pi_B$ are surjective. This implies that $P|_{\Gamma_i} = A_i$ $(1 \leq i \leq l)$ and $P|_{\Gamma} = B$. Since $B = P|_{\Gamma} = K|_{\Gamma} \perp L|_{\Gamma}$ is irreducible, we may assume that

\[
K|_{\Gamma} = B \quad \text{i.e.} \quad \Gamma = \text{supp}(B) \subseteq \text{supp}(K),
\]

\[
L|_{\Gamma} = E \quad \text{i.e.} \quad L \leq A = A_1 \perp \cdots \perp A_l.
\]

Suppose that $\Gamma \subseteq \text{supp}(K) \subseteq \Omega = \Gamma_1 \cup \cdots \cup \Gamma_l \cup \Gamma$. Then we may assume that $\emptyset \neq \text{supp}(K) \cap \Gamma_1 = \text{supp}(K|_{\Gamma_1})$, so that, $K|_{\Gamma_1} \neq E$. Since $A_1 = P|_{\Gamma_1} = K|_{\Gamma_1} \perp L|_{\Gamma_1}$ is irreducible, we have that

\[
K|_{\Gamma_1} = A_1 \quad \text{i.e.} \quad \Gamma_1 = \text{supp}(A_1) \subseteq \text{supp}(K) \quad \text{and} \quad \Gamma \cup \Gamma_1 \subseteq \text{supp}(K),
\]

\[
L|_{\Gamma_1} = E \quad \text{i.e.} \quad L \leq A_2 \perp \cdots \perp A_l.
\]

Repeating this process, we may assume that there exists $t < l$ such that

\[
\text{supp}(K) = \Gamma \cup \Gamma_1 \cup \cdots \cup \Gamma_t,
\]

\[
L \leq A_{t+1} \perp \cdots \perp A_l. \quad (\star)
\]

Note that if $t = l$ then $L = E$, a contradiction. Now $\pi_A : P = K \perp L \to A$ is surjective. Thus for any $a \in A_l$, there exist $(a_K, b_K) \in K \leq A \times B$ and $(a_L, e) \in L \leq A$ such that

\[
a = \pi_A((a_K, b_K) \times (a_L, e)) = a_K a_L.
\]

But by the above condition $(\star)$, $a_K \in A_1 \perp \cdots \perp A_t$ and $a_L \in A_{t+1} \perp \cdots \perp A_l$. Thus $a_K = e$ and $a = a_L \in L \leq P$. This implies $(a, e) \in P$ and $(aN_i)\theta = eN_2 = N_2$ by
the definition of $P$. Therefore $A_i \leq N_1$ which contradicts our assumption. The proof is complete. \qed

5.6. A strategy to determine $\mathcal{L}_\pi(\mathfrak{S}_n)^0$. In this section, we provide a method of determining $\mathcal{L}_\pi(\mathfrak{S}_n)^0$ which is focused on irreducible subgroups. So we introduce the notations

$$\text{IRR}(n)^0 := \{ E \neq H \leq \mathfrak{S}(\Omega) \mid H \text{ is irreducible such that } \text{fix}(H) = \emptyset \},$$

$$\mathcal{T}(n) := \{ E \neq H \leq \mathfrak{S}(\Omega) \mid H \text{ is transitive on } \Omega \} \subseteq \text{IRR}(n)^0.$$ 

Then, as in the following, we divide our work of determining $H \in \mathcal{L}_\pi(\mathfrak{S}_n)^0$ into two cases where $H$ is irreducible or not.

**A:** Determine $H \in \mathcal{L}_\pi(\mathfrak{S}_n)^0$ such that $H$ is not irreducible (see Theorem 5.12).

(Step A1) Give a non-trivial partition $n = (n_1 + \cdots + n_t) + \cdots + (n_s + \cdots + n_l)$ of $n$ such that $n_i \geq 2$ and $n_i > n_{i+1}$.

(Step B2) $H$ is $\mathfrak{S}_n$-conjugate to one of subgroups of the form $(H_1 \perp \cdots \perp H_t) \perp \cdots \perp (H_t \perp \cdots \perp H_l)$ where $H_i \in \mathcal{L}_\pi(\mathfrak{S}_n)^0$ for $1 \leq i \leq t$.

**B:** Determine $H \in \mathcal{L}_\pi(\mathfrak{S}_n)^0$ such that $H$ is irreducible.

(Step B1) Make a list of $\mathfrak{S}_n$-conjugate classes in $\mathcal{T}(n)$.

(Step B2) Describe subgroups in $\text{IRR}(n)^0 \setminus \mathcal{T}(n)$, namely intransitive irreducible subgroups $H$ having no fixed points (see Section 5.5). Indeed, we first give a non-trivial partition $n = n_1 + \cdots + n_{r-1} + n_r$ of $n$ such that $n_i \geq 2$. Let $A \leq \mathfrak{S}_{n-n_r}$ and $B \in \mathcal{T}(n_r)$ such that $A$ has $r-1$ orbits of lengths $n_i$ for $1 \leq i \leq r-1$. Calculate an irreducible pullback $H = A_1 \times^\theta B_1$ via a group isomorphism $\theta: A_1/A_2 \to B_1/B_2$ ($\neq \hat{E}$) where $A \cong A_1 > A_2$ and $B \geq B_1 \geq B_2$.

(Step B3) By the previous two Steps B1–B2, the set $\text{IRR}(n)^0$ is complete. Then, from $\text{IRR}(n)^0$, pick up subgroups belonging to $\mathcal{L}_\pi(\mathfrak{S}_n)$.

5.7. Examples $\mathcal{L}_\pi(\mathfrak{S}_n)^0$ ($n \leq 6$). According to a strategy introduced in Section 5.6, we determine $\mathcal{L}_\pi(\mathfrak{S}_n)$ for $4 \leq n \leq 6$. Let $\mathfrak{A}(\Omega) = \mathfrak{A}_n$ be the alternating group on $\Omega = \{1, \ldots, n\}$. For a prime number $p$ and a positive integer $m$, denote by $p^m$, $C_m$, $D_{2m}$ respectively the elementary abelian $p$-group of order $p^m$, cyclic group of order $m$, dihedral group of order $2m$. Set $\pi := \pi(\mathfrak{S}_n)$.

The cases of $\mathfrak{S}_2$ and $\mathfrak{S}_3$ are trivial as follows:

- $\text{IRR}(2)^0 = \mathcal{T}(2) = \mathcal{L}_\pi(\mathfrak{S}_2)^0 = \{ \mathfrak{S}_2 \cong C_2 \}$,
- $\text{IRR}(3)^0 = \mathcal{T}(3) = \{ \mathfrak{A}_3, \mathfrak{A}_3 \}$, and $\mathcal{L}_\pi(\mathfrak{S}_3)^0 = \{ \mathfrak{A}_3 \cong C_3 \}$.

The case of $\mathfrak{S}_4$:

(Steps A1–A2) A non-trivial partition of 4 not containing 1 as summands is only $4 = 2 + 2$. Then any non-irreducible subgroup $H$ in $\mathcal{L}_\pi(\mathfrak{S}_4)^0$ is conjugate to $H_1 \perp H_2$ where $H_i \in \mathcal{L}_\pi(\mathfrak{S}_4)^0$. Thus $H \sim \mathfrak{S}_4 \langle (1, 2) \rangle \perp \langle (3, 4) \rangle$.

(Step B1) It is easy to see that $T(4)/\sim_{\mathfrak{S}_4} = \{ \mathfrak{S}_4, \mathfrak{A}_4, \langle (1, 2, 3, 4), (2, 4) \rangle \cong D_8, V, \langle (1, 2, 3, 4) \rangle \cong C_4 \}$ where $V := \langle (1, 2)(3, 4), (1, 3)(2, 4) \rangle$ is the four group. In particular, $T(4)/\sim_{\mathfrak{S}_4} \cap \mathcal{L}_\pi(\mathfrak{S}_4) = \{ D_8, V, C_4 \}$.
(Step B2) A non-trivial partition of 4 not containing 1 as summands is $4 = 2 + 2$. There is the unique transitive subgroup $B := \langle (3, 4) \rangle \in T(2)$ on $\{3, 4\}$. Then we choose a transitive subgroup $A \in T(2)$ on $\{1, 2\}$ having a quotient $A/N$ of order 2, namely $(A, N) = \langle (1, 2) \rangle$. Define a group isomorphism $\theta: A/N \to B$. The pullback $A \times B = \langle (1, 2)(3, 4) \rangle \cong C_2$ is irreducible.

(Step B3) By Steps B1–B2, we have that

$$\text{IRR}(4)^0/\sim_{\mathfrak{S}_4} = T(4)/\sim_{\mathfrak{S}_4} \cup \{(1, 2)(3, 4)\}.$$  

Then $L_\pi(\mathfrak{S}_4)^0$ consists of 5-classes whose representatives are as follows:

<table>
<thead>
<tr>
<th>$H \in L_\pi(\mathfrak{S}<em>4)^0/\sim</em>{\mathfrak{S}_4}$</th>
<th>$\cong$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle (1, 2) \rangle \perp \langle (3, 4) \rangle$</td>
<td>$2^4$ non-irreducible</td>
</tr>
<tr>
<td>$\langle (1, 2, 3), (2, 4) \rangle$</td>
<td>$D_8$ irreducible and transitive</td>
</tr>
<tr>
<td>$\langle (1, 2, 3, 4) \rangle$</td>
<td>$2^2$</td>
</tr>
<tr>
<td>$\langle (1, 2)(3, 4) \rangle$</td>
<td>$C_4$ irreducible and intransitive</td>
</tr>
</tbody>
</table>

The case of $\mathfrak{S}_5$:

(Steps A1–A2) A non-trivial partition of 5 not containing 1 as summands is only $5 = 3 + 2$. Then any non-irreducible subgroup $H$ in $L_\pi(\mathfrak{S}_5)^0$ is conjugate to $H_1 \perp H_2$ where $H_1 \in L_\pi(\mathfrak{S}_3)^0$ and $H_2 \in L_\pi(\mathfrak{S}_2)^0$. Thus $H \sim_{\mathfrak{S}_5} \langle (1, 2, 3) \rangle \perp \langle (4, 5) \rangle$.

(Step B1) Since the order of a transitive group of degree 5 is divisible by 5, it is easy to see that $T(5)/\sim_{\mathfrak{S}_5} = \{\mathfrak{S}_5, \mathfrak{A}_5, C_5 \rtimes C_4, C_5 \rtimes C_2, C_5\}$. In particular, $T(5)/\sim_{\mathfrak{S}_5} \cap L_\pi(\mathfrak{S}_5) = \{\langle (1, 2, 3, 4, 5) \rangle \cong C_5\}$.

(Step B2) A non-trivial partition of 5 not containing 1 as summands is $5 = 3 + 2$. There is the unique transitive subgroup $B := \langle (4, 5) \rangle \in T(2)$ on $\{4, 5\}$. Then we choose a transitive subgroup $A \in T(3)$ on $\{1, 2, 3\}$ having a quotient $A/N$ of order 2, namely $(A, N) = (\mathfrak{A}_3, \mathfrak{S}_3)$. Define a group isomorphism $\theta: A/N \to B$. The pullback $A \times B = \langle (1, 2, 3), (1, 2)(4, 5) \rangle \cong \mathfrak{S}_5$ is irreducible.

(Step B3) By Steps B1–B2, we have that

$$\text{IRR}(5)^0/\sim_{\mathfrak{S}_5} = T(5)/\sim_{\mathfrak{S}_5} \cup \{(1, 2, 3), (1, 2)(4, 5)\}.$$  

Then $L_\pi(\mathfrak{S}_5)^0$ consists of 2-classes whose representatives are as follows:

<table>
<thead>
<tr>
<th>$H \in L_\pi(\mathfrak{S}<em>5)^0/\sim</em>{\mathfrak{S}_5}$</th>
<th>$\cong$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle (1, 2, 3) \rangle \perp \langle (4, 5) \rangle$</td>
<td>$C_2 \times C_2$ non-irreducible</td>
</tr>
<tr>
<td>$\langle (1, 2, 3, 4, 5) \rangle$</td>
<td>$C_5$ irreducible and transitive</td>
</tr>
</tbody>
</table>
The case of $\mathfrak{S}_6$:

(Steps A1–A2) Non-irreducible subgroups $H$ in $L_\pi(\mathfrak{S}_6)$ correspond to non-trivial partitions of 6 not containing 1 as summands. Thus those subgroups are determined as follows:

(i) $6 = 4 + 2$: $H \sim_{\mathfrak{S}_6} H_1 \perp H_2$ where $H_1 \in L_\pi(\mathfrak{S}_4)$ and $H_2 \in L_\pi(\mathfrak{S}_2)$, and thus

\[ H \sim_{\mathfrak{S}_6} D_8 \perp \langle (5, 6) \rangle, \quad V \perp \langle (5, 6) \rangle, \quad C_4 \perp \langle (5, 6) \rangle, \quad \langle (1, 2)(3, 4) \rangle \perp \langle (5, 6) \rangle. \]

(ii) $6 = 3 + 3$: $H \sim_{\mathfrak{S}_6} H_1 \perp H_2$ where $H_1 \in L_\pi(\mathfrak{S}_3)$, and thus

\[ H \sim_{\mathfrak{S}_6} \langle (1, 2, 3) \rangle \perp \langle (4, 5, 6) \rangle. \]

(iii) $6 = 2 + 2 + 2$: $H \sim_{\mathfrak{S}_6} H_1 \perp H_2 \perp H_3$ where $H_i \in L_\pi(\mathfrak{S}_2)$, and thus

\[ H \sim_{\mathfrak{S}_6} \langle (1, 2) \rangle \perp \langle (3, 4) \rangle \perp \langle (5, 6) \rangle. \]

(Step B1) We can find that there are 16-classes of transitive subgroups of $\mathfrak{S}_6$, and representatives are as follows:

\begin{align*}
T(6)/\sim_{\mathfrak{S}_6} = \{ & \mathfrak{S}_6, \mathfrak{A}_6, PGL(2, 5) \cong \mathfrak{S}_5, \mathfrak{A}_5, \mathfrak{S}_4, \\
& \mathfrak{S}_3 \rtimes \mathfrak{S}_2 \cong 3^2 \rtimes D_8, 3^2 \rtimes C_4, 3^2 \rtimes 2^2 \rtimes C_2, C_3 \times C_2, D_{12}, \mathfrak{S}_3, \\
& \mathfrak{S}_2 \rtimes \mathfrak{S}_3 \cong 2^3 \rtimes S_3, 2^3 \times C_3, 2^2 \times C_3, \mathfrak{S}_4 \}. 
\end{align*}

In particular, $T(6)/\sim_{\mathfrak{S}_6} \cap L_\pi(\mathfrak{S}_6) = \{ \langle (1, 2, 3, 4, 5, 6) \rangle \cong C_6 \}$.

(Step B2) In order to examine intransitive subgroups $H$ in $\text{IRR}(6)^0$, we consider pullbacks associated to non-trivial partitions of 6 not containing 1 as summands as follows:

(i) $6 = 4 + 2$: There is the unique transitive subgroup $B := \langle (5, 6) \rangle \in T(2)$ on $\{5, 6\}$. Then we choose a transitive subgroup $A \in T(4)$ on $\{1, 2, 3, 4\}$ having a quotient $A/N$ of order 2, so that, a group isomorphism $\theta : A/N \to B$ is defined.

<table>
<thead>
<tr>
<th>$\theta : A/N \to B$</th>
<th>$H = A \times^\theta B$</th>
<th>nilp.</th>
<th>$N_{\mathfrak{S}_6}(H)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathfrak{S}_3/\mathfrak{A}_4 \to B$</td>
<td>$\langle \mathfrak{A}_4, (1, 2)(5, 6) \rangle \cong \mathfrak{S}_4$</td>
<td>no</td>
<td>$D^{(1)} \times \langle (5, 6) \rangle$</td>
</tr>
<tr>
<td>$D_8/C_4 \to B$</td>
<td>$\langle (1, 2, 3, 4), (2, 4)(5, 6) \rangle \cong D_8$</td>
<td>yes</td>
<td>$D^{(1)} \times \langle (5, 6) \rangle$</td>
</tr>
<tr>
<td>$D_8/V \to B$</td>
<td>$\langle (1, 2)(3, 4), (1, 3)(2, 4), (2, 4)(5, 6) \rangle$</td>
<td>yes</td>
<td>$D^{(1)} \times \langle (5, 6) \rangle$</td>
</tr>
<tr>
<td>$D_8/\langle (1, 3), (2, 4) \rangle \to B$</td>
<td>$\langle (1, 3), (2, 4), (1, 2)(3, 4)(5, 6) \rangle$</td>
<td>yes</td>
<td>$D^{(1)} \times \langle (5, 6) \rangle$</td>
</tr>
<tr>
<td>$V/\langle (1, 2)(3, 4) \rangle \to B$</td>
<td>$\langle (1, 2)(3, 4), (1, 3)(2, 4)(5, 6) \rangle \cong 2^4$</td>
<td>yes</td>
<td>$D^{(2)} \times \langle (5, 6) \rangle$</td>
</tr>
<tr>
<td>$C_4/C_2 \to B$</td>
<td>$\langle (1, 3)(2, 4), (1, 2, 3, 4)(5, 6) \rangle \cong C_4$</td>
<td>yes</td>
<td>$D^{(1)} \times \langle (5, 6) \rangle$</td>
</tr>
</tbody>
</table>

where $D^{(1)} := \langle (1, 2, 3, 4), (2, 4) \rangle$ and $D^{(2)} := \langle (1, 3, 2, 4), (1, 2) \rangle$. 
(ii) $6 = 3 + 3$: There are three non-trivial quotients $A/N$ of transitive subgroups $A \in T(3)$, namely $(A, N) = (\mathfrak{S}_3, \mathfrak{A}_3), (\mathfrak{S}_3, E)$, and $(\mathfrak{A}_3, E)$.

\[
\begin{array}{|c|c|c|}
\hline
\theta: A/N \to A/N & H = A \times^n A & \text{nilp. } N_{\mathfrak{S}_6}(H) \\
\hline
\mathfrak{S}_3/\mathfrak{A}_3 \to \mathfrak{S}_3/\mathfrak{A}_3 & \langle(1, 2, 3), (4, 5, 6), (1, 2)(4, 5)\rangle \cong 3^2 \times C_2 & \text{no} \\
\mathfrak{S}_3/E \to \mathfrak{S}_3/E & \langle(1, 2, 3)(4, 5, 6), (1, 2)(4, 5)\rangle \cong \mathfrak{S}_3 & \text{no} \\
\mathfrak{A}_3/E \to \mathfrak{A}_3/E & \langle(1, 2, 3)(4, 5, 6)\rangle \cong C_3 & \text{yes } 3^2 \times C_2 \times C_2 \\
\hline
\end{array}
\]

(iii) $6 = (2 + 2) + 2$: There is the unique transitive subgroup $B := \langle(5, 6)\rangle \in T(2)$ on $\{5, 6\}$. Then we choose an intransitive subgroup $A \leq \mathfrak{S}_4$ on $\{1, 2, 3, 4\}$ which has two orbits of length 2. Namely $A$ is an irreducible subgroup $A_1 = \langle(1, 2)(3, 4)\rangle$ or non-irreducible subgroup $A_2 = \langle(1, 2)\rangle \perp \langle(3, 4)\rangle$. Each $A_i$ has a quotient of order 2.

\[
\begin{array}{|c|c|c|}
\hline
\theta: A/N \to B & H = A \times^n B & \text{nilp. } N_{\mathfrak{S}_6}(H) \\
\hline
A_1/E \to B & \langle(1, 2)(3, 4)(5, 6)\rangle \cong C_2 & \text{yes } \mathfrak{S}_2 \times \mathfrak{S}_3 \\
A_2/\langle(1, 2)(3, 4)\rangle \to B & \langle(1, 2, 3)(4, 5, 6)\rangle \cong 2^2 & \text{yes } \mathfrak{S}_2 \times \mathfrak{S}_3 \\
A_2/\langle(1, 2)\rangle \to B & \langle(1, 2)\rangle \perp \langle(3, 4)(5, 6)\rangle \cong 2^2 & \text{yes} \\
\hline
\end{array}
\]

Note that the last $\langle(1, 2)\rangle \perp \langle(3, 4)(5, 6)\rangle$ is the only non-irreducible subgroup among the above twelve subgroups in Step B2 (compare with Proposition 5.14). Thus there are 11-classes of intransitive subgroups in $\text{IRR}(6)^0$.

(Step B3) By Steps B1–B2, there are $(16 + 11)$-classes of subgroups in $\text{IRR}(6)^0$, and then $L_\pi(\mathfrak{S}_6)^0$ consists of 9-classes whose representatives are as follows:

\[
\begin{array}{|c|c|c|}
\hline
H \in L_\pi(\mathfrak{S}_6)^0/\sim_{\mathfrak{S}_6} & \cong & \\
\langle(1, 2, 3, 4), (2, 4)\rangle \perp \langle(5, 6)\rangle & D_8 \times C_2 & \text{non-irreducible} \\
V \perp \langle(5, 6)\rangle & 2^3 & \\
\langle(1, 2, 3, 4)\rangle \perp \langle(5, 6)\rangle & C_4 \times C_2 & \\
\langle(1, 2)(3, 4)\rangle \perp \langle(5, 6)\rangle & 2^2 & \\
\langle(1, 2, 3)\rangle \perp \langle(4, 5, 6)\rangle & 3^2 & \\
\langle(1, 2)\rangle \perp \langle(3, 4)\rangle \perp \langle(5, 6)\rangle & 2^3 & \\
\langle(1, 2, 3, 4, 5, 6)\rangle & C_2 \times C_3 & \text{irreducible and transitive} \\
\langle(1, 2, 3)(4, 5, 6)\rangle & C_3 & \\
\langle(1, 2)(3, 4)(5, 6)\rangle & C_2 & \text{irreducible and intransitive} \\
\hline
\end{array}
\]

Furthermore, Proposition 5.3 tells us that, since $2 \notin \pi$, the whole $L_\pi(\mathfrak{S}_6)$ is constructed by four parts $L_\pi(\mathfrak{S}_2)^0, L_\pi(\mathfrak{S}_3)^0, L_\pi(\mathfrak{S}_5)^0$, and $L_\pi(\mathfrak{S}_6)^0$. Therefore there are $(1 + 1 + 2 + 9)$-classes of subgroups in $L_\pi(\mathfrak{S}_6)$.
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References


Nobuo Iiyori
Department of Mathematics
Faculty of Education
Yamaguchi University
Yamaguchi 753-8511
Japan
e-mail: iiyori@yamaguchi-u.ac.jp

Masato Sawabe
Department of Mathematics
Faculty of Education
Chiba University
Inage-ku Yayoi-cho 1-33, Chiba 263-8522
Japan
e-mail: sawabe@faculty.chiba-u.jp