ON INSTABILITY OF GLOBAL PATH PROPERTIES OF SYMMETRIC DIRICHLET FORMS UNDER MOSCO-CONVERGENCE

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Abstract

We give sufficient conditions for Mosco convergences for the following three cases: symmetric locally uniformly elliptic diffusions, symmetric Lévy processes, and symmetric jump processes in terms of the $L^1(\mathbb{R}^d; dx)$-local convergence of the (elliptic) coefficients, the characteristic exponents and the jump density functions, respectively. We stress that the global path properties of the corresponding Markov processes such as recurrence/transience, and conservativeness/explosion are not preserved under Mosco convergences and we give several examples where such situations indeed happen.

1. Introduction

In the present paper, we are concerned with Mosco convergences of the following three types of the Dirichlet forms: symmetric strongly local Dirichlet forms satisfying the locally uniformly elliptic conditions, symmetric translation invariant Dirichlet forms, and symmetric jump-type Dirichlet forms. We give sufficient conditions for the Mosco convergences in the above three cases in terms of the L^1-local convergence of the corresponding coefficients, and show instability of global path properties under the Mosco convergences such as recurrence or transience, and conservativeness or explosion by giving several examples.

We find that the Mosco convergences follow from quite mild assumptions (see Assumption A, B and C), which are essentially $L^1(\mathbb{R}^d; dx)$-local convergences of the corresponding coefficients, which are diffusion coefficients, Lévy exponents and jump densities. Here dx denotes the Lebesgue measure on $\mathbb{R}^d$. Hereafter we fix our state space to $(\mathbb{R}^d, dx)$ and we write $L^p(\mathbb{R}^d)$ (or L^p) shortly for $L^p(\mathbb{R}^d; dx)$ ($1 \le p \le \infty$). Since the L^1-local convergence is one of the weakest notions of strong convergences, our results mean that the weakest convergence of the coefficients implies the Mosco convergences.

The Mosco convergence is a notion of convergences of closed forms on Hilbert spaces (see Definition 2.1), which was introduced by U. Mosco [11], originally to study the approximations of some variational inequalities. In [12], he showed that the Mosco convergence is equivalent to the strong convergences of the corresponding semigroups

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and resolvents. The strong convergence of semigroups implies the convergence of finite-dimensional distributions of the corresponding Markov processes when closed forms in question are regular Dirichlet forms. For such reasons, the Mosco convergence has been used to show the weak convergence of stochastic processes in the theory of Markov processes (see e.g. [18, 7, 16, 5, 4] and references therein). In [6], Kuwae and Shioya generalized the notion of the Mosco convergence, now is called the Mosco–Kuwae–Shioya convergence, as the basic L^2-spaces can change, while Hino considered the non-symmetric version of the Mosco convergence in [3]. Although both generalizations are quite important, in the present paper, we consider only symmetric cases and we fix a basic L^2-space as $L^2(\mathbb{R}^d)$.

Our another aim is to show that the Mosco convergence of Dirichlet forms does not preserve any global path properties for the corresponding processes of the Dirichlet forms in any respect. As stated above, the Mosco convergence is equivalent to the strong convergence of the corresponding semigroups, which implies only the convergences of finite-dimensional distributions of the corresponding Markov processes. Thus it is easy to imagine that global properties such as recurrence/transience and conservativeness/explosion are not preserved under the Mosco convergence. It seems, however, that those studies how to construct such examples concretely have not been investigated.

In this paper, we construct several examples whose global properties such as recurrence/transience and conservativeness/explosion are not preserved under the Mosco convergence. In constructing such examples, we use the results about sufficient conditions for Mosco convergences as explained in the second paragraph in this introduction.

To be more precise, let us first consider symmetric strongly local Dirichlet forms having the locally uniformly elliptic coefficients. Namely let $A_n(x) = (a_{ij}^n(x))$ be a sequence of $d \times d$ symmetric matrix valued functions where a_{ij}^n is a locally integrable Borel measurable function on $\mathbb{R}^d$ satisfying the following conditions:

ASSUMPTION A.   (A1) For any compact set $K \subset \mathbb{R}^d$, there exists $\lambda = \lambda(K) > 0$ so that for all $n \in \mathbb{N}$,

$$\lambda|\xi|^2 \le \langle A_n(x)\xi, \xi \rangle \le \lambda^{-1}|\xi|^2, \quad \text{a.e.} \quad x \in K, \quad \forall \xi \in \mathbb{R}^d.$$

(A2) For any compact set K,

$$\int_K \|A_n(x) - A(x)\| \, dx \to 0 \quad (n \to \infty),$$

where $\|A_n(x) - A(x)\|^2 := \sum_{i=1}^d \sum_{j=1}^d (a_{ij}^n(x) - a_{ij}(x))^2$, $x \in \mathbb{R}^d$.

Then, consider a sequence of symmetric quadratic forms

$$\mathcal{E}^n(u, v) = \int_{\mathbb{R}^d} \langle A_n(x)\nabla u(x), \nabla v(x)\rangle \, dx$$

and

$$\mathcal{E}(u, v) = \int_{\mathbb{R}^d} \langle A(x)\nabla u(x), \nabla v(x)\rangle \, dx$$

for u and v in $C_0^\infty(\mathbb{R}^d)$, where $C_0^\infty(\mathbb{R}^d)$ is the set of infinitely differentiable functions defined on $\mathbb{R}^d$ with compact support. Under Assumption A, it is known that $(\mathcal{E}^n, C_0^\infty(\mathbb{R}^d))$ (and $(\mathcal{E}, C_0^\infty(\mathbb{R}^d))$) are Markovian closable forms on $L^2(\mathbb{R}^d)$ (see [2, Section 3]). They become regular symmetric Dirichlet forms $(\mathcal{E}^n, \mathcal{F}^n)$ and $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathbb{R}^d)$. Our first result is the following:

Theorem 1.1. *Suppose that* Assumption A *holds. Then the Dirichlet forms* $(\mathcal{E}^n, \mathcal{F}^n)$ *converges to* $(\mathcal{E}, \mathcal{F})$ *on* $L^2(\mathbb{R}^d)$ *in the sense of Mosco.*

REMARK 1.2. (1) In [12], Mosco gave several examples for which the Dirichlet forms converge in Mosco's sense. In the case of our strongly local forms, assuming the convergence of the elliptic coefficients locally in $L^1(\mathbb{R}^d)$, he have shown the Γ-convergence. The Γ-convergence is weaker than Mosco convergence. He claimed that, in addition to the convergence of the elliptic coefficients locally in $L^1(\mathbb{R}^d)$, if the so-called "compactly imbedded" condition is satisfied, the Dirichlet forms converge in his sense. However, it is a bit harder to verify the "compactly imbedded" condition.
(2) In [3], Hino has given several equivalent conditions in order that the semigroups converge strongly in L^2 corresponding to time-dependent Dirichlet forms including both of symmetric and non-symmetric cases. In the case of our symmetric strongly local forms, his conditions required the diffusion coefficients $a_{ij}^n \in L^\infty(\mathbb{R}^d, dx)$ for any i, j, which is stronger than (A1) (see [3, Example 4.3]).
(3) In [8] and [14], they studied the convergence of quadratic forms under the uniformly elliptic condition and obtained the weak convergence of corresponding Markov processes. In Theorem 1.1 in the present paper, we only assume *the locally* uniformly elliptic condition.

We now consider the convergence of symmetric Lévy processes. Let $\{\varphi_n\}$ be a sequence of the characteristic functions defined by symmetric convolution semigroups $\{\nu_t^n, t > 0\}_{n\in\mathbb{N}}$:

$$e^{-t\varphi_n(x)} := \hat{\nu}_t^n(x) \quad (= \int_{\mathbb{R}^d} e^{i\langle x, y\rangle} \nu_t^n(dy)), \quad x \in \mathbb{R}^d.$$

According to the Lévy–Khinchin formula (Example 1.4.1 in [2]), we have the Follow-ing characterization of ν_t^n:

(1.1)
$$\varphi_n(x) = \frac{1}{2}\langle S_n x, x\rangle + \int_{\mathbb{R}^d}(1 - \cos(\langle x, y\rangle))n_n(dy),$$

where

(1.2) S_n is a non-negative definite symmetric $(d \times d)$-matrix

and

(1.3)
$$n_n(dy) \text{ is a symmetric Borel measure on } \mathbb{R}^d \setminus \{0\} \text{ so that}$$
$$\int_{\mathbb{R}^d \setminus \{0\}} |x|^2/(1 + |x|^2)n_n(dx) < \infty.$$

We consider the following condition:

Assumption B. φ_n converges to a function φ locally in $L^1(\mathbb{R}^d)$.

Under the assumption, we find that φ is also the characteristic function of a sym-metric convolution semigroup $\{\nu_t, t > 0\}$. Moreover the corresponding quadratic forms

$$\mathcal{E}^n(u, v) = \int_{\mathbb{R}^d} \hat{u}(x)\bar{\hat{v}}(x)\varphi_n(x)\,dx,$$

$$\mathcal{F}^n = \left\{u \in L^2(\mathbb{R}^d): \int_{\mathbb{R}^d}|\hat{u}(x)|^2\varphi_n(x)\,dx < \infty\right\}$$

and

$$\mathcal{E}(u, v) = \int_{\mathbb{R}^d} \hat{u}(x)\bar{\hat{v}}(x)\varphi(x)\,dx,$$

$$\mathcal{F} = \left\{u \in L^2(\mathbb{R}^d): \int_{\mathbb{R}^d}|\hat{u}(x)|^2\varphi(x)\,dx < \infty\right\}$$

are symmetric translation invariant Dirichlet forms on $L^2(\mathbb{R}^d)$. We show that $(\mathcal{E}^n, \mathcal{F}^n)$ converges to $(\mathcal{E}, \mathcal{F})$ in the sense of Mosco under Assumption B:

Theorem 1.3. *Assume that* Assumption B *holds. Then* $(\mathcal{E}^n, \mathcal{F}^n)$ *converges to* $(\mathcal{E}, \mathcal{F})$ *in the sense of Mosco.*

The point is that we only assume the convergence *locally* in $L^1(\mathbb{R}^d)$ of the respective quantities and no other further assumptions are needed.

We next consider the convergence of symmetric jump-type Dirichlet forms. Let $\tilde{J}(x, y)$ be a non-negative symmetric Borel measurable function on $\mathbb{R}^d \times \mathbb{R}^d \setminus \mathsf{diag}$ satisfying

$$(1.4) \qquad x \mapsto \int_{y \neq x} (1 \wedge d(x, y)^2) \tilde{J}(x, y) \, dy \in L^1_{\mathrm{loc}}(\mathbb{R}^d).$$

Here diag means that the diagonal set: $\mathsf{diag} = \{(x, x): x \in \mathbb{R}^d\}$. Consider the following quadratic form $\tilde{\mathcal{E}}$ on $L^2(\mathbb{R}^d)$:

$$\tilde{\mathcal{E}}(u, v) = \frac{1}{2} \iint_{x \neq y} (u(x) - u(y))(v(x) - v(y)) \tilde{J}(x, y) \, dx \, dy$$

for functions $u, v \in C^{\mathsf{lip}}_0(\mathbb{R}^d)$. Here $C^{\mathsf{lip}}_0(\mathbb{R}^d)$ is the set of all Lipschitz continuous functions on $\mathbb{R}^d$ with compact support. Under the condition (1.4), it is also known that $(\mathcal{E}, C^{\mathsf{lip}}_0(\mathbb{R}^d))$ is a closable Markovian symmetric form on $L^2(\mathbb{R}^d)$. Then the smallest closed extension $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form.

Now take $J_n(x, y)$ and $J(x, y)$ non-negative symmetric Borel measurable functions on $\mathbb{R}^d \times \mathbb{R}^d \setminus \mathsf{diag}$ satisfying (1.4) in place of $\tilde{J}(x, y)$ and then consider regular symmetric jump-type Dirichlet forms as follows:

$$\begin{cases} \mathcal{E}^n(u, v) = \dfrac{1}{2} \iint_{x \neq y} (u(x) - u(y))(v(x) - v(y)) J_n(x, y) \, dx \, dy, \\ \mathcal{F}^n = \overline{C^{\mathsf{lip}}_0(\mathbb{R}^d)}^{\sqrt{\mathcal{E}^n_1}}, \end{cases}$$

and

$$\begin{cases} \mathcal{E}(u, v) = \dfrac{1}{2} \iint_{x \neq y} (u(x) - u(y))(v(x) - v(y)) J(x, y) \, dx \, dy, \\ \mathcal{F} = \overline{C^{\mathsf{lip}}_0(\mathbb{R}^d)}^{\sqrt{\mathcal{E}_1}}, \end{cases}$$

where $\mathcal{E}_1(u, v) = \mathcal{E}(u, v) + (u, v)_{L^2(\mathbb{R}^d)}$. We make the following assumption.

ASSUMPTION C. (i) $J_n(x, y) \leq \tilde{J}(x, y)$ for $dx \otimes dy$-a.e. $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \setminus \mathsf{diag}$ and $\forall n \in \mathbb{N}$.
(ii) $\{J_n(x, y)\}$ converges to $J(x, y)$ *locally* in $L^1(\mathbb{R}^d \times \mathbb{R}^d \setminus \mathsf{diag}; dx \otimes dy)$.

Theorem 1.4. *Assume* Assumption C. *Then* $(\mathcal{E}^n, \mathcal{F}^n)$ *converges to* $(\mathcal{E}, \mathcal{F})$ *in the sense of Mosco.*

From now on, by using the above theorems, we construct several examples whose global path properties are not preserved under the Mosco convergence. We first consider the instability of conservativeness/explosion of the symmetric diffusion processes.

Under the same settings of Theorem 1.1, let the diffusion coefficients be diagonal $A_n(x) = a_n(x)I$, where I denotes the identity of $(d \times d)$-matrices. Then we have the following result:

Proposition 1.5. *The following results hold*:
(i)   (explosive ones to conservative one) *If we set*

$$\alpha_n(x) = (2 + |x|)^2(\log(2 + |x|))^{1+1/n}, \quad \alpha(x) = (2 + |x|)^2(\log(2 + |x|))$$

for $n \in \mathbb{N}$, then $(\mathcal{E}^n, \mathcal{F}^n)$ is explosive for any n and converges in the sense of Mosco to the conservative Dirichlet form $(\mathcal{E}, \mathcal{F})$.
(ii) (conservative ones to explosive one) *If we set*

$$\alpha_n(x) = (2 + |x|)^{2-1/n}(\log(2 + |x|))^2, \quad \alpha(x) = (2 + |x|)^2(\log(2 + |x|))^2$$

for $n \in \mathbb{N}$, then $(\mathcal{E}^n, \mathcal{F}^n)$ is conservative for any n and converges in the sense of Mosco to the explosive Dirichlet form $(\mathcal{E}, \mathcal{F})$.

We now consider the instability of recurrence/transience of the symmetric Lévy processes. Let α and α_n be measurable functions on $[0, \infty)$ satisfying that there exist positive constants $\underline{\alpha}$ and $\bar{\alpha}$ so that

$$0 < \underline{\alpha} \le \alpha_n(t) \le \bar{\alpha} < 2, \quad \text{a.e.} \quad t \in [0, \infty)$$

and define Lévy measures on $\mathbb{R}^d$ as follows:

$$n_n(dx) = |x|^{-d-\alpha_n(|x|)}\,dx, \quad n(dx) = |x|^{-d-\alpha(|x|)}\,dx.$$

Then the corresponding characteristic (Lévy) exponents are given by

$$\varphi_n(x) = \int_{\mathbb{R}^d}(1 - \cos(\langle x, \xi\rangle))n_n(d\xi), \quad \varphi(x) = \int_{\mathbb{R}^d}(1 - \cos(\langle x, \xi\rangle))n(d\xi),$$

respectively. Then the following proposition holds:

Proposition 1.6. *Let n_n and n be as above. Assume $d = 1$. Then the following results hold.*
(1)   (recurrent ones to transient one) *If we set*

$$\alpha_n(u) = 1 + 1/n - (\log(u + e^2))^{-1/2}, \quad \alpha(u) = 1 - (\log(u + e^2))^{-1/2}$$

for $u \ge 0$ and $n \in \mathbb{N}$, then $(\mathcal{E}^n, \mathcal{F}^n)$ is recurrent for any n and converges in the sense of Mosco to the transient Dirichlet form $(\mathcal{E}, \mathcal{F})$.

(2) (transient ones to recurrent one) *If we set*

$$\alpha_n(u) = 1 - (\log(u + e^2))^{-(1-1/n)}, \quad \alpha(u) = 1 - (\log(u + e^2))^{-1}$$

for $u \geq 0$ and $n \in \mathbb{N}$, then $(\mathcal{E}^n, \mathcal{F}^n)$ is transient for any n and converges in the sense of Mosco to the recurrent Dirichlet form $(\mathcal{E}, \mathcal{F})$.

The point is the sharp criterion of the recurrence/transience for the stable-type processes (see e.g. Theorem 3.3 in [20] and Theorem 4.2 in Appendix in the present paper).

REMARK 1.7.   We can give a rather simple example for which the symmetric Dirichlet forms corresponding to transient symmetric Lévy processes converge to the symmetric Dirichlet form corresponding to a recurrent symmetric Lévy process in the sense of Mosco:

Assume $d = 2$. Consider a function $\varphi_n(x) := |x|^{2-1/n}$, $x \in \mathbb{R}^2$ for each n. Then φ_n defines the characteristic exponent associated with a transient symmetric $(2-1/n)$-stable process on $\mathbb{R}^2$. Clearly $\varphi_n(x)$ converges to $\varphi(x) := |x|^2$ for all x and the limit $\varphi(x)$ is the characteristic exponent associated with a 2-dimensional Brownian motion that is recurrent. Note that this example shows not only the instability of (global) path properties but also the instability of path types. Namely, the jump processes converge to the diffusion process. We will discuss such instability of path types in a forthcoming paper.

We finally consider the instability of recurrence/transience of symmetric jump processes. Let α and α_n be measurable functions on $[0, \infty)$ satisfying that there exist positive constants $\underline{\alpha}$ and $\bar{\alpha}$ so that

$$0 < \underline{\alpha} \leq \alpha_n(u) \leq \bar{\alpha} < 2, \quad \text{a.e.} \quad u \in [0, \infty).$$

Let $c(x)$ be a measurable function on $\mathbb{R}^d$ satisfying that there exist $0 < c < C < \infty$ so that $c \leq c(x) \leq C$ for all $x \in \mathbb{R}^d$. We consider the following jump kernels:

$$J_n(x, y) = (c(x) + 1)|x - y|^{-d-\alpha_n(|x-y|)}, \quad x, y \in \mathbb{R}^d, \quad x \neq y.$$

and

$$J(x, y) = (c(x) + 1)|x - y|^{-d-\alpha(|x-y|)}, \quad x, y \in \mathbb{R}^d, \quad x \neq y.$$

We note that the corresponding jump processes are not necessarily Lévy processes because $c(x)$ is not necessarily translation-invariant. Even in this case, we have the following result similar to Proposition 1.6:

Proposition 1.8.   *Let J_n and J be as above. Assume $d = 1$. Then the following results hold.*

(i) (recurrent ones to transient one) *If we set*

$$\alpha_n(u) = 1 + 1/n - (\log(u + e^2))^{-1/2}, \quad \alpha(u) = 1 - (\log(u + e^2))^{-1/2}$$

for $u \geq 0$ and $n \in \mathbb{N}$, then $(\mathcal{E}^n, \mathcal{F}^n)$ is recurrent for each n and converges in the sense of Mosco to the transient Dirichlet form $(\mathcal{E}, \mathcal{F})$.
(ii) (transient ones to recurrent one) *If we set*

$$\alpha_n(u) = 1 - (\log(u + e^2))^{-(1-1/n)}, \quad \alpha(u) = 1 - (\log(u + e^2))^{-1}$$

for $u \geq 0$ and $n \in \mathbb{N}$, then $(\mathcal{E}^n, \mathcal{F}^n)$ is transient for each n and converges in the sense of Mosco to the recurrent Dirichlet form $(\mathcal{E}, \mathcal{F})$.

The organization of the paper is as follows. In the next section, we recall the Mosco convergence and give sufficient conditions for the Mosco convergence of the three types of Dirichlet forms. In Section 3, we give several examples where global path properties are not preserved under the Mosco convergence. In Appendix, we give a necessary and sufficient condition for the recurrence of a class of symmetric stable type Lévy processes.

2. Mosco convergence of symmetric Dirichlet forms on $L^2(\mathbb{R}^d)$

In the first part of this section, we briefly recall the notion of Mosco convergence following [12]. For a closed form $(\mathcal{E}, \mathcal{F})$ on a Hilbert space $\mathcal{H}$, let $\mathcal{E}(u, u) = \infty$ for every $u \in \mathcal{H} \setminus \mathcal{F}$. Here a closed form means a nonnegative definite symmetric closed form on $\mathcal{H}$, not necessarily densely defined.

DEFINITION 2.1. A sequence of closed forms $\mathcal{E}^n$ on a Hilbert space $\mathcal{H}$ is said to be convergent to $\mathcal{E}$ in the sense of Mosco if the following two conditions are satisfied:
(M1) for every u and every sequence $\{u_n\}$ converging to u *weakly* in $\mathcal{H}$,

$$\liminf_{n \to \infty} \mathcal{E}^n(u_n, u_n) \geq \mathcal{E}(u, u);$$

(M2) for every u there exists a sequence $\{u_n\}$ converging to u in $\mathcal{H}$ so that

$$\limsup_{n \to \infty} \mathcal{E}^n(u_n, u_n) \leq \mathcal{E}(u, u).$$

In [12], Mosco showed that a sequence of closed forms $\mathcal{E}^n$ on $\mathcal{H}$ is converging to $\mathcal{E}$ in the sense of Mosco if and only if the resolvents associated with $\mathcal{E}^n$ converges to the resolvent associated with $\mathcal{E}$ *strongly* on $\mathcal{H}$, and hence the semigroups associated with $\mathcal{E}^n$ converges to the semigroup associated with $\mathcal{E}$ *strongly* on $\mathcal{H}$.

2.1. Convergence of symmetric strongly local Dirichlet forms. Consider a sequence of forms

$$\mathcal{E}^n(u, v) = \int_{\mathbb{R}^d} \langle A_n(x) \nabla u(x), \nabla v(x) \rangle \, dx$$

for some functions u, v in $L^2(\mathbb{R}^d)$, where $A_n(x) = (a_{ij}^n(x))$ are $d \times d$ symmetric matrix valued functions satisfying Assumption A.

Under Assumption A, the forms $(\mathcal{E}^n, C_0^\infty(\mathbb{R}^d))$ for each n and $(\mathcal{E}, C_0^\infty(\mathbb{R}^d))$ are Markovian closable forms on $L^2(\mathbb{R}^d)$. They become regular symmetric Dirichlet forms $(\mathcal{E}^n, \mathcal{F}^n)$ and $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathbb{R}^d)$ (see [2]). Note that we set $\mathcal{E}^n(u, u) = \infty$ if $u \in L^2(\mathbb{R}^d) \setminus \mathcal{F}^n$. We first give a simple lemma which is used in showing that $\mathcal{E}^n$ converges to $\mathcal{E}$ in the sense of Mosco.

Lemma 2.2. *Suppose that* Assumption A *holds. For any compact set $K \subset \mathbb{R}^d$, there exists a subsequence $\{n_k\}_k$ so that $\int_K \|A_{n_k}^{1/2}(x) - A^{1/2}(x)\|^2 \, dx \to 0$ as $k \to \infty$.*

Proof. Since $A_n(x)$ is a non-negative definite matrix for each x, there exists a nonnegative definite matrix $A_n^{1/2}(x)$ so that $(A_n^{1/2}(x))^2 = A(x)$. Then by the uniform boundedness of $A_n^{1/2}$ on the compact set K, we have

$$\sup_{n \in \mathbb{N}} \operatorname{ess\,sup}_{x \in K} \|A_n^{1/2}(x) - A^{1/2}(x)\|^2 < \infty.$$

By (A2), there exists a subsequence $\{n_k\}_k$ so that $A_{n_k}(x) \to A(x)$ with respect to $\|\cdot\|$ for a.e. $x \in K$. By general theory of linear operators, we can check that $A_n^{1/2}(x) \to A^{1/2}(x)$ in a.e. $x \in K$ with respect to $\|\cdot\|^2$. Thus, by using the dominated convergence theorem, we finish the proof. $\square$

We now prove Theorem 1.1:

Proof of Theorem 1.1. We first show (M1): Take $u \in L^2(\mathbb{R}^d)$ and any sequence $\{u_n\} \subset L^2(\mathbb{R}^d)$ so that u_n converges to u in L^2 weakly. We may assume that $\liminf_{n \to \infty} \mathcal{E}^n(u_n, u_n) < \infty$. Taking a subsequence $\{n_k\}$, we have

$$(2.1) \qquad \infty > \liminf_{n \to \infty} \mathcal{E}^n(u_n, u_n) = \lim_{k \to \infty} \mathcal{E}^{n_k}(u_{n_k}, u_{n_k}) = \lim_{k \to \infty} \int_{\mathbb{R}^d} |A_{n_k}^{1/2} \nabla u_{n_k}(x)|^2 \, dx.$$

Let us set $A_n^{1/2}(x) = (b_{ij}^n(x))$ and $A^{1/2}(x) = (b_{ij}(x))$. By (2.1), there exists $w_i \in L^2(\mathbb{R}^d)$ so that, by taking a subsequence of $\{u_{n_k}\}$ if necessary, $\sum_{j=1}^d b_{ij}^{n_k} \partial_j u_{n_k}$ converges weakly to w_i in $L^2(\mathbb{R}^d)$ for each i.

We now show that $w_i = \sum_{j=1}^{d} b_{ij} \partial_j u$. To this end, take any $\eta \in C_0^\infty(\mathbb{R}^d)$. Then we find that

$$\int_{\mathbb{R}^d} \left(w_i - \sum_{j=1}^{d} b_{ij}(x)\partial_j u(x) \right) \eta(x)\, dx$$

$$= \sum_{j=1}^{d} \int_{\mathbb{R}^d} (w_i - b_{ij}^{n_k}(x)\partial_j u_{n_k}(x))\eta(x)\, dx$$

$$+ \sum_{j=1}^{d} \int_{\mathbb{R}^d} (b_{ij}^{n_k}(x)\partial_j u_{n_k}(x) - b_{ij}(x)\partial_j u_{n_k}(x))\eta(x)\, dx$$

$$+ \sum_{j=1}^{d} \int_{\mathbb{R}^d} (b_{ij}(x)\partial_j u_{n_k}(x) - b_{ij}(x)\partial_j u(x))\eta(x)\, dx =: (\mathrm{I})_k + (\mathrm{II})_k + (\mathrm{III})_k.$$

We know that $(\mathrm{I})_k$ converges to zero by definition. Now let us denote by K the support of the function η. Then

$$(\mathrm{II})_k = \sum_{j=1}^{d} \int_K (b_{ij}^{n_k}(x) - b_{ij}(x))\partial_j u_{n_k}(x)\eta(x)\, dx.$$

By Lemma 2.2, taking a subsequence if necessary, $b_{ij}^{n_k}$ converges to b_{ij} in $L^2(K)$. Since u_n converges weakly to u, $(\partial_j u_{n_k})$ also converges weakly to $(\partial_j u)$ in $L^2(\mathbb{R}^d)$. Thus $(\mathrm{II})_k$ converges to 0 as $k \to \infty$ by the Schwarz inequality and L^2-boundedness of the weakly convergent sequence $\{\partial_j u_n\}_n$. The third term $(\mathrm{III})_k$ converges to 0 since $\partial_i u_{n_k}$ converges weakly to $\partial_i u$ in $L^2(\mathbb{R}^d)$ and $b_{ij}\eta \in L^2(\mathbb{R}^d)$. Thus $w_i = \sum_{j=1}^{d} b_{ij}\partial_j u$ holds for each $i = 1, 2, \ldots, d$. Hence we have $\sum_{j=1}^{d} b_{ij}^{n_k}\partial_j u_{n_k}$ converges weakly to $\sum_{j=1}^{d} b_{ij}\partial_j u$, and we conclude that

$$\liminf_{n \to \infty} \mathcal{E}^n(u_n, u_n) = \lim_{k \to \infty} \int_{\mathbb{R}^d} |A_{n_k}^{1/2}\nabla u_{n_k}(x)|^2\, dx \geq \int_{\mathbb{R}^d} |A^{1/2}\nabla u(x)|^2\, dx = \mathcal{E}(u, u).$$

We second show (M2): It is enough to show for $u \in \mathcal{F}$. Since $C_0^\infty(\mathbb{R}^d)$ is a (common) core for the Dirichlet forms $(\mathcal{E}^n, \mathcal{F}^n)$, there exists a sequence $\{u_l\} \subset C_0^\infty(\mathbb{R}^d)$ so that

$$\lim_{l \to \infty} \mathcal{E}_1(u_l - u, u_l - u)$$

$$(2.2) \qquad = \lim_{l \to \infty} \left(\int_{\mathbb{R}^d} |A\nabla u_l(x) - A\nabla u(x)|^2\, dx + \int_{\mathbb{R}^d} |u_l(x) - u(x)|^2\, dx \right)$$

$$= 0.$$

Since any norms in the space of $(d \times d)$-matrices are equivalent, by Lemma 2.2 and taking a subsequence if necessary, it follows that for each $l \in \mathbb{N}$,

$$\int_{\mathbb{R}^d} |A_n^{1/2} \nabla u_l(x) - A^{1/2} \nabla u_l(x)|^2 \, dx \leq \int_{\tilde{K}_l} \|A_n^{1/2}(x) - A^{1/2}(x)\|_{\text{op}}^2 |\nabla u_l|^2(x) \, dx$$

$$\leq C \||\nabla u_l|\|_\infty^2 \int_{\tilde{K}_l} \|A_n^{1/2}(x) - A^{1/2}(x)\|^2 \, dx$$

$$\to 0 \quad \text{as} \quad n \to \infty,$$

where $C > 0$ denotes some constant such that $\| \cdot \|_{\text{op}} \leq C \| \cdot \|$ and $\|A\|_{\text{op}}$ means the operator norm of A: $\|A\|_{\text{op}} = \sup_{u \in \mathbb{R}^d : |u| \leq 1} |Au|/|u|$. This gives us that

$$\lim_{n \to \infty} \mathcal{E}^n(u_l, u_l) = \mathcal{E}(u_l, u_l), \quad l \in \mathbb{N}.$$

Thus, with (2.2), we have

$$\lim_{l \to \infty} \lim_{n \to \infty} \mathcal{E}^n(u_l, u_l) = \mathcal{E}(u, u).$$

This shows (M2) (see, e.g. Corollary 1.18 in [1], and the proof of Theorem 3.1 in [3]). □

2.2. Convergence of symmetric translation-invariant Dirichlet forms.

Let $\{\nu_t\}_{t>0}$ be a sequence of probability measures on $\mathbb{R}^d$ of a continuous symmetric convolution semigroup:

$$\begin{cases} \nu_t * \nu_s(A) = \nu_{t+s}(A), \quad t, s > 0, \quad A \in \mathcal{B}(\mathbb{R}^d), \\ \nu_t(A) = \nu_t(-A), \quad A \in \mathcal{B}(\mathbb{R}^d), \\ \nu_t \to \delta \quad \text{weakly}, \end{cases}$$

where $\nu_t * \nu_s$ denotes the convolution of ν_t and ν_s $(\nu_t * \nu_s(A) := \int \nu_t(A - x)\nu_s(dx), A \in \mathcal{B}(\mathbb{R}^d))$ and δ is the Dirac measure concentrated at the origin. Define the kernels by

$$p(t, x, A) := p_t(x, A) := \nu_t(A - x), \quad t > 0, \quad x \in \mathbb{R}^d, \quad A \in \mathcal{B}(\mathbb{R}^d),$$

then $\{p_t(x, A); t > 0, x \in \mathbb{R}^d, A \in \mathcal{B}(\mathbb{R}^d)\}$ is a Markovian transition function which is symmetric with respect to the Lebesgue measure in the following sense:

$$\int_{\mathbb{R}^d} p_t f(x)g(x) \, dx = \int_{\mathbb{R}^d} f(x)p_t g(x) \, dx, \quad f, g \in \mathcal{B}_+(\mathbb{R}^d).$$

According to the Lévy–Khinchin formula, we see that a continuous symmetric convolution semigroup $\{\nu_t, t > 0\}$ is characterized by a pair (S, n) satisfying (1.2) and (1.3) through the formula (1.1).

Now let $\{\varphi_n\}$ be a sequence of the characteristic functions defined by symmetric convolution semigroups $\{v_t^n, t > 0\}_{n \in \mathbb{N}}$:

$$e^{-t\varphi_n(x)} := \hat{v}_t^n(x) \quad (= \int_{\mathbb{R}^d} e^{i\langle x,y \rangle} v_t^n(dy)), \quad x \in \mathbb{R}^d.$$

Let φ be also a characteristic function defined by a symmetric convolution semigroup $\{v_t, t > 0\}$. The Dirichlet forms corresponding v_t^n are defined by

$$\begin{cases} \mathcal{E}^n(u, v) = \displaystyle\int_{\mathbb{R}^d} \hat{u}(x)\bar{\hat{v}}(x)\varphi_n(x)\,dx, \\ \mathcal{F}^n = \left\{ u \in L^2(\mathbb{R}^d) : \displaystyle\int_{\mathbb{R}^d} |\hat{u}(x)|^2 \varphi_n(x)\,dx < \infty \right\}. \end{cases}$$

We set that for each n, $\mathcal{E}^n(u, u) = \infty$ if $u \in L^2(\mathbb{R}^d) \setminus \mathcal{F}^n$. We assume Assumption B. Then we show Theorem 1.3:

Proof of Theorem 1.3. We first show (M1): Take any $u \in L^2(\mathbb{R}^d)$ and any sequence $\{u_n\} \subset L^2(\mathbb{R}^d)$ for which u_n converges to u weakly in $L^2(\mathbb{R}^d)$. We may assume $\liminf_{n\to\infty} \mathcal{E}^n(u_n, u_n) < \infty$.

According to the Parseval formula, note that u_n converges to u weakly in L^2 if and only if $\hat{u}_n$ converges to $\hat{u}$ weakly in L^2. Here $\hat{u}$ denotes the Fourier transform of u. Thus

$$\infty > \liminf_{n\to\infty} \mathcal{E}^n(u_n, u_n) = \liminf_{n\to\infty} \int_{\mathbb{R}^d} |\hat{u}_n(x)|^2 \varphi_n(x)\,dx$$

implies that there exist a subsequence $\{n_k\}$ and an element $w \in L^2(\mathbb{R}^d)$ so that $\hat{u}_{n_k} \cdot \sqrt{\varphi_{n_k}}$ converges to w weakly in $L^2(\mathbb{R}^d)$. We now show that $w = \hat{u} \cdot \sqrt{\varphi}$. For any $v \in C_0^\infty(\mathbb{R}^d)$, we see that

$$\left| \int_{\mathbb{R}^d} (w(x) - \hat{u}(x)\sqrt{\varphi(x)})v(x)\,dx \right|$$

$$\leq \left| \int_{\mathbb{R}^d} (w(x) - \hat{u}_{n_k}(x)\sqrt{\varphi_{n_k}(x)})v(x)\,dx \right|$$

$$+ \left| \int_{\mathbb{R}^d} (\hat{u}_{n_k}(x)\sqrt{\varphi_{n_k}(x)} - \hat{u}(x)\sqrt{\varphi_{n_k}(x)})v(x)\,dx \right|$$

$$+ \left| \int_{\mathbb{R}^d} (\hat{u}(x)\sqrt{\varphi_{n_k}(x)} - \hat{u}(x)\sqrt{\varphi(x)})v(x)\,dx \right|$$

$$=: (\mathrm{I})_k + (\mathrm{II})_k + (\mathrm{III})_k.$$

The first term $(\mathrm{I})_k$ converges to 0 by definition. For the second term $(\mathrm{II})_k$, using the condition (B) and the inequality $|\sqrt{a} - \sqrt{b}| \leq \sqrt{|a-b|}$, we have that $\sqrt{\varphi_n}v$ converges

to $\sqrt{\varphi} v$ in $L^2(\mathbb{R}^d)$. Thus the second term $(\mathrm{II})_k$ converges to zero by the Schwarz inequality and L^2-boundedness of $\{\hat{u}_{n_k}\}_k$. For the third term $(\mathrm{III})_k$,

$$(\mathrm{III})_k = \left| \int_{\mathbb{R}^d} (\sqrt{\varphi_{n_k}(x)} - \sqrt{\varphi(x)}) \hat{u}(x) v(x)\, dx \right|$$

$$\leq \|\hat{u}\|_{L^2} \sqrt{\int_{\mathbb{R}^d} (\sqrt{\varphi_{n_k}(x)} v(x) - \sqrt{\varphi(x)} v(x))^2\, dx} \to 0 \quad \text{as} \quad k \to \infty.$$

Thus we conclude that $w = \hat{u}\sqrt{\varphi}$. Hence we find that

$$\liminf_{n \to \infty} \mathcal{E}^n(u_n, u_n) = \lim_{k \to \infty} \mathcal{E}^{n_k}(u_{n_k}, u_{n_k}) = \lim_{k \to \infty} \int_{\mathbb{R}^d} |\hat{u}_{n_k}|^2 \varphi_{n_k}\, dx$$

$$\geq \int_{\mathbb{R}^d} |\hat{u}|^2 \varphi\, dx = \mathcal{E}(u, u).$$

This shows (M1).

We second show (M2): It is enough to show for $u \in \mathcal{F}$. Since $C_0^\infty(\mathbb{R}^d)$ is a (common) core for the Dirichlet forms, there exists a sequence $\{\tilde{u}_n\}$ of $C_0^\infty(\mathbb{R}^d)$ such that

$$\lim_{n \to \infty} \mathcal{E}_1(\tilde{u}_n - u, \tilde{u}_n - u)$$

(2.3)
$$= \lim_{n \to \infty} \left(\int_{\mathbb{R}^d} |\hat{\tilde{u}}_n(x) - \hat{u}(x)|^2 \varphi(x)\, dx + \int_{\mathbb{R}^d} (\tilde{u}(x) - u(x))^2\, dx \right)$$

$$= 0.$$

We now take a sequence $\{\chi_n\}$ of $C_0^\infty(\mathbb{R}^d)$ satisfying

$$\chi_n(x) = \chi_n(-x), \quad 0 \leq \chi_n(x) \leq \chi_{n+1}(x) \leq 1, \quad n \in \mathbb{N},$$

$$\lim_{n \to \infty} \chi_n(x) = 1, \quad x \in \mathbb{R}^d.$$

For any $n, l \in \mathbb{N}$, set $u_{n,l}(x) = \tilde{u}_n * \check{\chi}_l(x) = \int_{\mathbb{R}^d} \check{\chi}_l(x - y) \tilde{u}_n(y)\, dy$, $x \in \mathbb{R}^d$. Here the inverse Fourier transform of χ_l is denoted by $\check{\chi}_l$. Since

$$\|\tilde{u}_n * \check{\chi}_l - u * \check{\chi}_l\|_{L^2} = \|(\hat{\tilde{u}}_n - \hat{u}) \cdot \chi_l\|_{L^2}$$

$$\leq \|\hat{\tilde{u}}_n - \hat{u}\|_{L^2} = \|\tilde{u}_n - u\|_{L^2} \to 0 \quad \text{as} \quad n \to \infty$$

for each l and

$$\|u * \check{\chi}_l - u\|_{L^2} = \|\hat{u} \cdot \chi_l - \hat{u}\|_{L^2} \to 0 \quad \text{as} \quad l \to \infty,$$

then we have that

$$\lim_{l\to\infty}\lim_{n\to\infty}\|u_{n,l}-u\|_{L^2}=0.$$

On the other hand, we see that from (2.3), $\hat{\tilde{u}}_n\cdot\chi_l\sqrt{\varphi_n}=\widehat{\tilde{u}_n*\check{\chi}_l}\sqrt{\varphi_n}$ converges to $\hat{u}\cdot\chi_l\sqrt{\varphi}=\widehat{u*\check{\chi}_l}\sqrt{\varphi}$ in $L^2(\mathbb{R}^d)$ for any l. In fact, using the inequalities $(a-b)^2\le 2a^2+2b^2$, $|\sqrt{a}-\sqrt{b}|\le\sqrt{|a-b|}$ and the condition (B), we have

$$\int_{\mathbb{R}^d}(\hat{\tilde{u}}_n\cdot\chi_l\sqrt{\varphi_n}-\hat{u}\cdot\chi_l\sqrt{\varphi})^2\,dx$$

$$\le 2\int_{\mathbb{R}^d}(\hat{\tilde{u}}_n\cdot\chi_l\sqrt{\varphi_n}-\hat{\tilde{u}}_n\cdot\chi_l\sqrt{\varphi})^2\,dx+2\int_{\mathbb{R}^d}(\hat{\tilde{u}}_n\cdot\chi_l\sqrt{\varphi}-\hat{u}\cdot\chi_l\sqrt{\varphi})^2\,dx$$

$$\le 2\int_{\mathbb{R}^d}\hat{\tilde{u}}_n^2\chi_l^2|\varphi_n-\varphi|\,dx+2\int_{\mathbb{R}^d}(\hat{\tilde{u}}_n\cdot\chi_l\sqrt{\varphi}-\hat{u}\cdot\chi_l\sqrt{\varphi})^2\,dx$$

$$\to 0\quad\text{as}\quad n\to\infty.$$

Thus we find that

$$\lim_{n\to\infty}\mathcal{E}^n(u_{n,l},u_{n,l})=\lim_{n\to\infty}\int_{\mathbb{R}^d}|\widehat{\tilde{u}_n*\check{\chi}_l}|^2\varphi_n\,dx=\int_{\mathbb{R}^d}|\widehat{u*\check{\chi}_l}|^2\varphi\,dx$$

$$=\int_{\mathbb{R}^d}|\hat{u}|^2\chi_l^2\varphi\,dx$$

and

$$\lim_{l\to\infty}\int_{\mathbb{R}^d}|\hat{u}|^2\chi_l^2\varphi\,dx=\int_{\mathbb{R}^d}|\hat{u}|^2\varphi\,dx=\mathcal{E}(u,u).$$

These imply that

$$\lim_{l\to\infty}\lim_{n\to\infty}\mathcal{E}^n(u_{n,l},u_{n,l})=\mathcal{E}(u,u)\quad\text{and}\quad\lim_{l\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^d}|u_{n,l}-u|^2\,dx=0.$$

Therefore, by the diagonalization argument, we can find a sequence $\{l(n)\}_n$ so that

$$l(n)<l(n+1)\nearrow\infty\quad(n\to\infty),\quad\lim_{n\to\infty}\mathcal{E}^n(u_{n,l(n)},u_{n,l(n)})=\mathcal{E}(u,u)$$

and then (M2) is shown. □

2.3. Convergence of symmetric jump-type Dirichlet forms. Let $\tilde{J}(x,y)$ be a non-negative symmetric Borel-measurable function on $\mathbb{R}^d\times\mathbb{R}^d\setminus\mathsf{diag}$ satisfying

$$(2.4)\qquad x\mapsto\int_{y\neq x}(1\wedge d(x,y)^2)\tilde{J}(x,y)\,dy\in L^1_{\mathrm{loc}}(\mathbb{R}^d).$$

Consider the following quadratic form $\tilde{\mathcal{E}}$ on $L^2(\mathbb{R}^d)$:

$$\tilde{\mathcal{E}}(u, v) = \frac{1}{2} \iint_{x \neq y} (u(x) - u(y))(v(x) - v(y))\tilde{J}(x, y)\, dx\, dy$$

for some functions $u, v \in L^2(\mathbb{R}^d)$. Under the condition (2.4), it is known that $(\mathcal{E}, C_0^{\text{lip}}(\mathbb{R}^d))$ is a closable Markovian symmetric form on $L^2(\mathbb{R}^d)$. Thus taking the closure of $C_0^{\text{lip}}(\mathbb{R}^d)$ with respect to $\sqrt{\tilde{\mathcal{E}}_1}$, we find that $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ is a regular Dirichlet form.

Now take $J_n(x, y)$ and $J(x, y)$ non-negative symmetric Borel-measurable functions on $\mathbb{R}^d \times \mathbb{R}^d \setminus \text{diag}$ satisfying (2.4) in place of $\tilde{J}(x, y)$ and then consider regular symmetric Dirichlet forms $\mathcal{E}^n$ and $\mathcal{E}$ of pure jump type on $L^2(\mathbb{R}^d)$ as follows:

$$\begin{cases} \mathcal{E}^n(u, v) = \dfrac{1}{2} \iint_{x \neq y} (u(x) - u(y))(v(x) - v(y))J_n(x, y)\, dx\, dy, \\[2mm] \mathcal{F}^n = \overline{C_0^{\text{lip}}(\mathbb{R}^d)}^{\sqrt{\mathcal{E}_1^n}}, \end{cases}$$

and

$$\begin{cases} \mathcal{E}(u, v) = \dfrac{1}{2} \iint_{x \neq y} (u(x) - u(y))(v(x) - v(y))J(x, y)\, dx\, dy, \\[2mm] \mathcal{F} = \overline{C_0^{\text{lip}}(\mathbb{R}^d)}^{\sqrt{\mathcal{E}_1}}. \end{cases}$$

We assume Assumption C. Then we prove Theorem 1.4.

Proof of Theorem 1.4. We have to check the following two conditions:
(M1) For any $u \in L^2(\mathbb{R}^d)$ and $\{u_n\} \subset L^2(\mathbb{R}^d)$ which converges to u weakly in $L^2(\mathbb{R}^d)$,

$$\liminf_{n \to \infty} \mathcal{E}^n(u_n, u_n) \geq \mathcal{E}(u, u).$$

(M2) For any $u \in L^2(\mathbb{R}^d)$, there exists a sequence $\{u_n\} \subset L^2(\mathbb{R}^d)$ which converges to u in $L^2(\mathbb{R}^d)$ such that

$$\limsup_{n \to \infty} \mathcal{E}^n(u_n, u_n) \leq \mathcal{E}(u, u).$$

Proof of (M1). Suppose that
(1) u_n is weakly convergent to u in $L^2(\mathbb{R}^d)$ and
(2) $\liminf_{n \to \infty} \iint_{x \neq y} (u_n(x) - u_n(y))^2 J_n(x, y)\, dx\, dy < \infty$.
  We may assume that $\lim_{n \to \infty} \iint_{x \neq y} (u_n(x) - u_n(y))^2 J_n(x, y)\, dx\, dy < \infty$.

Then for each n, put $\bar{u}_n(x, y) = (u_n(x) - u_n(y))\sqrt{J_n(x, y)}$ for $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \setminus$ diag. Then $\{\bar{u}_n\}$ are bounded sequence in $L^2(\mathbb{R}^d \times \mathbb{R}^d \setminus \text{diag}; dx \otimes dy)$, and so there exists a subsequence $\{\bar{u}_{n_k}\}$ which converges to some element $\bar{u}$ *weakly* in $L^2(\mathbb{R}^d \times \mathbb{R}^d \setminus \text{diag}; dx \otimes dy)$.

We now claim that

$$\bar{u}(x, y) = (u(x) - u(y))\sqrt{J(x, y)}, \quad (dx \otimes dy)\text{-a.e.} \quad (x, y) \quad \text{with} \quad x \neq y.$$

For any nonnegative $v \in C_0(\mathbb{R}^d \times \mathbb{R}^d \setminus \mathsf{diag})$ and for any n_k, we see

$$\left| \iint_{x \neq y} (\bar{u}(x, y) - (u(x) - u(y))\sqrt{J(x, y)})v(x, y)\, dx\, dy \right|$$

$$\leq \left| \iint_{x \neq y} (\bar{u}(x, y) - (u_{n_k}(x) - u_{n_k}(y))\sqrt{J_{n_k}(x, y)})v(x, y)\, dx\, dy \right|$$

$$+ \left| \iint_{x \neq y} (u_{n_k}(x) - u_{n_k}(y))(\sqrt{J_{n_k}(x, y)} - \sqrt{J(x, y)})v(x, y)\, dx\, dy \right|$$

$$+ \left| \iint_{x \neq y} ((u_{n_k}(x) - u_{n_k}(y)) - (u(x) - u(y)))\sqrt{J(x, y)}v(x, y)\, dx\, dy \right|$$

$$=: (\mathrm{I})_{n_k} + (\mathrm{II})_{n_k} + (\mathrm{III})_{n_k}.$$

Since $\bar{u}_n$ converges to $\bar{u}$ weakly in $L^2(\mathbb{R}^d \times \mathbb{R}^d \setminus \mathsf{diag}; dx \otimes dy)$, we see $\lim_{k\to\infty}(\mathrm{I})_{n_k} = 0$. By making use of the Schwarz inequality and Assumption C and noting $\{u_{n_k}\}$ is a bounded sequence in $L^2(\mathbb{R}^d; dx)$, we see

$(\mathrm{II})_{n_k}$

$$\leq \sqrt{\iint_{x \neq y} (u_{n_k}(x) - u_{n_k}(y))^2 v(x, y)\, dx\, dy}$$

$$\times \sqrt{\iint_{x \neq y} (\sqrt{J_{n_k}(x, y)} - \sqrt{J(x, y)})^2 v(x, y)\, dx\, dy}$$

$$\leq \|v\|_\infty \|u_{n_k}\|_{L^2} \sqrt{2 \left(\left\| \int_{\{x : (x, \cdot) \in \mathrm{supp}[v]\}} |v(x, \cdot)|\, dx \right\|_\infty + \left\| \int_{\{y : (\cdot, y) \in \mathrm{supp}[v]\}} |v(\cdot, y)|\, dy \right\|_\infty \right)}$$

$$\times \sqrt{\iint_{\mathrm{supp}[v]} |J_{n_k}(x, y) - J(x, y)|\, dx\, dy}$$

$$\to 0 \quad \text{as} \quad n_k \to \infty.$$

Here we used elementary inequalities in the second inequality above: $(a - b)^2 \leq 2(a^2 + b^2)$ and $|\sqrt{a} - \sqrt{b}| \leq \sqrt{|a - b|}$ for $a, b \geq 0$. As for $(\mathrm{III})_{n_k}$, note that both

$$\varphi(x) = \int_{y \neq x} \sqrt{J(x, y)}v(x, y)\, dy, \quad x \in \mathbb{R}^d,$$

$$\psi(y) = \int_{y \neq x} \sqrt{J(x, y)}v(x, y)\, dx, \quad y \in \mathbb{R}^d$$

are in $L^2(\mathbb{R}^d)$. So we see

$$(\text{III})_{n_k} \leq \left| \int_{\mathbb{R}^d} (u_{n_k}(x) - u(x))\varphi(x)\, dx \right| + \left| \int_{\mathbb{R}^d} (u_{n_k}(y) - u(y))\psi(y)\, dy \right|$$

goes to 0 when $n_k \to \infty$. Thus we see

$$\bar{u}(x, y) = (u(x) - u(y))\sqrt{J(x, y)} \quad (dx \otimes dy)\text{-a.e.} \quad (x, y) \quad \text{with} \quad x \neq y.$$

Hence

$$\liminf_{n \to \infty} \mathcal{E}^n(u_n, u_n) \geq \iint_{x \neq y} (u(x) - u(y))J(x, y)\, dx\, dy = \mathcal{E}(u, u). \qquad \square$$

Proof of (M2).   Since $C_0^{\text{lip}}(\mathbb{R}^d)$ is a common core for $(\mathcal{E}^n, \mathcal{F}^n)$, it is enough to show (M2) for functions in $C_0^{\text{lip}}(\mathbb{R}^d)$ (see, e.g. Corollary 1.18 in [1], and the proof of Theorem 3.1 in [3]). Take any $u \in C_0^{\text{lip}}(\mathbb{R}^d)$. Put $u_n = u$ for each n, then u_n converges to u in $L^2(\mathbb{R}^d)$. Denote by K the support of u and take compact set F with $K \subset F$ and

$$(2.5) \qquad\qquad d(K, F^c) = \inf\{d(x, y) : x \in K, \ y \in F^c\} \geq 1.$$

Then

$$\begin{aligned}
\mathcal{E}^n(u_n, u_n) = \mathcal{E}^n(u, u) &= \iint_{x \neq y} (u(x) - u(y))^2 J_n(x, y)\, dx\, dy \\
&= \iint_{F \times F \setminus \text{diag}} (u(x) - u(y))^2 J_n(x, y)\, dx\, dy \\
&\quad + 2 \iint_{K \times F^c} (u(x) - u(y))^2 J_n(x, y)\, dx\, dy \\
&=: (\text{I})_n + 2(\text{II})_n.
\end{aligned}$$

We first estimate $(\text{I})_n$. For all $n \in \mathbb{N}$ and $(dx \otimes dy)$-a.e. $(x, y) \in K \times K \setminus \text{diag}$, we see that the integrand in $(\text{I})_n$, that is, $(u(x) - u(y))^2 J_n(x, y)$, is bounded by $M(1 \wedge d(x, y)^2)\tilde{J}(x, y)$ from above, where $M := \max\{\text{Lip}(u)^2, 4\|u\|_\infty^2\}$. Here $\text{Lip}(u)$ means the smallest Lipschitz constant of u. By the fact that the function $(1 \wedge d(x, y)^2)\tilde{J}(x, y)$ is integrable on the set $F \times F \setminus \text{diag}$ and Assumption C, we have

$$\lim_{n \to \infty} (\text{I})_n = \iint_{F \times F \setminus \text{diag}} (u(x) - u(y))^2 J(x, y)\, dx\, dy.$$

We next estimate $(\text{II})_n$. The integral $(\text{II})_n$ is the following:

$$\iint_{K \times F^c} u(x)^2 J_n(x, y)\, dx\, dy.$$

For $(dx \otimes dy)$-a.e. $(x, y) \in K \times F^c$, by Assumption C, we see

$$u(x)^2 J_n(x, y) \le \|u\|_\infty^2 \tilde{J}(x, y),$$

and the right hand side is integrable on the set $K \times F^c$ because of (2.4). Thus, by Assumption C, we have

$$
\begin{aligned}
\lim_{n \to \infty} (\mathrm{II})_n &= \lim_{n \to \infty} \iint_{K \times F^c} u(x)^2 J_n(x, y) \, dx \, dy \\
&= \iint_{K \times F^c} u(x)^2 J(x, y) \, dx \, dy.
\end{aligned}
$$

Combining these two estimates, we have

$$
\begin{aligned}
\lim_{n \to \infty} \mathcal{E}^n(u, u) &= \iint_{F \times F \backslash \mathrm{diag}} (u(x) - u(y))^2 J(x, y) \, dx \, dy \\
&\quad + 2 \iint_{K \times F^c} u(x)^2 J(x, y) \, dx \, dy \\
&= \iint_{x \ne y} (u(x) - u(y))^2 J(x, y) \, dx \, dy = \mathcal{E}(u, u).
\end{aligned}
$$

This concludes that (M2) holds. □

3. Instability of global path properties

3.1. Proof of Proposition 1.5.

Proof of Proposition 1.5. By [17, Theorem 2.2] and the Feller's test in [10], in the case of (i), $(\mathcal{E}^n, \mathcal{F}^n)$ is explosive and $(\mathcal{E}, \mathcal{F})$ is conservative, and, in the case of (ii), $(\mathcal{E}^n, \mathcal{F}^n)$ is conservative and $(\mathcal{E}, \mathcal{F})$ is explosive (see also, e.g., [2, p. 300]). By Theorem 1.1, $(\mathcal{E}^n, \mathcal{F}^n)$ converges to $(\mathcal{E}, \mathcal{F})$ in the sense of Mosco in the both cases (i) and (ii) and we finish the proof. □

3.2. Proof of Proposition 1.6. We first show the following lemma which is a sufficient condition for local L^1-convergence of φ_n to φ:

Lemma 3.1. *If $\alpha_n(t) \to \alpha(t)$ for every $t \in [0, \infty)$, then $\varphi_n \to \varphi$ locally in $L^1(\mathbb{R}^d)$.*

Proof. We show that, for any compact set $K \subset \mathbb{R}^d$, $\int_K |\varphi_n(x) - \varphi(x)| \, dx \to 0$. Since φ_n and φ are continuous functions and $|\varphi_n - \varphi|$ is uniformly bounded on K, making use of the dominated convergence theorem, it suffices to show that $\varphi_n(x) \to$

$\varphi(x)$ for a.e. $x \in K$. We see that

$$
\begin{aligned}
|\varphi_n(x) - \varphi(x)| &\leq \int_{\mathbb{R}^d} |1 - \cos(\langle x, \xi \rangle)| \cdot | \, |\xi|^{-1-\alpha_n(|\xi|)} - |\xi|^{-1-\alpha(|\xi|)}| \, d\xi \\
&\leq \int_{\mathbb{R}^d} |1 - \cos(\langle x, \xi \rangle)| \, |\xi|^{-1-\alpha_n(|\xi|)} \, d\xi \\
&\quad + \int_{\mathbb{R}^d} |1 - \cos(\langle x, \xi \rangle)| \, |\xi|^{-1-\alpha(|\xi|)} \, d\xi \\
&\leq \int_{\mathbb{R}^d} |1 - \cos(\langle x, \xi \rangle)| \max\{|\xi|^{-1-\bar{\alpha}}, |\xi|^{-1-\underline{\alpha}}\} \, d\xi \\
&\quad + \int_{\mathbb{R}^d} |1 - \cos(\langle x, \xi \rangle)| \max\{|\xi|^{-1-\bar{\alpha}}, |\xi|^{-1-\underline{\alpha}}\} \, d\xi \\
&< \infty.
\end{aligned}
$$

Since $\alpha_n(t) \to \alpha(t)$ as $n \to 0$ for any $t \in [0, \infty)$, it follows that

$$
(1 - \cos(\langle x, \xi \rangle))(|\xi|^{-1-\alpha_n(|\xi|)} - |\xi|^{-1-\alpha(|\xi|)}) \to 0
$$

as $n \to \infty$. By the dominated convergence theorem, we see that $\varphi_n(x) \to \varphi(x)$ for a.e. $x \in \mathbb{R}^d$. The proof is completed. $\square$

Now we show Proposition 1.6:

Proof of Proposition 1.6.   (i): By Theorem 4.1, we can verify that $(\mathcal{E}^n, \mathcal{F}^n)$ is recurrent for any n and $(\mathcal{E}, \mathcal{F})$ is transient. By Lemma 3.1, we have that $(\mathcal{E}^n, \mathcal{F}^n)$ converges to $(\mathcal{E}, \mathcal{F})$ in the sense of Mosco. (ii): The transience of $(\mathcal{E}^n, \mathcal{F}^n)$ and the recurrence of $(\mathcal{E}, \mathcal{F})$ follow directly from Theorem 4.2. By Lemma 3.1, we have that $(\mathcal{E}^n, \mathcal{F}^n)$ converges to $(\mathcal{E}, \mathcal{F})$ in the sense of Mosco and we finish the proof. $\square$

3.3.   Proof of Proposition 1.8.

Proof of Proposition 1.8.   We use Proposition 1.6 and the comparison theorems of Dirichlet forms [2, Theorem 1.6.4]. $\square$

4.   Appendix: Sharpness of recurrence criteria for symmetric Lévy processes

It is well-known that a translation invariant symmetric stable process with an index α ($0 < \alpha \leq 2$) is recurrent if and only if $d = 1 \leq \alpha \leq 2$ or $d = \alpha = 2$. The Lévy kernel is given by $n(dh) = c|h|^{-d-\alpha} \, dh$ for some constant $c = c(d, \alpha)$ if $0 < \alpha < 2$.

In this appendix, we give a recurrent criteria for a class of stable type Lévy processes having the Lévy measure $n(dh) = |h|^{-d-\alpha(|h|)} \, dh$, where α is a measurable function

defined on $[0, \infty)$. When α is a constant between 0 to 2, then this corresponds nothing but to a symmetric α stable process. Consider also the following quadratic form:

$$\mathcal{E}(u, v) = \iint_{h \neq 0} (u(x + h) - u(x))(v(x + h) - v(x))n(dh)\,dx$$

$$= \iint_{x \neq y} \frac{(u(y) - u(x))(v(y) - v(x))}{|x - y|^{d + \alpha(|x-y|)}}\,dx\,dy,$$

$$\mathcal{D}[\mathcal{E}] = \{u \in L^2(\mathbb{R}^d) \colon \mathcal{E}(u, u) < \infty\}.$$

Then it is known that $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$ is a symmetric, translation invariant Dirichlet form on $L^2(\mathbb{R}^d)$ under the following condition:

$$\int_{h \neq 0} (1 \wedge |h|^2)n(dh) = c_d \int_0^\infty (1 \wedge u^2)u^{-1-\alpha(u)}\,du < \infty.$$

In [20] (see also [9, 13]), we have shown the following theorem:

Theorem 4.1 (cf. Theorem 3.3 in [20]). *If the conditions*

$$\limsup_{R \to \infty} R^{-2+d} \int_0^R u^{1-\alpha(u)}\,du < \infty$$

and

$$\limsup_{R \to \infty} R^d \int_R^\infty u^{-1-\alpha(u)}\,du < \infty$$

hold, then the process is recurrent.

In the case where $d = 1$, we can show the following. Let $\varepsilon > 0$ and set

$$\alpha(u) = 1 - (\log(u + e^2))^{-\varepsilon}, \quad u \geq 0$$

for instance. Let us also consider the corresponding form:

$$\mathcal{E}(u, v) = \iint_{h \neq 0} (u(x + h) - u(x))(v(x + h) - v(x))n(dh)\,dx$$

$$= \iint_{x \neq y} \frac{(u(y) - u(x))(v(y) - v(x))}{|x - y|^{1 + \alpha(|x-y|)}}\,dx\,dy,$$

$$\mathcal{D}[\mathcal{E}] = \{u \in L^2(\mathbb{R}) \colon \mathcal{E}(u, u) < \infty\}.$$

Then we have the following criterion for the recurrence:

Theorem 4.2. *The form/process is recurrent if and only if $\varepsilon \geq 1$.*

Proof.   Though we have shown in [20] (cf. [19]) that the form is recurrent if $\varepsilon \geq 1$, we give the proof of it for reader's convenience. Namely, we estimate two integrals in the previous theorem in the case $d = 1$.

For $R > e$,

$$R^{-1} \int_0^R u^{1-\alpha(u)} \, du = R^{-1} \int_0^R u^{(\log(u+e^2))^{-\varepsilon}} \, du.$$

Since $\varepsilon \geq 1$, we find that

$$R^{-1} \int_0^R u^{(\log(u+e^2))^{-\varepsilon}} \, du$$

$$= R^{-1} \int_0^{\sqrt{R}-e^2} u^{(\log(u+e^2))^{-\varepsilon}} \, du + R^{-1} \int_{\sqrt{R}-e^2}^R u^{(\log(u+e^2))^{-\varepsilon}} \, du$$

$$\leq R^{-1} \int_0^{\sqrt{R}} u^{2-\varepsilon} \, du + R^{-1} \int_{\sqrt{R}-e^2}^R u^{((1/2)\log R)^{-\varepsilon}} \, du$$

$$\leq \frac{R^{2^{-\varepsilon-1}-1}}{2^{-\varepsilon}+1} + R^{-1+((1/2)\log R)^{-\varepsilon}}(R - \sqrt{R} + e^2)$$

$$= \frac{2^\varepsilon}{(1+2^\varepsilon)R^{1-2^{-\varepsilon-1}}} + R^{((1/2)\log R)^{-\varepsilon}}\left(1 - \frac{1}{\sqrt{R}} + \frac{e^2}{R}\right).$$

Since $\varepsilon \geq 1$, it follows that

$$\log R^{((1/2)\log R)^{-\varepsilon}} = \left(\frac{1}{2}\log R\right)^{-\varepsilon} \cdot \log R$$

$$= 2^\varepsilon (\log R)^{1-\varepsilon} \to \begin{cases} 0 & \text{if} \quad \varepsilon > 1, \\ 2^\varepsilon & \text{if} \quad \varepsilon = 1, \end{cases} \quad \text{as} \quad R \to \infty.$$

Thus we find that

$$\limsup_{R\to\infty} R^{-1} \int_0^R u^{1-\alpha(u)} \, du < \infty.$$

We now estimate the second condition: For $R > \sqrt{e}$,

$$R \int_R^\infty u^{-1-\alpha(u)} du = R \int_R^\infty u^{-2+(\log(u+e^2))^{-\varepsilon}} \, du \leq R \int_R^\infty u^{-2+(2\log R)^{-\varepsilon}} \, du$$

$$= R\left[\frac{1}{-1+(2\log R)^{-\varepsilon}} u^{-1+(2\log R)^{-\varepsilon}}\right]_R^\infty$$

$$= R\frac{R^{-1+(2\log R)^{-\varepsilon}}}{1-(2\log R)^{-\varepsilon}} = \frac{R^{(2\log R)^{-\varepsilon}}}{1-(2\log R)^{-\varepsilon}}.$$

Similar to the previous calculus, we find that $\log R^{(2\log R)^{-\varepsilon}} = 2^{-\varepsilon}(\log R)^{1-\varepsilon}$. Then it follows that

$$\limsup_{R\to\infty} R \int_R^\infty u^{-1-\alpha(u)}\,du < \infty.$$

Thus the process is recurrent for $\varepsilon \geq 1$.

Now we show that the process is transient if $0 < \varepsilon < 1$. In order to show this, recall that the characteristic function φ of the process is defined by

$$\varphi(\xi) = \int_{\mathbb{R}} (1 - \cos(\xi h))|h|^{-1-\alpha(|h|)}\,dh, \quad \xi \in \mathbb{R}$$

and the process is recurrent if and only if for some (or equivalently, for all) $r > 0$,

$$\int_{\{|\xi|<r\}} \frac{d\xi}{\varphi(\xi)} = \infty,$$

(see [15]). Then we will prove that $\int_{\{|\xi|<r\}}(1/\varphi(\xi))\,d\xi < \infty$ for some $0 < r \leq 1$ provided that $0 < \varepsilon < 1$. This means it is enough for us to estimate the function φ on $\{\xi \in \mathbb{R} : |\xi| < 1\}$.

Since $\varphi(0) = 0$, we only consider the case $0 < |\xi| < 1$.

First assume that $0 < \xi < 1$. Then

$$\begin{aligned}
\varphi(\xi) &= \int_{\mathbb{R}} (1 - \cos(\xi h))|h|^{-1-\alpha(|h|)}\,dh \\
&\geq \int_{\{\pi/2 < \xi x < \pi\}} (1 - \cos(\xi h))|h|^{-2+(\log(|h|+e^2))^{-\varepsilon}}\,dh \\
&= \int_{\{\pi/2 < u < \pi\}} (1 - \cos u)\left|\frac{u}{\xi}\right|^{-2+(\log(|u/\xi|+e^2))^{-\varepsilon}} \frac{du}{\xi} \quad (\xi h = u) \\
&\geq \xi^{1-(\log(\pi/\xi+e^2))^{-\varepsilon}} \int_{\{\pi/2 < u < \pi\}} (1 - \cos u)u^{-2}\,du \geq c\xi^{1-(\log(\pi/\xi+e^2))^{-\varepsilon}},
\end{aligned}$$

where c is a constant independent of ξ. Similarly, we can get a similar bound for $-1 < \xi < 0$:

$$\begin{aligned}
\varphi(\xi) &= \int_{\mathbb{R}} (1 - \cos(\xi h))|h|^{-1-\alpha(|h|)}\,dh \\
&\geq \int_{\{-\pi/2 > \xi x > -\pi\}} (1 - \cos(\xi h))|h|^{-1-\alpha(|h|)}\,dh \\
&\geq (-\xi)^{1-(\log(\pi/(-\xi)+e^2))^{-\varepsilon}} \int_{\{\pi/2 < u < \pi\}} (1 - \cos u)u^{-2}\,du \\
&\geq c(-\xi)^{1-(\log(\pi/(-\xi)+e^2))^{-\varepsilon}}.
\end{aligned}$$

Thus it follows that

$$\varphi(\xi) \geq c|\xi|^{1-(\log(\pi/|\xi|+e^2))^{-\varepsilon}}, \quad 0 < |\xi| < 1.$$

Then, noting $0 < \varepsilon < 1$, we find that

$$
\begin{aligned}
\int_{B(1)} \frac{d\xi}{\varphi(\xi)} &\leq c \int_{|\xi|<1} |\xi|^{-1+(\log(\pi/|\xi|+e^2))^{-\varepsilon}} \, d\xi \\
&\leq 2c \int_0^1 u^{-1+(\log(\pi/u+e^2))^{-\varepsilon}} \, du \quad (\pi/u + e^2 = t) \\
&= \pi c \int_{\pi+e^2}^\infty (\pi(t-e^2)^{-1})^{-1+(\log t)^{-\varepsilon}} (t-e^2)^{-2} \, dt \\
&\leq c' \int_{\pi+e^2}^\infty (t-e^2)^{-1-(\log t)^{-\varepsilon}} \, dt \quad (\log t = s) \\
&\leq c' \int_{\log(\pi+e^2)}^\infty (e^s - e^2)^{-1-s^{-\varepsilon}} \cdot e^s \, dt \leq c'' \int_2^\infty e^{-s^{1-\varepsilon}} ds \quad (s^{1-\varepsilon} = x) \\
&= \frac{c''}{1-\varepsilon} \int_{2^{1-\varepsilon}}^\infty e^{-x} x^{\varepsilon/(1-\varepsilon)} \, dx \leq c''' \Gamma\left(\frac{\varepsilon}{1-\varepsilon} + 1\right) < \infty,
\end{aligned}
$$

where c' and c'' are positive constants independent of t and s respectively. Therefore the form/process is transient for $0 < \varepsilon < 1$. □

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