Spaliński, J.
Osaka J. Math.
53 (2016), 293–307

A MODEL OF THE BOREL CONSTRUCTION ON THE FREE LOOPSPACE

JAN SPALIŃSKI

(Received December 5, 2014)

Abstract

Let X be a CW-complex with basepoint. We obtain a simple description of the Borel construction on the free loopspace of the suspension of X as a wedge of the classifying space of the circle and the homotopy colimit of a diagram consisting of products of a number of copies of X and the standard topological n-simplex. This is obtained by filtering the cyclic bar construction on the James model of the based loopspace by word length in order to express the homotopy type of the free loopspace as a colimit of powers of X and standard cyclic sets. It is shown that this colimit is in fact a homotopy colimit and commutativity of homotopy colimits is used to describe the Borel construction.

1. Introduction

Let X be a connected CW-complex with a basepoint $*$ and let $Y = \Sigma(X)$ be the reduced suspension of X. The first aim of this paper is to give a combinatorial model of the free loopspace on Y, which will be denoted by $L(Y)$. To achieve this, we start with the cyclic bar construction introduced by F. Waldhausen [21] which was applied to the monoid of Moore loops on Y by T. Goodwillie [11]. It is shown in the latter paper that we obtain a cyclic space (i.e. a cyclic object in the category of spaces) whose geometric realization is weakly equivalent to the free loopspace on Y in a way which preserves the S^1-action. Following an idea of W. Dwyer we replace the monoid of Moore loops on Y by $J(X)$—the I.M. James model of the based loopspace of X. By filtering the cyclic bar construction by word length, we obtain a simple description of the free loopspace in terms of the standard cyclic sets indexed by cartesian products of X (Theorem 1).

The next section describes the Borel construction on the free loopspace of a suspension (Theorem 6). The proof is based on the observation that the colimit describing the free loopspace obtained in the earlier section is in fact a homotopy colimit.

The Borel construction on the free loopspace has been studied earlier by C.-F. Bödigheimer [3], Bödigheimer and I. Madsen [4] as well G. Carlsson and R.L. Cohen [5]. We will return to the relation later on in the paper.

2010 Mathematics Subject Classification. Primary 55P91; Secondary 55P35, 57T30.

We thank William Dwyer for the idea of filtering the cyclic bar construction by word length and to David Blanc for a number of helpful conversations.

2. The cyclic bar construction on the James model

We work in the category Top of compactly generated Hausdorff topological spaces as described in Steenrod [18]. We use the symbol $\approx$ to denote a homeomorphism, the symbol $\simeq$ to denote a homotopy equivalence, and the symbol $\overset{w}{\simeq}$ to denote a weak homotopy equivalence.

We recall the main facts about cyclic sets. The cyclic category Λ was introduced by A. Connes in [7]. Our notation and terminology will be consistent with the fundamental reference by J.-L. Loday [14]. The objects are the finite ordered sets $[n] = \{0, 1, \ldots, n\}$, where $n \geq 0$. The morphisms are generated by the order preserving maps $\delta_i\colon [n-1] \to [n]$—skip element i, the order preserving maps $\sigma_i\colon [n+1] \to [n]$—repeat element i for $i = 0, \ldots, n$ (as in the simplicial category Δ) and the cyclic order preserving maps $\tau_n : [n] \to [n]$, $\tau_n(i) = i - 1 \bmod (n + 1)$. With respect to the simplicial face and degeneracy maps, the cyclic operators satisfy the following relations:

$$\tau_n\delta_i = \begin{cases} \delta_{i-1}\tau_{n-1}, & \text{for} \quad 1 \leq i \leq n, \\ \delta_n, \text{for} \quad i = 0, \end{cases}$$

$$\tau_n\sigma_i = \begin{cases} \sigma_{i-1}\tau_{n+1}, & \text{for} \quad 1 \leq i \leq n, \\ \sigma_n\tau_{n+1}^2, & \text{for} \quad i = 0. \end{cases}$$

A cyclic space is a contravariant functor from Λ to the category of spaces (a cyclic set is a special case of a cyclic space, when all the spaces in the range of X have discrete topology).

There is a forgetful functor which associates with each cyclic space the underlying simplicial space. For $n \geq 0$ let Δ^n be the closed topological n-simplex, i.e. $\big\{(x_0, \ldots, x_n) \in \mathbb{R}^n \bigm| \sum_{i=0}^n x_i = 1 \text{ and } x_i \geq 0\big\}$. The geometric realization of a cyclic space is by definition the geometric realization of the underlying simplicial space,

$$|X| = \left(\coprod_{n\geq 0} X_n \times \Delta^n\right) \Big/ \sim,$$

where $\sim$ ranges over the simplicial relations: $(x, \phi_*(u)) \sim (\phi^*(x), u)$ for $\phi \in \Delta$. The key property of cyclic spaces is the fact that the geometric realization of the underlying simplicial space has a canonical action of S^1, see W.G. Dwyer, M. Hopkins and D. Kan [9] and Jones [13].

In what follows, a key role is played by the standard cyclic sets defined for $n \geq 0$ as

$$\Lambda[n] = \mathrm{Hom}_{\Lambda^{\mathrm{op}}}([n], -).$$

We denote the dual elements of the generators of the cyclic category—δ_i, σ_i and τ_n—by d_i, s_i and t_n. If an element of $\Lambda[n]$ needs to be explicitly mentioned, it will be understood as the composite of the latter. In what follows we will make use of the map $\Lambda[n]$ to $\Lambda[n-1]$, where $n \geq 1$, obtained by precomposing an element θ of $\Lambda[n]$ with the generator $s_0\colon [n-1] \to [n]$ (i.e. $\theta \mapsto \theta s_0$). Since the cyclic structure maps—d_i, s_i, t_m—are applied on the left, this map $\Lambda[n] \to \Lambda[n-1]$ is a map of cyclic sets (denoted by $(s_0)^*$ in what follows). We let ι_n denote the identity map $[n] \to [n]$ in Λ^{op}, which is a generator of $\Lambda[n]$. The realization of $\Lambda[n]$ is homeomorphic to $S^1 \times \Delta^n$, with S^1 acting by rotation of the first coordinate ([9] or [13]). There is also an action of the cyclic group $\mathbb{Z}/(n+1)$ on $\Lambda[n]$, obtained by precomposition with the cyclic operator t_n, i.e. $\theta \mapsto \theta\, t_n$. On the realization it takes the following form (J. Jones [13]):

$$\tau_n(\theta, u_0, \ldots, u_n) = (\theta - u_0, u_1, \ldots, u_n, u_0),$$

where the first coordinate θ belongs to $S^1 = \mathbb{R}/\mathbb{Z}$, with the action of the generator of $\mathbb{Z}$ on $\mathbb{R}$ given by $t \mapsto t+1$.

It follows from the simplicial relations that every element of $\Lambda[n]$ can be written as a composite $t_m^k\theta$, where $\theta \in \Delta^{\mathrm{op}}$. In particular, since there is a unique morphism $[0] \to [m]$ in Δ^{op}, $\Lambda[0]$ in dimension m is the cyclic group of order $m+1$ generated by t_m.

We define $\Lambda[-1]$ to be the constant cyclic space on a point.

The cyclic bar construction on a topological monoid was introduced by Waldhausen [21]. Let G be a topological monoid with identity element 1. Put $\Gamma_n(G) = G^{n+1} = G \times \cdots \times G$ (cartesian product of $n+1$ copies of G). We denote an element of $\Gamma_n(G)$ by $(g_0 \mid \cdots \mid g_n)$. The following maps give the collection $\Gamma_*(G)$ the structure of a cyclic space.

$$d_i(g_0 \mid \cdots \mid g_n) = \begin{cases} (g_0 \mid \cdots \mid g_i g_{i+1} \mid \cdots \mid g_n) & \text{for} \quad 0 \leq i < n, \\ (g_n g_0 | g_1 \mid \cdots \mid g_{n-1}) & \text{for} \quad i = n, \end{cases}$$

$$s_i(g_0 \mid \cdots \mid g_n) = (g_0 \mid \cdots \mid g_i \mid 1 \mid g_{i+1} \mid \cdots \mid g_n) \quad \text{for} \quad 0 \leq i \leq n,$$

$$t_n(g_0 \mid \cdots \mid g_n) = (g_n \mid g_0 \mid \cdots \mid g_{n-1}).$$

Proposition 1. *Let X be a based, connected CW-complex. There exists a canonical homotopy equivalence*

$$\phi\colon |\Gamma.J(X)| \to L\Sigma X$$

which is S^1-equivariant.

Proof. Let Y be a pathconnected pointed space and ΩY denote the monoid of Moore loops on Y. Goodwillie [11] shows that there is a canonical weak equivalence

$$\gamma\colon |\Gamma.\Omega Y| \to LY$$

which is S^1-equivariant. Applying this to $Y = \Sigma(X)$, we obtain a weak equivalence

$$|\Gamma.\Omega\Sigma(X)| \to L\Sigma(X).$$

Combining this with the weak equivalence $J(X) \to \Omega\Sigma(X)$ given by the James model [12], we obtain a weak equivalence

$$\phi\colon |\Gamma.J(X)| \to L\Sigma(X)$$

which is S^1-equivariant. Since we assume X is a CW-complex, the domain of the above map is a CW-complex. By Milnor [17], the space $L\Sigma(X)$ has the homotopy type of a CW-complex. Hence the above map is a homotopy equivalence by Whitehead's theorem. $\square$

3. A filtration of the cyclic bar construction by word length

The goal of this section is to relate $\Gamma_* J(X)$ to the standard cyclic sets $\Lambda[n]$. In order to do so, we introduce a filtration of $\Gamma_* J(X)$ by word length (since the James model $J(X)$ consists of words in X of finite length).

Let $\|w\|$ denote the length of a word w in JX. We define the total word length of an element $(w_0 \mid \cdots \mid w_m)$ in $\Gamma_m J(X)$ as $\|w_0\| + \cdots + \|w_m\|$. For $n \geq 0$, let F_n denote the subset of $\Gamma_* J(X)$ consisting of elements of total word length at most n.

Lemma 1. *For $n \geq 0$ the subset F_n is a cyclic subspace of $\Gamma_* J(X)$.*

Proof. This follows from the observation that each of the structure maps in the cyclic bar construction preserves total word length. $\square$

Hence we have a filtration:

$$F_0 \subset F_1 \subset F_2 \subset \cdots \subset \Gamma_* J(X).$$

We can describe the first two filtration elements directly. The cyclic space F_0 consists of a single element $\underbrace{(* \mid \cdots \mid *)}_{n+1}$ in dimension n. This cyclic space is isomorphic to $\Lambda[-1]$. The cyclic space F_1 has the following description. Each point $x \in X \setminus \{*\}$

generates a cyclic subset of F_1 of the following form.

$$\begin{tikzcd}
2: & (\mathbf{x} \mid * \mid *) \arrow[rr, bend right, "t_2"'] & & (* \mid \mathbf{x} \mid *) \arrow[rr, bend right, "t_2"'] & & (* \mid * \mid \mathbf{x}) \arrow[llll, bend right, "t_2"'] \\
1: & & (\mathbf{x} \mid *) \arrow[rr, leftrightarrow, bend right, "t_1"'] & & (* \mid \mathbf{x}) & \\
0: & & & (\mathbf{x}) & &
\end{tikzcd}$$

Clearly, this is a copy of $\Lambda[0]$. Inside F_1 we also have a copy of F_0 generated by the basepoint $\{*\}$. We will let the symbol X also denote the constant cyclic space generated by the CW-complex X (i.e. all structure maps are equal to the identity), and similarly for the symbol $\{*\}$. Hence F_1 is isomorphic, as a cyclic space, to the colimit of the following diagram:

$$\begin{tikzcd}
\{*\} \times \Lambda[0] \arrow[r, "\bar{\mu}_0"] \arrow[d, "\bar{\lambda}_0"'] & X \times \Lambda[0] \\
\Lambda[-1] &
\end{tikzcd}$$

where $\bar{\lambda}_0$ is the unique map to $\Lambda[-1]$ (the cyclic model for the one point space) and $\bar{\mu}_0$ is the product of the inclusion of the basepoint and the identity on $\Lambda[0]$. These maps will appear again in the second definition below. There are two reasons why the description of the filtration element F_2 needs to be somewhat more complicated. First, although a typical element (x, y) in $X \times X$ generates a copy of $\Lambda[1]$, if either x or y is the base point $*$, then the pair generates a copy of $\Lambda[0]$ accounted for in F_1. Second, given a pair of points $x, y \in X \setminus \{*\}$, the pair (x, y) generates the same cyclic subset as (y, x). In order to make the bookkeeping transparent, we introduce the category $\mathcal{L}$ and the functor G whose values are products of a number of copies of X (i.e. a constant cyclic space on the CW-complex X) and the standard cyclic set (of an appropriate dimension). The colimit operation applied to the diagram $G(\mathcal{L})$ gives the cyclic bar construction.

DEFINITION. Let $\mathcal{L}$ be the category whose objects are the following pairs of non-negative integers:

$$(n, n) \quad \text{and} \quad (n, n+1), \quad \text{where} \quad n \geq 0.$$

The morphisms are freely generated by:

- $\lambda_n \colon (n, n+1) \to (n, n)$, for $n \geq 0$,
- $\mu_n \colon (n, n+1) \to (n+1, n+1)$, for $n \geq 0$,

- $\gamma_{n-1}\colon (n,n) \to (n,n)$, for $n \geq 2$,

modulo the relation $(\gamma_{n-1})^n = \mathrm{id}$ for $n \geq 2$.

The initial part of $\mathcal{L}$ is described by the following diagram:

$$\begin{tikzcd}[column sep=small]
& (0,1) \arrow[dl, "\lambda_0"'] \arrow[dr, "\mu_0"] & & (1,2) \arrow[dl, "\lambda_1"'] \arrow[dr, "\mu_1"] & & (2,3) \arrow[dl, "\lambda_2"'] \arrow[dr, "\mu_2"] & & \cdots \arrow[dl, "\lambda_3"'] \\
(0,0) & & (1,1) & & (2,2) \arrow[loop below, "\gamma_1"] & & (3,3) \arrow[loop below, "\gamma_2"] &
\end{tikzcd}$$

DEFINITION. Let $G\colon \mathcal{L} \to \mathrm{Top}^{\Lambda^{\mathrm{op}}}$ be the functor defined as follows:

$$G(n,m) = X^n \times \Lambda[m-1],$$

where X^n denotes the constant cyclic space on the CW-complex X^n (i.e. with identity structure maps). On morphisms, we set $G(\lambda_0) = \bar{\lambda}_0$, $G(\mu_0) = \bar{\mu}_0$; for $n \geq 1$, $\theta \in \Lambda[n]$:

$$G(\lambda_n)(x_0,\ldots,x_{n-1},\theta) = (x_0,\ldots,x_{n-1},\theta s_0),$$
$$G(\mu_n)(x_0,\ldots,x_{n-1},\theta) = (x_0,*,x_1,\ldots,x_{n-1};\theta);$$

and for $n \geq 2$ and $\phi \in \Lambda[n-1]$:

$$G(\gamma_{n-1})(x_0,x_1,\ldots,x_{n-1},\phi) = (x_1,\ldots,x_{n-1},x_0,\phi t_{n-1}).$$

Note that the cyclic structure on $X^n \times \Lambda[n-1]$ and $X^n \times \Lambda[n]$ is obtained by applying the cyclic operators (d_i, s_i, t_n) on the *left* to the last coordinate (i.e. θ and ϕ in the formulas above). The maps $G(\lambda_n)$ and $G(\gamma_{n-1})$ are defined using the action of the cyclic operators s_0 and t_{n-1} on the right. For this reason these are maps of cyclic spaces.

Theorem 1. *The colimit of the functor G is homeomorphic to the cyclic bar construction on $J(X)$:*

$$\mathtt{colim}_{\mathcal{L}}\, G \approx \Gamma_* J(X).$$

Moreover, if we restrict to the N-skeleton of $\mathcal{L}$ (i.e. pairs with both entries less than or equal to N), the colimit is isomorphic to F_N, the N-th word length filtration of $\Gamma_* J(X)$.

Proof. The proof is based on the following three observations:

- For $n \geq 1$, the element $(x_0, \ldots, x_{n-1}, \iota_{n-1}) \in X^n \times \Lambda[n-1]_{n-1}$ represents $(x_0 \mid \cdots \mid x_{n-1}) \in \Gamma_{n-1} J(X)$.
- Face maps applied to this element represent removal of bars. (Simplicial dimension in $\Lambda[n-1]$ corresponds to dimension in $\Gamma_* J(X)$.)
- The role of the elements in the upper row of the diagram (cyclic spaces $G(n, n+1)$) is to ensure that elements of the form $(x_0, \ldots, x_{n-1}, \iota_n)$ with some $x_i = *$ for $i > 0$ represent degenerate simplices in the colimit.

In order to establish a homeomorphism of the colimit with $\Gamma_* J(X)$ (in each dimension), for $n \geq 0$ we construct maps of cyclic spaces

$$\alpha_n \colon G(n, n) \to \Gamma_* J(X),$$
$$\beta_n \colon G(n, n+1) \to \Gamma_* J(X),$$

which are compatible with the maps in the diagram $G(\mathcal{L})$.

We set $\alpha_0(*, \iota_{-1}) = (*)$ and $\beta_0(*, \iota_0) = (*)$. For $n \geq 1$, $\phi \in \Lambda[n-1]$ and $\theta \in \Lambda[n]$ we let

$$\alpha_n(x_0, \ldots, x_{n-1}, \phi) = \phi(x_0 \mid \cdots \mid x_{n-1}),$$
$$\beta_n(x_0, \ldots, x_{n-1}, \theta) = \theta(x_0 \mid * \mid x_1 \mid \cdots \mid x_{n-1}).$$

First we need to check that the following diagram commutes where $n \geq 1$ (the commutativity of the $n = 0$ analogue of this diagram is obvious):

$$\begin{tikzcd}[column sep=small]
 & X^n \times \Lambda[n] \arrow[dl, "G(\lambda_n)"'] \arrow[dr, "G(\mu_n)"] & \\
X^n \times \Lambda[n-1] \arrow[dr, "\alpha_n"'] & & X^{n+1} \times \Lambda[n] \arrow[dl, "\alpha_{n+1}"] \\
 & \Gamma_* J(X) &
\end{tikzcd}$$

$$\begin{aligned}
\alpha_n G(\lambda_n)(x_0, \ldots, x_{n-1}, \theta) &= \alpha_n(x_0, \ldots, x_{n-1}, \theta s_0) \\
&= \alpha_n \theta s_0(x_0, \ldots, x_{n-1}, \iota_{n-1}) \\
&= \theta s_0 \alpha_n(x_0, \ldots, x_{n-1}, \iota_{n-1}) \\
&= \theta s_0(x_0 \mid \cdots \mid x_{n-1}) \\
&= \theta(x_0 \mid * \mid x_1 \mid \cdots \mid x_{n-1}), \\
\alpha_{n+1} G(\mu_n)(x_0, \ldots, x_{n-1}, \theta) &= \alpha_{n+1}(x_0, *, x_1, \ldots, \ldots, x_{n-1}, \theta) \\
&= \theta(x_0 \mid * \mid x_1 \mid \cdots \mid x_{n-1}).
\end{aligned}$$

Moreover, each of the above composites must be equal to β_n, which it clearly is.

Second, we need to check that the following diagram commutes for $n \geq 2$:

$$
\begin{tikzcd}
X^n \times \Lambda[n-1] \arrow[rr, "G(\gamma_{n-1})"] \arrow[dr, "\alpha_n"'] & & X^n \times \Lambda[n-1] \arrow[dl, "\alpha_n"] \\
& \Gamma_* J(X) &
\end{tikzcd}
$$

$$
\begin{aligned}
\alpha_n G(\gamma_{n-1})(x_0, \ldots, x_{n-1}, \phi) &= \alpha_n(x_1, \ldots, x_{n-1}, x_0, \phi t_{n-1}) \\
&= \alpha_n(\phi t_{n-1}(x_1, \ldots, x_{n-1}, x_0, \iota_{n-1})) \\
&= \phi t_{n-1}\alpha_n(x_1, \ldots, x_{n-1}, x_0, \iota_{n-1}) \\
&= \phi t_{n-1}(x_1 \mid \cdots \mid x_{n-1}|x_0) \\
&= \phi(x_0 \mid \cdots \mid x_{n-1}) = \alpha_n(x_0, \ldots, x_{n-1}, \phi).
\end{aligned}
$$

Hence the maps α_n and β_n assemble to a map from the diagram $G(\mathcal{L})$ to $\Gamma_* J(X)$, and therefore induce a unique map

$$\alpha\colon \mathtt{colim}_{\mathcal{L}}\, G \to \Gamma_* J(X).$$

Since every nondegenerate element in $\Gamma_* J(X)$ is an iterated face map of an element of the form $(x_0 \mid \cdots \mid x_{n-1})$, we conclude that the map α is onto.

It is a little harder to show that the map α is one to one. We will call two elements

$$(x_0, \ldots, x_{n-1}, \phi) \in X^n \times \Lambda[n-1] \quad \text{and} \quad (y_0, \ldots, y_{m-1}, \psi) \in X^m \times \Lambda[m-1]$$

equivalent if they determine the same element in the colimit. Hence we need to show that if two elements as above map by α to the same element in $\Gamma_* J(X)$, then they are equivalent.

First, note that if $(x_0, \ldots, x_{n-1}, \phi)$ is such that $x_0 = \cdots = x_{n-1} = *$, then the equality $\alpha_n(x_0, \ldots, x_{n-1}, \phi) = \alpha_m(y_0, \ldots, y_{m-1}, \psi)$ implies $y_0 = \cdots = y_{m-1} = *$ and the two tuples determine the same element in the colimit, i.e. are equivalent.

Hence from now on we assume that at least one x_i and at least one y_j is different than $*$. Therefore, by changing ϕ or ψ if necessary (replacing a given x-tuple or y-tuple by an equivalent one), we may assume $x_0 \neq *$ and $y_0 \neq *$.

Next, replacing a given x-tuple or a given y-tuple by an equivalent one if necessary, we may assume that all the x_i and all the y_j are different than $*$. (Essentially, the role of the diagram $G(\mathcal{L})$ is to allow replacement of tuples containing $*$ in position other than the first from the left by equivalent shorter tuples.)

Hence the proof of injectivity reduces to showing that for arbitrary $(x_0, \ldots, x_{n-1}, \phi)$ and $(y_0, \ldots, y_{m-1}, \psi)$ with all x_i and y_j different than $*$ if we have

$$\alpha_n(x_0, \ldots, x_{n-1}, \phi) = \alpha_m(y_0, \ldots, y_{m-1}, \psi),$$

then $(x_0, \dots, x_{n-1}, \phi)$ and $(y_0, \dots, y_{m-1}, \psi)$ are equivalent.

By definition of α_n, the above equality can be restated as

$$\phi(x_0 \mid \cdots \mid x_{n-1}) = \psi(y_0 \mid \cdots \mid y_{m-1}).$$

It is not hard to check that any element in the cyclic category Λ^{op} can be uniquely written as TSD where T is some power of a cyclic operator, S is a composite of degeneracy maps, and D is a composite of face maps (for details see Loday [14], 6.1.8 and May [16], formula (3) on p. 4).

Hence we can write

$$\phi = T_x S_x D_x \quad \text{and} \quad \psi = T_y S_y D_y.$$

By applying an appropriate power of $G(\gamma_{n-1})$, we can replace the above x-tuple by an equivalent one in which T_x is the identity. Similarly for the y-tuple. We will retain the earlier notation, i.e. we have two tuples $(x_0,\dots,x_{n-1},S_x D_x)$ and $(y_0,\dots,y_{m-1},S_y D_y)$ which α maps to the same element in $\Gamma_* J(X)$. Since the coordinates equal to $*$ in the image $\Gamma_* J(X)$ are determined by S_x and S_y, we conclude that these degeneracy maps must be equal.

It remains to show that if

$$(x_0, \dots, x_{n-1}, D_x) \quad \text{and} \quad (y_0, \dots, y_{m-1}, D_y)$$

determine the same element in $\Gamma_* J(X)$, then they must be equivalent.

Since none of the x_i or y_j equals the base point $*$, we must have $n = m$, $x_i = y_i$ for $i = 0, \dots, n$, and finally $D_x = D_y$.

We conclude that the original x-tuple and y-tuple are also equivalent.

We have obtained a continuous bijection $\alpha\colon \mathtt{colim}_{\mathcal{L}}\, G \to \Gamma_* J(X)$ (i.e. this map is a continuous bijection in each simplicial dimension). Since we work in the category of compactly generated Hausdorff spaces, to conclude that it is a homeomorphism, it is enough to establish that it is proper, i.e. inverse images of compact sets are compact (see e.g. N.P. Strickland [19], Proposition 3.17). Fix a simplicial dimension $p \geq 0$. By Lemma 9.3 in Steenrod [18], a compact subset C of $\Gamma_* J(X)_p$ is contained in a filtration element F_m for some nonnegative integer m. Next, observe that the CW-structure of X determines a canonical CW-structure in $(F_m)_p$, as well as $\mathtt{colim}_{\mathcal{L},n,k\leq m}\, G(n,k)$. Moreover, the map α establishes a bijection on the cells of the two structures, hence it is a proper map. $\square$

Note that the cyclic subspace generated by $\alpha_n(x_0, \dots, x_{n-1}, \iota_{n-1})$ need not be isomorphic to the standard cyclic set $\Lambda[n-1]$. Namely, if some of the elements $(x_0,\dots,x_{n-1})$ coincide, then this cyclic subspace will be isomorphic to the quotient of the standard cyclic set $\Lambda[n-1]$ by a subgroup of $\mathbb{Z}/n$ (the action given by precomposition with t_{n-1}).

This interesting property of cyclic spaces does not have a counterpart in the setting of simplicial spaces.

For example, if x_0 is not the basepoint $*$, the cyclic subset $\alpha_2(x_0, x_0, \iota_1)$ is homeomorphic to $\Lambda[1]/(\mathbb{Z}/2)$, whose realization is homeomorphic to the Möbius band. More precisely, $\Lambda[1]/(\mathbb{Z}/2)$ is a cyclic set with a single element in dimension zero, two nondegenerate elements in dimension 1 and a single nondegenerate element in dimension 2. The cyclic operator t_1 switches the degenerate and one nondegenerate element, and these give a copy of a circle with a free S^1-action in the realization. The remaining nondegenerate element in dimension one is fixed by the cyclic operator t_1, and this element generates a copy of a circle in the realization which is fixed by the copy of $\mathbb{Z}/2$ sitting inside the group S^1.

The geometric realization is a left adjoint, hence commutes with colimits (see e.g. S. Mac Lane, Chapter 5, §5, [15]). Hence we have the following corollary.

Corollary 1. *The free loopspace on the suspension of a based CW-complex has the homotopy type of the colimit of the following diagram*:

$$\begin{tikzcd}
 & \{*\} \times S^1 \arrow[dl] \arrow[dr] & & X \times S^1 \times \Delta^1 \arrow[dl] \arrow[dr] & & X^2 \times S^1 \times \Delta^2 \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \\
\{*\} & & X \times S^1 & & X^2 \times S^1 \times \Delta^1 \arrow[loop below] & & X^3 \times S^1 \times \Delta^2 \arrow[loop below] &
\end{tikzcd}$$

The above model is related to the configuration space model (or the May–Milgram–McDuff-Bödigheimer model) in the following way. Let X be a connected CW-complex with basepoint $*$ and S_n be the symmetric group on n elements. By Bödigheimer [3] or Carlsson–Cohen [5] we have, up to homotopy,

$$L\,\Sigma(X) \simeq \left(\coprod_{n=0}^{\infty} \tilde{C}_n(S^1) \times_{S_n} X^n \right) \Big/ \sim$$

where $\tilde{C}_n(S^1)$ is the configuration space of ordered n-tuples of pairwise distinct points on the circle and $\sim$ is the standard basepoint condition:

$$(m_1, \ldots, m_n) \times_{S_n} (x_1, \ldots, x_n) \sim (m_1, \ldots, \hat{m}_i, \ldots, m_n) \times_{S_n} (x_1, \ldots, \hat{x}_i, \ldots, x_n),$$

if $x_i = *$. We can observe that $\tilde{C}_n(S^1)$ is homeomorphic to $S^1 \times \tilde{C}_{n-1}(I)$, where I denotes the open unit interval. Moreover, $\tilde{C}_{n-1}(I)/S_{n-1} = \Delta_o^{n-1}$ (see e.g. C. Westerland [22]). However here Δ_o^{n-1} denotes the open topological n-simplex, not the closed simplex Δ^n in our description.

4. The Borel construction

Let G be a topological group and X a space on which G acts. A. Borel [2] introduces the space

$$X_{hG} = EG \times_G X.$$

This space is currently referred to as the Borel construction or the homotopy orbit space of X by G. This space encodes the way G acts on X and invariants of this space can be used to study the action. For an interesting short modern account, see Dwyer and H.-W. Henn [8].

The main properties are as follows:

- The Borel construction on a point is the classifying space: $\{*\}_{hG} = BG$.
- There is a fibration sequence $X \to X_{hG} \to BG$.
- A G-map $f\colon X \to Y$ which is a homotopy equivalence induces a homotopy equivalence of the Borel constructions $f_{hG}\colon X_{hG} \to Y_{hG}$.

For our next proposition, we need the following two standard results.

Theorem 2. *The pushout of the diagram $A \leftarrow B \to C$ in which one of the two maps* (*i.e.* $B \to A$ *or* $B \to C$) *is a cofibration is a homotopy pushout* (*i.e. such a diagram is cofibrant*).

Theorem 3. *Given a group G, a free G-CW-complex is a cofibrant diagram in* Top^G, *and hence the homotopy orbit space coincides with the ordinary orbit space.*

For a proof of the above, see e.g. J. Strom [20], Theorem 7.11 and Theorem 8.72.

Proposition 2. *The colimit which describes $\Gamma_* J(X)$ in* Theorem 1 *is in fact a homotopy colimit, i.e.*

$$\mathtt{colim}_{\mathcal{L}}\, G \overset{w}{\simeq} \mathtt{hocolim}_{\mathcal{L}}\, G.$$

Proof. Let $G_n = \mathtt{colim}_{(k,l)\in\mathcal{L},k,l\leq n}\, G(k,l)$. Clearly, G_{n+1} is the colimit of the following diagram:

$$\begin{tikzcd} & G(n,n+1) \arrow[dl, "G(\lambda_n)"'] \arrow[dr, "G(\mu_n)"] & \\ G_n & & G(n+1,n+1) \arrow[loop below, "\gamma_2"] \end{tikzcd}$$

and there are natural maps $G_n \to G_{n+1}$ induced by the colimit. Moreover,

$$\mathtt{colim}_{\mathcal{L}}\, G \overset{w}{\simeq} \mathtt{colim}(G_0 \to G_1 \to G_2 \to \cdots).$$

We know that a colimit over a sequential category is in fact a homotopy colimit (see e.g. [8]). Hence

$$\mathtt{colim}_{\mathcal{L}}\, G \overset{w}{\simeq} \mathtt{hocolim}(G_0 \to G_1 \to G_2 \to \cdots).$$

Next, we observe that the colimit diagram representing G_{n+1} above can be decomposed into two diagrams: a pushout and a diagram representing a cyclic group action (via γ_n). Since the right diagonal arrow in the above diagram is a cofibration (a product of the basepoint inclusion with a number of identity maps), this pushout is in fact a homotopy pushout. Next, since the action of $\mathbb{Z}/(n+1)$ on the result of the pushout is free (the action is free on the $\Lambda[n]$ factor), we conclude that the colimit diagram defining G_{n+1} in terms of G_n is in fact a homotopy colimit. Hence we see that the calculation of $\mathtt{colim}_{\mathcal{L}}\, G$ can be decomposed into three stages: a pushout, colimit over a diagram representing a free finite group action, a sequential colimit. In our specific case we conclude that these are homotopy colimits, and hence the original colimit is in fact a homotopy colimit. $\square$

We need the following two preliminary results.

Theorem 4 (Fiedorowicz and Loday, 5.12 in [10])**.** *For any cyclic space X there is a natural weak equivalence*:

$$\mathtt{hocolim}_{\Lambda^{\mathrm{op}}}\, X \overset{w}{\simeq} ES^1 \times_{S^1} |X|.$$

Theorem 5. *Let $\mathbf{C}$ and $\mathbf{C}'$ be small categories and F be a $\mathbf{C} \times \mathbf{C}'$ diagram of spaces, i.e. a functor $F\colon \mathbf{C} \times \mathbf{C}' \to \mathrm{Top}$. Then*

$$\mathtt{hocolim}_{\mathbf{C}}\, \mathtt{hocolim}_{\mathbf{C}'}\, F \overset{w}{\simeq} \mathtt{hocolim}_{\mathbf{C}'}\, \mathtt{hocolim}_{\mathbf{C}}\, F.$$

For a proof see e.g. D.J. Benson and S.D. Smith [1], Theorem 4.5.20. Now we can make the following calculation.

$$\begin{aligned} |\Gamma_* J(X)|_{hS^1} &= ES^1 \times_{S^1} |\Gamma_* J(X)| \\ &\overset{w}{\simeq} \mathtt{hocolim}_{\Lambda^{\mathrm{op}}}\, \Gamma_* J(X) \\ &\overset{w}{\simeq} \mathtt{hocolim}_{\Lambda^{\mathrm{op}}}\, \mathtt{hocolim}_{(n,m)\in\mathcal{L}}\, G(n,m) \\ &\overset{w}{\simeq} \mathtt{colim}_{(n,m)\in\mathcal{L}}\, \mathtt{hocolim}_{\Lambda^{\mathrm{op}}}\, G(n,m). \end{aligned}$$

The first weak equivalence above follows from the above theorem of Fiedorowicz and Loday, the second from our description of the cyclic bar construction at the beginning of this section and the third from the commutativity of iterated homotopy colimits stated in the theorem immediately above.

We let $\tilde{G}(n, m) = \mathtt{hocolim}_{\Lambda^{op}} G(n, m)$. Observe that for $m \geq 1$ we have:

$$\begin{aligned}\tilde{G}(n, m) &\overset{w}{\simeq} ES^1 \times_{S^1} |G(n, m)| \\ &\overset{w}{\simeq} ES^1 \times_{S^1} (X^n \times |\Lambda[m-1]|) \\ &\overset{w}{\simeq} ES^1 \times_{S^1} (X^n \times S^1 \times \Delta^{m-1}) \\ &\overset{w}{\simeq} ES^1 \times X^n \times \Delta^{m-1} \\ &\overset{w}{\simeq} X^n \times \Delta^{m-1}.\end{aligned}$$

Here is the justification of each weak equivalence above:

1. The theorem of Fiedorowicz and Loday.
2. The definition of $G(n, m)$.
3. The known structure of the realization of the standard cyclic set stated in the introduction (Proposition 2.7 in [9] or Theorem 3.4 in [13]).
4. The realization of the standard cyclic set has a free S^1-action (again Proposition 2.7 in [9] or Theorem 3.4 in [13]).
5. Contractibility of ES^1.

Moreover, we have

$$\tilde{G}(0, 0) \overset{w}{\simeq} ES^1 \times_{S^1} |G(0, 0)| \overset{w}{\simeq} ES^1 \times_{S^1} * \overset{w}{\simeq} BS^1.$$

For $n \geq 1$, the map $\tilde{G}(\lambda_n)\colon \tilde{G}(n, n+1) \to \tilde{G}(n, n)$ is the map

$$X^n \times \Delta^n \to X^n \times \Delta^{n-1}$$

induced by s_0^*, i.e. is the product of the identity map on X^n and the affine map from the simplex to itself which identifies the 0-th and the 1-st simplex.

The map $\tilde{G}(\lambda_0)\colon \tilde{G}(0, 1) \to \tilde{G}(0, 0)$ is a map sending the basepoint $*$ to BS^1.

For $n \geq 1$, the map $\tilde{G}(\mu_n)\colon \tilde{G}(n, n+1) \to \tilde{G}(n+1, n+1)$ is the map

$$X^n \times \Delta^n \to X^{n+1} \times \Delta^n$$

which is the product of the inclusion map $X^n \to X^{n+1}$, $(x_0, \ldots, x_{n-1}) \mapsto (x_0, *, x_1, \ldots, x_{n-1})$ and the identity map on Δ^n.

The map $\tilde{G}(\mu_0)\colon \tilde{G}(0,1) \to \tilde{G}(1,1)$ is a map sending the basepoint $*$ to the natural basepoint of $X \times \Delta^0$.

For $n \geq 2$, the map $\tilde{G}(\gamma_{n-1})\colon \tilde{G}(n, n) \to \tilde{G}(n, n)$ is the map

$$X^n \times \Delta^{n-1} \to X^n \times \Delta^{n-1}$$

which is

$$(x_0, \dots, x_{n-1}, u_0, \dots, u_{n-1}) \mapsto (x_1, \dots, x_{n-1}, x_0, u_1, \dots, u_{n-1}, u_0).$$

We have established the following result.

Theorem 6. *For a connected CW-complex X with basepoint $*$ the Borel construction on the free loopspace of the suspension has the following description*:

$$ES^1 \times_{S^1} L\Sigma(X) \overset{w}{\simeq} BS^1 \vee \mathtt{hocolim}_{(n,m)\in\mathcal{L},\, n,m\geq 1} X^n \times \Delta^{m-1}$$

where the maps between the various $X^n \times \Delta^{m-1}$ are the ones described immediately above.

References

[1] D.J. Benson and S.D. Smith: Classifying Spaces of Sporadic Groups, Mathematical Surveys and Monographs **147**, Amer. Math. Soc., Providence, RI, 2008.

[2] A. Borel: Seminar on transformation groups, Annals of Mathematics Studies **46**, Princeton Univ. Press, Princeton, NJ, 1960.

[3] C.-F. Bödigheimer: *Stable splittings of mapping spaces*; in Algebraic Topology (Seattle, Wash., 1985), Lecture Notes in Math. **1286**, Springer, Berlin, 1987, 174–187.

[4] C.-F. Bödigheimer and I. Madsen: *Homotopy quotients of mapping spaces and their stable splitting*, Quart. J. Math. Oxford Ser. (2) **39** (1988), 401–409.

[5] G.E. Carlsson and R.L. Cohen: *The cyclic groups and the free loop space*, Comment. Math. Helv. **62** (1987), 423–449.

[6] R.L. Cohen: *A model for the free loop space of a suspension*; in Algebraic Topology (Seattle, Wash., 1985), Lecture Notes in Math. **1286**, Springer, Berlin, 1987, 193–207.

[7] A. Connes: *Cohomologie cyclique et foncteurs* Ext^n, C.R. Acad. Sci. Paris Sér. I Math. **296** (1983), 953–958.

[8] W.G. Dwyer and H.-W. Henn: Homotopy Theoretic Methods in Group Cohomology, Advanced Courses in Mathematics, CRM Barcelona, Birkhäuser, Basel, 2001.

[9] W.G. Dwyer, M.J. Hopkins and D.M. Kan: *The homotopy theory of cyclic sets*, Trans. Amer. Math. Soc. **291** (1985), 281–289.

[10] Z. Fiedorowicz and J.-L. Loday: *Crossed simplicial groups and their associated homology*, Trans. Amer. Math. Soc. **326** (1991), 57–87.

[11] T.G. Goodwillie: *Cyclic homology, derivations, and the free loopspace*, Topology **24** (1985), 187–215.

[12] I.M. James: *Reduced product spaces*, Ann. of Math. (2) **62** (1955), 170–197.

[13] J.D.S. Jones: *Cyclic homology and equivariant homology*, Invent. Math. **87** (1987), 403–423.

[14] J.-L. Loday: Cyclic Homology, Springer, Berlin, 1992.

[15] S. MacLane: Categories for the Working Mathematician, Springer, New York, 1971.

[16] J.P. May: Simplicial Objects in Algebraic Topology, Univ. of Chicago Press, Chicago, 1967.

[17] J. Milnor: *On spaces having the homotopy type of a CW-complex*, Trans. Amer. Math. Soc. **90** (1959), 272–280.

[18] N.E. Steenrod: *A convenient category of topological spaces*, Michigan Math. J. **14** (1967), 133–152.

[19] N.P. Strickland: *The category of CGWH spaces*, preprint, 2009.

[20] J. Strom: Modern Classical Homotopy Theory, Graduate Studies in Mathematics **127**, Amer. Math. Soc., Providence, RI, 2011.

[21] F. Waldhausen: *Algebraic K-theory of topological spaces* II; in Algebraic Topology, Aarhus 1978 (Proc. Sympos., Univ. Aarhus, Aarhus, 1978), Lecture Notes in Math. **763**, Springer, Berlin, 1979, 356–394.

[22] C. Westerland: *Configuration spaces in topology and geometry*, Austral. Math. Soc. Gaz. **38** (2011), 279–283.

Faculty of Mathematics and Information Science
Warsaw University of Technology
Koszykowa 75, 00-662 Warsaw
Poland
e-mail: jan.spalinski@gmail.com