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*Osaka University*
ALMOST RELATIVE INJECTIVE MODULES

SURJEET SINGH

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Abstract

The concept of a module $M$ being almost $N$-injective, where $N$ is some module, was introduced by Baba (1989). For a given module $M$, the class of modules $N$, for which $M$ is almost $N$-injective, is not closed under direct sums. Baba gave a necessary and sufficient condition under which a uniform, finite length module $U$ is almost $V$-injective, where $V$ is a finite direct sum of uniform, finite length modules, in terms of extending properties of simple submodules of $V$. Let $M$ be a uniform module and $V$ be a finite direct sum of indecomposable modules. Some conditions under which $M$ is almost $V$-injective are determined, thereby Baba’s result is generalized. A module $M$ that is almost $M$-injective is called an almost self-injective module. Commutative indecomposable rings and von Neumann regular rings that are almost self-injective are studied. It is proved that any minimal right ideal of a von Neumann regular, almost right self-injective ring, is injective. This result is used to give an example of a von Neumann regular ring that is not almost right self-injective.

Introduction

Let $M_R$, $N_R$ be two modules. As defined by Baba [4], $M$ is said to be almost $N$-injective, if for any homomorphism $f: A \to M$, $A \subseteq N$, either $f$ extends to a homomorphism $g: N \to M$ or there exist a decomposition $N = N_1 \oplus N_2$ with $N_1 \neq 0$ and a homomorphism $h: M \to N_1$ such that $hf(x) = \pi(x)$ for any $x \in A$, where $\pi: N \to N_1$ is a projection with kernel $N_2$. A module $M$ that is almost $M$-injective, is called an almost self-injective module. For a module $M$, the class of those modules $N$ for which $M$ is almost $N$-injective, is not closed under direct sums. Let $\{U_k: 0 \leq k \leq n\}$ be a finite family of uniform modules of finite composition lengths, and $U = \bigoplus \sum_{k=1}^{n} U_k$. Baba [4] has given a characterization for $U_0$ to be almost $U$-injective in terms of the property of simple submodules of $U$ being contained in uniform summands of $U$. Let $M$ be a uniform module and $V$ be a finite direct sum of indecomposable modules. In Section 1, we investigate conditions under which $M$ is almost $V$-injective. The main result is given in Theorem 1.12 and it generalizes the result by Baba. An alternative short proof of a result by Harada [10] is given in Theorem 1.16. It is well known that a (commutative) integral domain $R$ is almost self-injective if and only if it is a valuation domain. Let $R$ be a commutative ring having no non-trivial idempotent and $Q$ be its classical quotient ring. In Section 2, it is proved that $R_Q$ is almost self-injective.
if and only if for any elements \( a, b \in R \) with \( \text{ann}(a) \subseteq \text{ann}(b) \), either \( bR \subseteq aR \) or \( aR \preceq bR \) with \( a = bc \) for some regular element \( c \), and \( Q_R \) is injective and uniform. It follows that any commutative, indecomposable ring \( R \) that is almost self-injective but not self-injective, is local. In Section 3, von Neumann regular rings \( R \) with \( R_R \) almost self-injective are studied. A characterization of such rings is given in Theorem 3.1. It is proved that any von Neumann regular ring \( R \) that is either commutative or right \( C_S \) is almost right self-injective. In Theorem 3.4, it is proved that any minimal right ideal of a von Neumann regular ring \( R \) that is almost right self-injective, is injective. This result is used to give an example of a von Neumann regular ring that is not almost right self-injective.

**Preliminaries**

All rings considered here are with unity and all modules are unital right modules unless otherwise stated. Let \( M \) a module. Then \( E(M) \), \( J(M) \) denote its injective hull, radical respectively. The symbols \( N \leq M \), \( N < M \), \( N \subseteq e \) \( M \) denote that \( N \) is a submodule of \( M \), \( N \) is a submodule different from \( M \), \( N \) is an essential submodule of \( M \) respectively. A module \( M \) whose ring of endomorphisms \( \text{End}(M) \) is local, is called a \( LE \) module. A module \( M \) such that its complement submodules are summands of \( M \), is called a \( CS \) module (or a module satisfying condition \((C_1)\)). If a module \( M \) is such that for any two summands \( A, B \) of \( M \) with \( A \cap B = 0 \), \( A + B \) is a summand of \( M \), then it is said to satisfy condition \((C_3)\). A module \( M \) satisfying conditions \((C_1)\), \((C_3)\) is called a quasi-continuous module. The terminology used here is available in standard text books like [3], [6].

**1. Direct sums of uniform modules**

**Definition 1.1.** Let \( M_R \) and \( N_R \) be any two modules. Then \( M \) is said to be almost \( N \)-injective, if given any \( R \)-homomorphism \( f: A \to M \), \( A \leq N \) either \( f \) extends to an \( R \)-homomorphism from \( N \) to \( M \) or there exist a decomposition \( N = N_1 \oplus N_2 \) with \( N_1 \neq 0 \), and an \( R \)-homomorphism \( h: M \to N_1 \) such that \( hf(x) = \pi(x) \) for any \( x \in A \), where \( \pi: N \to N_1 \) is a projection with kernel \( N_2 \).

One can easily prove the following two results. (See [2])

**Proposition 1.2.** (i) A module \( M_R \) is almost \( N_R \)-injective, if and only if for any \( R \)-homomorphism \( f: L \to M \), \( L \preceq N \) which is maximal with respect to the property that it cannot be extended from \( N \) to \( M \), there exist a decomposition \( N = N_1 \oplus N_2 \) with \( N_1 \neq 0 \), and an \( R \)-homomorphism \( h: M \to N_1 \) such that \( hf(x) = \pi(x) \) for any \( x \in L \), where \( \pi: N \to N_1 \) is a projection with kernel \( N_2 \).

(ii) If a module \( M \) is almost \( N \)-injective and \( N \) is indecomposable, then any \( R \)-homomorphism \( f: L \to M \), \( L \subseteq e \) \( N \) with \( \ker f \neq 0 \) extends to an \( R \)-homomorphism from \( N \) to \( M \).
**Proposition 1.3.** Let $A_R, B_R$ any two modules and $f : L \to B$, $L < A$ be an $R$-homomorphism that is maximal with respect to the property that it cannot be extended from $A$ to $B$. If $C$ is a summand of $A$ and $L \cap C < C$, then $f_1 = f \mid L \cap C$ from $L \cap C$ to $B$ is a maximal homomorphism that cannot be extended from $C$ to $B$.

The following is well known. (See [12])

**Proposition 1.4.** Let $M_R, N_R$ be any two modules such that $M$ is almost $N$-injective.
(i) Any summand $K$ of $M$ is almost $N$-injective.
(ii) If $W$ is a summand of $N$, then $M$ is almost $W$-injective.
(iii) If $N = N_1 \oplus N_2$ and $M$ is not $N$-injective, then $M$ is either not $N_1$-injective or not $N_2$-injective.

**Lemma 1.5.** Let $M_R$ and $N_R$ be any two modules such that $M$ is almost $N$-injective, and $f : L \to M$, $L < N$ be a maximal homomorphism which cannot be extended from $N$ to $M$. Let $N = N_1 \oplus N_2$ with $N_1 \neq 0$ and $h : M \to N_1$ be a homomorphisms such that $hf(x) = \pi(x)$ for $x \in L$, where $\pi : N \to N_1$ is a projection with kernel $N_2$. Then the following hold.
(i) $f$ is monic on $L \cap N_1$ and $f(L \cap N_1)$ is a closed submodule of $M$.
(ii) $\ker h$ is a complement of $f(N_1 \cap L)$.
(iii) $f(N_2 \cap L) \subseteq \ker h$.
(iv) If $M$ is a CS module, then $f(N_1 \cap L)$ and $\ker h$ are summands of $M$.

Proof. (i) Now $hf(x) = x$ for any $x \in L \cap N_1$, which gives $f(L \cap N_1) \cap \ker h = 0$. We get a complement $H$ of $\ker h$ containing $f(L \cap N_1)$. Then $h \mid H$ is monic and $N_1 \cap L \subseteq h(H) \subseteq N_1$. Define $\lambda : h(H) \to H$, $\lambda(h(y)) = y$ for any $y \in H$. Then $\lambda$ extends $f \mid (L \cap N_1)$. By Proposition 1.3, $h(H) = L \cap N_1$. Which proves that $f(N_1 \cap L) = H$. Hence $f(L \cap N_1)$ is a closed submodule of $M$ and is a complement of $\ker h$.
(ii) Let $K$ be a complement of $f(N_1 \cap L)$ containing $\ker h$. Then $\ker h \subseteq K$. Let $x \in K$. Suppose $h(x) \neq 0$. As $h(x) \in N_1$, there exists an $r \in R$ such that $0 \neq h(xr) \in L \cap N_1$. Thus $h(xr) = h(y)$ for some $y \in f(L \cap N_1)$, $xr - y \in \ker h \subseteq K$. Which gives $y \in K \cap f(L \cap N_1) = 0$. Therefore, $h(xr) = h(y) = 0$, which is a contradiction. Hence $K = \ker h$.

The last two parts are obvious.

**Theorem 1.6.** Let $M_R$ be a quasi-continuous module and $N_R$ any module. Then $M$ is almost $N$-injective if and only if for any homomorphism $f : L \to M$, $L < N$ which is maximal such that it cannot be extended to a homomorphism from $N$ to $M$, the following hold.
(i) There exist decompositions $N = N_1 \oplus N_2$, $M = M_1 \oplus M_2$ with $N_1 \neq 0$.
(ii) $f$ is monic on $L \cap N_1$ and $f(N_1 \cap L) = M_1$. 

\[\square\]
(iii) \( f(N_2 \cap L) \subseteq M_2 \).
(iv) \( L = (L \cap N_1) \oplus (L \cap N_2) \).

Proof. (i) Let \( M \) be almost \( N \)-injective. By Lemma 1.5, there exist a decomposition \( N = N_1 \oplus N_2 \) and a homomorphism \( h: M \to N_1 \) such that \( N_1 \neq 0 \), \( f \) is monic on \( N_1 \cap L \), \( M_1 = f(N_1 \cap L) \) and \( M_2 = \ker h \) are summands of \( M \), and \( hf(x) = \pi(x) \) for \( x \in L \), where \( \pi: N \to N_1 \) is a projection with kernel \( N_2 \). As \( M_1, M_2 \) are complements of each other and \( M \) satisfies \((C_1)\), we get \( M = M_1 \oplus M_2 \). Thus \( h(M) = h(M_1) \).

(ii) It is proved in Lemma 1.5.

(iii) Let \( z \in L \). Then \( z = x_1 + x_2 \) for some \( x_1 \in N_1, x_2 \in N_2 \). Then \( x_1 = hf(z) \in h(M_1) = hf(N_1 \cap L) = N_1 \cap L \), which also gives \( x_2 \in N_2 \cap L \). Hence \( L = (L \cap N_1) \oplus (L \cap N_2) \).

Conversely, let the above conditions hold. Define \( h: M \to N_1 \) as follows. Let \( y \in M \). Then \( y = y_1 + y_2 \) for some \( y_1 \in M_1, y_2 \in M_2 \). Now \( y_1 = f(x_1) \) for some \( x_1 \in N_1 \cap L \). Set \( h(y) = x_1 \).

Corollary 1.7. Let \( M_R \) be a uniform module and \( N_R \) any module.
(i) \( M \) is almost \( N \)-injective if and only if for any homomorphism \( f: L \to M, L < N \) which is maximal such that it cannot be extended from \( N \) to \( M \), there exists a decomposition \( N = N_1 \oplus N_2 \) such that \( f(N_1 \cap L) = M, N_2 = \ker f \) and \( L = (L \cap N_1) \oplus N_2 \).
(ii) \( M \) is almost \( N \)-injective if and only if for any homomorphism \( f: L \to M, L < N \) which is maximal such that it cannot be extended from \( N \) to \( M \), there exists a decomposition \( N = N_1 \oplus N_2 \) such that \( f \) is monic on \( N_1 \cap L \), \( f(N_1 \cap L) = M \) and \( L = (L \cap N_1) \oplus N_2 \).
(iii) Let \( D \) be an (commutative) integral domain and \( F \) be its quotient field. Then \( D \) is almost \( F_D \)-injective.

Proof. Clearly, \( M \) is quasi-continuous. (i) Suppose \( M \) is almost \( N \)-injective. By Theorem 1.6, \( N = N_1 \oplus N_2 \), \( N_1 \neq 0 \), \( f \) is monic on \( N_1 \cap L \), \( f(N_1 \cap L) = M \), and \( f(N_2 \cap L) = 0 \). As \( f \mid N_2 \cap L = 0 \), it can be extended from \( N_2 \) to \( M \), therefore by Proposition 1.3, \( N_2 = N_2 \cap L \). Hence \( L = (N_1 \cap L) \oplus N_2 \). The converse is immediate from Theorem 1.6.

(ii) Suppose the given condition holds. We get a homomorphism \( \lambda: N_2 \to (N_1 \cap L) \) such that for any \( x \in N_2, \lambda(x) = y \), whenever \( f(x) = f(y) \). Then \( N_2' = \{x - \lambda(x): x \in N_2\} \subseteq \ker f \) and \( N = N_1 \oplus N_2' \). After this (i) proves the result.

(iii) Let \( f: L \to D, L < F_D \) be a homomorphism that cannot be extended from \( F \) to \( D \). Then \( F \neq D \). However \( F_D \) is injective, so \( f \) extends to an automorphism \( g \) of \( F_D \). Let \( K = g^{-1}(D) \). Then \( K = cD \) for some \( c \in F \) such that \( g(c) = 1 \). Clearly, \( L \subseteq K \). \( g(K) = D \). The maximality of \( f \) gives \( L = K \). By (i), \( D \) is almost \( F_D \)-injective.
Lemma 1.8. Let $M_R$ be uniform module and be almost $N_R$-injective. If $N$ has a uniform summand $N_1$ such that $M$ is not $N_1$-injective, then for any uniform submodule $V$ of $N$, there exists a proper summand $K_2$ of $N$ such that $K_2 \cap V \neq 0$.

If $N = N_1 \oplus N_2$ with $N_2$ also uniform, then $K_2$ is uniform.

Proof. Now $M$ is almost $N_1$-injective. So there exists a maximal $R$-monomorphism $\lambda: T \to M$, $T < N_1$, which cannot be extended from $N_1$ to $M$. By Corollary 1.7, $\lambda(T) = M$. Now $N = N_1 \oplus N_2$ for some $N_2 < N$. This gives a maximal $R$-homomorphism $f: L \to M$, $L < N$ which extends $\lambda$ and $N_2 = \ker f$. We can take $V \subseteq T \oplus N_2$. We need only to discuss the case, when $V \cap N_1 = 0 = V \cap N_2$. We take $V = xR$, $x = x_1 + x_2$ with $x_1 \in T$, $x_2 \in N_2$. We get an isomorphism $g: x_2 R \to x_1 R$, $g(x_2) = x_1$. Define a mapping $\mu: x_1 R \oplus x_2 R \to M$, $\mu(x_1 r_1 + x_2 r_2) = f(x_1 r_1 - g(x_2 r_2)) = f(x_1 (r_1 - r_2))$. It is one-to-one on $x_1 R$ and it equals $f$ on $x_1 R$. So we have a maximal extension $\eta: K \to M$, $K \leq N$, of $\mu$, which also extends $f | T$. As $\lambda = f | T$ has no extension from $N_1$ to $M$, $K < N$. By Corollary 1.7, we have $N = K_1 \oplus K_2$ such that with $K_2 = \ker \eta$. As $x_1 + x_2 \in \ker \mu \subseteq \ker \eta$, we get $x_1 + x_2 \in K_2$, which shows that $V \cap K_2 \neq 0$. The last part is obvious. \[\square\]

Remark. In the above proof, $K_2$ need not be uniform.

Theorem 1.9. Let $M_R$ be uniform, $N_R$ a module that is not indecomposable and $M$ be almost $N$-injective for any proper summand $T$ of $N$. Then $M$ is almost $N$-injective if and only if given any uniform summand $K$ of $N$ and uniform submodule $V$ of $N$ such that $M$ is not $K$-injective and $V$ embeds in $K$, there exists a proper summand $K'$ of $N$ such that $K' \cap V \neq 0$.

Proof. If $M$ is almost $N$-injective, by Lemma 1.8, $M$ satisfies the given condition. Conversely, let the given condition hold. Let $f: L \to M$, $L < N$ be a maximal homomorphism that cannot be extended from $N$ to $M$. By the hypothesis, there exists a decomposition $N = N_1 \oplus N_2$ with $0 < N_1 < N$. Set $f_1 = f | N_1 \cap L$. Suppose $f_1: N_1 \cap L \to M$ cannot be extended from $N_1$ to $M$. As $M$ is almost $N_1$-injective, $N_1 = N_{11} \oplus N_{12}$, such that $f_1$ is monic on $N_{11} \cap L$, $f(N_{11} \cap L) = M$ and $N_{12} = \ker f_1$.

Case 1. $N_2 = N_2 \cap L$. We get an $R$-homomorphism $\lambda: N_2 \to N_{11}$ such that for any $x \in N_2$, $\lambda(x) = y \in (N_{11} \cap L)$ whenever $f(x) = f(y)$, i.e. $f(x - y) = 0$. Set $K_2 = \{x - \lambda(x): x \in N_2\}$. Then $K_2 \subseteq \ker f$, $N = N_{11} \oplus N_{12}$, $N_2 = N_{11} \oplus N_{12} \oplus K_2 = N_{11} \oplus \ker f$. In this case we finish.

Case 2. $N_2 \cap L < N_2$. Then we also have $N_2 = N_{21} \oplus N_{22}$ such that $f_2 = f | N_{21}$ is monic on $N_{21}$, $f(N_{21} \cap L) = M$ and $N_{22} = \ker f_2$. As $f(N_{11} \cap L) = M = f(N_{21} \cap L)$, we have an isomorphism $\lambda: N_{21} \cap L \to N_{11} \cap L$ such that for any $x \in (N_{21} \cap L)$, $y \in (N_{11} \cap L)$, $\lambda(x) = y$ if and only if $f(x) = f(y)$. Then $V = \{x - \lambda(x): x \in N_{21} \cap L\} \subseteq N_{11} \oplus N_{21}$, $V$ is embeddable in $N_{11}$ and $V \subseteq \ker f$. \[\square\]
Now $N_{11}, N_{21}$ are uniform. If $K = N_{11} \oplus N_{21} < N$, then by the hypothesis, $M$ is almost $K$-injective. Therefore $K = U_1 \oplus U_2$ such that $U_1$ is uniform, $f$ is monic on $U_1 \cap L$ and $U_2 \subseteq \ker f$, which gives $N = U_1 \oplus \ker f$, as already seen $N_{12} \oplus N_{22} \subseteq \ker f$.

Now suppose $N = N_{11} \oplus N_{21}$. By the hypothesis, $N = U_1 \oplus U_2$ such that $0 < U_2 < N$ and $V \cap U_2 \neq 0$ for the $V$ defined above. As $U_2$ is uniform, $\ker f \cap U_2 \neq 0$. Thus $f \mid U_2$ is not monic, it follows from Corollary 1.7 that $f \mid U_2 \cap L$ can be extended from $U_2$ to $M$. Therefore $U_2 \subset L$. Which gives $U_1 \cap L < U_1$, $f$ is monic on $U_1 \cap L$ and $f(U_1 \cap L) = M$. We get a homomorphism $\mu: U_2 \to U_1$ such that $\mu(x) = y$ for any $x \in U_2$, $y \in U_1 \cap L$ whenever $f(x) = f(y)$. Then $V_2 = \{x - \mu(x); x \in U_2\} \subseteq \ker f$. We get $N = U_1 \oplus \ker f$.

Hence in any case $N = U \oplus \ker f$ for some uniform submodule $U$, $f$ is monic on $U \cap L$ and $f(U \cap L) = M$. By Corollary 1.7, $M$ is almost $N$-injective.

Lemma 1.10. Let $N_R = N_1 \oplus N_2$, where $N_i$ are indecomposable and their rings of endomorphisms are local. Let $M_R$ be uniform and almost $N$-injective, $f: L \to M$, $L < N$ be a maximal homomorphism that cannot be extended from $N$ to $M$ and $N_1 \cap L < N_1$.

(i) If $g: W \to N_1 \cap L$, $W \leq N_2 \cap L$ is a non-zero homomorphism, then either $g$ extends from $N_2$ to $N_1$, or $g$ is monic and $g^{-1}$ on $g(W)$ extends from $N_1$ to $N_2$.

(ii) If $V$ is a uniform submodule of $N$ such that $V \subseteq (N_1 \cap L) \oplus (N_2 \cap L)$ and it naturally embeds in $N_2$, then there exists a proper summand $U$ of $N$ containing $V$.

(iii) For any uniform submodule $V_1$ of $N$, there exists a proper summand $U$ of $N$ such that $V_1 \cap U \neq 0$.

Proof. (i) Now $N_1 \cap L < N_1$ and $f \mid (N_1 \cap L)$ cannot be extended from $N_1$ to $M$. As $M$ is almost $N_1$-injective, by Corollary 1.7, $f$ is monic on $N_1 \cap L$ and $f(N_1 \cap L) = M$, which gives that $N_1$ is uniform. Let $W_1 = (N_1 \cap L) \oplus W$. Define $f': W_1 \to M$, $f'(x + y) = f(x - g(y)), x \in N_1 \cap L, y \in W$. Then $\ker f' = \{x + y; y \in W, x = g(y)\} \neq 0$. We get a maximal homomorphism $f_1: L_1 \to M$, $L_1 \equiv N$ which extends $f'$ and $f \mid N_1 \cap L$. Then $L_1 < N$ and $N = U_1 \oplus U_2$, where $U_1$ is uniform and $U_2 = \ker f_1$. In particular, $\ker f' \subseteq U_2$. By Krull–Schmidt–Azumaya theorem, we can get $N = N_1 \oplus U_2$ or $N = N_2 \oplus U_2$.

Case 1. $N = N_1 \oplus N_2 = N_1 \oplus U_2$. Let $\pi_1: N \to N_1$ be associated projections. Then $\pi_2(U_2) = N_2$. Let $\lambda = \pi_2 \mid U_2$. We have $\lambda^{-1}: N_2 \to U_2$. Let $y \in W$. By definition $g(y) + y \in (N_1 \cap L) \oplus (N_2 \cap L)$ and $g(y) + y \in \ker f' \subseteq U_2$. Thus $\lambda(g(y) + y) = y$, which gives $\lambda^{-1}(y) = g(y) + y$. Under the projection $\pi_1: N \to N_1$, $\pi_1 \lambda^{-1}(y) = g(y)$. Thus $\pi_1 \lambda^{-1}: N_2 \to N_1$ extends $g$.

Case 2. $N = N_1 \oplus N_2 = N_2 \oplus U_2$. Then $\pi_1(U_2) = N_1$. Let $\lambda_1 = \pi_1 \mid U_2$. Then $\lambda_1(g(y) + y) = g(y)$, and as $\lambda_1$ is monic, $g(y) = 0$ if and only if $y = 0$, i.e. $g$ monic. Now $\lambda_1^{-1}(g(y)) = g(y) + y, \pi_2 \lambda_1^{-1}(g(y)) = y$. Thus $\pi_2 \lambda_1^{-1}: N_1 \to N_2$ extends $g^{-1}$ on $g(W)$. 
(ii) Suppose \( V \) is a uniform submodule of \( N \) such that \( V \subseteq (N_1 \cap L) \oplus (N_2 \cap L) \) and \( V \) naturally embeds in \( N_2 \). Let \( W = \pi_2(V) \). We get a homomorphism \( g: W \to N_1 \cap L, \ g(\pi_2(x)) = \pi_1(x), \ x \in V. \) If \( g \) extends to an \( R \)-homomorphism \( g' \) from \( N_2 \) to \( N_1 \), then \( U = \{x + g'(x): x \in N_2\} \) is a summand of \( N \) containing \( V \). If \( g \) does not extend from \( N_2 \) to \( N_1 \), by Case 2, \( g \) is monic and \( g^{-1} \) on \( g(W) \) extends to a homomorphism \( g': N_1 \to N_2 \). In this case \( U' = \{x + g'(x): x \in N_1\} \) contains \( V \) and is a summand of \( N \) isomorphic to \( N_1 \).

Take any uniform submodule \( V_1 \) of \( N \) such that \( V_1 \cap N_1 = 0 \). Then \( V_1 \) embeds in \( N_2 \). As \( L \cap N_2 \subseteq N_2 \), there exists a non-zero \( x = x_1 + x_2 \in V_1 \) with \( x_1 \in N_1 \), \( x_2 \in N_2 \cap L \). Once again as \( N_1 \cap L \subseteq N_1 \), we can choose \( x \) to be also have \( x_1 \in N_1 \cap L \). Then \( V = xR \subseteq (N_1 \cap L) \oplus (N_2 \cap L) \), which, by (ii), is contained in a proper summand \( K \) of \( N \). Clearly, \( V_1 \cap K \neq 0. \)

**Theorem 1.11.** Let \( N_R = N_1 \oplus N_2 \), where \( N_i \) are indecomposable and their rings of endomorphisms are local. Let \( M_R \) be uniform. Then \( M \) is almost \( N \)-injective if and only if either \( M \) is \( N \)-injective or \( M \) is almost \( N_i \)-injective for \( i = 1, 2 \), but is not \( N_j \)-injective for some \( j \), say for \( j = 1 \), and any uniform submodule \( V \) of \( N \) has non-zero intersection with some indecomposable summand of \( N \).

**Proof.** In view of Lemma 1.10, we only need to prove the converse. Suppose the given conditions holds. Let \( f: L \to M, \ L < N \) be a maximal \( R \)-homomorphism that cannot be extended from \( N \) to \( M \).

Let \( L \cap N_1 < N_1 \). Then \( f \) is monic on \( L \cap N_1 \), \( f(L \cap N_1) = M \), which gives that \( V = \{x - y: x \in N_1 \cap L, y \in N_2 \cap L \) and \( f(x) = f(y)\} \neq 0, V \subseteq \ker f \) and it embeds in \( N_1 \). Suppose \( f \mid (N_2 \cap L) \) is monic. Then \( V \) naturally embeds in \( N_1 \), therefore \( V \) is uniform. By the hypothesis, \( N = U_1 \oplus U_2 \) with \( V \cap U_2 \neq 0 \). As \( M \) is almost \( U_2 \)-injective and \( \ker f \cap U_2 \neq 0, U_2 \subseteq L \). Then \( L \cap U_1 < U_1 \) and \( f \) is monic on \( U_1 \), \( f(U_1 \cap L) = M \). We get \( K = \{x - y: x \in U_1 \cap L, y \in U_2 \) and \( f(x) = f(y)\} \equiv U_2 \) and \( K \subseteq \ker f \). Trivially, \( N = U_1 \oplus \ker f \). If \( f \mid N_2 \cap L \) is not monic, then \( N_2 \subseteq L \), as above we get \( N = N_1 \oplus \ker f \).

Let \( L \cap N_1 = N_1 \). Then \( L \cap N_2 < N_2 \) and once again, we continue as before. Hence \( M \) is almost \( N \)-injective.

**Theorem 1.12.** Let \( M_R \) be a uniform module and \( N_R = N_1 \oplus N_2 \oplus \cdots \oplus N_k \) a finite direct sum of modules whose rings of endomorphisms are local. Then \( M \) is almost \( N \)-injective if and only if \( M \) is almost \( N_i \)-injective for every \( i \), and if for some \( i, M \) is not \( N_j \)-injective, then for every \( j \neq i \), \( N_i \oplus N_j \) has the property that for any uniform submodule \( V \) of \( N_i \oplus N_j \), there exists a proper summand \( U \) of \( N_i \oplus N_j \) such that \( U \cap V \neq 0 \).

**Proof.** In view of Theorem 1.11, we only need to prove the converse. Let \( f: L \to M, \ L < N \) be a maximal homomorphism that cannot be extended from \( N \) to \( M \). Then
for some $i$, say for $i = 1$, $f_1 = f | (N_1 \cap L)$: \((N_1 \cap L) \to M\) cannot be extended from $N_1$ to $M$. As $M$ is almost $N_1$-injective, $f_1$ is monic and $f(N_1 \cap L) = M$. Consider any $j \neq 1$ and $f_j = f | (N_j \cap L)$. By Theorem 1.11, $M$ is $N_1 \oplus N_j$-injective. By Corollary 1.7, $N_1 \oplus N_j = U_1 \oplus U_2$ for some uniform submodules $U_1$ and $U_2 \subseteq \ker f$. Thus $U_2 \subseteq L$ and $L \cap U_1 < U_1$. This proves that in the decomposition $N_R = N_1 \oplus N_2 \oplus \cdots \oplus N_k$, we can replace $N_1 \oplus N_j$ by a $U_1 \oplus U_2$ with $U_2 \subseteq \ker f$. This proves that $N = V \oplus \ker f$ for some uniform submodule $V$. By Corollary 1.7, $M$ is almost $N$-injective.

The above theorem generalizes the following result by Baba [4].

**Theorem 1.13.** Let $U_k$ be a uniform module of finite composition length for $k = 0, 1, \ldots, n$. Then the following two conditions are equivalent.

1. $U_0$ is almost $\bigoplus \sum_{k=1}^n U_k$-injective.
2. $U_0$ is almost $U_k$-injective for $k = 1, 2, \ldots, n$ and if $\soc(U_0) \cong \soc(U_k) \cong \soc(U_l)$ for some $k, l \in \{1, 2, \ldots, n\}$ with $k \neq l$, then
   (i) $U_0$ is $U_k$ and $U_l$-injective or
   (ii) $U_k \oplus U_l$ is extending for simple modules, in the sense that any simple submodule of $U_k \oplus U_l$ is contained in a uniform summand of $U_k \oplus U_l$.

The following is known.

**Lemma 1.14.** Let $\{N, V_i\}$ be a family of modules over a ring $R$. Then $M = \bigoplus \sum_{i=1}^n V_i$ is almost $N$-injective if and only if every $V_i$ is almost $N$-injective.

**Lemma 1.15.** Let $U_1, U_2$ be two uniform modules such that $U_2$ is almost $U_1$-injective. Let $V$ be a uniform submodule of $N = U_1 \oplus U_2$ such that $V \cap U_2 = 0$. Then there exists a uniform summand $K$ of $N$ isomorphic to $U_1$ or $U_2$, which contains $V$. Any uniform submodule of $N$ has non-zero intersection with some uniform summand of $N$.

Proof. Let $\pi_i : N \to U_i$ be associated projections. The hypothesis gives a homomorphism $\sigma : \pi_1(V) \to \pi_2(V)$, $\sigma(\pi_1(x)) = \pi_2(x)$ for any $x \in V$. We get a maximal homomorphism $\eta : L \to U_2$, $L \leq U_1$ extending $\sigma$. Then either $L = U_1$, or $\eta$ is monic and $\eta(L) = U_2$. In the former case, take $K = \{y + \eta(y) : y \in U_1\}$ and in the later case, take $K = \{y + \eta(y) : y \in L\}$. The second part is immediate.

We get an alternative proof of the following result by Harada [10].

**Theorem 1.16.** Let $M = M_1 \oplus M_2 \oplus \cdots \oplus M_k$, where each $M_i$ has its ring of endomorphisms local. Then the following are equivalent.

(i) $M$ is almost self-injective.
(ii) For any $i, j$, $M_i$ is almost $M_j$-injective.
Proof. Suppose $M$ is almost self-injective. Then each $M_i$ is almost self-injective. Therefore each $M_i$ is uniform. As $M_i$ is almost $M$-injective, by Lemma 1.14, condition (ii) holds. Fix an $i$, $1 \leq i \leq k$. Consider any $1 \leq r, s \leq k$. By the hypothesis, $M_s$ is almost $M_r$-injective. By Lemma 1.15, given any uniform submodule $V$ of $W = M_r \oplus M_s$, there exists a uniform summand $K$ of $W$ such that $V \cap K \neq \emptyset$. By Theorem 1.12, $M_i$ is almost $M$-injective. As $M$ is a direct sum of $M_i$’s, it follows from Lemma 1.14 that $M$ is almost self-injective.

2. Commutative rings

**Proposition 2.1.** Let $R$ be any commutative indecomposable ring and $Q$ be its quotient ring. If $R$ is almost self-injective. Then the following hold.

(i) If $a, b \in R$ and $\text{ann}(a) \subseteq \text{ann}(b)$, then $bR \subseteq aR$, or $aR < bR$ and $a = bc$ for some regular element $c \in R$.

(ii) If $a, b \in R$ are regular, then either $aR \subseteq bR$ or $bR \subseteq aR$.

(iii) $Q_R$ is injective and uniform.

Conversely, if $R$ satisfies conditions (i) and (iii), then $R$ is almost self-injective.

Proof. Let $a$, $b$ be two elements of $R$ such that $\text{ann}(a) \subseteq \text{ann}(b)$. We have a homomorphism $\sigma : aR \to bR$, $\sigma(a) = b$. If $\sigma$ extends to an endomorphism $\eta$ of $R_R$, then $b = ac$, where $c = \eta(1)$, which gives $bR \subseteq aR$. Suppose $\sigma$ does not extend to an endomorphism of $R_R$. Then $b \notin aR$. As $R_R$ is uniform, by Corollary 1.7, there exists a maximal extension $\eta : L \to R$. $L < R$ of $\sigma$ such that it is monic and $\eta(L) = R$. Thus $L = cR$ where $c$ is such that $\eta(c) = 1$. This $c$ is regular, non-unit and $a = bc$. This proves (i). Now (ii) is immediate from (i).

(iii) Let $\sigma : A \to Q_R$, $A < R_R$ be a homomorphism. Suppose $\sigma(A) \subseteq R$. If it extends to an $\eta \in \text{End}(R_R)$ and $\eta(1) = c$, then multiplication by $c$ gives an endomorphism of $Q_R$ extending $\sigma$. Otherwise for some regular element $c \in R$ we have an $\eta; cR \to R$ with $\eta(c) = 1$, which extends $\sigma$. Then $c^{-1} \in Q$ and multiplication by $c^{-1}$ gives an $R$-endomorphism of $Q_R$ extending $\sigma$. This proves that if $\sigma(A) \subseteq R$, then $\sigma$ extends to an endomorphism of $Q_R$.

Suppose $\sigma(A) \not\subseteq R$. Let $S$ be the set of regular elements of $R$. Then $Q = R_S$. Set $B = \sigma(A)$. Let $B' = B \cap R$. Then $B \subseteq B'$.

Let $A' = \sigma^{-1}(B')$ and $\sigma_1 = \sigma | A'$. Then $\sigma_1(A') = B' \subseteq R$. Therefore $\sigma_1$ extends to an endomorphism $\eta$ of $Q_R$. Let $x \in A$. Then $\sigma(x) = y\sigma^{-1}$ for some regular element $c \in R$, $y \in B'$. Which gives $\sigma(xc) = y$, $xc \in A'$, $\eta(xc) = y$, If $\eta(x) = z$, then $y = zc$, $\sigma(x) = z$. Hence $\eta$ extends $\sigma$. This proves that $Q_R$ is injective. It also gives that $Q_R = E(R_R)$. As $R$ is uniform, $Q_R$ is uniform.

Conversely, let $R$ satisfy the given conditions. Let $f : A \to R_R$, $A < R$ be a homomorphism that cannot be extended in $\text{End}(R_R)$. By (iii), $\sigma$ extends to an $R$-endomorphism $\eta$ of $Q$. It follows from (ii) that if an $x \in Q$ is regular, then $x \in R$ or $x^{-1} \in R$. Now $\eta(1) = ac^{-1}$ for some $a, c \in R$ with $c$ regular.

**Case 1.** $a$ is regular. It follows from (ii) that $\eta(1) \in R$ or $\eta(1)^{-1} \in R$. In the former case, $\eta | R$ is an extension in $\text{End}(R_R)$ of $\sigma$. Suppose $\eta(1)^{-1} \in R$, but $\eta(1) \notin R$. 


Then for any \( x \in A \), \( \sigma(x) = \eta(1)x \), gives \( x = \sigma(x)\eta(1)^{-1} \). So that \( A \subseteq \eta(1)^{-1}R < R \). We have an isomorphism \( \lambda : \eta(1)^{-1}R \rightarrow R \) with \( \lambda(\eta(1)^{-1}) = 1 \). Then \( \lambda \) extends \( \sigma \).

**Case 2.** \( a \) is not regular. By (i), \( a = cr \) for some \( r \in R \), therefore \( \eta(1) = ac^{-1} = r \) and \( \eta \mid R \) is an extension in \( \text{End}(R_R) \) of \( \sigma \). Hence \( R \) is almost self-injective. \( \square \)

**Theorem 2.2.** Let \( R \) be a commutative, indecomposable, almost self-injective ring. Let \( Q \) be the quotient ring of \( R \).

(i) Either \( Q = R \) or there exists a prime ideal \( P \) in \( R \) such that \( Q = R_P \).

(ii) \( R \) is a local ring.

Proof. Suppose \( Q \neq R \). Then \( R \) has a regular element that is not a unit. Let \( a \in R \) be regular but not a unit. We claim that \( A = \bigcup_{k=1}^{\infty} a^kR \) is the unique maximal prime ideal such that \( a \notin A \). And we also prove that any element in \( R \setminus A \) is regular. Let \( b \in R \setminus A \). Then for some \( k, b \notin a^kR \). It follows from Proposition 2.1 that \( b \) is regular and \( a^kR < bR \). Thus \( A \) is a prime ideal of \( R \). As \( a^kR < aR, a \notin A \). Let \( P' \) be a maximal prime ideal in \( R \) such that \( a \notin P' \). Then there exists a \( b \in P' \) such that \( b \notin A \). Then, as seen above, \( a^k \in bR \subseteq P' \) for some \( k \geq 1 \), which gives \( a \in P' \), which is a contradiction. Hence \( A = P' \). Thus to each regular non-unit \( a \in R \), is associated a unique maximal prime ideal \( P_a = \bigcap_{k=1}^{\infty} a^kR \) such that \( a \notin P_a \).

Every element of \( R \setminus P_a \) is regular. It follows from Proposition 2.1 (ii) that the family of \( P_a \) is linearly ordered. Let \( P \) be the intersection of these \( P_a \)’s. Then \( R \setminus P \) is the set of all regular elements in \( R \). Hence \( Q = R_P \).

Let \( P' \) be a prime ideal of \( R \) other than \( P \). Suppose \( P' \not\subseteq P \). As \( R \setminus P \) consists of regular elements, there exists a regular element \( a \in P' \). Then \( P_a \subseteq P' \), so \( P \subseteq P_a \subseteq P' \).

Let \( P_1, P_2 \) be two prime ideals not contained in \( P \). Suppose \( P_1 \not\subseteq P_2 \). Then there exists an \( a \in P_1 \setminus P_2 \). As \( a \notin P \), it is regular. Let \( b \in P_2 \). By Proposition 2.1 (i), \( b \in a^kR \) for any \( k \geq 1 \). It follows that \( P_2 \subseteq P_a \). Trivially, \( P_a \subseteq P_1 \). Hence \( P_2 \subseteq P_1 \). It follows that the family \( F \) of those prime ideals of \( R \) that are not contained in \( P \) is linearly ordered and each member of \( F \) contains \( P \). Hence \( R \) is local. \( \square \)

An indecomposable, commutative, almost self-injective ring need not be a valuation ring.

**Example 1.** Let \( F \) be a field and \( Q = F[x, y] \) with \( x^2 = 0 = y^2 \). Then \( Q = F + Fx + Fy + Fxy \) is a local, self-injective ring. Choose \( F \) to be the quotient field of a valuation domain \( T \neq F \). Set \( R = T + Fx + Fy + Fxy \subset Q \). Any \( 0 \neq a \in F \) is such that either \( a \in T \) or \( a^{-1} \in T \), \( J(Q) = Fx + Fy + Fxy \subset R \) and is nilpotent. Any element of \( R \) not in \( J(Q) \) is regular and is of the form \( au \) with \( a \in T \) and \( u \) a unit in \( R \). By using this it follows that \( Q \) is the classical quotient ring of \( R \). Let \( A \) be a non-zero ideal of \( R \). Then \( \{a \in F : axy \in A\} \) is a non-zero \( T \)-submodule of \( F \), which shows that \( R_R \) is uniform. The ideals \( Fx + Fxy, Fy + Fxy \) in \( R \) are not comparable. Therefore \( R_R \) is not uniserial. If \( A \not\subset J(Q) \), then some \( au \in A \) with \( 0 \neq a \in F, u \) a unit in \( R \), so \( a \in A \); which gives \( J(Q) = aJ(Q) \subset A \).
Let $\sigma : A \to R$, $A < R_R$ be an $R$-homomorphism. Now $A' = \{ \alpha v : \alpha \in F, v \in A \}$ is an ideal of $Q$ containing $A$, $\eta : A' \to Q$, such that for any $c \in F$, $v \in A$, $\eta(cv) = c\sigma(v)$ is a $Q$-homomorphism. As $Q$ is self-injective, there exists an $\omega \in Q$ such that $\eta(cv) = \omega cv$ for any $cv \in A'$. If $\omega \in R$, obviously $\sigma$ extends to an endomorphism of $R_R$. Suppose $\omega \notin R$. Then $\omega = c^{-1}u$ for some non-zero $c \in T$ which is not a unit in $T$, and $u$ is a unit in $R$. Thus $g = cu^{-1} \in R$. For any $v \in A$, $\sigma(v) = g^{-1}v \in R$, $v = g\sigma(v) \in gR$. Thus $A < gR$ and $\lambda : gR \to R$, $\lambda(g) = 1$, extends $\sigma$. Hence $R$ is a local ring that is almost self-injective and $R_R$ is not uniserial.

**Lemma 2.3.** Let $A$, $B$ be two rings such that $A$ is local and $M$ be an $(A,B)$-bimodule. Let $R = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$. Then $e_{11}R$ is uniform if and only if $M_B$ is uniform and $A_M$ is faithful.

**Proof.** Let $e_{11}R$ be uniform. Let $x = a_{11}e_{11} + a_{12}e_{12}$, $y = b_{11}e_{11} + b_{12}e_{22}$ be two non-zero elements in $e_{11}R$. Then for some $r = r_{11}e_{11} + r_{12}e_{12} + r_{22}e_{22}$, $s = s_{11}e_{11} + s_{12}e_{12} + s_{22}e_{22} \in R$, $xr = ys \neq 0$. Which gives $a_{11}r_{11} = b_{11}s_{11}$, $a_{11}r_{12} + a_{12}r_{22} = b_{11}s_{12} + b_{12}s_{22}$.

**Case 1.** $a_{11} = 0$ and $b_{11} = 0$. Then $a_{12}r_{22} = b_{12}s_{22} \neq 0$, which gives that $M_B$ is uniform.

**Case 2.** $a_{11} \neq 0$, $b_{11} = 0$, $a_{12} = 0$, $b_{12} \neq 0$. Then $a_{11}r_{12} = b_{12}s_{22} \neq 0$. Therefore $a_{11}M \neq 0$. Hence $A_M$ is faithful.

Conversely, let $M_B$ be uniform and $A_M$ be faithful. Then $e_{12}M$ is a uniform right ideal of $R$, and for any $x \neq 0$ in $e_{11}R$, $xR \cap e_{12}M \neq 0$. Hence $e_{11}R$ is uniform. \[\square\]

The above lemma helps to get examples of non-commutative, almost self-injective rings.

**Example 2.** Let $A$ be a valuation domain and $K$ be its quotient field. Let $R = \begin{bmatrix} A & K \\ 0 & B \end{bmatrix}$, where $B$ is a valuation ring contained in $K$ such that $K$ is a quotient field of $B$. By Lemma 2.3, $e_{11}R$ is uniform. Let $f : L \to e_{11}R$, $L < e_{11}R$ be a maximal homomorphism that cannot be extended to an endomorphism of $e_{11}R$. Now $e_{12}K$ is a quasi-injective $R$-module and $f(L \cap e_{12}K) \subseteq e_{12}K$. Therefore $f \mid (L \cap e_{12}K)$ can be extended to an $R$-endomorphism $g$ of $e_{12}K$. As $f$ is monic on $L \cap e_{12}K$, $f$ is monic. Then $f' : L + e_{12}K \to e_{11}R$, $f'(x + y) = f(x) + g(y)$ for any $x \in L$, $y \in e_{12}K$ extends $f$. Which gives $e_{12}K \subseteq L$, $L = (e_{11}A \cap L) \oplus (e_{12}K)$ as an abelian group. Now $f(e_{12}) = e_{12}b$ for some $b \in K$. Then $f(e_{12}c) = e_{12}cb$ for every $c \in K$. Let $x = a_{11}e_{11} + a_{12}e_{12} \in L$ with $a_{11} \neq 0$. Then $e_{11}a_{11} \in L$ and $f(e_{11}a_{11}) = e_{11}a_{11}u$ for some $u \in K$. We get $f(e_{12}a_{11}) = f(e_{11}a_{11})e_{12} = e_{12}u$. On the other hand, $f(e_{12}a_{11}) = e_{12}a_{11}b$. Hence $u = a_{11}b$. Thus $f(x) = xb = (e_{11}b)x$ for every $x \in L$. If $b \in A$, $f$ can be extended to an $R$-endomorphism of $e_{11}R$ given by left multiplication by $e_{11}b$. Suppose $b \notin A$. Then $b^{-1} \in A$. Then the $R$-endomorphism $h$ of $e_{11}R$ given by left
multiplication by $e_{11}b^{-1}$ is such that $hf(z) = z$ for every $z \in L$. Hence $e_{11}R$ is almost self-injective.

Any $R$-homomorphism $\lambda : L \to e_{11}R$, $L < e_{22}R$ is such that $f(L) \subseteq e_{11}K$. As $e_{22}R = e_{22}B$, $\lambda$ can be extended from $e_{22}R$ to $e_{11}R$. It follows that $e_{11}R$ is $e_{22}$-injective. Let $f : L \to e_{22}R$, $L < e_{11}R$ be a non-zero homomorphism. Now $L \cap e_{12}K \neq 0$. As $e_{22}R = e_{22}B$ it follows that for some $b \in K$, $g = f | L \cap e_{12}K$ is such that $g(e_{12}x) = e_{22}xb$ for any $e_{12}x \in L \cap e_{12}K$, therefore $f$ is monic. If an $x = a_{11}e_{11} \in L$, then $f(x) = 0$. This proves that $L \subseteq e_{12}K$. Then $h : e_{22}R \to e_{11}R$, $h(e_{11}x) = e_{12}xb^{-1}$, $x \in B$ is such that $hf(u) = u$ for every $u \in L$. Hence $e_{22}R$ is almost $e_{11}$-injective. By Theorem 1.16, $R$ is almost self-injective.

By using Theorem 1.16, one can easily prove that the ring $T_p(D)$ of upper triangular matrices over a division ring $D$ is almost right self-injective.

3. Von Neumann regular rings

Theorem 3.1. Let $R$ be a von Neumann regular ring. Then $R$ is almost right self-injective if and only if for any maximal homomorphism $\sigma : A \to R$, $A < R$ which cannot be extended to an $R$-endomorphism of $R$, there exist non-zero idempotents $e, f \in R$, such that $eR \subseteq A$, $\sigma | eR$ is a monomorphism, $\sigma(eR) = fR$, $\sigma(A \cap (1 - e)R) \subseteq (1 - f)R$.

Proof. Let $R$ be almost right self-injective, Let $\sigma : L \to R$ be a maximal $R$-homomorphism that cannot be extended to an endomorphism of $R$. By definition, $R = eR \oplus (1 - e)R$ and there exists an $R$-homomorphism $h : R \to eR$ such that $hf(x) = ex$ for every $x \in L$. There exists $u^2 = u \neq 0$ in $L \cap eR$ such that $eR = uR \oplus (e - u)R$, and $e - u$ is an idempotent orthogonal to $u$. Let $u : eR \to uR$ be a projection with kernel $(e - u)R$. Then $\pi h(x) = ux$. So we take $e = u$ and $h = \pi h$. As $h(R) = eR$, $R = gR \oplus (1 - g)R$ for some idempotent $g \in R$ such that $\ker h = (1 - g)R$. Now $h(R) = eR = h\sigma(eR)$, we get $R = \sigma(eR) \oplus \ker h$. Thus, there exists an idempotent $f \in R$, such that $R = fR \oplus (1 - f)R$, $\sigma(eR) = fR$, $\ker h = (1 - f)R$ and $h | fR$ is the inverse of $\sigma | eR$. Clearly, for any $x \in (1 - e)R \cap A$, $h\sigma(x) = 0$ gives $\sigma(x) \in (1 - f)R$.

Conversely, let $R$ satisfy the given conditions. Let $\sigma : L \to R$ be a maximal homomorphism that cannot be extended to an endomorphism of $R$. Then there exist non-zero idempotents $e, f \in R$ such that $L = eR \oplus (L \cap (1 - e)R)$, $\sigma$ is monic on $eR$, $\sigma(eR) = fR, \sigma((1 - e)R \cap L) \subseteq (1 - f)R$. We define $h : R \to eR$ as follows. Let $y \in R$. Then $y = f(y) + (1 - f)y$. Now $f(y) = \sigma(ex)$ for some uniquely determined $ex \in eR$. Set $h(y) = ex$. If follows that for any $x \in L$, $h\sigma(x) = ex$. Hence $R$ is almost right self-injective. ☐

Corollary 3.2. Any von Neumann regular ring $R$ that is right CS, is almost right self-injective.
Proof. Let \( \sigma : A \to R, A < R \) be a non-zero \( R \)-homomorphism. As \( \ker \sigma \) is not large in \( A \), there exists a non-zero idempotent \( e \in A \) such that \( eR \cap \ker \sigma = 0 \). Then \( \sigma(eR) = fR \) for some idempotent \( f \in R \). Let \( B \) be a complement of \( eR \) in \( R \).

**Remark.** Any von Neumann regular ring that is right \( CS \) is right continuous. In [7], examples of continuous commutative von Neumann regular rings that are not self-injective are given. Hence a von Neumann regular almost right self-injective need not be right self-injective.

**Proposition 3.3.** Any von Neumann regular ring in which all idempotents are central, is almost self-injective.

Proof. Let \( \sigma : A \to R, A < R \) be a non-zero homomorphism. We get a non-zero idempotent \( e \in A \) such that \( f \mid eR \) is monic. Let \( \sigma(e) = x \), then \( x = xe = ex \) gives \( \sigma(eR) \subseteq eR \). Suppose \( \eta = \sigma \mid eR \). Now \( \sigma(eR) = xR = fR \) for some idempotent \( f \in eR \). Therefore \( x = xf \), \( \eta(e - f) = xf(e - f) = 0 \), \( e = f \). Hence \( \sigma(eR) = eR \).

It also follows that \( \sigma(A \cap (1-e)R) \subseteq A \cap (1-e)R \). Hence \( R \) is almost right self-injective.

The following result determines a class of von Neumann regular rings that are not almost right-injective.

**Theorem 3.4.** Let \( R \) be an almost right self-injective, von Neumann regular ring.

(i) Any complement of a minimal right ideal of \( R \) is principal

(ii) Any minimal right ideal of \( R \) is injective.

Proof. Let \( A \) be a minimal right ideal of \( R \). Then \( A = eR \) for some indecomposable idempotent \( e \in R \). Let \( C \) be a complement of \( eR \). We get a maximal homomorphism \( \sigma : L \to R, L \leq R \) such that \( eR \oplus C \subseteq L \). \( \sigma \) is identity on \( eR \), and is zero on \( C \).

**Case 1.** \( L = R \). Then \( R = fR \oplus \ker \sigma \), hence minimal. As \( e \notin \ker \sigma \), we get \( R = eR \oplus \ker \sigma \). We get \( C = \ker \sigma \), a principal right ideal.

**Case 2.** \( L < R \). By Theorem 3.1, there exist non-zero idempotents \( f \in L, g \in R \) such that \( \sigma(f) = gR \). \( \sigma(L \cap (1-f)R) \subseteq (1-g)R \). Now \( C \subseteq \ker \sigma \subseteq (1-f)R \). Thus \( fR \) is simple, as in Case 1. \( L = eR \oplus ((1-f)R \cap L) \) and \( eR \nsubseteq (1-f)R \). As \( C \subseteq (1-f)R \), we get \( C = (1-f)R \).
Suppose $A$ is not injective. Let $E = E(A)$. We get $x \in E \setminus A$. Then $A < xR$. Let $C = \text{ann}_R(x)$. As $xR$ is non-singular, $C$ is a closed right ideal of $R$ and its complement in $R_R$ is uniform. If $C$ were principal, we would get $R = B \oplus C$ with $B$ simple, which is not possible, as $xR$ is not simple. Hence $C$ is not principal. Let $H$ be a complement of $C$. As $H$ is uniform, it is simple. This contradicts (i), Hence $A$ is injective.

**Example 3.** Let $F$ be any field and $R$ be the ring of column finite matrices over $F$, indexed by the set $\mathbb{N}$ of positive integers. This ring is right self-injective. Let $S$ be subring of $R$ consisting of matrices that are also row finite. Then $R$ is a maximal right quotient ring of $S$. Consider the matrix unit $e_{11}$. Then $e_{11}S$ is a minimal right ideal of $S$. However $e_{11}S < e_{11}R$ and $e_{11}R$, as a right $S$-module is injective hull of $e_{11}S$. Hence $S$ is not almost right self-injective.

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**References**


House No. 424, Sector 35 A
Chandigarh-160036
India
e-mail: ossinghpal@yahoo.co.in