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INERTIA GROUPS AND SMOOTH STRUCTURES OF ($n-1$)-CONNECTED $2n$-MANIFOLDS

KASILINGAM RAMESH

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Abstract

Let M^{2n} denote a closed $(n-1)$-connected smoothable topological $2n$-manifold. We show that the group $\mathcal{C}(M^{2n})$ of concordance classes of smoothings of M^{2n} is isomorphic to the group of smooth homotopy spheres $\bar{\Theta}_{2n}$ for $n=4$ or 5, the concordance inertia group $I_c(M^{2n})=0$ for $n=3$, 4, 5 or 11 and the homotopy inertia group $I_h(M^{2n})=0$ for $n=4$. On the way, following Wall's approach [16] we present a new proof of the main result in [9], namely, for $n=4$, 8 and $H^n(M^{2n};\mathbb{Z})\cong\mathbb{Z}$, the inertia group $I(M^{2n})\cong\mathbb{Z}_2$. We also show that, up to orientation-preserving diffeomorphism, M^8 has at most two distinct smooth structures; M^{10} has exactly six distinct smooth structures and then show that if M^{14} is a π-manifold, M^{14} has exactly two distinct smooth structures.

1. Introduction

We work in the categories of closed, oriented, simply-connected *Cat*-manifolds M and N and orientation preserving maps, where *Cat* $=$ *Diff* for smooth manifolds or *Cat* $=$ *Top* for topological manifolds. Let $\bar{\Theta}_m$ be the group of smooth homotopy spheres defined by M. Kervaire and J. Milnor in [6]. Recall that the collection of homotopy spheres Σ which admit a diffeomorphism $M\to M\#\Sigma$ form a subgroup $I(M)$ of $\bar{\Theta}_m$, called the inertia group of M, where we regard the connected sum $M\#\Sigma^m$ as a smooth manifold with the same underlying topological space as M and with smooth structure differing from that of M only on an m-disc. The homotopy inertia group $I_h(M)$ of M^m is a subset of the inertia group consisting of homotopy spheres Σ for which the identity map id: $M\to M\#\Sigma^m$ is homotopic to a diffeomorphism. Similarly, the concordance inertia group of M^m, $I_c(M^m)\subseteq\bar{\Theta}_m$, consists of those homotopy spheres Σ^m such that M and $M\#\Sigma^m$ are concordant.

The paper is organized as following. Let M^{2n} denote a closed $(n-1)$-connected smoothable topological $2n$-manifold. In Section 2, we show that the group $\mathcal{C}(M^{2n})$ of concordance classes of smoothings of M^{2n} is isomorphic to the group of smooth homotopy spheres $\bar{\Theta}_{2n}$ for $n=4$ or 5, the concordance inertia group $I_c(M^{2n})=0$ for $n=3$, 4, 5 or 11 and the homotopy inertia group $I_h(M^{2n})=0$ for $n=4$.

In Section 3, we present a new proof of the following result in [9].

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Theorem 1.1. *Let M^{2n} be an $(n-1)$-connected closed smooth manifold of dimension $2n \neq 4$ such that $H^n(M;\mathbb{Z}) \cong \mathbb{Z}$. Then the inertia group $I(M^{2n}) \cong \mathbb{Z}_2$.*

In Section 4, we show that, up to orientation-preserving diffeomorphism, M^8 has at most two distinct smooth structures; M^{10} has exactly six distinct smooth structures and if M^{14} is a π-manifold, then M^{14} has exactly two distinct smooth structures.

2. Concordance inertia groups of $(n-1)$-connected $2n$-manifolds

We recall some terminology from [6]:

Definition 2.1. (a) A homotopy m-sphere Σ^m is a closed oriented smooth manifold homotopy equivalent to the standard unit sphere $\mathbb{S}^m$ in $\mathbb{R}^{m+1}$.
(b) A homotopy m-sphere Σ^m is said to be exotic if it is not diffeomorphic to $\mathbb{S}^m$.

Definition 2.2. Define the m-th group of smooth homotopy spheres Θ_m as follows. Elements are oriented h-cobordism classes $[\Sigma]$ of homotopy m-spheres Σ, where Σ and Σ' are called (oriented) h-cobordant if there is an oriented h-cobordism $(W, \partial_0 W, \partial_1 W)$ together with orientation preserving diffeomorphisms $\Sigma \to \partial_0 W$ and $(\Sigma')^- \to \partial_1 W$. The addition is given by the connected sum. The zero element is represented by $\mathbb{S}^m$. The inverse of $[\Sigma]$ is given by $[\Sigma^-]$, where Σ^- is obtained from Σ by reversing the orientation. M. Kervaire and J. Milnor [6] showed that each Θ_m is a finite abelian group ($m \geq 1$).

Definition 2.3. Two homotopy m-spheres Σ_1^m and Σ_2^m are said to be equivalent if there exists an orientation preserving diffeomorphism $f\colon \Sigma_1^m \to \Sigma_2^m$.

The set of equivalence classes of homotopy m-spheres is denoted by $\bar{\Theta}_m$. The Kervaire–Milnor [6] paper worked rather with the group Θ_m of smooth homotopy spheres up to h-cobordism. This makes a difference only for $m = 4$, since it is known, using the h-cobordism theorem of Smale [12], that $\Theta_m \cong \bar{\Theta}_m$ for $m \neq 4$. However the difference is important in the four dimensional case, since Θ_4 is trivial, while the structure of $\bar{\Theta}_4$ is a great unsolved problem.

Definition 2.4. Let M be a closed topological manifold. Let (N, f) be a pair consisting of a smooth manifold N together with a homeomorphism $f\colon N \to M$. Two such pairs (N_1, f_1) and (N_2, f_2) are concordant provided there exists a diffeomorphism $g\colon N_1 \to N_2$ such that the composition $f_2 \circ g$ is topologically concordant to f_1, i.e., there exists a homeomorphism $F\colon N_1 \times [0,1] \to M \times [0,1]$ such that $F_{|N_1 \times 0} = f_1$ and $F_{|N_1 \times 1} = f_2 \circ g$. The set of all such concordance classes is denoted by $\mathcal{C}(M)$.

We will denote the class in $\mathcal{C}(M)$ of $(M^m \# \Sigma^m, \mathrm{id})$ by $[M^m \# \Sigma^m]$. (Note that $[M^n \# \mathbb{S}^n]$ is the class of (M^n, id).)

DEFINITION 2.5. Let M^m be a closed smooth m-dimensional manifold. The inertia group $I(M) \subset \bar{\Theta}_m$ is defined as the set of $\Sigma \in \bar{\Theta}_m$ for which there exists a diffeomorphism $\phi\colon M \to M \# \Sigma$.

Define the homotopy inertia group $I_h(M)$ to be the set of all $\Sigma \in I(M)$ such that there exists a diffeomorphism $M \to M \# \Sigma$ which is homotopic to $\mathrm{id}\colon M \to M \# \Sigma$.

Define the concordance inertia group $I_c(M)$ to be the set of all $\Sigma \in I_h(M)$ such that $M \# \Sigma$ is concordant to M.

REMARK 2.6. (1) Clearly, $I_c(M) \subseteq I_h(M) \subseteq I(M)$.
(2) For $M = \mathbb{S}^m$, $I_c(M) = I_h(M) = I(M) = 0$.

Now we have the following:

Theorem 2.7. *Let M^{2n} be a closed smooth $(n-1)$-connected $2n$-manifold with $n \geq 3$.*
(i) *If n is any integer such that Θ_{n+1} is trivial, then $I_c(M^{2n}) = 0$.*
(ii) *If n is any integer greater than 3 such that Θ_n and Θ_{n+1} are trivial, then*

$$\mathcal{C}(M^{2n}) = \{[M^{2n} \# \Sigma] \mid \Sigma \in \bar{\Theta}_{2n}\} \cong \bar{\Theta}_{2n}.$$

(iii) *If $n = 8$ and $H^n(M; \mathbb{Z}) \cong \mathbb{Z}$, then $M^{2n} \# \Sigma^{2n}$ is not concordant to M^{2n}, where $\Sigma^{2n} \in \bar{\Theta}_{2n}$ is the exotic sphere. In particular, $\mathcal{C}(M^{2n})$ has at least two elements.*
(iv) *If n is any even integer such that Θ_n and Θ_{n+1} are trivial, then $I_h(M) = 0$.*

Proof. Let $Cat = Top$ or G, where Top and G are the stable spaces of self homeomorphisms of $\mathbb{R}^n$ and self homotopy equivalences of $\mathbb{S}^{n-1}$ respectively. For any degree one map $f_M\colon M \to \mathbb{S}^{2n}$, we have a homomorphism

$$f_M^*\colon [\mathbb{S}^{2n}, Cat/O] \to [M, Cat/O].$$

By Wall [15], M has the homotopy type of $X = \left(\bigvee_{i=1}^k \mathbb{S}_i^n\right) \cup_g \mathbb{D}^{2n}$, where k is the n-th Betti number of M, $\bigvee_{i=1}^k \mathbb{S}_i^n$ is the wedge sum of n-spheres which is the n-skeleton of M and $g\colon \mathbb{S}^{2n-1} \to \bigvee_{i=1}^k \mathbb{S}_i^n$ is the attaching map of $\mathbb{D}^{2n}$. Let $\phi\colon M \to X$ be a homotopy equivalence of degree one and $q\colon X \to \mathbb{S}^{2n}$ be the collapsing map obtained by identifying $\mathbb{S}^{2n}$ with $X/\bigvee_{i=1}^k \mathbb{S}_i^n$ in an orientation preserving way. Let $f_M = q \circ \phi\colon M \to \mathbb{S}^{2n}$ be the degree one map.

Consider the following Puppe's exact sequence for the inclusion $i\colon \bigvee_{i=1}^k \mathbb{S}_i^n \hookrightarrow X$ along Cat/O:

(2.1)

$$\cdots \to \left[\bigvee_{i=1}^k S\mathbb{S}_i^n, Cat/O\right] \xrightarrow{(S(g))^*} [\mathbb{S}^{2n}, Cat/O] \xrightarrow{q^*} [X, Cat/O] \xrightarrow{i^*} \left[\bigvee_{i=1}^k \mathbb{S}_i^n, Cat/O\right],$$

where $S(g)$ is the suspension of the map $g\colon \mathbb{S}^{2n-1} \to \bigvee_{i=1}^k \mathbb{S}_i^n$.

Using the fact that

$$\left[\bigvee_{i=1}^{k} S\mathbb{S}_i^n, Cat/O\right] \cong \prod_{i=1}^{k} [\mathbb{S}_i^{n+1}, Cat/O]$$

and

$$\left[\bigvee_{i=1}^{k} \mathbb{S}_i^n, Cat/O\right] \cong \prod_{i=1}^{k} [\mathbb{S}_i^{n}, Cat/O],$$

the above exact sequence (2.1) becomes

$$\cdots \to \prod_{i=1}^{k} [\mathbb{S}_i^{n+1}, Cat/O] \xrightarrow{(S(g))^*} [\mathbb{S}^{2n}, Cat/O] \xrightarrow{q^*} [X, Cat/O] \xrightarrow{i^*} \prod_{i=1}^{k} [\mathbb{S}_i^{n}, Cat/O].$$

(i): If n is any integer such that Θ_{n+1} is trivial and $Cat = Top$ in the above exact sequence (2.1), by using the fact that

$$[\mathbb{S}^m, Top/O] = \bar{\Theta}_m \quad (m \neq 3, 4)$$

and $[\mathbb{S}^4, Top/O] = 0$ ([10, pp. 200–201]), we have $q^*\colon [\mathbb{S}^{2n}, Top/O] \to [X, Top/O]$ is injective. Hence $f_M^* = \phi^* \circ q^*\colon \bar{\Theta}_{2n} \to [M, Top/O]$ is injective. By using the identifications $\mathcal{C}(M^{2n}) = [M, Top/O]$ given by [10, pp. 194–196], $f_M^*\colon \bar{\Theta}_{2n} \to \mathcal{C}(M^{2n})$ becomes $[\Sigma^{2n}] \to [M \# \Sigma^{2n}]$. $I_c(M)$ is exactly the kernel of f_M^*, and so $I_c(M) = 0$. This proves (i).

(ii): If $n > 3$, Θ_n and Θ_{n+1} are trivial, and $Cat = Top$ then, from the above exact sequence (2.1) we have $q^*\colon [\mathbb{S}^{2n}, Top/O] \to [X, Top/O]$ is an isomorphism. This shows that $f_M^* = \phi^* \circ q^*\colon \bar{\Theta}_{2n} \to \mathcal{C}(M^{2n})$ is an isomorphism and hence

$$\mathcal{C}(M^{2n}) = \{[M^{2n} \# \Sigma] \mid \Sigma \in \bar{\Theta}_{2n}\}.$$

This proves (ii).

(iii): If $n = 8$ and $H^n(M; \mathbb{Z}) \cong \mathbb{Z}$, then M^{2n} has the homotopy type of $X = \mathbb{S}^n \cup_g \mathbb{D}^{2n}$, where $g\colon \mathbb{S}^{2n-1} \to \mathbb{S}^n$ is the attaching map. In order to prove $M^{2n} \# \Sigma^{2n}$ is not concordant to M^{2n}, by the above exact sequence (2.1) for $Cat = Top$, it suffices to prove $q^*\colon [\mathbb{S}^{16}, Top/O] \to [X, Top/O]$ is monic, which is equivalent to saying that $(S(g))^*\colon [S\mathbb{S}^8, Top/O] \to [\mathbb{S}^{16}, Top/O]$ is the zero homomorphism. For the case $g = p$, where $p\colon \mathbb{S}^{15} \to \mathbb{S}^8$ is the Hopf map, $(S(g))^*$ is the zero homomorphism, which was proved in the course of the proof of Lemma 1 in [2, pp. 58–59]. This proof works verbatim for any map $g\colon \mathbb{S}^{2n-1} \to \mathbb{S}^n$ as well. This proves (iii).

(iv): If n is any even integer such that Θ_n and Θ_{n+1} are trivial, then $\pi_{n+1}(G/O) = 0$. This shows that from the above exact sequence (2.1) for $Cat = G$, $q^*\colon [\mathbb{S}^{2n}, G/O] \to [X, G/O]$ is injective. Then $f_M^* = \phi^* \circ q^*\colon [\mathbb{S}^{2n}, G/O] \to [M, G/O]$ is injective. From the surgery exact sequences of M and $\mathbb{S}^{2n}$, we get the following commutative diagram ([3, Lemma 3.4]):

$$\begin{tikzcd}
L_{2n+1}(e) \arrow[r] \arrow[d, "="] & \bar{\Theta}_{2n} \arrow[r, "\eta_{\mathbb{S}^{2n}}"] \arrow[d, "f_M^{\bullet}"] & \pi_{2n}(G/O) \arrow[r] \arrow[d, "f_M^*"] & L_{2n}(e) \arrow[d, "="] \\
L_{2n+1}(e) \arrow[r] & \mathcal{S}^{Diff}(M) \arrow[r, "\eta_M"] & [M, G/O] \arrow[r] & L_{2n}(e)
\end{tikzcd} \tag{2.2}$$

By using the facts that $L_{2n+1}(e) = 0$, injectivity of $\eta_{\mathbb{S}^{2n}}$ and η_M follow from the diagram, and combine with the injectivity of f_M^* to show that $f_M^{\bullet}\colon \bar{\Theta}_{2n} \to \mathcal{S}^{Diff}(M)$ is injective. $I_h(M)$ is exactly the kernel of $f_M^{\bullet}$, and so $I_h(M) = 0$. This proves (iv). $\square$

REMARK 2.8. (i) By M. Kervaire and J. Milnor [6], $\Theta_m = 0$ for $m = 1, 2, 3, 4,$ 5,6 or 12. If M^{2n} is a closed smooth $(n-1)$-connected $2n$-manifold, by Theorem 2.7 (i) and (ii), $I_c(M^{2n}) = 0$ for $n = 3, 4, 5$ or 11 and $\mathcal{C}(M^{2n}) \cong \bar{\Theta}_{2n}$ for $n = 4$ or 5.
(ii) If M has the homotopy type of $\mathbb{O}\mathbf{P}^2$, by Theorem 1.1 and Theorem 2.7 (iii), we have $I_c(M) = 0 \neq I(M)$.
(iii) By Theorem 2.7 (iv), if M has the homotopy type of $\mathbb{H}\mathbf{P}^2$, then $I_h(M) = 0$.

DEFINITION 2.9. Let M and N are smooth manifolds. A smooth map $f\colon M \to N$ is called tangential if for some integers k, l, $f^*(T(N)) \oplus \epsilon_M^k \cong T(M) \oplus \epsilon_M^l$.

DEFINITION 2.10. Let M be a topological manifold. Let (N, f) be a pair consisting of a smooth manifold N together with a tangential homotopy equivalence of degree one $f\colon N \to M$. Two such pairs (N_1, f_1) and (N_2, f_2) are equivalent provided there exists a diffeomorphism $g\colon N_1 \to N_2$ such that $f_2 \circ g$ is homotopic to f_1. The set of all such equivalence classes is denoted by $\theta(M)$.

For $M = \mathbb{H}\mathbf{P}^2$, [5, Theorem 4] shows $\theta(\mathbb{H}\mathbf{P}^2)$ contains at most two elements. Now by Remark 2.8 (iii), we have the following:

Corollary 2.11. $\theta(\mathbb{H}\mathbf{P}^2)$ *contains exactly two elements, with representatives given by* $(\mathbb{H}\mathbf{P}^2, \mathrm{id})$ *and* $(\mathbb{H}\mathbf{P}^2 \# \Sigma^8, \mathrm{id})$, *where* Σ^8 *is the exotic* 8*-sphere.*

3. Inertia groups of projective plane-like manifolds

In [15], C.T.C. Wall assigned to each closed oriented $(n-1)$-connected $2n$-dimensional smooth manifold M^{2n} with $n \geq 3$, a system of invariants as follows:
(1) $H = H^n(M; \mathbb{Z}) \cong \mathrm{Hom}(H_n(M; \mathbb{Z}), \mathbb{Z}) \cong \bigoplus_{j=1}^k \mathbb{Z}$, the cohomology group of M, with k the n-th Betti number of M,
(2) $I\colon H \times H \to \mathbb{Z}$, the intersection form of M which is unimodular and n-symmetric, defined by

$$I(x, y) = \langle x \cup y, [M] \rangle,$$

where the homology class $[M]$ is the orientation class of M,
(3) A map $\alpha\colon H^n(M;\mathbb{Z}) \to \pi_{n-1}(SO_n)$ that assigns each element $x \in H^n(M;\mathbb{Z})$ to the characteristic map $\alpha(x)$ for the normal bundle of the embedded n-sphere $\mathbb{S}^n_x$ representing x.
Denote by $\chi = S \circ \alpha\colon H^n(M;\mathbb{Z}) \to \pi_{n-1}(SO_{n+1}) \cong \widetilde{KO}(\mathbb{S}^n)$, where $S\colon \pi_{n-1}(SO_n) \to \pi_{n-1}(SO_{n+1})$ is the suspension map. Then

$$\chi = S \circ \alpha \in H^n(M;\widetilde{KO}(\mathbb{S}^n)) = \mathrm{Hom}(H^n(M;\mathbb{Z});\widetilde{KO}(\mathbb{S}^n))$$

can be viewed as an n-dimensional cohomology class of M, with coefficients in $\widetilde{KO}(\mathbb{S}^n)$. The obstruction to triviality of the tangent bundle over the n-skeleton is the element $\chi \in H^n(M;\widetilde{KO}(\mathbb{S}^n))$ [15]. By [15, pp. 179–180], the Pontrjagin class of M^{2n} is given by

$$p_m(M^{2n}) = \pm a_m(2m-1)!\,\chi, \tag{3.1}$$

where $n = 4m$ and

$$a_m = \begin{cases} 1 & \text{if} \quad 4m \equiv 0 \pmod 8, \\ 2 & \text{if} \quad 4m \equiv 4 \pmod 8. \end{cases}$$

Define $\Theta_n(k)$ to be the subgroup of $\bar{\Theta}_n$ consisting of those homotopy n-sphere Σ^n which are the boundaries of k-connected $(n+1)$-dimensional compact manifolds, $1 \le k < [n/2]$. Thus, $\Theta_n(k)$ is the kernel of the natural map $i_k\colon \bar{\Theta}_n \to \Omega_n(k)$, where $\Omega_n(k)$ is the n-dimensional group in k-connective cobordism theory [13] and i_k sends Σ^n to its cobordism class. Using surgery, we see $\Omega_*(1)$ is the usual oriented cobordism group. So $\bar{\Theta}_n = \Theta_n(1)$. Similarly, $\Omega_n(2) \cong \Omega_n^{Spin}$ $(n \ge 7)$; since $BSpin$ is, in fact, 3-connected, for $n \ge 8$, $\Omega_n(2) \cong \Omega_n(3)$ and $\Theta_n(2) = \Theta_n(3) = bSpin_n$. Here $bSpin_n$ consists of homotopy n-sphere which bound spin manifolds.

In [16], C.T.C. Wall defined the Grothendieck group $\mathcal{G}_n^{2n+1}$, a homomorphism $\vartheta\colon \mathcal{G}_n^{2n+1} \to \bar{\Theta}_{2n}$ such that $\vartheta(\mathcal{G}_n^{2n+1}) = \Theta_{2n}(n-1)$ and proved the following theorem:

Theorem 3.1 (Wall). *Let M^{2n} be a closed smooth $(n-1)$-connected $2n$-manifold and Σ^{2n} be a homotopy sphere in $\bar{\Theta}_{2n}$. Then $M \# \Sigma^{2n}$ is an orientation-preserving diffeomorphic to M if and only if*
(i) $\Sigma^{2n} = 0$ *in* $\bar{\Theta}_{2n}$ *or*
(ii) $\chi \not\equiv 0 \pmod 2$ *and* $\Sigma^{2n} \in \vartheta(\mathcal{G}_n^{2n+1}) = \Theta_{2n}(n-1)$

We also need the following result from [1]:

Theorem 3.2 (Anderson, Brown, Peterson). *Let $\eta_n\colon \bar{\Theta}_n \to \Omega_n^{Spin}$ be the homomorphism such that η_n sends Σ^n to its spin cobordism class. Then $\eta_n \ne 0$ if and only if $n = 8k+1$ or $8k+2$.*

Proof of Theorem 1.1. Let ξ be a generator of $H^n(M^{2n};\mathbb{Z})$. Consider the case $n=4$. Then by Itiro Tamura [14] and (3.1), the Pontrjagin class of M^{2n} is given by

$$p_1(M^{2n}) = 2(2h+1)\xi = \pm 2\chi,$$

where $h \in \mathbb{Z}$. This implies that

$$\chi = \pm(2h+1)\xi.$$

Likewise, for $n=8$, we have

$$p_2(M^{2n}) = 6(2k+1)\xi = \pm 6\chi,$$

where $k \in \mathbb{Z}$. This implies that

$$\chi = \pm(2k+1)\xi.$$

Therefore in either case, $\chi \not\equiv 0 \pmod 2$. Now by Theorem 3.1, it follows that

$$I(M^{2n}) = \Theta_{2n}(n-1).$$

Since $\Theta_{2n}(n-1)$ is the kernel of the natural map $i_{n-1}\colon \bar{\Theta}_{2n} \to \Omega_{2n}(n-1)$, where $\Omega_{2n}(n-1) \cong \Omega_8^{Spin}$ for $n=4$ and $\Omega_{2n}(n-1) \cong \Omega_{16}^{String} \cong \mathbb{Z} \oplus \mathbb{Z}$ for $n=8$ [4]. Now by Theorem 3.2 and using the fact that $\bar{\Theta}_{16} \cong \mathbb{Z}_2$ [6], we have $i_{n-1} = 0$ for $n=4$ and 8. This shows that $\Theta_{2n}(n-1) = \bar{\Theta}_{2n}$. This implies that

$$I(M^{2n}) \cong \mathbb{Z}_2.$$

This completes the proof of Theorem 1.1. □

4. Smooth structures of $(n-1)$-connected $2n$-manifolds

DEFINITION 4.1 (*Cat* = *Diff* or *Top*-structure sets, [3]). Let M be a closed *Cat*-manifold. We define the *Cat*-structure set $\mathcal{S}^{Cat}(M)$ to be the set of equivalence classes of pairs (N, f) where N is a closed *Cat*-manifold and $f\colon N \to M$ is a homotopy equivalence. And the equivalence relation is defined as follows:

$(N_1, f_1) \sim (N_2, f_2)$ if there is a *Cat*-isomorphism $\phi\colon N_1 \to N_2$
such that $f_2 \circ h$ is homotopic to f_1.

We will denote the class in $\mathcal{S}^{Cat}(M)$ of (N, f) by $[(N, f)]$. The base point of $S^{Cat}(M)$ is the equivalence class $[(M, \mathrm{id})]$ of $\mathrm{id}\colon M \to M$.

The forgetful maps $F_{Diff}\colon \mathcal{S}^{Diff}(M) \to \mathcal{S}^{Top}(M)$ and $F_{Con}\colon \mathcal{C}(M) \to \mathcal{S}^{Diff}(M)$ fit into a short exact sequence of pointed sets [3]:

$$\mathcal{C}(M) \xrightarrow{F_{Con}} \mathcal{S}^{Diff}(M) \xrightarrow{F_{Diff}} \mathcal{S}^{Top}(M).$$

Theorem 4.2. *Let n be any integer greater than 3 such that Θ_n and Θ_{n+1} are trivial and M^{2n} be a closed smooth $(n-1)$-connected $2n$-manifold. Let $f: N \to M$ be a homeomorphism where N is a closed smooth manifold. Then*
(i) *there exists a diffeomorphism $\phi: N \to M \# \Sigma^{2n}$, where $\Sigma^{2n} \in \bar{\Theta}_{2n}$ such that the following diagram commutes up to homotopy*:

$$\begin{tikzcd} N \arrow[r, "\phi"] \arrow[dr, "f"'] & M \# \Sigma^{2n} \arrow[d, "\mathrm{id}"] \\ & M \end{tikzcd}$$

(ii) *If $I_h(M) = \bar{\Theta}_{2n}$, then $f: N \to M$ is homotopic to a diffeomorphism.*

Proof. Consider the short exact sequence of pointed sets

$$\mathcal{C}(M) \xrightarrow{F_{Con}} \mathcal{S}^{Diff}(M) \xrightarrow{F_{Diff}} \mathcal{S}^{Top}(M).$$

By Theorem 2.7 (ii), we have

$$\mathcal{C}(M) = \{[M \# \Sigma] \mid \Sigma \in \bar{\Theta}_{2n}\} \cong \bar{\Theta}_{2n}.$$

Since $[(N, f)] \in F_{Diff}^{-1}([(M, \mathrm{id})])$, we obtain

$$[(N, f)] \in \mathrm{Im}(F_{Con}) = \{[M \# \Sigma] \mid \Sigma \in \bar{\Theta}_{2n}\}.$$

This implies that there exists a homotopy sphere $\Sigma^{2n} \in \bar{\Theta}_{2n}$ such that $(N, f) \sim (M \# \Sigma^{2n}, \mathrm{id})$ in $\mathcal{S}^{Diff}(M)$. This implies that there exists a diffeomorphism $\phi: N \to M \# \Sigma^{2n}$ such that f is homotopic to $\mathrm{id} \circ \phi$. This proves (i).

If $I_h(M) = \bar{\Theta}_{2n}$, then $\mathrm{Im}(F_{Con}) = \{[(M,\mathrm{id})]\}$ and hence $(N, f) \sim (M,\mathrm{id})$ in $\mathcal{S}^{Diff}(M)$. This shows that $f: N \to M$ is homotopic to a diffeomorphism $N \to M$. This proves (ii). $\square$

Theorem 4.3. *Let n be any integer greater than 3 such that Θ_n and Θ_{n+1} are trivial and M^{2n} be a closed smooth $(n-1)$-connected $2n$-manifold. Then the number of distinct smooth structures on M^{2n} up to diffeomorphism is less than or equal to the cardinality of $\bar{\Theta}_{2n}$. In particular, the set of diffeomorphism classes of smooth structures on M^{2n} is $\{[M \# \Sigma] \mid \Sigma \in \bar{\Theta}_{2n}\}$.*

Proof. By Theorem 4.2 (i), if N is a closed smooth manifold homeomorphic to M, then N is diffeomorphic to $M \# \Sigma^{2n}$ for some homotopy $2n$-sphere Σ^{2n}. This implies that the set of diffeomorphism classes of smooth structures on M^{2n} is $\{[M \# \Sigma] \mid \Sigma \in \bar{\Theta}_{2n}\}$. This shows that the number of distinct smooth structures on M^{2n} up to diffeomorphism is less than or equal to the cardinality of $\bar{\Theta}_{2n}$. $\square$

REMARK 4.4. (1) By Theorem 4.3, every closed smooth 3-connected 8-manifold has at most two distinct smooth structures up to diffeomorphism.
(2) If M^8 is a closed smooth 3-connected 8-manifold such that $H^4(M;\mathbb{Z}) \cong \mathbb{Z}$, then by Theorem 1.1, $I(M) \cong \mathbb{Z}_2$. Now by Theorem 4.3, M has a unique smooth structure up to diffeomorphism.
(3) If $M = \mathbb{S}^4 \times \mathbb{S}^4$, then by Theorem 4.3, $\mathbb{S}^4 \times \mathbb{S}^4$ has at most two distinct smooth structures up to diffeomorphism, namely, $\{[\mathbb{S}^4\times\mathbb{S}^4],[\mathbb{S}^4\times\mathbb{S}^4\#\Sigma]\}$, where Σ is the exotic 8-sphere. However, by [11, Theorem A], $I(\mathbb{S}^4 \times \mathbb{S}^4) = 0$. This implies that $\mathbb{S}^4 \times \mathbb{S}^4$ has exactly two distinct smooth structures.

Theorem 4.5. *Let M be a closed smooth 3-connected 8-manifold with stable tangential invariant $\chi = S\circ\alpha\colon H_4(M;\mathbb{Z}) \to \pi_3(SO) = \mathbb{Z}$. Then M has exactly two distinct smooth structures up to diffeomorphism if and only if* $\operatorname{Im}(S \circ \alpha) \subseteq 2\mathbb{Z}$.

Proof. Suppose M has exactly two distinct smooth structures up to diffeomorphism. Then by Theorem 4.3, M and $M \# \Sigma$ are not diffeomorphic, where Σ is the exotic 8-sphere. Since $\bar{\Theta}_8 = \Theta_8(3)$, by Theorem 3.1, the stable tangential invariant χ is zero (mod 2) and hence $\operatorname{Im}(S \circ \alpha) \subseteq 2\mathbb{Z}$. Conversely, suppose $\operatorname{Im}(S \circ \alpha) \subseteq 2\mathbb{Z}$. Now by Theorem 3.1, M can not be diffeomorphic to $M \# \Sigma$, where Σ is the exotic 8-sphere. Now by Theorem 4.3, M has exactly two distinct smooth structures up to diffeomorphism. $\square$

REMARK 4.6. If $n = 2, 3, 5, 6, 7 \pmod 8$ or the stable tangential invariant χ of M^{2n} is zero (mod 2), then by [16, Corollary, p. 289] and Theorem 3.1, we have $I(M^{2n}) = 0$. So, by Theorem 4.3, we have the following:

Theorem 4.7. *Let n be any integer greater than 3 such that Θ_n and Θ_{n+1} are trivial and M^{2n} be a closed smooth $(n-1)$-connected $2n$-manifold. If $n = 2, 3, 5, 6, 7$ (mod 8) or the stable tangential invariant χ of M^{2n} is zero (mod 2), then the set of diffeomorphism classes of smooth structures on M^{2n} is in one-to-one correspondence with group $\bar{\Theta}_{2n}$.*

REMARK 4.8. (1) By Theorem 4.7, every closed smooth 4-connected 10-manifold has exactly six distinct smooth structures, namely, $\{[M \# \Sigma] \mid \Sigma \in \bar{\Theta}_{10} \cong \mathbb{Z}_6\}$.
(2) If M^{2n} is n-parallelisable, almost parallelisable or π-manifold, then the stable tangential invariant χ of M is zero [15]. Then by Theorem 4.7, we have the following:

Corollary 4.9. *Let n be any integer greater than 3 such that Θ_n and Θ_{n+1} are trivial and M^{2n} be a closed smooth $(n-1)$-connected $2n$-manifold. If M^{2n} is n-parallelisable, almost parallelisable or π-manifold, then the set of diffeomorphism classes of smooth structures on M^{2n} is in one-to-one correspondence with group $\bar{\Theta}_{2n}$.*

DEFINITION 4.10 ([8]). The normal k-type of a closed smooth manifold M is the fibre homotopy type of a fibration $p\colon B \to BO$ such that the fibre of the map p

is connected and its homotopy groups vanish in dimension $\geq k+1$, admitting a lift of the normal Gauss map $\nu_M\colon M \to BO$ to a map $\bar{\nu}_M\colon M \to B$ such that $\bar{\nu}_M\colon M \to B$ is a $(k+1)$-equivalence, i.e., the induced homomorphism $\bar{\nu}_M\colon \pi_i(M) \to \pi_i(B)$ is an isomorphism for $i \leq k$ and surjective for $i = k+1$. We call such a lift a normal k-smoothing.

Theorem 4.11. *Let $n = 5, 7$ and let M_0 and M_1 be closed smooth $(n-1)$-connected $2n$-manifolds with the same Euler characteristic. Then*
(i) *There is a homotopy sphere $\Sigma^{2n} \in \bar{\Theta}_{2n}$ such that M_0 and $M_1 \# \Sigma^{2n}$ are diffeomorphic.*
(ii) *Let M^{2n} be a closed smooth $(n-1)$-connected $2n$-manifold such that $[M] = 0 \in \Omega_{2n}^{String}$ and let Σ be any exotic $2n$-sphere in $\bar{\Theta}_{2n}$. Then M and $M \# \Sigma$ are not diffeomorphic.*

Proof. (i): M_0 and M_1 are $(n-1)$-connected, and n is 5 or 7; therefore, $p_1/2$ and the Stiefel–Whitney classes ω_2 vanish. So, M_0 and M_1 are *BString*-manifolds. Let $\bar{\nu}_{M_j}\colon M_j \to BString$ be a lift of the normal Gauss map $\nu_{M_j}\colon M_j \to BO$ in the fibration $p\colon BString = BO\langle 8\rangle \to BO$, where $j = 0$ and 1. Since *BString* is 7-connected, $p_{\#}\colon \pi_i(BString) \to \pi_i(BO)$ is an isomorphism for all $i \geq 8$. This shows that $\bar{\nu}_{M_j}\colon M_j \to BString$ is an n-equivalence and hence the normal $(n-1)$-type of M_0 and M_1 is $p\colon BString \to BO$. We know that $\Omega_{2n}^{String} \cong \bar{\Theta}_{2n}$, where the group structure is given by connected sum [4]. This implies that there always exists $\Sigma^{2n} \in \bar{\Theta}_{2n}$ such that M_0 and $M_1 \# \Sigma^{2n}$ are *BString*-bordant. Since M_0 and $M_1 \# \Sigma^{2n}$ have the same Euler characteristic, by [8, Corollary 4], M_0 and $M_1 \# \Sigma^{2n}$ are diffeomorphic.

(ii): Since the image of the standard sphere under the isomorphism $\bar{\Theta}_{2n} \cong \Omega_{2n}^{String}$ represents the trivial element in Ω_{2n}^{String}, we have $[M^{2n}] \neq [M \# \Sigma]$ in Ω_{2n}^{String}. This implies that M and $M \# \Sigma$ are not *BString*-bordant. By obstruction theory, M^{2n} has a unique string structure. This implies that M and $M \# \Sigma$ are not diffeomorphic. □

Theorem 4.12. *Let M be a closed smooth 6-connected 14-dimensional π-manifold and Σ is the exotic 14-sphere. Then $M \# \Sigma$ is not diffeomorphic to M. Thus, $I(M) = 0$. Moreover, if N is a closed smooth manifold homeomorphic to M, then N is diffeomorphic to either M or $M \# \Sigma$.*

Proof. It follows from results of Anderson, Brown and Peterson on spin cobordism [1] that the image of the natural homomorphism $\Omega_{14}^{framed} \to \Omega_{14}^{Spin}$ is 0 and $\Omega_{14}^{String} \cong \Omega_{14}^{Spin} \cong \mathbb{Z}_2$ [4]. This shows that $[M] = 0 \in \Omega_{14}^{String}$. Now by Theorem 4.11 (ii), $M \# \Sigma$ is not diffeomorphic to M. If N is a closed smooth manifold homeomorphic to M, then N and M have the same Euler characteristic. Then by Theorem 4.11 (i), N is diffeomorphic to either M or $M \# \Sigma$. □

REMARK 4.13. By the above Theorem 4.12, the set of diffeomorphism classes of smooth structures on a closed smooth 6-connected 14-dimensional π-manifold M is

$$\{[M], [M \# \Sigma]\} \cong \mathbb{Z}_2,$$

where Σ is the exotic 14-sphere. So, the number of distinct smooth structures on M is 2.

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Department of Mathematics
Indian Institute of Technology, Bombay
Mumbai-76
India
e-mail: rameshk@math.iitb.ac.in
e-mail: mathsramesh1984@gmail.com