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# INERTIA GROUPS AND SMOOTH STRUCTURES OF ( $n-1$ )-CONNECTED $2 n$-MANIFOLDS 

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#### Abstract

Let $M^{2 n}$ denote a closed ( $n-1$ )-connected smoothable topological $2 n$-manifold. We show that the group $\mathcal{C}\left(M^{2 n}\right)$ of concordance classes of smoothings of $M^{2 n}$ is isomorphic to the group of smooth homotopy spheres $\bar{\Theta}_{2 n}$ for $n=4$ or 5 , the concordance inertia group $I_{c}\left(M^{2 n}\right)=0$ for $n=3,4,5$ or 11 and the homotopy inertia group $I_{h}\left(M^{2 n}\right)=0$ for $n=4$. On the way, following Wall's approach [16] we present a new proof of the main result in [9], namely, for $n=4,8$ and $H^{n}\left(M^{2 n} ; \mathbb{Z}\right) \cong \mathbb{Z}$, the inertia group $I\left(M^{2 n}\right) \cong \mathbb{Z}_{2}$. We also show that, up to orientation-preserving diffeomorphism, $M^{8}$ has at most two distinct smooth structures; $M^{10}$ has exactly six distinct smooth structures and then show that if $M^{14}$ is a $\pi$-manifold, $M^{14}$ has exactly two distinct smooth structures.


## 1. Introduction

We work in the categories of closed, oriented, simply-connected Cat-manifolds $M$ and $N$ and orientation preserving maps, where Cat $=$ Diff for smooth manifolds or Cat $=$ Top for topological manifolds. Let $\bar{\Theta}_{m}$ be the group of smooth homotopy spheres defined by M. Kervaire and J. Milnor in [6]. Recall that the collection of homotopy spheres $\Sigma$ which admit a diffeomorphism $M \rightarrow M \# \Sigma$ form a subgroup $I(M)$ of $\bar{\Theta}_{m}$, called the inertia group of $M$, where we regard the connected sum $M \# \Sigma^{m}$ as a smooth manifold with the same underlying topological space as $M$ and with smooth structure differing from that of $M$ only on an $m$-disc. The homotopy inertia group $I_{h}(M)$ of $M^{m}$ is a subset of the inertia group consisting of homotopy spheres $\Sigma$ for which the identity map id: $M \rightarrow M \# \Sigma^{m}$ is homotopic to a diffeomorphism. Similarly, the concordance inertia group of $M^{m}, I_{c}\left(M^{m}\right) \subseteq \bar{\Theta}_{m}$, consists of those homotopy spheres $\Sigma^{m}$ such that $M$ and $M \# \Sigma^{m}$ are concordant.

The paper is organized as following. Let $M^{2 n}$ denote a closed ( $n-1$ )-connected smoothable topological $2 n$-manifold. In Section 2, we show that the group $\mathcal{C}\left(M^{2 n}\right)$ of concordance classes of smoothings of $M^{2 n}$ is isomorphic to the group of smooth homotopy spheres $\bar{\Theta}_{2 n}$ for $n=4$ or 5 , the concordance inertia group $I_{c}\left(M^{2 n}\right)=0$ for $n=3$, 4,5 or 11 and the homotopy inertia group $I_{h}\left(M^{2 n}\right)=0$ for $n=4$.

In Section 3, we present a new proof of the following result in [9].

[^0]Theorem 1.1. Let $M^{2 n}$ be an ( $n-1$ )-connected closed smooth manifold of dimension $2 n \neq 4$ such that $H^{n}(M ; \mathbb{Z}) \cong \mathbb{Z}$. Then the inertia group $I\left(M^{2 n}\right) \cong \mathbb{Z}_{2}$.

In Section 4, we show that, up to orientation-preserving diffeomorphism, $M^{8}$ has at most two distinct smooth structures; $M^{10}$ has exactly six distinct smooth structures and if $M^{14}$ is a $\pi$-manifold, then $M^{14}$ has exactly two distinct smooth structures.

## 2. Concordance inertia groups of $(n-1)$-connected $2 n$-manifolds

We recall some terminology from [6]:
DEFINITION 2.1. (a) A homotopy $m$-sphere $\Sigma^{m}$ is a closed oriented smooth manifold homotopy equivalent to the standard unit sphere $\mathbb{S}^{m}$ in $\mathbb{R}^{m+1}$.
(b) A homotopy $m$-sphere $\Sigma^{m}$ is said to be exotic if it is not diffeomorphic to $\mathbb{S}^{m}$.

DEFINITION 2.2. Define the $m$-th group of smooth homotopy spheres $\Theta_{m}$ as follows. Elements are oriented $h$-cobordism classes [ $\Sigma$ ] of homotopy $m$-spheres $\Sigma$, where $\Sigma$ and $\Sigma^{\prime}$ are called (oriented) $h$-cobordant if there is an oriented $h$-cobordism $\left(W, \partial_{0} W, \partial_{1} W\right)$ together with orientation preserving diffeomorphisms $\Sigma \rightarrow \partial_{0} W$ and $\left(\Sigma^{\prime}\right)^{-} \rightarrow \partial_{1} W$. The addition is given by the connected sum. The zero element is represented by $\mathbb{S}^{m}$. The inverse of $[\Sigma]$ is given by $\left[\Sigma^{-}\right]$, where $\Sigma^{-}$is obtained from $\Sigma$ by reversing the orientation. M. Kervaire and J. Milnor [6] showed that each $\Theta_{m}$ is a finite abelian group ( $m \geq 1$ ).

Definition 2.3. Two homotopy $m$-spheres $\Sigma_{1}^{m}$ and $\Sigma_{2}^{m}$ are said to be equivalent if there exists an orientation preserving diffeomorphism $f: \Sigma_{1}^{m} \rightarrow \Sigma_{2}^{m}$.

The set of equivalence classes of homotopy $m$-spheres is denoted by $\bar{\Theta}_{m}$. The Kervaire-Milnor [6] paper worked rather with the group $\Theta_{m}$ of smooth homotopy spheres up to $h$-cobordism. This makes a difference only for $m=4$, since it is known, using the $h$-cobordism theorem of Smale [12], that $\Theta_{m} \cong \bar{\Theta}_{m}$ for $m \neq 4$. However the difference is important in the four dimensional case, since $\Theta_{4}$ is trivial, while the structure of $\bar{\Theta}_{4}$ is a great unsolved problem.

Definition 2.4. Let $M$ be a closed topological manifold. Let $(N, f)$ be a pair consisting of a smooth manifold $N$ together with a homeomorphism $f: N \rightarrow M$. Two such pairs $\left(N_{1}, f_{1}\right)$ and $\left(N_{2}, f_{2}\right)$ are concordant provided there exists a diffeomorphism $g: N_{1} \rightarrow N_{2}$ such that the composition $f_{2} \circ g$ is topologically concordant to $f_{1}$, i.e., there exists a homeomorphism $F: N_{1} \times[0,1] \rightarrow M \times[0,1]$ such that $F_{\mid N_{1} \times 0}=f_{1}$ and $F_{\mid N_{1} \times 1}=f_{2} \circ g$. The set of all such concordance classes is denoted by $\mathcal{C}(M)$.

We will denote the class in $\mathcal{C}(M)$ of $\left(M^{m} \# \Sigma^{m}\right.$, id) by $\left[M^{m} \# \Sigma^{m}\right]$. (Note that [ $\left.M^{n} \# \mathbb{S}^{n}\right]$ is the class of $\left.\left(M^{n}, \mathrm{id}\right).\right)$

Definition 2.5. Let $M^{m}$ be a closed smooth $m$-dimensional manifold. The inertia group $I(M) \subset \bar{\Theta}_{m}$ is defined as the set of $\Sigma \in \bar{\Theta}_{m}$ for which there exists a diffeomorphism $\phi: M \rightarrow M$ \# $\Sigma$.

Define the homotopy inertia group $I_{h}(M)$ to be the set of all $\Sigma \in I(M)$ such that there exists a diffeomorphism $M \rightarrow M \# \Sigma$ which is homotopic to id: $M \rightarrow M \# \Sigma$.

Define the concordance inertia group $I_{c}(M)$ to be the set of all $\Sigma \in I_{h}(M)$ such that $M \# \Sigma$ is concordant to $M$.

Remark 2.6. (1) Clearly, $I_{c}(M) \subseteq I_{h}(M) \subseteq I(M)$.
(2) For $M=\mathbb{S}^{m}, I_{c}(M)=I_{h}(M)=I(M)=0$.

Now we have the following:
Theorem 2.7. Let $M^{2 n}$ be a closed smooth $(n-1)$-connected $2 n$-manifold with $n \geq 3$.
(i) If $n$ is any integer such that $\Theta_{n+1}$ is trivial, then $I_{c}\left(M^{2 n}\right)=0$.
(ii) If $n$ is any integer greater than 3 such that $\Theta_{n}$ and $\Theta_{n+1}$ are trivial, then

$$
\mathcal{C}\left(M^{2 n}\right)=\left\{\left[M^{2 n} \# \Sigma\right] \mid \Sigma \in \bar{\Theta}_{2 n}\right\} \cong \bar{\Theta}_{2 n} .
$$

(iii) If $n=8$ and $H^{n}(M ; \mathbb{Z}) \cong \mathbb{Z}$, then $M^{2 n} \# \Sigma^{2 n}$ is not concordant to $M^{2 n}$, where $\Sigma^{2 n} \in \bar{\Theta}_{2 n}$ is the exotic sphere. In particular, $\mathcal{C}\left(M^{2 n}\right)$ has at least two elements.
(iv) If $n$ is any even integer such that $\Theta_{n}$ and $\Theta_{n+1}$ are trivial, then $I_{h}(M)=0$.

Proof. Let $C a t=T o p$ or $G$, where Top and $G$ are the stable spaces of self homeomorphisms of $\mathbb{R}^{n}$ and self homotopy equivalences of $\mathbb{S}^{n-1}$ respectively. For any degree one map $f_{M}: M \rightarrow \mathbb{S}^{2 n}$, we have a homomorphism

$$
f_{M}^{*}:\left[\mathbb{S}^{2 n}, \mathrm{Cat} / \mathrm{O}\right] \rightarrow[M, \mathrm{Cat} / \mathrm{O}] .
$$

By Wall [15], $M$ has the homotopy type of $X=\left(\bigvee_{i=1}^{k} \mathbb{S}_{i}^{n}\right) \cup_{g} \mathbb{D}^{2 n}$, where $k$ is the $n$-th Betti number of $M, \bigvee_{i=1}^{k} \mathbb{S}_{i}^{n}$ is the wedge sum of $n$-spheres which is the $n$-skeleton of $M$ and $g: \mathbb{S}^{2 n-1} \rightarrow \bigvee_{i=1}^{k} \mathbb{S}_{i}^{n}$ is the attaching map of $\mathbb{D}^{2 n}$. Let $\phi: M \rightarrow X$ be a homotopy equivalence of degree one and $q: X \rightarrow \mathbb{S}^{2 n}$ be the collapsing map obtained by identifying $\mathbb{S}^{2 n}$ with $X / \bigvee_{i=1}^{k} \mathbb{S}_{i}^{n}$ in an orientation preserving way. Let $f_{M}=q \circ$ $\phi: M \rightarrow \mathbb{S}^{2 n}$ be the degree one map.

Consider the following Puppe's exact sequence for the inclusion $i: \bigvee_{i=1}^{k} \mathbb{S}_{i}^{n} \hookrightarrow X$ along $\mathrm{Cat} / \mathrm{O}$ :

$$
\begin{equation*}
\cdots \rightarrow\left[\bigvee_{i=1}^{k} S \mathbb{S}_{i}^{n}, \mathrm{Cat} / O\right] \xrightarrow{(S(g))^{*}}\left[\mathbb{S}^{2 n}, \mathrm{Cat} / \mathrm{O}\right] \xrightarrow{q^{*}}[X, \mathrm{Cat} / \mathrm{O}] \xrightarrow{i^{*}}\left[\bigvee_{i=1}^{k} \mathbb{S}_{i}^{n}, \mathrm{Cat} / O\right], \tag{2.1}
\end{equation*}
$$

where $S(g)$ is the suspension of the map $g: \mathbb{S}^{2 n-1} \rightarrow \bigvee_{i=1}^{k} \mathbb{S}_{i}^{n}$.
Using the fact that

$$
\left[\bigvee_{i=1}^{k} S \mathbb{S}_{i}^{n}, \text { Cat } / O\right] \cong \prod_{i=1}^{k}\left[\mathbb{S}_{i}^{n+1}, \text { Cat } / O\right]
$$

and

$$
\left[\bigvee_{i=1}^{k} \mathbb{S}_{i}^{n}, \operatorname{Cat} / O\right] \cong \prod_{i=1}^{k}\left[\mathbb{S}_{i}^{n}, \operatorname{Cat} / O\right]
$$

the above exact sequence (2.1) becomes

$$
\cdots \rightarrow \prod_{i=1}^{k}\left[\mathbb{S}_{i}^{n+1}, \mathrm{Cat} / \mathrm{O}\right] \xrightarrow{(S(g))^{*}}\left[\mathbb{S}^{2 n}, \mathrm{Cat} / \mathrm{O}\right] \xrightarrow{q^{*}}[X, \mathrm{Cat} / \mathrm{O}] \xrightarrow{i^{*}} \prod_{i=1}^{k}\left[\mathbb{S}_{i}^{n}, \mathrm{Cat} / \mathrm{O}\right]
$$

(i): If $n$ is any integer such that $\Theta_{n+1}$ is trivial and Cat $=T o p$ in the above exact sequence (2.1), by using the fact that

$$
\left[\mathbb{S}^{m}, T o p / O\right]=\bar{\Theta}_{m} \quad(m \neq 3,4)
$$

and $\left[\mathbb{S}^{4}\right.$, Top $\left./ O\right]=0\left(\left[10\right.\right.$, pp. 200-201]), we have $q^{*}:\left[\mathbb{S}^{2 n}\right.$, Top $\left./ O\right] \rightarrow[X$, Top $/ O]$ is injective. Hence $f_{M}^{*}=\phi^{*} \circ q^{*}: \bar{\Theta}_{2 n} \rightarrow[M, T o p / O]$ is injective. By using the identifications $\mathcal{C}\left(M^{2 n}\right)=[M, T o p / O]$ given by [10, pp. 194-196], $f_{M}^{*}: \bar{\Theta}_{2 n} \rightarrow \mathcal{C}\left(M^{2 n}\right)$ becomes $\left[\Sigma^{2 n}\right] \rightarrow\left[M \# \Sigma^{2 n}\right] . I_{c}(M)$ is exactly the kernel of $f_{M}^{*}$, and so $I_{c}(M)=0$. This proves (i).
(ii): If $n>3, \Theta_{n}$ and $\Theta_{n+1}$ are trivial, and Cat $=$ Top then, from the above exact sequence (2.1) we have $q^{*}:\left[\mathbb{S}^{2 n}, T o p / O\right] \rightarrow[X, T o p / O]$ is an isomorphism. This shows that $f_{M}^{*}=\phi^{*} \circ q^{*}: \bar{\Theta}_{2 n} \rightarrow \mathcal{C}\left(M^{2 n}\right)$ is an isomorphism and hence

$$
\mathcal{C}\left(M^{2 n}\right)=\left\{\left[M^{2 n} \# \Sigma\right] \mid \Sigma \in \bar{\Theta}_{2 n}\right\}
$$

This proves (ii).
(iii): If $n=8$ and $H^{n}(M ; \mathbb{Z}) \cong \mathbb{Z}$, then $M^{2 n}$ has the homotopy type of $X=$ $\mathbb{S}^{n} \cup_{g} \mathbb{D}^{2 n}$, where $g: \mathbb{S}^{2 n-1} \rightarrow \mathbb{S}^{n}$ is the attaching map. In order to prove $M^{2 n} \# \Sigma^{2 n}$ is not concordant to $M^{2 n}$, by the above exact sequence (2.1) for Cat $=T o p$, it suffices to prove $q^{*}:\left[\mathbb{S}^{16}, T o p / O\right] \rightarrow[X, T o p / O]$ is monic, which is equivalent to saying that $(S(g))^{*}:\left[S \mathbb{S}^{8}, T o p / O\right] \rightarrow\left[\mathbb{S}^{16}, T o p / O\right]$ is the zero homomorphism. For the case $g=p$, where $p: \mathbb{S}^{15} \rightarrow \mathbb{S}^{8}$ is the Hopf map, $(S(g))^{*}$ is the zero homomorphism, which was proved in the course of the proof of Lemma 1 in [2, pp.58-59]. This proof works verbatim for any map $g: \mathbb{S}^{2 n-1} \rightarrow \mathbb{S}^{n}$ as well. This proves (iii).
(iv): If $n$ is any even integer such that $\Theta_{n}$ and $\Theta_{n+1}$ are trivial, then $\pi_{n+1}(G / O)=$ 0 . This shows that from the above exact sequence (2.1) for $C a t=G, q^{*}:\left[\mathbb{S}^{2 n}, G / O\right] \rightarrow$ $[X, G / O]$ is injective. Then $f_{M}^{*}=\phi^{*} \circ q^{*}:\left[\mathbb{S}^{2 n}, G / O\right] \rightarrow[M, G / O]$ is injective. From
the surgery exact sequences of $M$ and $\mathbb{S}^{2 n}$, we get the following commutative diagram ([3, Lemma 3.4]):


By using the facts that $L_{2 n+1}(e)=0$, injectivity of $\eta_{\mathbb{S}^{2} n}$ and $\eta_{M}$ follow from the diagram, and combine with the injectivity of $f_{M}^{*}$ to show that $f_{M}^{\bullet}: \bar{\Theta}_{2 n} \rightarrow \mathcal{S}^{\text {Diff }}(M)$ is injective. $I_{h}(M)$ is exactly the kernel of $f_{M}^{\bullet}$, and so $I_{h}(M)=0$. This proves (iv).

Remark 2.8. (i) By M. Kervaire and J. Milnor [6], $\Theta_{m}=0$ for $m=1,2,3,4$, 5,6 or 12. If $M^{2 n}$ is a closed smooth ( $n-1$ )-connected $2 n$-manifold, by Theorem 2.7 (i) and (ii), $I_{c}\left(M^{2 n}\right)=0$ for $n=3,4,5$ or 11 and $\mathcal{C}\left(M^{2 n}\right) \cong \bar{\Theta}_{2 n}$ for $n=4$ or 5 .
(ii) If $M$ has the homotopy type of $\mathbb{O} \mathbf{P}^{2}$, by Theorem 1.1 and Theorem 2.7 (iii), we have $I_{c}(M)=0 \neq I(M)$.
(iii) By Theorem 2.7 (iv), if $M$ has the homotopy type of $\mathbb{H} \mathbf{P}^{2}$, then $I_{h}(M)=0$.

Definition 2.9. Let $M$ and $N$ are smooth manifolds. A smooth map $f: M \rightarrow$ $N$ is called tangential if for some integers $k, l, f^{*}(T(N)) \oplus \epsilon_{M}^{k} \cong T(M) \oplus \epsilon_{M}^{l}$.

Definition 2.10. Let $M$ be a topological manifold. Let $(N, f)$ be a pair consisting of a smooth manifold $N$ together with a tangential homotopy equivalence of degree one $f: N \rightarrow M$. Two such pairs $\left(N_{1}, f_{1}\right)$ and $\left(N_{2}, f_{2}\right)$ are equivalent provided there exists a diffeomorphism $g: N_{1} \rightarrow N_{2}$ such that $f_{2} \circ g$ is homotopic to $f_{1}$. The set of all such equivalence classes is denoted by $\theta(M)$.

For $M=\mathbb{H} \mathbf{P}^{2},\left[5\right.$, Theorem 4] shows $\theta\left(\mathbb{H} \mathbf{P}^{2}\right)$ contains at most two elements. Now by Remark 2.8 (iii), we have the following:

Corollary 2.11. $\theta\left(\mathbb{H} \mathbf{P}^{2}\right)$ contains exactly two elements, with representatives given by $\left(\mathbb{H} \mathbf{P}^{2}, \mathrm{id}\right)$ and $\left(\mathbb{H} \mathbf{P}^{2} \# \Sigma^{8}, \mathrm{id}\right)$, where $\Sigma^{8}$ is the exotic 8 -sphere.

## 3. Inertia groups of projective plane-like manifolds

In [15], C.T.C. Wall assigned to each closed oriented ( $n-1$ )-connected $2 n$ dimensional smooth manifold $M^{2 n}$ with $n \geq 3$, a system of invariants as follows:
(1) $H=H^{n}(M ; \mathbb{Z}) \cong \operatorname{Hom}\left(H_{n}(M ; \mathbb{Z}), \mathbb{Z}\right) \cong \bigoplus_{j=1}^{k} \mathbb{Z}$, the cohomology group of $M$, with $k$ the $n$-th Betti number of $M$,
(2) $I: H \times H \rightarrow \mathbb{Z}$, the intersection form of $M$ which is unimodular and $n$-symmetric, defined by

$$
I(x, y)=\langle x \cup y,[M]\rangle
$$

where the homology class $[M]$ is the orientation class of $M$,
(3) A map $\alpha: H^{n}(M ; \mathbb{Z}) \rightarrow \pi_{n-1}\left(S O_{n}\right)$ that assigns each element $x \in H^{n}(M ; \mathbb{Z})$ to the characteristic map $\alpha(x)$ for the normal bundle of the embedded $n$-sphere $\mathbb{S}_{x}^{n}$ representing $x$.
Denote by $\chi=S \circ \alpha: H^{n}(M ; \mathbb{Z}) \rightarrow \pi_{n-1}\left(S O_{n+1}\right) \cong \widetilde{K O}\left(\mathbb{S}^{n}\right)$, where $S: \pi_{n-1}\left(S O_{n}\right) \rightarrow$ $\pi_{n-1}\left(S O_{n+1}\right)$ is the suspension map. Then

$$
\chi=S \circ \alpha \in H^{n}\left(M ; \widetilde{K O}\left(\mathbb{S}^{n}\right)\right)=\operatorname{Hom}\left(H^{n}(M ; \mathbb{Z}) ; \widetilde{K O}\left(\mathbb{S}^{n}\right)\right)
$$

can be viewed as an $n$-dimensional cohomology class of $M$, with coefficients in $\widetilde{K O}\left(\mathbb{S}^{n}\right)$. The obstruction to triviality of the tangent bundle over the $n$-skeleton is the element $\chi \in$ $H^{n}\left(M ; \widetilde{K O}\left(\mathbb{S}^{n}\right)\right)$ [15]. By [15, pp. 179-180], the Pontrjagin class of $M^{2 n}$ is given by

$$
\begin{equation*}
p_{m}\left(M^{2 n}\right)= \pm a_{m}(2 m-1)!\chi \tag{3.1}
\end{equation*}
$$

where $n=4 m$ and

$$
a_{m}=\left\{\begin{array}{lll}
1 & \text { if } & 4 m \equiv 0(\bmod 8) \\
2 & \text { if } & 4 m \equiv 4(\bmod 8)
\end{array}\right.
$$

Define $\Theta_{n}(k)$ to be the subgroup of $\bar{\Theta}_{n}$ consisting of those homotopy $n$-sphere $\Sigma^{n}$ which are the boundaries of $k$-connected ( $n+1$ )-dimensional compact manifolds, $1 \leq k<[n / 2]$. Thus, $\Theta_{n}(k)$ is the kernel of the natural map $i_{k}: \bar{\Theta}_{n} \rightarrow \Omega_{n}(k)$, where $\Omega_{n}(k)$ is the $n$-dimensional group in $k$-connective cobordism theory [13] and $i_{k}$ sends $\Sigma^{n}$ to its cobordism class. Using surgery, we see $\Omega_{*}(1)$ is the usual oriented cobordism group. So $\bar{\Theta}_{n}=\Theta_{n}(1)$. Similarly, $\Omega_{n}(2) \cong \Omega_{n}^{\text {Spin }}(n \geq 7)$; since BSpin is, in fact, 3 -connected, for $n \geq 8, \Omega_{n}(2) \cong \Omega_{n}(3)$ and $\Theta_{n}(2)=\Theta_{n}(3)=b$ Spin $_{n}$. Here $b$ Spin $_{n}$ consists of homotopy $n$-sphere which bound spin manifolds.

In [16], C.T.C. Wall defined the Grothendieck group $\mathcal{G}_{n}^{2 n+1}$, a homomorphism $\vartheta: \mathcal{G}_{n}^{2 n+1} \rightarrow \bar{\Theta}_{2 n}$ such that $\vartheta\left(\mathcal{G}_{n}^{2 n+1}\right)=\Theta_{2 n}(n-1)$ and proved the following theorem:

Theorem 3.1 (Wall). Let $M^{2 n}$ be a closed smooth $(n-1)$-connected $2 n$-manifold and $\Sigma^{2 n}$ be a homotopy sphere in $\bar{\Theta}_{2 n}$. Then $M \# \Sigma^{2 n}$ is an orientation-preserving diffeomorphic to $M$ if and only if
(i) $\Sigma^{2 n}=0$ in $\bar{\Theta}_{2 n}$ or
(ii) $\chi \not \equiv 0(\bmod 2)$ and $\Sigma^{2 n} \in \vartheta\left(\mathcal{G}_{n}^{2 n+1}\right)=\Theta_{2 n}(n-1)$

We also need the following result from [1]:
Theorem 3.2 (Anderson, Brown, Peterson). Let $\eta_{n}: \bar{\Theta}_{n} \rightarrow \Omega_{n}^{\text {Spin }}$ be the homomorphism such that $\eta_{n}$ sends $\Sigma^{n}$ to its spin cobordism class. Then $\eta_{n} \neq 0$ if and only if $n=8 k+1$ or $8 k+2$.

Proof of Theorem 1.1. Let $\xi$ be a generator of $H^{n}\left(M^{2 n} ; \mathbb{Z}\right)$. Consider the case $n=4$. Then by Itiro Tamura [14] and (3.1), the Pontrjagin class of $M^{2 n}$ is given by

$$
p_{1}\left(M^{2 n}\right)=2(2 h+1) \xi= \pm 2 \chi,
$$

where $h \in \mathbb{Z}$. This implies that

$$
\chi= \pm(2 h+1) \xi .
$$

Likewise, for $n=8$, we have

$$
p_{2}\left(M^{2 n}\right)=6(2 k+1) \xi= \pm 6 \chi
$$

where $k \in \mathbb{Z}$. This implies that

$$
\chi= \pm(2 k+1) \xi
$$

Therefore in either case, $\chi \not \equiv 0(\bmod 2)$. Now by Theorem 3.1, it follows that

$$
I\left(M^{2 n}\right)=\Theta_{2 n}(n-1) .
$$

Since $\Theta_{2 n}(n-1)$ is the kernel of the natural map $i_{n-1}: \bar{\Theta}_{2 n} \rightarrow \Omega_{2 n}(n-1)$, where $\Omega_{2 n}(n-1) \cong \Omega_{8}^{\text {Spin }}$ for $n=4$ and $\Omega_{2 n}(n-1) \cong \Omega_{16}^{\text {String }} \cong \mathbb{Z} \oplus \mathbb{Z}$ for $n=8$ [4]. Now by Theorem 3.2 and using the fact that $\bar{\Theta}_{16} \cong \mathbb{Z}_{2}[6]$, we have $i_{n-1}=0$ for $n=4$ and 8. This shows that $\Theta_{2 n}(n-1)=\bar{\Theta}_{2 n}$. This implies that

$$
I\left(M^{2 n}\right) \cong \mathbb{Z}_{2}
$$

This completes the proof of Theorem 1.1.

## 4. Smooth structures of $(n-1)$-connected $2 n$-manifolds

Definition 4.1 (Cat = Diff or Top-structure sets, [3]). Let $M$ be a closed Catmanifold. We define the Cat-structure set $\mathcal{S}^{C a t}(M)$ to be the set of equivalence classes of pairs $(N, f)$ where $N$ is a closed Cat-manifold and $f: N \rightarrow M$ is a homotopy equivalence. And the equivalence relation is defined as follows:
$\left(N_{1}, f_{1}\right) \sim\left(N_{2}, f_{2}\right)$ if there is a Cat-isomorphism $\phi: N_{1} \rightarrow N_{2}$ such that $f_{2} \circ h$ is homotopic to $f_{1}$.

We will denote the class in $\mathcal{S}^{C a t}(M)$ of $(N, f)$ by $[(N, f)]$. The base point of $S^{C a t}(M)$ is the equivalence class $[(M, \mathrm{id})]$ of id: $M \rightarrow M$.

The forgetful maps $F_{\text {Diff }}: \mathcal{S}^{\text {Diff }}(M) \rightarrow \mathcal{S}^{\text {Top }}(M)$ and $F_{C o n}: \mathcal{C}(M) \rightarrow \mathcal{S}^{\text {Diff }}(M)$ fit into a short exact sequence of pointed sets [3]:

$$
\mathcal{C}(M) \xrightarrow{F_{\text {Con }}} \mathcal{S}^{\text {Diff }}(M) \xrightarrow{F_{\text {Diff }}} \mathcal{S}^{T o p}(M) .
$$

Theorem 4.2. Let $n$ be any integer greater than 3 such that $\Theta_{n}$ and $\Theta_{n+1}$ are trivial and $M^{2 n}$ be a closed smooth ( $n-1$ )-connected $2 n$-manifold. Let $f: N \rightarrow M$ be a homeomorphism where $N$ is a closed smooth manifold. Then
(i) there exists a diffeomorphism $\phi: N \rightarrow M \# \Sigma^{2 n}$, where $\Sigma^{2 n} \in \bar{\Theta}_{2 n}$ such that the following diagram commutes up to homotopy:

(ii) If $I_{h}(M)=\bar{\Theta}_{2 n}$, then $f: N \rightarrow M$ is homotopic to a diffeomorphism.

Proof. Consider the short exact sequence of pointed sets

$$
\mathcal{C}(M) \xrightarrow{F_{\text {Con }}} \mathcal{S}^{\text {Diff }}(M) \xrightarrow{F_{D i f f}} \mathcal{S}^{T o p}(M)
$$

By Theorem 2.7 (ii), we have

$$
\mathcal{C}(M)=\left\{[M \# \Sigma] \mid \Sigma \in \bar{\Theta}_{2 n}\right\} \cong \bar{\Theta}_{2 n} .
$$

Since $[(N, f)] \in F_{\text {Diff }}^{-1}([(M$, id $)])$, we obtain

$$
[(N, f)] \in \operatorname{Im}\left(F_{C o n}\right)=\left\{[M \# \Sigma] \mid \Sigma \in \bar{\Theta}_{2 n}\right\} .
$$

This implies that there exists a homotopy sphere $\Sigma^{2 n} \in \bar{\Theta}_{2 n}$ such that $(N, f) \sim(M$ \# $\Sigma^{2 n}$, id) in $\mathcal{S}^{\text {Diff }}(M)$. This implies that there exists a diffeomorphism $\phi: N \rightarrow M \# \Sigma^{2 n}$ such that $f$ is homotopic to id $\circ \phi$. This proves (i).

If $I_{h}(M)=\bar{\Theta}_{2 n}$, then $\operatorname{Im}\left(F_{\text {Con }}\right)=\{[(M$, id $)]\}$ and hence $(N, f) \sim(M$, id $)$ in $\mathcal{S}^{\text {Diff }}(M)$. This shows that $f: N \rightarrow M$ is homotopic to a diffeomorphism $N \rightarrow M$. This proves (ii).

Theorem 4.3. Let $n$ be any integer greater than 3 such that $\Theta_{n}$ and $\Theta_{n+1}$ are trivial and $M^{2 n}$ be a closed smooth $(n-1)$-connected $2 n$-manifold. Then the number of distinct smooth structures on $M^{2 n}$ up to diffeomorphism is less than or equal to the cardinality of $\bar{\Theta}_{2 n}$. In particular, the set of diffeomorphism classes of smooth structures on $M^{2 n}$ is $\left\{[M \# \Sigma] \mid \Sigma \in \bar{\Theta}_{2 n}\right\}$.

Proof. By Theorem 4.2 (i), if $N$ is a closed smooth manifold homeomorphic to $M$, then $N$ is diffeomorphic to $M \# \Sigma^{2 n}$ for some homotopy $2 n$-sphere $\Sigma^{2 n}$. This implies that the set of diffeomorphism classes of smooth structures on $M^{2 n}$ is $\{[M \# \Sigma] \mid$ $\left.\Sigma \in \bar{\Theta}_{2 n}\right\}$. This shows that the number of distinct smooth structures on $M^{2 n}$ up to diffeomorphism is less than or equal to the cardinality of $\bar{\Theta}_{2 n}$.

REMARK 4.4. (1) By Theorem 4.3, every closed smooth 3-connected 8-manifold has at most two distinct smooth structures up to diffeomorphism.
(2) If $M^{8}$ is a closed smooth 3-connected 8-manifold such that $H^{4}(M ; \mathbb{Z}) \cong \mathbb{Z}$, then by Theorem $1.1, I(M) \cong \mathbb{Z}_{2}$. Now by Theorem $4.3, M$ has a unique smooth structure up to diffeomorphism.
(3) If $M=\mathbb{S}^{4} \times \mathbb{S}^{4}$, then by Theorem $4.3, \mathbb{S}^{4} \times \mathbb{S}^{4}$ has at most two distinct smooth structures up to diffeomorphism, namely, $\left\{\left[\mathbb{S}^{4} \times \mathbb{S}^{4}\right],\left[\mathbb{S}^{4} \times \mathbb{S}^{4} \# \Sigma\right]\right\}$, where $\Sigma$ is the exotic 8 -sphere. However, by $[11$, Theorem $A], I\left(\mathbb{S}^{4} \times \mathbb{S}^{4}\right)=0$. This implies that $\mathbb{S}^{4} \times \mathbb{S}^{4}$ has exactly two distinct smooth structures.

Theorem 4.5. Let $M$ be a closed smooth 3-connected 8 -manifold with stable tangential invariant $\chi=S \circ \alpha: H_{4}(M ; \mathbb{Z}) \rightarrow \pi_{3}(S O)=\mathbb{Z}$. Then $M$ has exactly two distinct smooth structures up to diffeomorphism if and only if $\operatorname{Im}(S \circ \alpha) \subseteq 2 \mathbb{Z}$.

Proof. Suppose $M$ has exactly two distinct smooth structures up to diffeomorphism. Then by Theorem 4.3, $M$ and $M \# \Sigma$ are not diffeomorphic, where $\Sigma$ is the exotic 8sphere. Since $\bar{\Theta}_{8}=\Theta_{8}(3)$, by Theorem 3.1, the stable tangential invariant $\chi$ is zero $(\bmod 2)$ and hence $\operatorname{Im}(S \circ \alpha) \subseteq 2 \mathbb{Z}$. Conversely, suppose $\operatorname{Im}(S \circ \alpha) \subseteq 2 \mathbb{Z}$. Now by Theorem 3.1, $M$ can not be diffeomorphic to $M \# \Sigma$, where $\Sigma$ is the exotic 8 -sphere. Now by Theorem 4.3, $M$ has exactly two distinct smooth structures up to diffeomorphism.

REMARK 4.6. If $n=2,3,5,6,7(\bmod 8)$ or the stable tangential invariant $\chi$ of $M^{2 n}$ is zero $(\bmod 2)$, then by [16, Corollary, p. 289] and Theorem 3.1, we have $I\left(M^{2 n}\right)=0$. So, by Theorem 4.3, we have the following:

Theorem 4.7. Let $n$ be any integer greater than 3 such that $\Theta_{n}$ and $\Theta_{n+1}$ are trivial and $M^{2 n}$ be a closed smooth $(n-1)$-connected $2 n$-manifold. If $n=2,3,5,6,7$ $(\bmod 8)$ or the stable tangential invariant $\chi$ of $M^{2 n}$ is zero $(\bmod 2)$, then the set of diffeomorphism classes of smooth structures on $M^{2 n}$ is in one-to-one correspondence with group $\bar{\Theta}_{2 n}$.

REMARK 4.8. (1) By Theorem 4.7, every closed smooth 4-connected 10-manifold has exactly six distinct smooth structures, namely, $\left\{[M \# \Sigma] \mid \Sigma \in \bar{\Theta}_{10} \cong \mathbb{Z}_{6}\right\}$.
(2) If $M^{2 n}$ is $n$-parallelisable, almost parallelisable or $\pi$-manifold, then the stable tangential invariant $\chi$ of $M$ is zero [15]. Then by Theorem 4.7, we have the following:

Corollary 4.9. Let $n$ be any integer greater than 3 such that $\Theta_{n}$ and $\Theta_{n+1}$ are trivial and $M^{2 n}$ be a closed smooth $(n-1)$-connected $2 n$-manifold. If $M^{2 n}$ is $n$-parallelisable, almost parallelisable or $\pi$-manifold, then the set of diffeomorphism classes of smooth structures on $M^{2 n}$ is in one-to-one correspondence with group $\bar{\Theta}_{2 n}$.

DEFINITION 4.10 ([8]). The normal $k$-type of a closed smooth manifold $M$ is the fibre homotopy type of a fibration $p: B \rightarrow B O$ such that the fibre of the map $p$
is connected and its homotopy groups vanish in dimension $\geq k+1$, admitting a lift of the normal Gauss map $\nu_{M}: M \rightarrow B O$ to a map $\bar{\nu}_{M}: M \rightarrow B$ such that $\bar{\nu}_{M}: M \rightarrow B$ is a $(k+1)$-equivalence, i.e., the induced homomorphism $\bar{\nu}_{M}: \pi_{i}(M) \rightarrow \pi_{i}(B)$ is an isomorphism for $i \leq k$ and surjective for $i=k+1$. We call such a lift a normal $k$-smoothing.

Theorem 4.11. Let $n=5,7$ and let $M_{0}$ and $M_{1}$ be closed smooth ( $n-1$ )-connected $2 n$-manifolds with the same Euler characteristic. Then
(i) There is a homotopy sphere $\Sigma^{2 n} \in \bar{\Theta}_{2 n}$ such that $M_{0}$ and $M_{1} \# \Sigma^{2 n}$ are diffeomorphic.
(ii) Let $M^{2 n}$ be a closed smooth $(n-1)$-connected $2 n$-manifold such that $[M]=0 \in$ $\Omega_{2 n}^{\text {String }}$ and let $\Sigma$ be any exotic $2 n$-sphere in $\bar{\Theta}_{2 n}$. Then $M$ and $M \# \Sigma$ are not diffeomorphic.

Proof. (i): $M_{0}$ and $M_{1}$ are $(n-1)$-connected, and $n$ is 5 or 7 ; therefore, $p_{1} / 2$ and the Stiefel-Whitney classes $\omega_{2}$ vanish. So, $M_{0}$ and $M_{1}$ are BString-manifolds. Let $\bar{\nu}_{M_{j}}: M_{j} \rightarrow B$ String be a lift of the normal Gauss map $\nu_{M_{j}}: M_{j} \rightarrow B O$ in the fibration $p: B$ String $=B O\langle 8\rangle \rightarrow B O$, where $j=0$ and 1. Since BString is 7connected, $p_{\#}: \pi_{i}(B$ String $) \rightarrow \pi_{i}(B O)$ is an isomorphism for all $i \geq 8$. This shows that $\bar{\nu}_{M_{j}}: M_{j} \rightarrow$ BString is an $n$-equivalence and hence the normal ( $n-1$ )-type of $M_{0}$ and $M_{1}$ is $p$ : BString $\rightarrow B O$. We know that $\Omega_{2 n}^{\text {String }} \cong \bar{\Theta}_{2 n}$, where the group structure is given by connected sum [4]. This implies that there always exists $\Sigma^{2 n} \in \bar{\Theta}_{2 n}$ such that $M_{0}$ and $M_{1} \# \Sigma^{2 n}$ are BString-bordant. Since $M_{0}$ and $M_{1} \# \Sigma^{2 n}$ have the same Euler characteristic, by [8, Corollary 4], $M_{0}$ and $M_{1} \# \Sigma^{2 n}$ are diffeomorphic.
(ii): Since the image of the standard sphere under the isomorphism $\bar{\Theta}_{2 n} \cong \Omega_{2 n}^{\text {String }}$ represents the trivial element in $\Omega_{2 n}^{\text {String }}$, we have $\left[M^{2 n}\right] \neq[M \# \Sigma]$ in $\Omega_{2 n}^{\text {String }}$. This implies that $M$ and $M \# \Sigma$ are not BString-bordant. By obstruction theory, $M^{2 n}$ has a unique string structure. This implies that $M$ and $M \# \Sigma$ are not diffeomorphic.

Theorem 4.12. Let $M$ be a closed smooth 6 -connected 14 -dimensional $\pi$-manifold and $\Sigma$ is the exotic 14 -sphere. Then $M \# \Sigma$ is not diffeomorphic to $M$. Thus, $I(M)=0$. Moreover, if $N$ is a closed smooth manifold homeomorphic to $M$, then $N$ is diffeomorphic to either $M$ or $M \# \Sigma$.

Proof. It follows from results of Anderson, Brown and Peterson on spin cobordism [1] that the image of the natural homomorphism $\Omega_{14}^{\text {framed }} \rightarrow \Omega_{14}^{\text {Spin }}$ is 0 and $\Omega_{14}^{\text {String }} \cong$ $\Omega_{14}^{\text {Spin }} \cong \mathbb{Z}_{2}$ [4]. This shows that $[M]=0 \in \Omega_{14}^{\text {String }}$. Now by Theorem 4.11 (ii), $M \# \Sigma$ is not diffeomorphic to $M$. If $N$ is a closed smooth manifold homeomorphic to $M$, then $N$ and $M$ have the same Euler characteristic. Then by Theorem 4.11 (i), $N$ is diffeomorphic to either $M$ or $M \# \Sigma$.

REMARK 4.13. By the above Theorem 4.12, the set of diffeomorphism classes of smooth structures on a closed smooth 6-connected 14 -dimensional $\pi$-manifold $M$ is

$$
\{[M],[M \# \Sigma]\} \cong \mathbb{Z}_{2}
$$

where $\Sigma$ is the exotic 14 -sphere. So, the number of distinct smooth structures on $M$ is 2 .

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