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# ON ALGEBRAS WHICH RESEMBLE THE LOCAL WEYL ALGEBRA

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## 1. Introduction

Let $K$ be an algebraically closed field of characteristic zero and let $\hat{\mathcal{O}}_n(K)=K[[x_1, \cdots, x_n]]$ be the formal power series ring over $K$ in $n$ variables. According to Björk [1], we denote by $\hat{D}_n(K)$ the subring of $\mathrm{End}_K(\hat{\mathcal{O}}_n(K))$ generated over $K$ by the left multiplications by elements of $\hat{\mathcal{O}}_n(K)$ and partial differentials $\partial_i=\partial/\partial x_i$,

$$\hat{D}_n(K) = \hat{\mathcal{O}}_n(K)\langle\partial_1, \cdots, \partial_n\rangle$$

where $\partial_i x_j - x_j \partial_i = \delta_{ij}$ (Kronecker's delta) and $\partial_i \partial_j = \partial_j \partial_i$. The ring $\hat{D}_n(K)$, called the *local Weyl algebra*, has the $\Sigma$-filtration $\{\Sigma_v\}_{v\geq 0}$ such that $\Sigma_0=\hat{\mathcal{O}}_n(K)$ and $\Sigma_v=\{\Sigma_\alpha f_\alpha \partial^\alpha; f_\alpha \in \hat{\mathcal{O}}_n(K)$ and $\partial^\alpha=\partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}$ with $|\alpha|=\alpha_1+\cdots+\alpha_n\leq v\}$ and that the associated graded ring $\mathrm{gr}_\Gamma(\hat{D}_n(K))$ is a polynomial ring over $\hat{\mathcal{O}}_n(K)$ in $n$ variables. Moreover, $\hat{D}_n(K)$ has weak global dimension $n$, i.e., $\mathrm{w.gl.dim}(\hat{D}_n(K)) = n$.

These are ring-theoretic, algebraic properties which the local Weyl algebra $\hat{D}_n(K)$ has. In the present article, we consider whether or not these properties are sufficient to characterize the ring $\hat{D}_n(K)$. For this purpose, we introduce the notion of pre-$W$-algebra and $W$-algebra (see below for the definition) and show that a $W$-algebra, which satisfies the above-listed properties $\hat{D}_n(K)$ has and one additional condition, i.e., $L=\Sigma_1/\Sigma_0$ is essentially abelian, is realized as a subalgebra of some $\hat{D}_n(K)$. After all, we are successful only in the case $n=1$. We are, however, convinced that our approach of computing the weak global dimension of a $W$-algebra will be useful to study locally a vector field at a smooth point on an algebraic variety.

We employ the terminology and notation in [1].

## 2. Structure theorems

To simplify the notation, we denote $\hat{\mathcal{O}}_n(K)$ by $R$. Let $A$ be a (not necessarily commutative) $K$-algebra containing $R$ generated by finitely many elements over $R$. Consider the following three conditions on $A$:

(i) $A$ has a $\Sigma$-filtration $\{\Sigma_v\}_{v\geq 0}$ such that $\Sigma_v (v\geq 0)$ is a two-sided $R$-submodule of $A$, $\Sigma_0=R$, $\Sigma_1$ generates $A$ over $R$, $\Sigma_v\cdot\Sigma_w\subset\Sigma_{v+w}$ for any $v, w\geq 0$ and $A=\bigcup_{v\geq 0}\Sigma_v$;

(ii) The associated graded ring $\mathrm{gr}_\Sigma(A):=\oplus_{v\geq 0}\Sigma_v/\Sigma_{v-1}$ is a polynomial ring $R[y_1, \cdots, y_m]$ in $m$ variables;

(iii) $\mathrm{w.gl.dim}\,(A)=n$.

If $A$ satisfies the above conditions (i) and (ii), we call it a *pre-W-algebra over* $R$. We denote by $L$ the free $R$-module $\Sigma_1/\Sigma_0=\oplus_{i=1}^m Ry_i$.

**Lemma 2.1.** *Let $A$ be a pre-W-algebra over $R$. Then we have the following:*

(1) *Let $Y_1, \cdots, Y_m$ be elements of $\Sigma_1$ such that $y_i\equiv Y_i \pmod{\Sigma_0}$ for any $i$. Then $A$ is generated by $Y_1, \cdots, Y_m$ over $R$, which we write as $A=R\langle Y_1, \cdots, Y_m\rangle$.*

(2) *For any $y\in L$ and $a\in R$, define $y[a]$ by*

$$y[a] = Ya-aY$$

*for $Y\in\Sigma_1$ with $y\equiv Y\pmod{\Sigma_0}$. Then $y[a]$ is independent of the choice of $Y$, and $y$ is considered as a $K$-derivation on $R$. So, we have an $R$-linear map $\rho: L\to \mathrm{Der}_K(R)$; we write $y[a]$ as $\rho(y)(a)$ as well and we use this map $\rho$ in the subsequent discussions without referring explicitly to this lemma.*

(3) *Define a bracket product $[y, z]$ on $L$ by*

$$[y, z] \equiv YZ-ZY \pmod{\Sigma_0}$$

*for $Y, Z\in\Sigma_1$ with $y\equiv Y \pmod{\Sigma_0}$ and $z\equiv Z \pmod{\Sigma_0}$. Then $[y, z]$ is well-defined and $\rho$ is a Lie-algebra homomorphism, i.e., $\rho([y, z])=[\rho(y), \rho(z)]$.*

Proof. (1) For any $f\in A$, we define $\nu(f)$ as the smallest integer $r$ with $f\in\Sigma_r$. If $\nu(f)=r$, there exists $F_r(y_1, \cdots, y_m)\in R[y_1, \cdots, y_m]_r=$the $r$-th homogeneous part of $\mathrm{gr}_\Sigma(A)$ such that $f-F_r(Y_1, \cdots, Y_m)\in\Sigma_{r-1}$. By induction on $\nu(f)$, we can verify the assertion straightforwardly.

(2) Replace $Y$ be $Y+b$ with $b\in R$. Then we have

$$(Y+b)a-a(Y+b) = Ya-aY,$$

whence $y[a]$ is independent of the choice of $Y$. Furthermore, we have

$$\begin{aligned} y[ab] &= Y(ab)-(ab)Y = (aY+y[a])b-abY \\ &= a(Yb-bY)+y[a]b = ay[b]+y[a]b. \end{aligned}$$

So, $y[\ \ ]$ is a $K$-derivation on $R$.

(3) The assertion can be verified by a straightforward computation.

Q.E.D.

The structure of a pre-$W$-algebra over $R$ is given in the following:

**Theorem 2.2.** (1) *Let $A$ be a pre-$W$-algebra over $R$. Let $Y_1, \cdots, Y_m$ be elements of $\Sigma_1$ as chosen in the previous lemma. Write*

$$Y_i Y_j - Y_j Y_i = \sum_{k=1}^m \rho_{ij,k} Y_k + \sigma_{ij}, \quad 1 \leq i, j \leq m, \tag{2.0}$$

*where $\rho_{ij,k}, \sigma_{ij} \in R$. Then we have the following equalities:*

$$\begin{aligned}\sum_{l=1}^m (\rho_{ij,l}\,\rho_{lk,s} + \rho_{jk,l}\,\rho_{li,s} + \rho_{ki,l}\,\rho_{lj,s}) \\ = y_i[\rho_{jk,s}] + y_j[\rho_{ki,s}] + y_k[\rho_{ij,s}], \quad 1 \leq i, j, k, s \leq m\end{aligned} \tag{2.1}$$

$$\begin{aligned}\sum_{l=1}^m (\rho_{ij,l}\,\sigma_{lk} + \rho_{jk,l}\,\sigma_{li} + \rho_{ki,l}\,\sigma_{lj}) \\ = y_i[\sigma_{jk}] + y_j[\sigma_{ki}] + y_k[\sigma_{ij}], \quad 1 \leq i, j, k \leq m\end{aligned} \tag{2.2}$$

$$\rho_{ij,k} = -\rho_{ji,k}, \ \sigma_{ij} = -\sigma_{ji}, \quad 1 \leq i, j, k \leq m. \tag{2.3}$$

*The elements $\{\rho_{ij,k};\ 1 \leq i, j, k \leq m\}$ are determined uniquely by the Lie algebra $L$ and the choice of $R$-free basis $\{y_1, \cdots, y_m\}$ of $L$.*

(2) *Suppose we are given as in Lemma 2.1 the Lie algebra $L$ and an $R$-linear map $\rho: L \to \mathrm{Der}_K R$ which is a Lie-algebra homomorphism. For an $R$-free basis $\{y_1, \cdots, y_m\}$ of $L$, suppose we are given elements $\{\sigma_{ij};\ 1 \leq i, j \leq m\}$ satisfying the conditions (2.2) and (2.3) above. Then there exists a $K$-algebra $A$ with a $\Sigma$-filtration $\{\Sigma_v\}_{v \geq 0}$ such that*

(i) *$A$ is generated over $R$ by elements $Y_1, \cdots, Y_m$;*
(ii) *The equalities (2.0)-(2.3) hold;*
(iii) *$\Sigma_v = \{\Sigma_\alpha f_\alpha Y^\alpha; f_\alpha \in R,\ Y^\alpha = Y_1^{\alpha_1} \cdots Y_m^{\alpha_m},\ |\alpha| \leq v\}$ for any $v \geq 0$;*
(iv) *$\mathrm{gr}_\Sigma(A) \cong R[y_1, \cdots, y_m] :=$ the symmetric algebra of $L$ over $R$.*

Proof. (1) By the definition of $[y_i, y_j]$ in Lemma 2.1, $\{\rho_{ij,k};\ 1 \leq i, j, k \leq m\}$ are the multiplication constants of the Lie algebra $L$. Hence they are uniquely determined by the choice of the $R$-free basis $\{y_1, \cdots, y_m\}$ of $L$. If one chooses $\{Y_1, \cdots, Y_m\}$ as in Lemma 2.1, then $\{1, Y_1, \cdots, Y_m\}$ is an $R$-free basis of $\Sigma_1$. Then the equalities (2.1) and (2.2) follow from the Jacobi identity:

$$[[Y_i, Y_j], Y_k] + [[Y_j, Y_k], Y_i] + [[Y_k, Y_i], Y_j] = 0,$$

where $[Y_i, Y_j] = Y_i Y_j - Y_j Y_i$.

(2) Let $\{Y_1, \cdots, Y_m\}$ be indeterminates and let $A$ be the free $K$-algebra generated by $Y_1, \cdots, Y_m$ over $R$ modulo the two-sided ideal $I$ generated by

$$\{Y_i Y_j - Y_j Y_i - \sum_{k=1}^m \rho_{ij,k} Y_k - \sigma_{ij};\ 1 \leq i, j, k \leq m\}$$

and

$$\{Y_i f - f Y_i - \rho(y_i)(f);\ 1 \leq i \leq m,\ \forall f \in R\}.$$

We write $y_i[f]=\rho(y_i)(f)$ by identifying $Y_i$'s with $y_i$'s in $L$, We can employ the proof of the Poincaré-Birkoff-Witt theorem (cf. Jacobson [2]) without major changes in the present situation to show that every element of $A$ is written uniquely as a linear combination of standard monomials in $Y_1, \cdots, Y_m$ with coefficients in $R$. In particular, the equalities (2.1) and (2.2) imply that $\Sigma_1$ (with the notation in (iii)) is a free $R$-module generated by 1, $Y_1, \cdots, Y_m$. Note that there is a surjective homomorphism $\theta: R[y_1, \cdots, y_m] \to \mathrm{gr}_{\Sigma}(A)$. Its kernel is generated by the relations $y_i y_j - y_j y_i$ and $y_i f - f y_i$, $1 \le i, j \le m$. But these elements are already zero in $R[y_1, \cdots, y_m]$. Hence $\mathrm{gr}_{\Sigma}(A) \cong R[y_1, \cdots, y_m]$.

Q.E.D.

Let $A$ be a pre-$W$-algebra over $R$. We are interested in the existence of an algebra homomorphism from $A$ to the local Weyl algebra $\hat{D}_n(K)$, which is the identity homomorphism when restricted on the subalgebra $R$. We call it a *K-algebra homomorphism over R*.

**Theorem 2.3.** *Let A be a pre-W-algebra over R. Then the following conditions on A are equivalent:*

(1) *There is a K-algebra homomorphism* $\tilde{\rho}: A \to \hat{D}_n(K)$ *over R such that* $\tilde{\rho}(\Sigma_v) \subset \Sigma_v$ *for all* $v \ge 0$ *and* $\tilde{\rho}|_{\Sigma_1}$ *induces the Lie-algebra homomorphism* $\rho: L := \Sigma_1/\Sigma_0 \to \mathrm{Der}_K(R)$ (*cf. Lemma* 2.1).

(2) *There exists a lifting* $\{Y_1, \cdots, Y_m\}$ *of the R-free basis* $\{y_1, \cdots, y_m\}$ *in* $\Sigma_1$ *for which* $\sigma_{ij}=0$, $1 \le i, j \le m$.

(3) *There exist* $\{a_i\}_{1\le i\le m}$ *in R such that*

$$\sigma_{ij} = \sum_{l=1}^{m} \rho_{ij,l}\, a_l + y_j[a_i] - y_i[a_j]\,, \quad 1 \le i, j \le m\,. \tag{2.4}$$

(4) *There exists an R-free submodule* $\tilde{L}$ *of* $\Sigma_1$ *such that* $\tilde{L}$ *is closed under the bracket product* $[Y, Z] = YZ - ZY$ *and the natural residue homomorphism* $\pi: \Sigma_1 \to L$ *induces a Lie-algebra isomorphism* $\pi|_{\tilde{L}}: \tilde{L} \to L$.

Proof.

(1)$\Rightarrow$(2). Note that $\hat{D}_n(K)$ acts on $R$ in the natural fashion. So, $A$ acts on $R$ via the homomorphism $\tilde{\rho}$. For $Y \in \Sigma_1$, let $a = \tilde{\rho}(Y)\cdot 1$ and let $Y' = Y - a$. Then, since $\tilde{\rho}(Y) \in \Sigma_1 := \bigoplus_{i=1}^{n} R\partial/\partial x_i + R$, we know that $\tilde{\rho}(Y') \in \mathrm{Der}_K(R)$. In particular, $\tilde{\rho}(Y')\cdot 1 = 0$. Now, for the given lifting $\{Y_1, \cdots, Y_m\}$, we set $Y_i' = Y_i - \tilde{\rho}(Y_i)\cdot 1$, $1 \le i \le m$. Then $\{Y_1', \cdots, Y_m'\}$ is a lifting of $\{y_1, \cdots, y_m\}$ in $\Sigma_1$. We assume from the beginning that $Y_i' = Y_i$, $1 \le i \le m$. Then the equality (2.0) implies $\sigma_{ij} = 0$ $(1 \le i, j \le m)$ because $\tilde{\rho}(Y_i) \in \mathrm{Der}_K(R)$.

(2)$\Rightarrow$(3). Suppose $\{Y_1, \cdots, Y_m\}$ is the given lifting of $\{y_1, \cdots, y_m\}$ and $\{Y_1', \cdots, Y_m'\}$ is a lifting for which $\sigma_{ij}' = 0$ when we write

$$Y_i' Y_j' - Y_j' Y_i' = \sum_{k=1}^{m} \rho_{ij,k}\, Y_k' + \sigma_{ij}'\,, \quad 1 \le i, j \le m\,. \tag{2.0$'$}$$

Then $Y_i'=Y_i+a_i$ with $a_i \in R$. Replacing $Y_i'$ in (2.0)′ by this expression, we obtain the equality (2.4).

(3)⇒(2). Conversely, if we are given $\{a_i\}_{1\le i\le m}$ satisfying (2.4), set $Y_i'=Y_i+a_i$. Then $\{Y_1', \cdots, Y_m'\}$ is a lifting of $\{y_1, \cdots, y_m\}$ for which $\sigma_{ij}'=0$.

(2)⇒(4). Let $\{Y_1, \cdots, Y_m\}$ be as in (2) above. Let $\tilde{L}$ be the $R$-submodule of $\Sigma_1$ generated by $Y_1, \cdots, Y_m$. Then $\tilde{L}$ is a free $R$-module. Since $\sigma_{ij}=0$, we readily verify that $[Y, Z]\in\tilde{L}$ for any $Y, Z\in\tilde{L}$. Clearly, $\pi$ induces an isomorphism between $\tilde{L}$ and $L$.

(4)⇒(1). Define $\tilde{\rho}$: $\tilde{L}\to \mathrm{Der}_K(R)$ by $\tilde{\rho}(Y)=\rho(\pi(Y))$. Extend this to $\Sigma_1$ in a natural fashion by putting $\tilde{\rho}|_{\Sigma_0}=\mathrm{id}_R$. Furthermore, we extend $\tilde{\rho}$ to the free $K$-algebra $F$ generated over $R$ by $Y_1, \cdots, Y_m$ as follows. For an element $Y_{i_1}f_{i_1}\cdots Y_{i_r}f_{i_r}$ of $F$ with $Y_{i_j}\in\{Y_1, \cdots, Y_m\}$ and $f_{i_j}\in R$, define

$$Y_{i_1}f_{i_1}\cdots Y_{i_r}f_{i_r}\cdot(a) = y_{i_1}[f_{i_1}[y_{i_2}[\cdots[f_{i_r}a]\cdots]]],$$

where $y_{ij}=\pi(Y_{ij})$ and $f[b]:=fb\in R$. In view of (2) of Theorem 2.2, $A$ is identified with the residue ring of $F$ by the two-sided ideal $I$ considered in Theorem 2.2. So, in order to have $\tilde{\rho}$ as above, we have only to show that

$$y_i[y_j[a]]-y_j[y_i[a]] = \sum_{k=1}^{m}\rho_{ij,k}\,y_k[a] \quad\text{and}\quad y_i[fa]=fy_i[a]+y_i[f]\,a$$

for $a\in R$. These equations hold, in fact, because $\rho$: $L\to\mathrm{Der}_K(R)$ being a Lie-algebra homomorphism implies

$$y_i[y_j[a]]-y_j[y_i[a]] = [y_i, y_j]\,[a] = \sum_{k=1}^{m}\rho_{ij,k}\,y_k[a]$$

and the second equality above. Q.E.D.

If a pre-$W$-algebra $A$ over $R$ satisfies one of the equivalent conditions in Theorem 2.3, we call *$A$ a $W$-algebra over $R$*.

REMARK 2.4. (1) Suppose that $\rho$: $L\to\mathrm{Der}_K(R)$ is an isomorphism. Then, as an $R$-free basis $\{y_1, \cdots, y_m\}$ of $L$, we can take $y_i=\rho^{-1}(\partial/\partial x_i)$. Then $\rho_{ij,k}=0$ for all $1\le i, j, k\le m$. So the case with all $\rho_{ij,k}=0$ can take place. We then say that $L$ is *essentially abelian*.

(2) Suppose $L$ is essentially abelian. Let $\{y_1, \cdots, y_m\}$ be an $R$-free basis of $L$ such that $[y_i, y_j]=0$, $1\le i, j\le m$ and let $\{Y_1, \cdots, Y_m\}$ be such that $y_i\equiv Y_i$ (mod $\Sigma_0$) and $Y_i\,Y_j-Y_j\,Y_i=\sigma_{ij}\in R$. Suppose we can take $\sigma_{ij}=c_{ij}\in K^*=K-(0)$ for $1\le i, j\le m$ and $i\neq j$ and that $\rho(y_i)(\mathcal{M})\subset\mathcal{M}$, where $\mathcal{M}$ is the maximal ideal of $R$. Then we cannot find $\{a_i\}_{1\le i\le m}$ so that the equality (2.4) holds. There exists a $K$-algebra $A$ over $R$ satisfying these conditions. In fact, we take $m=n$, $\rho$: $L\to\mathrm{Der}_K(R)$ to be a homomorphism such that $\rho(y_i)=\partial/\partial x_i$, $1\le i\le n$, and $A$ to be the residue ring of a free $K$-algebra $F$ over $R$ generated by $Y_1, \cdots, Y_n$ modulo the two-sided ideal $I$ as considered in Theorem 2.2, (2). Then $\rho$ cannot

be extended to a $K$-algebra homomorphism $\tilde{\rho}$: $A \to \tilde{D}_n(K)$ over $R$ as considered in Theorem 2.3.

## 3. Case $L$ is essentially abelian

We begin with the following:

**Lemma 3.1.** *Let $A$ be a $W$-algebra over $R$ with a $K$-algebra homomorphism $\tilde{\rho}$: $A \to \hat{D}_n(K)$ over $R$ which is an extension of the Lie-algebra homomorphism $\rho$: $L \to \mathrm{Der}_K(R)$. Then we have* $\mathrm{w.gl.dim}\,(A) \geq n$.

Proof. Note that any element $\xi$ of $A$ can be expressed as $\xi = \Sigma_\alpha f_\alpha Y^\alpha$, where $f_\alpha \in R$ and $Y^\alpha = Y_1^{\alpha_1} \cdots Y_m^{\alpha_m}$ (cf. the equality $Ya - aY = y[a]$ in Lemma 2.1). Furthermore, this expression is unique. Indeed, if we have a nontrivial expression $\Sigma_\alpha f_\alpha Y^\alpha = 0$ then this yields a homogeneous nontrivial relation

$$\sum_{|\alpha|=v} f_\alpha y^\alpha = 0\,, \quad y^\alpha = y_1^{\alpha_1} \cdots y_m^{\alpha_m}$$

where $v = \max\{|\alpha|\,; f_\alpha \neq 0\}$. This contradicts the hypothesis that $\mathrm{gr}_\Sigma(A)$ is a polynomial ring in $y_1, \cdots, y_m$ over $R$. Hence $A$ is a free $R$-module, whence $A$ is $R$-flat as a left $R$-module. Similarly, $\xi$ can be expressed uniquely as $\xi = \Sigma_\beta Y^\beta g_\beta$. So, $A$ is $R$-flat as a right $R$-module. Hence $A$ is $R$-flat as a ring. In view of Björk [1, Cor.2.9, p.42], we have

$$(*) \quad \mathrm{w.dim}_R(A \otimes_R M) \leq \mathrm{w.dim}_A(A \otimes_R M)$$

for any left $R$-module $M$. Take an $R$-module $K = R/\mathcal{M}$ with $\mathcal{M} = (x_1, \cdots, x_n) R$. Then, by the theory of syzyzy, we know that $\mathrm{w.dim}_R(K) = n$; in fact, $\mathrm{Tor}_n^R(K, K) = K \neq (0)$. Then the above inequality $(*)$ implies that $\mathrm{w.dim}_A(A \otimes_R K) \geq n$. Hence $\mathrm{w.gl.dim}\,(A) \geq n$. Q.E.D.

We shall be concerned with the condition $\mathrm{w.gl.dim}(A) = n$ for a $W$-algebra over $R$.

**Theorem 3.2.** *Let $A$ be a $W$-algebra over $R$ with a $K$-algebra homomorphism $\tilde{\rho}$: $A \to \hat{D}_n(K)$ over $R$. Suppose that $L$ is essentially abelian and $A$ has* $\mathrm{w.gl.dim}(A) = n$. *Then $\tilde{\rho}$ is an injection.*

Proof. Let $\tilde{\rho}_1 := \tilde{\rho}\,|\,_{\tilde{L}}$, where $\tilde{L}$ is an $R$-free submodule of $\Sigma_1$ isomorphic to $L$ as a Lie algebra (cf. Theorem 2.3). Then there exists an $R$-free basis $\{Y_1, \cdots, Y_m\}$ of $\tilde{L}$ such that $Y_i Y_j = Y_j Y_i$ for $1 \leq i, j \leq m$. Let $\tilde{L}_0 = \oplus_{i=1}^m KY_i$ and let $Q = \mathrm{Ker}(\tilde{\rho}_1|_{\tilde{L}_0})$. Then $\tilde{L}_0 \cong Q \oplus \tilde{\rho}_1(\tilde{L}_0)$ is a direct sum as Lie algebras and $Q$ is contained in the center of $A$. Let $B$ be the $R$-subalgebra of $\hat{D}_n(K)$ generated by $\tilde{\rho}_1(\tilde{L}_0)$ and let $J$ be the two-sided ideal of $A$ generated by $Q$. Then $B \cong A/J$ and $B$ is a $W$-algebra over $R$. Indeed, we may take $\{Y_1, \cdots, Y_m\}$ so that $\{Y_{r+1}, \cdots, Y_m\}$ is a $K$-basis of $Q$. Let $\bar{Y}_i = \tilde{\rho}_1(Y_i)$, $1 \leq i \leq r$. Then $B$ is generated by $\bar{Y}_1, \cdots, \bar{Y}_r$ over $R$ which act on $R$ via the derivations $\delta_i = y_i[\ \ ]$, $1 \leq i \leq r$. Note that $\{\bar{Y}_1, \cdots, \bar{Y}_r\}$ are linearly independent over $R$. So, $r \leq n$. We claim:

**Lemma 3.3.** *$\{\delta_1, \cdots, \delta_r\}$ are algebraically independent over $R$. Namely, if $\Sigma_\gamma f_\gamma \delta^\gamma = 0$ with $f_\gamma \in R$ and $\delta^\gamma = \delta_1^{\gamma_1} \cdots \delta_r^{\gamma_r}$ then $f_\gamma = 0$ for all $\gamma$.*

Proof. Denote by $Q(R)$ the quotient field of $R$. We can find $\Delta_1, \cdots, \Delta_r \in \oplus_{i=1}^r Q(R)\, \delta_i$ satisfying the following conditions:

(1) $\oplus_{i=1}^r Q(R)\, \delta_i = \oplus_{i=1}^r Q(R)\, \Delta_i$;

(2) We can express $\Delta_i = \sum_{j=1}^n a_{ij}\, \partial_j$ with $a_{ij} \in R$ and $\partial_j = \partial/\partial x_j$, and if we define $s_i$ as $\min\{j;\ a_{ij} \neq 0\}$ then $s_1 < s_2 < \cdots < s_r$.

Suppose we have a nontrivial relation $\Sigma_\gamma f_\gamma \delta^\gamma = 0$. Let $v = \max\{|\gamma|;\ f_\gamma \neq 0\}$. Expressing $\delta_i$ as a $Q(R)$-linear combination of $\Delta_j$'s and substituting it for $\delta_i$ in $\Sigma_\gamma f_\gamma \delta^\gamma = 0$, we obtain a nontrivial relation $\Sigma_\gamma g_\gamma \Delta^\gamma = 0$ with $\max\{|\gamma|;\ g_\gamma \neq 0\} = v$. Expressing then $\Delta^\gamma$ in terms of $\partial^\beta = \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$, we obtain

$$(*) \qquad \sum_{|\gamma| = v} \Big(g_\gamma \prod_{i=1}^{r} (a_{is_i})^{\gamma_i}\Big)\, \partial^{\tilde{\gamma}} + \cdots = 0\,,$$

where $\tilde{\gamma}$, as an $n$-tuple, has $\gamma_i$ at the $s_i$-th entry for $1 \leq i \leq r$ and 0 elsewhere if $\gamma = (\gamma_1, \cdots, \gamma_r)$. Among $g_\gamma$'s with $|\gamma| = v$ and $g_\gamma \neq 0$, let $(\alpha_1, \cdots, \alpha_r)$ be the smallest with respect to the lexicographic relation: $(\gamma_1, \cdots, \gamma_r) \leq (\gamma_1', \cdots, \gamma_r')$ if and only if $\gamma_1 = \gamma_1', \cdots, \gamma_{t-1} = \gamma_{t-1}', \gamma_t \leq \gamma_t'$. Then $(g_\alpha \prod_{i=1}^r (a_{is_i})^{\alpha_i})\, \partial^{\tilde{\alpha}}$ has no other terms in $(*)$ to cancel with. This is a contradiction. Q.E.D.

Proof of Theorem 3.2 resumed. The above lemma implies that $B$ is isomorphic to a $W$-algebra over $R$ generated by $Y_1, \cdots, Y_r$. Since any element $\xi$ of $A$ is expressed uniquely in the form

$$(**) \qquad \xi = \sum_\gamma f_\gamma Y^\gamma + \eta\,, \quad f_\gamma \in R \quad \text{and} \quad \eta \in J\,,$$

where $Y^\gamma = Y_1^{\gamma_1} \cdots Y_r^{\gamma_r}$, we know that $A/J$ is isomorphic to $B$.

Now we can easliy show that $A \cong B[Y_{r+1}, \cdots, Y_m]$, a polynomial ring in $Y_{r+1}, \cdots, Y_m$ over $B$ (cf. the above expression $(**)$ of $\xi$). By Björk [1, Th. 3.4, p.43], we have $\text{w.gl.dim}(A) = \text{w.gl.dim}(B) + (m-r) \geq n + m - r$ (cf. Lemma 3.1). By the hypothesis $\text{w.gl.dim}(A) = n$, we have $m = r$. This implies $J = (0)$. Hence $A \cong B$. Q.E.D.

A $W$-algebra $A$ over $R$ is called a *$W$-subalgebra of $\hat{D}_n(K)$* provided $\tilde{\rho}$ is injective.

**Theorem 3.4.** *There is a one-to-one correspondence between the set of $W$-subalgebras of $\hat{D}_n(K)$ and the set of $R$-submodules $\tilde{L}$ of $\mathrm{Der}_K(R)$ satisfying the conditions:*

(L-1) *$\tilde{L}$ is a free R-submodule of* $\mathrm{Der}_K(R)$;
(L-2) *$\tilde{L}$ is closed under the bracket product of* $\mathrm{Der}_K(R)$.

Proof. Let $A$ be a $W$-subalgebra of $\hat{D}_n(K)$. Then we can find an $R$-free submodule $\tilde{L}$ of $\Sigma_1$ which is isomorphic to $L:=\Sigma_1/\Sigma_0$. Since $\tilde{\rho}$ is injective, so is $\rho: L\to\mathrm{Der}_K(R)$. Hence $\tilde{L}$ is an $R$-free submodule of $\mathrm{Der}_K(R)$. Since $\rho\cdot(\pi|_{\tilde{L}})$ is a Lie-algebra homomorphism, $\tilde{L}$ is closed under the bracket product of $\mathrm{Der}_K(R)$ (cf. Theorem 2.3). Conversely, let $\tilde{L}$ be an $R$-submodule of $\mathrm{Der}_K(R)$ satisfying the conditions (L-1) and (L-2). Let $\{Y_1, \cdots, Y_m\}$ be an $R$-free basis of $\tilde{L}$. Then we have:
(1) $Y_i Y_j - Y_j Y_i = \sum_{k=1}^{m} \rho_{i j,k} Y_k$, $1\leq i,j\leq m$,
(2) $Y_i f - f Y_i = Y_i[f]$ for $f\in R$ and $1\leq i\leq m$.

Construct a $K$-algebra $A$ over $R$ as in Theorem 2.2, (2). Then the natural $K$-algebra homomorphism $A\to\hat{D}_n(K)$ over $R$ is injective (cf. the proof of Lemma 3.3). Q.E.D.

A $W$-subalgebra $A$ of $\hat{D}_n(K)$ is said to be *of maximal rank if* rank $\tilde{L}=n$. We shall consider the case $n=1$. Then $L$ is essentially abelian. Hence there exists a $K$-algebra homomorphism $\tilde{\rho}$: $A\to\hat{D}_1(K)$ over $R$ which must be injective by virtue of Theorem 3.4. We set $Y=Y_1$, a free generator of the $R$-module $\tilde{L}$ (cf. Theorem 2.3). Then we have $Yx-xY=f$, where $f=x^r u$ with $u\in R^*$. Replacing $Y$ by $u^{-1}Y$, we may assume that $f=x^r$. We shall show:

**Lemma 3.5.** $\mathrm{Tor}_2^A(K,K)=K$ *if* $r\geq 2$, *while it is zero if* $r=1$. $\mathrm{Tor}_1^A(K,K)=K$ *if* $r=1$.

Proof. Suppose $r>0$. Then $K$ is a two-sided $A$-module. As a right $A$-module $K$ has the following free $A$-module resolution:

$$0\to e_2 A \xrightarrow{\varphi_1} e_1 A\oplus e_1' A \xrightarrow{\varphi_0} e_0 A \xrightarrow{\varepsilon} K\to 0\,,$$

where $\varepsilon$ is the natural residue homomorphism and $\varphi_i (i=0, 1)$ is given as:

$$\varphi_0(e_1)=e_0 Y,\ \ \varphi_0(e_1')=e_0 x \quad\text{and}\quad \varphi_1(e_2)=e_1 x - e_1'(Y+x^{r-1})\,.$$

Take the tensor product of this sequence with a left $A$-module $K=Av$ to obtain the complex:

$$0\to e_2 A\otimes_A Av \xrightarrow{\bar{\varphi}_1} (e_1 A\otimes_A Av)\oplus(e_1' A\otimes_A Av) \xrightarrow{\bar{\varphi}_0} e_0 A\otimes_A Av\to 0\,,$$

where we can identify $e_i A\otimes_A Av$ with $e_i\otimes Kv$ for $e_i=e_0, e_1, e_1'$ and $e_2$. Then it is clear that $\bar{\varphi}_1=\bar{\varphi}_0=0$ if $r\geq 2$. Hence $\mathrm{Tor}_2^A(K,K)=K$ if $r\geq 2$. If $r=1$, then $\bar{\varphi}_1(e_2\otimes v)=-e_1'\otimes v$, whence $\bar{\varphi}_1$ is injective. So, $\mathrm{Tor}_2^A(K,K)=0$ if $r=1$. If $r=1$, $\mathrm{Tor}_1^A(K,K)=K$ because $\bar{\varphi}_0=0$. Q.E.D.

**Corollary 3.6.** *Let $A$ be a $W$-subalgebra of $\hat{D}_1(K)$ with* w.gl.dim $(A)=1$. *Then $A=\hat{D}_1(K)$.*

Proof. With the same notations as in Lemma 3.5, it suffices to show that w.gl.dim$(A)=2$ if $r=1$. Suppose $r=1$ and consider the following exact sequence

$$0 \to e_2 A \xrightarrow{\varphi_1} e_1 A \oplus e_1' A \xrightarrow{\varphi_0} \operatorname{Im} \varphi_0 \to 0 .$$

Suppose that w.gl.dim$(A)=1$. Then $\operatorname{Im}\varphi_0$ is a projective $A$-module in view of the free $A$-module resolution of $K$ given in the proof of Lemma 3.5. So, the above sequence must split. Hence there exists an $A$-homomorphism $\psi$: $e_1 A \oplus e_1' A \to e_2 A$ such that $\psi\varphi_1=id_{e_2 A}$, Write $\psi(e_1)=e_2 a$ and $\psi(e_1')=e_2 b$ for some $a, b$ of $A$. Then we have $ax-b(Y+1)=1$. We claim, however, that $Ax+A(Y+1)$ is a proper left ideal of $A$. Indeed, $Ax=xA$ (cf. Lemma 3.7 below) and $A/Ax$ is isomorphic to a polynomial ring $K[Y]$. Hence $A/Ax+A(Y+1)=K$ and our claim is proved. This is a contradiction. Consequently, we have w.gl.dim $(A)=2$. Q.E.D.

We still remain in the case $n=r=1$. A simple right or left $A$-module $M$ is said to be *unfaithful* if $\operatorname{ann}_A(M)\neq 0$. For $\alpha\in K$, define $K_\alpha=A/xA+(Y-\alpha)A$. Then we have the following:

**Lemma 3.7.** *The following assertions hold true:*

(1) *$K_\alpha$ is a simple right $A$-module as well as a simple left $A$-module.*

(2) *$K_\alpha\cong K_\beta$ if and only if $\alpha=\beta$.*

(3) *Every unfaithful simple right or left $A$-module is isomorphic to $K_\alpha$ for some $\alpha\in K$.*

(4) *Let $S_A$ and $_AT$ be unfaithful simple right and left $A$-modules, respectively. Then* $\operatorname{Tor}_1^A(S, T)=0$.

Proof. The first three assertions can be proved as in the case of a skew polynomial ring or in the case of the universal enveloping algebra of a two-dimensional Lie algebra over $K$. For the convenience of the readers, we shall sketch the proof.

(1) By the relation $Yx-xY=x$, we have

$$(Y-\alpha)\,x-x\,(Y-\alpha)=x \quad \text{for} \quad \alpha\in K$$

This implies that

$$xA=Ax \quad \text{and} \quad xA+(Y-\alpha)\,A=Ax+A(Y-\alpha)$$

Since $K_\alpha\cong K[Y]/(Y-\alpha)$, $K_\alpha$ is simple as right and left $A$-modules.

(2) This easily follows from the first assertion.

(3) Since $xA\subset\operatorname{ann}_A(K_\alpha)$, $K_\alpha$ is unfaithful. Let $I$ be a nonzero two-sided

ideal of $A$. Then $x^n \in I$ for some $n$. Indeed, let $\xi$ be a nonzero element of $I$ and write it as

$$\xi = \sum_{i=0}^{r} f_i Y^i \quad \text{with} \quad f_i \in R \quad \text{and} \quad f_r \neq 0 .$$

Then $\xi x - x\xi = rxf_r Y^{r-1} +$(terms of lower degree) is an element of $I$. Since $rxf_r \neq 0$, we can continue this step of finding an element of $I$ with lower degree in $Y$. After the $r$-steps repeated, we find an element $x^r f_r$ of $I$. Multiplying to this element a unit in $R$, we find $x^r \in I$. Let $S$ be an unfaithful simple right $A$-module. Set $I = \mathrm{ann}_A(S) \neq 0$. Then $x^n \in I$ and $x^{n-1} \notin I$ for some $n$. Since $Sx^{n-1} \neq 0$, there exists $s \in S$ such that $sx^{n-1} \neq 0$. Since $S$ is simple, we have $S = sx^{n-1}A = sAx^{n-1}$, whence $Sx = sAx^n = 0$. Hence $x \in I$. So, $xA \subset I$. It is clear that $I$ is a prime ideal of $A$ in the sense that $J_1 J_2 \subset I$ for two-sided ideals $J_1, J_2$ of $A$ implies $J_1 \subset I$ or $J_2 \subset I$. Let $\bar{A} = A/xA \cong K[Y]$ and $\bar{I}$ the image of $I$ in $\bar{A}$. Since $\bar{I}$ is a prime ideal of $K[Y]$, we have $\bar{I} = (Y-\alpha)K$ for some $\alpha \in K$. Hence $I = xA + (Y-\alpha)A$ and $S \cong A/I = K_\alpha$. A similar argument applies to a simple left $A$-module.

(4) In order to prove the assertion, we have to show

$$\mathrm{Tor}_1^A(K_\alpha, K_\beta) = 0 \quad \text{for} \quad \alpha, \beta \in K .$$

We can easily show this result by replacing $Y$ by $Y-\alpha$ in the proof of Lemma 3.5. Q.E.D.

If $n \geq 2$, we know little on $W$-subalgebras of $\hat{D}_n(K)$ even if it is of maximal rank. We shall give two partial results.

**Proposition 3.8.** *Let $A$ be a $W$-subalgebra of maximal rank of $\hat{D}_n(K)$ corresponding to a Lie subalgebra $\tilde{L} = \oplus_{i=1}^n RY_i$ wiih $Y_i = x_i^{r_i}\partial/\partial x_i$ and $r_i \geq 1$. Then we have*

$$\mu := \max\{v;\ \mathrm{Tor}_v^A(K,K) \neq 0\} = 2\#\{i;\ r_i \geq 2\} + \#\{i;\ r_i = 1\} .$$

*Hence $r_i = 1$ for all $i$ provided* w.gl.dim$(A) = n$.

Proof. Let $S_i$ be the free algebra generated by $Y_i$ over a one-dimensional polynomial ring $K[x_i]$ modulo the two-sided ideal generated by $Y_i x_i - x_i Y_i = x_i^{r_i}$. Since $Y_i Y_j = Y_j Y_i$ and $x_i Y_j = Y_j x_i$ if $i \neq j$, $A$ is isomorphic to

$$(S_1 \otimes_K S_2 \otimes_K \cdots \otimes_K S_n) \otimes_{K[x_1,\cdots,x_n]} R ,$$

when $S_1 \otimes_K \cdots \otimes_K S_n$ is regarded as an algebra over $K[x_1, \cdots, x_n]$. Consider a complex

$$(\tilde{C}_i^\bullet):\ 0 \to e_2^{(i)} S_i \xrightarrow{\varphi_1} e_1^{(i)} S_i \oplus e_1'^{(i)} S_i \xrightarrow{\varphi_0} e_0^{(i)} S_i \xrightarrow{\varepsilon} K \to 0 ,$$

which is defined in the same fashion as in the proof of Lemma 3.5 with $A$ replaced by $S_i$. It is a resolution of the two-sided $S_i$-module $K$ by free right $S_i$-modules. The complex $\tilde{C}^{\bullet}:=(\tilde{C}_1^{\bullet}\otimes_K\cdots\otimes_K\tilde{C}_n^{\bullet})\otimes_{K[x_1,\cdots,x_n]}R$ is a resolution of the two-sided $A$-module $K$ by free right $A$-modules. Let $C_i^{\bullet}$ (resp. $C^{\bullet}$) be the complex obtained from $\tilde{C}_i^{\bullet}$ (resp. $\tilde{C}^{\bullet}$) by replacing $K$ by 0. Then, taking the tensor products with the left $A$-module $K$, we obtain $\bar{C}^{\bullet}:=C^{\bullet}\otimes_A K\cong\bar{C}_1^{\bullet}\otimes_K\cdots\otimes_K\bar{C}_n^{\bullet}$, where $\bar{C}_i^{\bullet}=C_i^{\bullet}\otimes_A K$. By the Künneth formula for homologies, we have

$$\mathrm{Tor}_v^A(K,K)\cong\oplus_{v_1+\cdots+v_n=v}\mathrm{Tor}_{v_1}^{S_1}(K,K)\otimes_K\cdots\otimes_K\mathrm{Tor}_{v_n}^{S_n}(K,K).$$

Hence we obtain the stated formula in view of Lemma 3.5. Q.E.D.

**Proposition 3.9.** *Let $A$ be a W-subalgebra of maximal rank of $\hat{D}_2(K)$ corresponding to a Lie subalgebra $\tilde{L}=RY_1+RY_2$ with $Y_1=h\partial/\partial x_i$, where $h=x_1f+x_2g\in Rx_1+Rx_2$. Suppose that $h$ is a homogeneous polynomial in $x_1$ and $x_2$. Then* $\mathrm{Tor}_3^A(K,K)\neq 0$ *and* $\mathrm{Tor}_4^A(K,K)=0$.

Proof. We have the following relations:

$$\begin{aligned} Y_1Y_2-Y_2Y_1&=-h_{x_2}Y_1+h_{x_1}Y_2\\ Y_1x_1-x_1Y_1&=h=Y_2x_2-x_2Y_2\\ Y_1x_2-x_2Y_1&=0=Y_2x_1-x_1Y_2,\end{aligned}$$

where $h_{x_i}=\partial h/\partial x_i$. Construct a complex of right $A$-modules:

$$0\to e_3A\xrightarrow{\varphi_2}e_2A\oplus e_2'A\oplus e_2''A\oplus e_2'''A\xrightarrow{\varphi_1}$$
$$e_1A\oplus e_1'A\oplus e_1''A\oplus e_1'''A\xrightarrow{\varphi_0}e_0A\xrightarrow{\varepsilon}K\to 0,$$

where:

(0) $K$ is the two-sided $A$-module with $x_i\cdot 1=Y_i\cdot 1=0$ for $i=1,2$;

(i) $\varepsilon(e_0)=1$;

(ii) $\varphi_0(e_1)=e_0Y_1$, $\varphi_0(e_1')=e_0x_1$, $\varphi_0(e_1'')=e_0Y_2$, $\varphi_0(e_1''')=e_0x_2$;

(iii) $\varphi_1(e_2)=e_1x_1-e_1'(Y_1+f)-e_1'''g$, $\varphi_1(e_2')=-e_1'f+e_1''x_2-e_1'''(Y_2+g)$, $\varphi_1(e_2'')=e_1x_2-e_1'''Y_1$, $\varphi_1(e_2''')=-e_1'Y_2+e_1''x_1$;

(iv) $\varphi_2(e_3)=e_2x_2(Y_2+g+h_{x_2})+e_2'x_1(Y_1+f+h_{x_1})-e_2''x_1(Y_2+g+h_{x_2})-e_2'''x_2(Y_1+f+h_{x_1})$.

It is straightforward to show that this complex is a resolution of $K$ by right free $A$-modules. The stated result follows from this observation. Q.E.D.

## References

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