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<td>Spaggiari, Fulvia</td>
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Osaka University
ON THE STABLE CLASSIFICATION
OF SPIN FOUR-MANIFOLDS

FULVIA SPAGGIARI

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1. Introduction

The stable classification of closed connected topological respectively smooth four-manifolds (with orientation or spin structure) via bordism theory is a very nice result in topology of manifolds, and can be found in [11] and [23]. Here stably means that one allows additional connected sums with copies of $\mathbb{S}^2 \times \mathbb{S}^2$ on both sides. In [19] the closed oriented 4-manifolds with finitely presentable fundamental group $\pi$ were classified modulo connected sum with simply connected closed 4-manifolds. More precisely, the stable equivalence classes of these manifolds are bijective to the quotient $H_4(B\pi;\mathbb{Z})/(\text{Aut}\ \pi)_*$ via the map $M \rightarrow f_*[M]$, where $[M] \in H_4(M)$ is the fundamental class, and $f: M \rightarrow B\pi$ is the classifying map for the universal covering of $M$ (see [19, Theorem 1]). The proof of this theorem is based on some facts concerning the cobordism groups $\Omega_4(M)$, $\Omega_4(B\pi)$, and $\Omega_4$ (see for example [7] and [28]). Recently, this result has been extended to the non-orientable case in [18] at least for abelian fundamental groups.

The aim of the present paper is to study the stable classification of closed connected oriented spin smooth 4-manifolds by using techniques of Kervaire-Milnor surgery, as explained for example in [4], [5], [6], and [20]. Then we reproduce a nice result of Kurazono and Matumoto [19] for such manifolds under the assumption that the fundamental group is finitely presentable and has vanishing second and third homology with $\mathbb{Z}_2$-coefficients.

Let $\mathcal{M}_\pi$ (resp. $\mathcal{M}_\pi^{\text{Spin}}$) be the set of closed connected oriented smooth (resp. spin) 4-manifolds with finitely presentable fundamental group $\pi$, which are considered up to (resp. spin) stable equivalence. We say that two manifolds in $\mathcal{M}_\pi$ (resp. $\mathcal{M}_\pi^{\text{Spin}}$) are (resp. spin) stably equivalent if they become diffeomorphic (resp. spin preserving diffeomorphic) after taking connected sums with copies of $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{S}^2 \times \mathbb{S}^2$ (resp. $\mathbb{S}^2 \times \mathbb{S}^2$) on both sides. The first result of the paper is the following
Theorem A. There are bijective maps
\[ C : \Omega_4(B\pi)/(\text{Aut } \pi)_* \rightarrow M_{\pi} \]
and
\[ C^{\text{Spin}} : \Omega_4^{\text{Spin}}(B\pi)/(\text{Aut } \pi)_* \rightarrow M_{\pi}^{\text{Spin}}. \]
The inverse maps are given by sending \( \{M\} \in M_{\pi} \) and \( \{M\}^{\text{Spin}} \in M_{\pi}^{\text{Spin}} \) to \( \{(M, f)\} \in \Omega_4(B\pi)/(\text{Aut } \pi)_* \) and \( \{(M, \sigma_M, f)\} \in \Omega_4^{\text{Spin}}(B\pi)/(\text{Aut } \pi)_* \), respectively. Here \( \sigma_M \) denotes the spin structure on \( M \), and \( f : M \rightarrow B\pi \) is the classifying map.

The statement for the map \( C \) was proved in [19], while that for the map \( C^{\text{Spin}} \) follows from the results given in the next section. Then we can consider the Hurewicz homomorphisms
\[ \mu : \Omega_4(B\pi) \rightarrow H_4(B\pi; \mathbb{Z}) \]
and
\[ \mu^{\text{Spin}} : \Omega_4^{\text{Spin}}(B\pi) \rightarrow H_4(B\pi; \mathbb{Z}) \]
defined by the correspondences \((M, f) \rightarrow f_*[M]\) and \((M, \sigma_M, f) \rightarrow f_*[M]\), respectively. By [11] and [19] the map \( \mu \) is surjective with kernel isomorphic to \( \mathbb{Z} \) and generated by \( \mathbb{C}P^2 \). So there is a decomposition
\[ \Omega_4(B\pi) \cong \Omega_4 \oplus \Omega_4^{\text{Spin}}(B\pi) \cong \mathbb{Z} \oplus H_4(B\pi; \mathbb{Z}) \]
where \( \Omega_4^{\text{Spin}}(B\pi) \cong H_4(B\pi; \mathbb{Z}) \) is the cokernel of the monomorphism \( i : \Omega_4 \rightarrow \Omega_4(B\pi) \), and the isomorphism \( \Omega_4 \cong \mathbb{Z} \) is given by the signature. In §3 we will prove that if \( H_2(B\pi; \mathbb{Z}_2) \cong H_2(B\pi; \mathbb{Z}_2) \cong 0 \), then the map \( \mu^{\text{Spin}} \) is surjective with kernel isomorphic to \( 16\mathbb{Z} \) and generated by the Kummer surface \( K^4 \). The last result permits to obtain a stable decomposition theorem analogous to that proved in [19] for the class of spin smooth 4-manifolds whose fundamental group satisfies the above homological conditions. For this, we say that two closed connected oriented spin smooth 4-manifolds are spin weakly stably equivalent if they become spin preserving diffeomorphic after taking connected sums with copies of the Kummer surface \( K^4 \) and \( S^2 \times S^2 \). Then the second result, we will prove, is the following

Theorem B. Let \( \pi \) be a finitely presentable group which has vanishing second and third homology with \( \mathbb{Z}_2 \)-coefficients. Then the spin weak stable equivalence classes of closed connected oriented spin smooth 4-manifolds \( M \) with fundamental group \( \pi \) one-to-one correspond with the elements of \( H_4(B\pi; \mathbb{Z})/(\text{Aut } \pi)_* \) via the map \((M, \sigma_M, f) \rightarrow f_*[M]\), where \( \sigma_M \) is the spin structure on \( M \), and \( f : M \rightarrow B\pi \) is the
classifying map. In particular, if \( f_*[M] = 0 \), then \( M \) is spin weakly stably equivalent to the boundary of the regular neighborhood of an embedded finite 2-complex, realizing \( \pi \), in 5-space.

For the proof we treat with the spin cobordism groups \( \Omega^\text{Spin}_4(B\pi) \). For the definition of spin cobordism groups we refer to [7] and [28]. A spin structure \( \sigma_M \) on a manifold \( M \) is best thought of as a choice of trivialization of the tangent bundle of \( M \) over the 2-skeleton [27].

The following corollary is related with some papers concerning the homotopy type and the stable classification of closed 4-manifolds with free fundamental group (see [2], [3], [13], [15], [16] and [17]).

**Corollary.** Let \( M \) be a closed connected oriented spin smooth 4-manifold whose fundamental group \( \pi_1(M) \) is a free product \( G_1 \ast \cdots \ast G_p \) such that \( H_2(BG_i; \mathbb{Z}_2) = H_3(BG_i; \mathbb{Z}_2) = 0 \) for any \( i = 1, \ldots, p \). Then \( M \# lK^4\# k(\mathbb{S}^2 \times \mathbb{S}^2) \) is spin preserving diffeomorphic to a connected sum \( M_1 \# \cdots \# M_p \) of closed connected oriented spin smooth 4-manifolds \( M_i \) with \( \pi_1(M_i) \cong G_i \) for some non-negative integers \( l \) and \( k \). The decomposition is spin stably unique.

2. The map \( c^{\text{Spin}} \)

In this section we prove Theorem A for the class of closed connected spin smooth 4-manifolds with finitely presentable fundamental group \( \pi \). We use only simple techniques of Kervaire-Milnor surgery (see for example [4], [5], [6], and [20]).

**Lemma 1.** If \( \pi \) is finitely presented, any element \( \omega \) in \( \Omega^\text{Spin}_4(B\pi) \) gives a closed oriented spin smooth 4-manifold \( (N, \sigma_N) \) with \( \pi_1(N) \cong \pi \) and a map \( g: N \to B\pi \) such that \( g \) induces an isomorphism on \( \pi_1 \) and \( [(N, \sigma_N, g)] = \omega \) in \( \Omega^\text{Spin}_4(B\pi) \).

Proof. The proof goes in the same way as that of Lemma 5 of [19]. We have only to keep the spin structures as in [4], [5], [6] and [20]. We can arrange that \( f \) induces an epimorphism on \( \pi_1 \) by redefining \( M \) to be \( M_1 \# k(\mathbb{S}^1 \times \mathbb{S}^3) \) and redefining \( f \) (we continue to use the same notation). It is easy to see that \( f \) extends in the desired way as does the spin structure, also denoted \( \sigma_M \) (see for example [6, Proposition 4.2]). Now perform surgery on embedded circles in \( \text{Int} M \) which represent elements of the kernel of \( f_* \) to get a new spin 4-manifold \( (N, \sigma_N) \) (see [25, Lemma 5]). Indeed, \( \sigma_M \) extends to a spin structure \( \sigma_N \) on the surgery manifold \( N \). Since \( \pi \) is finitely presented, it is possible, by a finite number of surgeries, to obtain a closed oriented spin smooth 4-manifold \( (N, \sigma_N) \) and a map \( g: N \to B\pi \) which induces an isomorphism on \( \pi_1 \). Furthermore, we have \( [(N, \sigma_N, g)] = \omega \) in \( \Omega^\text{Spin}_4(B\pi) \) since \( k(\mathbb{S}^1 \times \mathbb{S}^3) \) represents the trivial class in \( \Omega^\text{Spin}_4(B\pi) \).

\( \square \)
Corollary 2. If the pairs \((M, \sigma_M, f)\) and \((N, \sigma_N, g)\) represent the same element of \(\Omega^\text{Spin}_4(B\pi)\) such that the induced maps on \(\pi_1\) are isomorphic, then there exist a compact oriented smooth cobordism \((W, F)\) and a spin structure \(\sigma_W\) on \(W\) extending those on \(\partial W = M \cup (-N)\) such that both inclusions \(M \subset W\) and \(N \subset W\) induce isomorphisms on \(\pi_1\).

**Lemma 3.** Let \((W, \sigma_W, F)\) be a compact oriented smooth spin cobordism between \((M, \sigma_M, f)\) and \((N, \sigma_N, g)\) such that both inclusions \(M \subset W\) and \(N \subset W\) induce isomorphisms on \(\pi_1\). Then \(M\#k(S^2 \times S^2)\) is spin preserving diffeomorphic to \(N\#h(S^2 \times S^2)\) for some non-negative integers \(k\) and \(h\).

Proof. We can simplify the handle decomposition of \(W\) relative to \(M\) so that it has only 2-handles and 3-handles as in the usual proof of \(s\)-cobordism theorem in higher dimension. Then the feet of 2-handles are isotopic to the trivial one because it should represent the zero element in \(\pi_1\) by the assumption. So the middle level manifold is a connected sum of \(M\) and some copies of \(S^2 \times S^2\) since the cobordism is spin. By thinking from the other direction, it is also spin preserving diffeomorphic to a connected sum of \(N\) and some copies of \(S^2 \times S^2\).

These results together imply that the map \(c^\text{Spin}\) is bijective, as claimed.

3. Spin cobordism group

Let \((M, \sigma_M)\) be a closed connected oriented spin smooth 4-manifold with finite presentable fundamental group \(\pi\). Then we have a map \(f: M \to B\pi\) from \(M\) to the classifying space \(B\pi\). The map is unique up to homotopy if we fix the induced isomorphism on \(\pi\). The map determines the oriented spin cobordism class \([(M, \sigma_M, f)]\) in \(\Omega^\text{Spin}_4(B\pi)\). On the other hand, any element \(\omega\) of \(\Omega^\text{Spin}_4(B\pi)\) gives a closed connected oriented spin smooth 4-manifold \((N, \sigma_N)\) and a map \(g: N \to B\pi\) with \(g_*: \pi_1(N) \to \pi\) (see Lemma 1 in §2). The manifolds \(M\) and \(N\) will be shown to be spin weakly stably equivalent provided \(H_2(B\pi; \mathbb{Z}_2) \cong H_3(B\pi; \mathbb{Z}_2) \cong 0\). For this we need some results which describe the properties of the Hurewicz homomorphism \(\mu^\text{Spin}\).

**Lemma 4.** Let \(X\) be a CW-complex such that \(H_2(X; \mathbb{Z}_2) = H_3(X; \mathbb{Z}_2) = 0\). Then the map

\[
\mu^\text{Spin}: \Omega^\text{Spin}_4(X) \to H_4(X; \mathbb{Z}),
\]

defined by

\[
\mu^\text{Spin}[(M, \sigma_M, f)] = f_*[M],
\]
is surjective and \( \text{Ker } \mu^\text{Spin} \cong \Omega_4^\text{Spin} \). Moreover, the restriction of \( \mu^\text{Spin} \) on
\[
\widetilde{\Omega}_4^\text{Spin} (X) = \text{Ker} (\Omega_4^\text{Spin} (X) \to \Omega_4^\text{Spin} (*))
\]
is an isomorphism.

Proof. The Atiyah-Hirzebruch spectral sequence
\[
E_{p,q}^2 : H_p (X; \Omega_q^{\text{Spin}}) \Rightarrow \Omega_p^{\text{Spin}}(X)
\]
has vanishing \( E^2 \) terms for \( p+q \leq 4 \) except for \( E_{0,4}^2 \) and \( E_{4,0}^2 \). In fact, recall that \( \Omega_n^{\text{Spin}} \) is \( \mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_2, 0, \) and \( \mathbb{Z} \) for \( n = 0, 1, 2, 3, 4 \) (see [26]), and hence \( E_{3,1}^2 = H_3(X; \Omega_4^{\text{Spin}}) \), \( E_{2,2}^2 = H_2(X; \Omega_2^{\text{Spin}}) \) and \( E_{1,3}^2 = H_1(X; \Omega_3^{\text{Spin}}) \) vanish (under our hypothesis). In general, \( E_{p,q}^\infty \cong J_{p,q}/J_{p-1,q+1} \), where
\[
J_{p,q} = \text{Im} (\Omega_{p+q}^{\text{Spin}} (X^{(p)}, X^{(p-1)}) \to \Omega_{p+q}^{\text{Spin}}(X)).
\]
Thus \( E_{0,4}^\infty \) is the image of the split monomorphism \( \Omega_4^{\text{Spin}}(*) \to \Omega_4^{\text{Spin}}(X) \) whose cokernel is \( E_{4,0}^\infty \subset H_4(X; \mathbb{Z}) \). By dimensional reasoning
\[
\alpha^p : E_{p,q}^r \to E_{p-r,q+r-1}^r
\]
and by comparing with the spectral sequence for \( \Omega_{p+q}^{\text{Spin}}(*) \), it follows that every element in \( E_{0,4}^2 \) and \( E_{4,0}^2 \) is a permanent cycle. So we have \( E_{0,4}^\infty = E_{4,0}^\infty \cong H_4(X; \mathbb{Z}) \) and \( E_{0,4}^\infty = E_{4,0}^\infty \cong H_0(X; \mathbb{Z}) \cong \mathbb{Z} \cong \Omega_4^{\text{Spin}}(*) \). Then we get the exact sequence
\[
0 \to E_{0,4}^2 \cong \Omega_4^{\text{Spin}}(*) \to \Omega_4^{\text{Spin}}(X) \to E_{4,0}^2 \cong H_4(X; \mathbb{Z}) \to 0.
\]
The map \( \mu^\text{Spin} : \Omega_4^{\text{Spin}}(X) \to H_4(X; \mathbb{Z}) \) induces a map from the spectral sequence for \( \Omega_{p+q}^{\text{Spin}}(X) \) to the spectral sequence for \( H_{p+q}(X; \mathbb{Z}) \) and coincides with the map \( \Omega_4^{\text{Spin}}(X) \to E_{4,0}^2 \cong H_4(X; \mathbb{Z}) \) of the sequence above for \( n = 4 \). Finally, we note that the kernel of this map is \( E_{0,4}^2 \cong \Omega_4^{\text{Spin}} \cong \mathbb{Z} \), which is generated by the Kummer surface \( K^4 \).

\[\square\]

**Corollary 5.** If \( H_2(B\pi; \mathbb{Z}_2) = H_3(B\pi; \mathbb{Z}_2) = 0 \), then the map
\[
\mu^\text{Spin} : \Omega_4^{\text{Spin}}(B\pi) \to H_4(B\pi; \mathbb{Z})
\]
is an epimorphism, and \( \text{Ker } \mu^\text{Spin} \cong \Omega_4^{\text{Spin}} \) is generated by the Kummer surface. Then there is a decomposition
\[
\Omega_4^{\text{Spin}}(B\pi) \cong \Omega^{\text{Spin}} \oplus \widetilde{\Omega}_4^{\text{Spin}}(B\pi) \cong 16\mathbb{Z} \oplus H_4(B\pi; \mathbb{Z}).
\]
where \( \tilde{\Omega}^\text{Spin}_{4}(B\pi) \cong H_4(B\pi; \mathbb{Z}) \) denotes the cokernel of the split monomorphism
\[
\iota^\text{Spin} : \Omega^\text{Spin}_{4} \to \tilde{\Omega}^\text{Spin}_{4}(B\pi),
\]
and the isomorphism \( \Omega^\text{Spin}_{4} \cong 16\mathbb{Z} \) is given by the signature.

As a consequence of Corollary 5, we get the following useful results first proved in [5, Theorem 5.2] and [6, Proposition 5.1], respectively.

**Corollary 6.** If \( H_8(B\pi; \mathbb{Z}_2) = H_8(B\pi; \mathbb{Z}_2) = 0 \), then an oriented spin cobordism class \([\langle M, \sigma_M, f \rangle]\) is zero in \( \tilde{\Omega}^\text{Spin}_{4}(B\pi) \) if and only if the signature of \( M \) vanishes, and \( f_*[M] = 0 \) in \( H_4(B\pi; \mathbb{Z}) \).

**Corollary 7.** Suppose that \( H_8(B\pi; \mathbb{Z}_2) = H_8(B\pi; \mathbb{Z}_2) = 0 \). Then \( \tilde{\Omega}^\text{Spin}_{4}(B\pi) \) is trivial if and only if \( H_4(B\pi; \mathbb{Z}) = 0 \).

Now we are going to prove Theorem B. Let \( \pi \) be a finitely presented group which has vanishing second and third homology with \( \mathbb{Z}_2 \)-coefficients. A closed connected oriented spin 4-manifold \((M, \sigma_M)\) with fundamental group \( \pi \) carries a classifying map \( f : M \to B\pi \). The triple \((M, \sigma_M, f)\) determines an oriented spin cobordism class \([\langle M, \sigma_M, f \rangle]\) in \( \Omega^\text{Spin}_{4}(B\pi) \), and an element \( \mu^\text{Spin}[\langle M, \sigma_M, f \rangle] = f_*[M] \) in \( H_4(B\pi; \mathbb{Z}) \).

Of course, spin weakly stably equivalent 4-manifolds determine the same element of \( H_4(B\pi; \mathbb{Z})/(\text{Aut } \pi)_+ \). Conversely, take any element of \( H_4(B\pi; \mathbb{Z}) \). Then it gives an element of
\[
\tilde{\Omega}^\text{Spin}_{4}(B\pi) = \text{Ker}(\Omega^\text{Spin}_{4}(B\pi) \to \Omega^\text{Spin}_{4}(\ast))
\]
by Corollary 5. It comes from a closed connected spin smooth 4-manifold \((N, \sigma_N)\) with \( \pi_1(N) \cong \pi \) and a map \( g : N \to B\pi \) by Lemma 1. Let \((M', \sigma_{M'}, f')\) be another triple with \( \pi_1(M) \cong \pi \) and a map \( f : M \to B\pi \) such that \( f_*[M] = g_*[N] \). Then for some \( l \) and \( m \) we have
\[
[\langle M\#lK^4, \sigma_{M}', f' \rangle] = [\langle N\#mK^4, \sigma_N', g' \rangle]
\]
in \( \Omega^\text{Spin}_{4}(B\pi) \) by Corollary 5, and the fact that \( \Omega^\text{Spin}_{4}(\ast) \) is generated by the Kummer surface \( K^4 \) (Here \( f' \) and \( g' \) are maps sending \( K^4 \)'s to one point). Therefore the manifolds \( M \) and \( N \) are spin weakly stably equivalent by Corollary 2, and Lemma 3, i.e. \( M\#lK^4\#k(S^2 \times S^2) \) is spin preserving diffeomorphic to \( N\#mK^4\#h(S^2 \times S^2) \) for some \( l, m, h \) and \( k \).

4. Some applications

(1) If \( \pi \) is a free group of rank \( p \), then \( B\pi \cong \bigvee_p S^1 \), so we get in particular \( H_i(B\pi; \mathbb{Z}_2) \cong 0 \) for \( i = 2, 3 \), and \( H_4(B\pi; \mathbb{Z}) \cong 0 \). Thus we have \( \tilde{\Omega}^\text{Spin}_{4}(B\pi) \cong 16\mathbb{Z} \).
and the isomorphism is given by the signature. Theorem A implies that if \( M \) is a closed connected oriented spin 4-manifold with signature zero and \( \pi_1(M) \cong \pi \), then \( M \) is spin stably homeomorphic to \( #p(S^1 \times S^3) \) (see \[2\], \[3\], \[13\], and \[15\]). Theorem B says that a closed connected oriented spin 4-manifold \( M \) with \( \pi_1(M) \cong \pi \) becomes homeomorphic to \( #p(S^1 \times S^3) \) after taking connected sums with copies of \( K^4 \) and \( S^2 \times S^2 \). We recall that there exists a closed oriented topological 4-manifold with fundamental group \( \mathbb{Z} \) which is not the connected sum of \( S^1 \times S^3 \) with a simply connected 4-manifold (see \[12\]).

(2). Let \( \pi \) be a group with a presentation of deficiency one which is an extension of \( \mathbb{Z} \) by a finitely generated normal subgroup. It was shown in \[14\] that the canonical 2-complex corresponding to that presentation is aspherical, hence \( \pi \) has geometric dimension at most 2. Furthermore, the Euler characteristic of \( B\pi \) vanishes. Suppose that \( H_i(B\pi; \mathbb{Z}_2) \cong \mathbb{Z}_2 \) (examples are given by knot like groups, i.e., groups having abelianization \( \mathbb{Z} \) and deficiency one). Since \( \chi(B\pi) = 0 \), it follows that \( H_i(B\pi; \mathbb{Z}_2) = 0 \) for \( i = 2, 3 \), and \( H_4(B\pi; \mathbb{Z}) = 0 \). Thus we obtain \( \Omega_4^{\text{Spin}}(B\pi) \cong 16\mathbb{Z} \), as before. We recall that an algebraic characterization of certain 4-manifolds called exact manifolds with infinite cyclic first homology was given in nice recent papers of Kawauchi (see \[16\] and \[17\]).

(3). If \( \pi \cong \mathbb{Z}_p \oplus \mathbb{Z} \) where \( p \) is a prime number, \( p > 2 \), then \( H_i(B\pi; \mathbb{Z}) \cong \mathbb{Z}_p \). Since \( \text{Aut} \pi \) identifies all the non-zero elements of \( H_i(B\pi; \mathbb{Z}) \), we get that \( H_i(B\pi; \mathbb{Z})/(\text{Aut} \pi) \), is isomorphic to \( \mathbb{Z}_2 \) (see \[19\]). Further, we have \( H_i(B\pi; \mathbb{Z}_2) \cong 0 \) for \( i = 2, 3 \), hence \( S_4^{\text{Spin}}(B\pi) \cong 16\mathbb{Z} \oplus \mathbb{Z}_p \). Let \( Y^4 \) be the boundary of a regular neighbourhood of an embedded finite 2-complex \( X^2 \) realizing \( \pi \) in the standard 5-space. The induced homomorphism \( H_i(Y; \mathbb{Z}) \to H_i(B\pi; \mathbb{Z}) \) is trivial since it factorizes through \( H_i(X; \mathbb{Z}) = 0 \). Thus \( [Y] \) goes to zero in \( H_i(B\pi; \mathbb{Z})/(\text{Aut} \pi) \), \( \cong \mathbb{Z}_2 \). Of course, \( Y^4 \) is spin and has trivial signature since it embeds smoothly in \( \mathbb{R}^5 \). Let \( \Sigma_p \) be the product \( L(p, 1) \times S^1 \), where \( L(p, 1) \) is the usual lens space. Then \( [\Sigma_p] \) goes to a nontrivial element of \( H_i(B\pi; \mathbb{Z}) \). Theorem B says that any closed connected oriented spin smooth 4-manifold \( M \) becomes spin stably equivalent to either \( \Sigma_p \) or \( Y^4 \).

(4). If \( \pi \) is a cyclic group \( \mathbb{Z}_p \) of odd order, then \( H_i(B\pi; \mathbb{Z}_2) = 0 \) for \( i = 2, 3 \), and \( H_4(B\pi; \mathbb{Z}) = 0 \), hence \( \Omega_4^{\text{Spin}}(B\pi) \cong 16\mathbb{Z} \). Let \( \Sigma_p \) be the closed spin 4-manifold obtained from \( \Sigma_p \) by killing the generator of \( \mathbb{Z} \subset \pi_1(\Sigma_p) = \mathbb{Z}_p \oplus \mathbb{Z} \). By Theorem B any closed connected oriented spin 4-manifold \( M \) with \( \pi_1(M) \cong \mathbb{Z}_p \) becomes diffeomorphic to \( \Sigma_p \) after stabilization with copies of \( K^4 \) and \( S^2 \times S^2 \) (compare with Theorem 2.5 of \[11\]). Further examples of smooth 4-manifolds with cyclic fundamental groups were constructed in \[8\] by using the knot surgery construction.

(5). Let \( \pi \) be the fundamental group of a closed aspherical 4-manifold \( Q^4 \) which is a rational homology 4-sphere. The existence of such a manifold was proved for example in \[24\]. If further \( H_2(B\pi; \mathbb{Z}_2) = 0 \), then the condition \( \chi(B\pi) = 2 \) implies that the Betti numbers \( \beta_i \) vanish (mod 2) for \( i = 1, 3 \), hence \( H_3(B\pi; \mathbb{Z}_2) = 0 \). Of course, we also have \( H_4(B\pi; \mathbb{Z}) \cong \mathbb{Z} \), hence \( H_i(B\pi; \mathbb{Z})/(\text{Aut} \pi) \) is isomorphic to either \( \mathbb{Z} \) or \( \mathbb{Z}/\{\pm 1\} \).
Finally, we obtain $\mathbb{S}^{\text{Spin}}_4(B\pi) \cong 16\mathbb{Z} \oplus \mathbb{Z}$.

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References

stable classification of spin 4-manifolds


Dipartimento di Matematica
Università di Modena and Reggio Emilia
Via Campi 213/B, 41100 Modena, Italy
e-mail: spaggiari.fulvia@unimo.it