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A NOTE ON SEMIPRIMARY PP-RINGS

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A ring R with identity is called a left PP-ring if every principal left ideal of R is R-projective. In [1] Harada gave a characterization of semiprimary left PP-rings in terms of his generalized triangular matrix rings and used this to show that a left PP-ring which is semiprimary is also a right PP-ring. The purpose of this note is to give a more direct and somewhat less complicated proof of this result of Harada.

Recall that a *Baer ring* is a ring with identity in which the left annihilator of every subset is generated by an idempotent. As was observed by Kaplansky [2], left may be replaced by right in this definition. To see this, let $l(X)$ and $r(X)$ denote the left and right annihilators of the subset X in R, then $r(X)=r\ l\ r\ (X)$; hence if $l(r(X))=Re$, $e^2=e$, then $r(X)=r(Re)=(1-e)R$.

Theorem. *Let R be a semiprimary ring with identity. Then, the following are equivalent*

(i) *R is a left PP-ring.*
(ii) *R is a Baer ring.*
(iii) *R is a right PP-ring.*

Proof. (i) *implies* (ii): If $a\in R$, then Ra is projective and therefore the exact sequence $R\overset{d}{\to}Ra\to 0$ splits (where $d(r)=ra$) and hence Ker $d=l(a)$ is a direct summand of ${}_RR$. Since R has an identity one easily shows that $l(a)=Re$ for some idempotent e in R. Now by an argument due to Maeda [3], we can extend this to two elements: Let a, $b\in R$. If $l(a)=Re$ and $l(b)=Rf$, then $l(a)=l(1-e)$ and $l(b)=l(1-f)$. As we have just shown there is an idempotent g such that $l(e(1-f))=Rg$. It is straightforward now to show that ge is an idempotent and $l(1-e,\ 1-f)=Rge$. Hence, $l(a,\ b)=Rge$.

Since $l(1-e)=Re$ when $e^2=e$, we have by induction that if X is finite, $l(X)=Re$ for some idempotent e. Now, since $l(X)=\cap\ \{l(x): x\in X\}$, to establish (ii) it clearly suffices to prove that a semiprimary ring satisfies the descending

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chain condition on principal left ideals which are generated by idempotents. But this is clear since $Rf \subset Re$ implies that $Re = Rf \oplus R(e-ef)$.

By the above comment on the left-right symmetry of Baer rings it suffices to show that (ii) implies (i): Let $a \in R$, then $l(a) = Re$, $e^2 = e$. Hence ${}_RR = R(1-e) \oplus l(a)$, and since $l(a) = \text{Ker}\,(r \rightarrow ra)$ we have ${}_RR \cong Ra \oplus l(a)$ and so Ra is projective.

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References

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