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## A NOTE ON SEMIPRIMARY PP-RINGS

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A ring  $R$  with identity is called a left PP-ring if every principal left ideal of  $R$  is  $R$ -projective. In [1] Harada gave a characterization of semiprimary left PP-rings in terms of his generalized triangular matrix rings and used this to show that a left PP-ring which is semiprimary is also a right PP-ring. The purpose of this note is to give a more direct and somewhat less complicated proof of this result of Harada.

Recall that a *Baer ring* is a ring with identity in which the left annihilator of every subset is generated by an idempotent. As was observed by Kaplansky [2], left may be replaced by right in this definition. To see this, let  $l(X)$  and  $r(X)$  denote the left and right annihilators of the subset  $X$  in  $R$ , then  $r(X)=r\ l(X)$ ; hence if  $l(r(X))=Re$ ,  $e^2=e$ , then  $r(X)=r(Re)=(1-e)R$ .

**Theorem.** *Let  $R$  be a semiprimary ring with identity. Then, the following are equivalent*

- (i)  $R$  is a left PP-ring.
- (ii)  $R$  is a Baer ring.
- (iii)  $R$  is a right PP-ring.

Proof. (i) *implies* (ii): If  $a \in R$ , then  $Ra$  is projective and therefore the exact sequence  $R \xrightarrow{d} Ra \rightarrow 0$  splits (where  $d(r)=ra$ ) and hence  $\text{Ker } d=l(a)$  is a direct summand of  ${}_R R$ . Since  $R$  has an identity one easily shows that  $l(a)=Re$  for some idempotent  $e$  in  $R$ . Now by an argument due to Maeda [3], we can extend this to two elements: Let  $a, b \in R$ . If  $l(a)=Re$  and  $l(b)=Rf$ , then  $l(a)=l(1-e)$  and  $l(b)=l(1-f)$ . As we have just shown there is an idempotent  $g$  such that  $l(e(1-f))=Rg$ . It is straightforward now to show that  $ge$  is an idempotent and  $l(1-e, 1-f)=Rge$ . Hence,  $l(a, b)=Rge$ .

Since  $l(1-e)=Re$  when  $e^2=e$ , we have by induction that if  $X$  is finite,  $l(X)=Re$  for some idempotent  $e$ . Now, since  $l(X)=\cap \{l(x): x \in X\}$ , to establish (ii) it clearly suffices to prove that a semiprimary ring satisfies the descending

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chain condition on principal left ideals which are generated by idempotents. But this is clear since  $Rf \subset Re$  implies that  $Re = Rf \oplus R(e - ef)$ .

By the above comment on the left-right symmetry of Baer rings it suffices to show that (ii) implies (i): Let  $a \in R$ , then  $l(a) = Re$ ,  $e^2 = e$ . Hence  ${}_R R = R(1 - e) \oplus l(a)$ , and since  $l(a) = \text{Ker}(r \rightarrow ra)$  we have  ${}_R R \cong Ra \oplus l(a)$  and so  $Ra$  is projective.

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### References

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- [3] S. Maeda: *On a ring whose principal right ideals generated by idempotents form a lattice*, J. Sci. Hiroshima Univ. Ser. A, **24** (1960), 509–525.