<table>
<thead>
<tr>
<th><strong>Title</strong></th>
<th>On quasiinvariants of $S_n$ of hook shape</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Author(s)</strong></td>
<td>Tsuchida, Tadayoshi</td>
</tr>
<tr>
<td><strong>Citation</strong></td>
<td>Osaka Journal of Mathematics. 47(2) P.461–P.485</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>2010-06</td>
</tr>
<tr>
<td><strong>Text Version</strong></td>
<td>publisher</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="https://doi.org/10.18910/5910">https://doi.org/10.18910/5910</a></td>
</tr>
<tr>
<td><strong>DOI</strong></td>
<td>10.18910/5910</td>
</tr>
<tr>
<td><strong>rights</strong></td>
<td></td>
</tr>
<tr>
<td><strong>Note</strong></td>
<td></td>
</tr>
</tbody>
</table>
ON QUASIINVARIANTS OF $S_n$ OF HOOK SHAPE

TADAYOSHI TSUCHIDA

(Received July 11, 2008, revised January 8, 2009)

Abstract

O. Chalykh, A.P. Veselov and M. Feigin introduced the notion of quasiinvariants of Coxeter groups, which is a generalization of invariants. In [2], Bandlow and Musiker showed that for the symmetric group $S_n$ of order $n$, the space of quasiinvariants has a decomposition indexed by standard tableaux. They gave a description of a basis for the components indexed by standard tableaux of shape $(n-1, 1)$. In this paper, we generalize their results to a description of a basis for the components indexed by standard tableaux of arbitrary hook shape.

1. Introduction

In [3] and [5], O. Chalykh, A.P. Veselov and M. Feigin introduced the notion of quasiinvariants for Coxeter groups, which is a generalization of invariants. For any Coxeter group $G$, the quasiinvariants are determined by a multiplicity $m$ which is a $G$-invariant map from the set of reflections to non-negative integers.

We denote by $S_n$ the symmetric group of order $n$. In the case of $S_n$, the multiplicity is a constant function. Take a non-negative integer $m$. A polynomial $P \in \mathbb{Q}[x_1, x_2, \ldots, x_n]$ is called an $m$-quasiinvariant if the difference

$$(1 - (i, j))P(x_1, \ldots, x_n)$$

is divisible by $(x_i - x_j)^{2m+1}$ for any transposition $(i, j) \in S_n$.

The notion of quasiinvariants appeared in the study of the quantum Calogero Moser system. In the case of $S_n$, this system is determined by the following differential operator (the generalized Calogero–Moser Hamiltonian):

$$L_m = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - 2m \sum_{1 \leq i < j \leq n} \frac{1}{x_i - x_j} \left( \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right)$$

where $m$ is a real number.

Let $G$ be a Coxeter group. We denote by $S^G$ the sub ring generated by invariant polynomials for $G$ and by $I^G$ the ideal of the ring of quasiinvariants generated by the

2000 Mathematics Subject Classification. 68R05, 05E10.
invariant polynomials of positive degree. For a generic multiplicity, there exists an isomorphism from the ring $S^G$ to the ring of $G$-invariant quantum integrals of the generalized Calogero–Moser Hamiltonian (sometimes called Harish-Chandra isomorphism). We denote by $L_1, L_2, \ldots, L_n$ the operators corresponding to fundamental invariant polynomials $\sigma_1, \sigma_2, \ldots, \sigma_n$. The generalized Calogero–Moser Hamiltonian is a member of this ring (see for example [5], [6]).

In the case of non-negative integer multiplicities, Chalykh and Veselov showed that there exists a homomorphism from the ring of quasiinvariants to the commutative ring of differential operators whose coefficients are rational functions (see e.g. [3]). It is shown that the restriction of such homomorphism onto $S^G$ induces the Harish-Chandra isomorphism. In the case of non-negative integer multiplicities there are much more quantum integrals.

Let $m$ be a non-negative integer multiplicity. In [5], Feigin and Veselov introduced the notion of $m$-harmonics which are defined as the solutions of the following system:

$$L_1 \psi = 0,$$
$$L_2 \psi = 0,$$
$$\cdots$$
$$L_n \psi = 0.$$

Feigin and Veselov also showed that the solutions of such system are polynomials. They also showed that the space of $m$-harmonic polynomials is a subspace of the space of $m$-quasiinvariants and has dimension $|G|$. In [7], G. Felder and Veselov gave a formula of the Hilbert series of the space of $m$-harmonic polynomials.

In [4], P. Etingof and V. Ginzburg proved the following:

(i) the ring of quasiinvariants of $G$ is a free module over $S^G$, Cohen–Macaulay and Gorenstein,

(ii) there is an isomorphism from the quotient space of quasiinvariants by $I^G$ to the dual space of $m$-harmonic polynomials,

(iii) the Hilbert series of the quotient space of the quasiinvariants by $I^G$ is equal to that of $m$-harmonic polynomials.

Let $I_2(N)$ be the dihedral group of regular N-gon. In [5], Feigin and Veselov considered quasiinvariants of $I_2(N)$ for any constant multiplicity. Since $I_2(N)$ has rank 2, quasiinvariants can be expressed as a polynomial in $z$ and $\bar{z}$. Feigin and Veselov gave generators over $S^{I_2(N)}$ by a direct calculation. In [6], Feigin studied quasiinvariants of $I_2(N)$ for any non-negative integer multiplicity. He gave a free basis of the module of quasiinvariants over $S^{I_2(N)}$ using the above mentioned results of Etingof and Ginzburg. An explicit description of basis of the quotient space of quasiinvariants for $S_3$ is contained in [5]. Another description is given in [1]. In [7], for $S_n$ Felder and Veselov provided integral expressions for the lowest degree non-symmetric quasiinvariant polynomials (the degree $nm + 1$). However, for any integer $n \geq 4$ a basis of the quotient
space of quasiinvariants of $S_n$ is not known.

In this paper, we consider the quasiinvariants of $S_n$. In this case, $m$ is a non-negative integer. We denote by $\text{QI}_m$ the ring of quasiinvariants and by $\Lambda_n$ the ring of symmetric polynomials. We define $\text{QI}_m^*$ as the quotient space of $\text{QI}_m$ by the ideal generated by the homogeneous symmetric polynomials of positive degree.

In [2], J. Bandlow and G. Musiker showed that the space $\text{QI}_m$ has a decomposition into subspaces indexed by standard tableaux. Each component has a $\text{BF}_n$ module structure. The quotient space $\text{QI}_m^*$ is also decomposed in the same way. They constructed an explicit basis of the submodules of $\text{QI}_m^*$ indexed by standard tableaux of shape $(n-1, 1)$.

In this paper, we extend the result in [2]. We construct a basis of the submodules of $\text{QI}_m^*$ indexed by standard tableaux of shape $(n - k + 1, 1^{k-1})$ (a hook) (see Theorem 3.8). The elements of our basis are expressed as determinants of a matrix with entries similar to elements of basis introduced in [2]. We also show that our basis is a free basis of the submodule of $\text{QI}_m^*$ indexed by a hook $(n - k + 1, 1^{k-1})$ over $\text{BF}_n$ (Corollary 3.11).

We also show how the operator $L_m$ acts on our basis. In [5], it is proved that the operator $L_m$ preserves $\text{QI}_m$. In [2], it is obtained explicit formulas of the action of $L_m$ on their basis. We extend these formulas to those of our basis (Theorem 4.4).

2. Preliminaries

2.1. Symmetric group and Young diagram. We denote $\mathbb{Q}[x_1, x_2, \ldots, x_n]$ by $K_n$ and the symmetric group on $\{1, 2, \ldots, n\}$ by $S_n$. For a finite set $X$, we denote the symmetric group on $X$ by $S_X$.

The symmetric group $S_n$ acts on $K_n$ by

$$\sigma P(x_1, \ldots, x_n) = P(x_{\sigma(1)}, \ldots, x_{\sigma(n)}), \quad \sigma \in S_n.$$ 

A polynomial $P(x_1, x_2, \ldots, x_n)$ is called a symmetric polynomial when for any $\sigma \in S_n$, $P(x_1, x_2, \ldots, x_n)$ satisfies

$$\sigma P(x_1, \ldots, x_n) = \sigma(x_1, \ldots, x_n).$$

We denote by $\Lambda_n$ the subspace spanned by symmetric polynomials and by $\Lambda_n^d$ the subspace of $\Lambda_n$ spanned by homogeneous polynomials of degree $d$. We set $\Lambda_n^d = [0]$ if $d < 0$. The $i$-th elementary symmetric polynomial is denoted by $e_i$. For a partition $\nu = (\nu_1, \nu_2, \ldots)$, we define $e_\nu = \prod_i e_{\nu_i}$. A basis of $\Lambda_n$ is given by $\{e_\nu\}$.

The group ring of $S_n$ over $\mathbb{Q}$ is denoted by $\mathbb{Q}S_n$. The action of $S_n$ on $K_n$ is naturally extended to that of $\mathbb{Q}S_n$. For a subgroup $H$ of $S_n$, we define $[H], [H]'$ in $\mathbb{Q}S_n$ by

$$[H] = \sum_{\sigma \in H} \sigma,$$

$$[H]' = \sum_{\sigma \in H} \text{sgn}(\sigma) \sigma.$$
Let \( \lambda = (\lambda_1, \lambda_2, \ldots) \) be a partition. When \( \lambda \) is a partition of a positive integer \( n \), we denote this by \( \lambda \vdash n \). We define \( l(\lambda) = \#\{i \mid \lambda_i \neq 0\} \) and \( |\lambda| = \sum_i \lambda_i \). They are called the length and the size of \( \lambda \) respectively.

For a partition \( \lambda \), the Young diagram of shape \( \lambda \) is a diagram such that its \( i \)-th row has \( \lambda_i \) boxes and it is arranged in left-justified rows. For example, the Young diagram of shape \((4, 3, 1)\) is

\[
\begin{array}{ccc}
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\end{array}
\]

We denote by \((i, j)\) a box on the \((i, j)\)-th position of the diagram. For instance, the box \((2, 3)\) of the Young diagram of shape \((4, 3, 1)\) is

\[
\begin{array}{ccc}
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\end{array}
\]

We identify the Young diagram of shape \( \lambda \) with the partition \( \lambda \).

Let \( k, n \) be integers such that \( k \geq 2 \) and \( n \geq k \). We define \( \eta(n, k) = (n-k+1, 1^{k-1}) \). We have \( l(\eta(n, k)) = k \) and \( |\eta(n, k)| = n \). We call \( \eta(n, k) \) (also the Young diagram of \( \eta(n, k) \)) the hook.

For \( \lambda \vdash n \), we define the arm length \( a(i, j) \) for box \((i, j) \in \lambda \) as

\[
a(i, j) = \#\{(i, l) \mid j < l, (i, l) \in \lambda\}.
\]

We also define the leg length \( l(i, j) \) for box \((i, j) \) as

\[
l(i, j) = \#\{(k, j) \mid i < k, (k, j) \in \lambda\}.
\]

We define \( h(i, j) = a(i, j) + l(i, j) + 1 \) called the hook length for box \((i, j) \in \lambda \).

A tableau of shape \( \lambda \) is obtained by assigning a positive integer to each box of the Young diagram \( \lambda \). In this paper, we assume that entries of boxes are different each other. For a tableau \( D \), we denote by \( D_{i,j} \) the entry in the box \((i, j)\) of \( D \). We define

\[
\text{mem}(D) = \{D_{i,j} \mid (i, j) \in \lambda\}.
\]

A tableau \( T \) is called a standard tableau if \( T \) satisfies \( \text{mem}(T) = \{1, 2, \ldots, n\} \) and

\[
T_{i,j} < T_{k,j}, \quad T_{i,j} < T_{l,i}, \quad i < k, \quad j < l.
\]

We denote by \( ST(\lambda) \) the set of all standard tableaux of shape \( \lambda \) and by \( ST(n) \) the set of all standard tableaux with \( n \) boxes.
For a tableau $D$ of shape $\lambda$, we define
\[
C(D) = \{[\sigma \in S_{\text{mem}(D)}] \mid \sigma \text{ preserves each column of } D\},
\]
\[
R(D) = \{[\sigma \in S_{\text{mem}(D)}] \mid \sigma \text{ preserves each row of } D\},
\]
\[
f_\lambda = \#ST(\lambda),
\]
\[
\gamma_D = \frac{f_\lambda C(D) R(D)}{n!},
\]
\[
V_D = \prod_{(i,j) \in C_D} (x_i - x_j)
\]
where $C_D = \{(i,j) \mid i < j \text{ and } i, j \text{ are entries in a same column of } D\}$. The element
\[
\gamma_D \in \mathbb{Q}S_{\text{mem}(D)}
\]
satisfies $\gamma_D^2 = \gamma_D$.

**Definition 2.1.** Let $s_1, s_2, \ldots, s_n$ be mutually distinct positive integers.

1. We denote by $D(s_1, s_2, \ldots, s_k; s_1, s_{k+1}, \ldots, s_n)$ the tableau of shape $\eta(n, k)$ such that the entries in the first column and in the first row are $s_1, s_2, \ldots, s_k$ and $s_1, s_{k+1}, \ldots, s_n$ in order, respectively.

2. A tableau $D(s_1, s_2, \ldots, s_k; s_1, s_{k+1}, \ldots, s_n)$ is a standard tableau of shape $\eta(n, k)$ if and only if the following holds:
\[
\begin{align*}
&\text{$s_1, s_2, \ldots, s_n$ is a permutation of $1, 2, \ldots, n$,} \\
&s_1 = 1, s_2 \leq \cdots \leq s_k, s_{k+1} \leq \cdots \leq s_n.
\end{align*}
\]
Then we simply write $D(s_1, s_2, \ldots, s_k; s_1, s_{k+1}, \ldots, s_n)$ as $T(1, s_2, \ldots, s_k)$.

3. Let $i$ be an integer such that $1 \leq i \leq k$ (resp. $k + 1 \leq i \leq n$). We set $D = D(s_1, s_2, \ldots, s_k; s_1, s_{k+1}, \ldots, s_n)$. We define
\[
D^{\uparrow i} = D(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_k; s_1, s_{k+1}, \ldots, s_n)
\]
(resp. $D^{\downarrow i} = D(s_1, \ldots, s_k; s_1, s_{k+1}, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$).

For example, the standard tableau $T(1, 3, 4) = D(1, 3, 4; 1, 2, 5, 6)$ of shape $(4, 1, 1)$ is
\[
\begin{array}{cccc}
1 & 2 & 5 & 6 \\
3 & & & \\
4 & & & \\
\end{array}
\]
The tableau $T(1, 3, 4)^1$ is
\[
\begin{array}{cccc}
3 & 2 & 5 & 6 \\
4 & & & \\
\end{array}
\]
and $T(1, 3, 4)^2$ is
\[
\begin{array}{c}
1 & 5 & 6 \\
3 & & \\
4 & & \\
\end{array}
\]
We have the following propositions.

**Proposition 2.2** ([2]). For any \( f = \sum_{\sigma \in S_n} f_{\sigma} \sigma \in QS_n, P \in \Lambda_n \) and \( Q \in K_n \), we have \( f(PQ) = Pf(Q) \).

**Proposition 2.3** ([2]). Let \( i_1, i_2, \ldots, i_n \) be a permutation of \( 1, 2, \ldots, n \). Then \([S_n] \) and \([S_n] \)' are expressed as follows:

\[
[S_n] = (1 + (i_1, i_n) + \cdots + (i_{n-1}, i_n)) \cdots (1 + (i_1, i_3) + (i_2, i_3))(1 + (i_1, i_2)),
\]

\[
[S_n]' = (1 - (i_1, i_n) - \cdots -(i_{n-1}, i_n)) \cdots (1 - (i_1, i_3) - (i_2, i_3))(1 - (i_1, i_2)).
\]

### 2.2. The quasiinvariants of \( S_n \)

We recall the definition and the notation of \( m \)-quasiinvariants. Take a non-negative integer \( m \). A polynomial \( P \in K_n \) is called an \( m \)-quasiinvariant if the difference

\[
(1 - (i, j))P(x_1, \ldots, x_n)
\]

is divisible by \((x_i - x_j)^{2m+1}\) for any transposition \((i, j) \in S_n\). We denote by \( QI_m \) the ring of quasiinvariants and by \( \Lambda_n \) the space of symmetric polynomials. We denote by \( I_m \) the ideal of \( QI_m \) generated by \( e_1, \ldots, e_n \). We set \( QI_m^* = QI_m/I_m \).

We recall results in [2].

**Lemma 2.4** ([2]). The ring \( QI_m \) of quasiinvariants has following decomposition:

\[
QI_m = \bigoplus_{T \in ST(n)} \gamma_T(QI_m).
\]

The space \( \gamma_T(QI_m) \) has following description:

\[
(2.1) \quad \gamma_T(QI_m) = \gamma_T(K_n) \cap V_T^{2m+1} K_n.
\]

For \( \lambda \vdash n \), the vector space \( \bigoplus_{T \in ST(\lambda)} \gamma_T(QI_m) \) is called the \( \lambda \)-isotypic component of \( QI_m \).

Let \( K \) be a polynomial ring. We denote by \( K[i] \) the subspace spanned by homogeneous polynomials of degree \( i \) in \( K \). The Hilbert series of \( K \) is defined as a formal power series \( \sum_{i=0}^{\infty} \dim(K[i])t^i \). We denote it by \( H(K, t) \).

For \([f] \in QI_m^*\), we define the degree of \([f]\) as the minimal degree in the class \([f]\). In [4] and [7], the Hilbert series of \( QI_m^* \) is given as follows:

**Theorem 2.5** ([4], [7]).

\[
H(QI_m^*, t) = n! t^{mn(n-1)/2} \sum_{\lambda \vdash n} \prod_{i,j \in \lambda} \prod_{k=1}^{n} t^{|w(i,j,m)|} \frac{1 - t^k}{h(i,j)(1 - t^{h(i,j)})}
\]
where we set \( w(i, j; m) = m((i, j) - a(i, j)) + l(i, j) \).

In particular, for \( T \in ST(\lambda) \) the Hilbert series of \( \gamma_T(Q^{m}_{I_m}) \) is given as follows:

\[
H(\gamma_T(Q^{m}_{I_m}); t) = t^{m(n-1)/2} \prod_{(i,j) \in \lambda} \prod_{k=1}^{n} \frac{1 - t^{k}}{1 - t^{h(i,j)}}.
\]

Let \( s_1, s_2, \ldots, s_n \) be mutually distinct positive integers. We set \( D = D(s_1, s_2; s_1, s_3, \ldots, s_n) \). We define the following polynomial in \( \mathbb{Q}[x_{s_1}, \ldots, x_{s_n}] \):

\[
Q^{l,m}_{D} = \int_{x_{s_1}}^{x_{s_2}} t^l \prod_{i=1}^{n}(t - x_{s_i})^m dt.
\]

Recall that we define \( \eta(n, k) = (n - k + 1, 1^{k-1}) \). In [2], J. Bandlow and G. Musiker found an explicit basis of \( \gamma_T(Q^{m}_{I_m}) \) when \( T \in ST(\eta(n, 2)) \).

**Theorem 2.6** ([2]). Let \( T \in ST(\eta(n, 2)) \). The set \( \{Q^{0,0}_{T}, Q^{1,0}_{T}, \ldots, Q^{n-2,m}_{T} \} \) is a basis of \( \gamma_T(Q^{m}_{I_m}) \).

**Remark 2.7.** In [2], it is shown that \( Q^{l,m}_{T} \) is divisible by \( V_T = (x_1 - x_j)^{2m+1} \). We can similarly show that \( Q^{l,m}_{D} \) is divisible by \( V_D = (x_{s_i} - x_{s_j})^{2m+1} \).

Let \( f \in \mathbb{Q}[x_{s_1}, x_{s_2}, \ldots, x_{s_n}] \). We denote by \( \deg_{x_{s_i}}(f) \) the degree of \( f \) as the polynomial in \( x_{s_i} \). The leading term of \( f \) in \( x_{s_i} \) means the highest term of \( f \) in \( x_{s_i} \) and the leading coefficient of \( f \) in \( x_{s_i} \) means the coefficient of the leading term of \( f \) in \( x_{s_i} \). For a homogeneous polynomial \( g \), we define \( \deg(g) \) as the degree of \( g \).

The polynomials \( Q^{l,m}_{D} \) have the following properties, which we will use to show Proposition 3.3.

**Proposition 2.8.** Let \( s_1, s_2, \ldots, s_n \) be mutually distinct positive integers. Let \( l \) be a non-negative integer and take a tableau \( D = D(s_1, s_2; s_1, s_3, \ldots, s_n) \) of shape \( \eta(n, 2) \).

The polynomial \( Q^{l,m}_{D} \) is a homogeneous polynomial of degree \( nm + l + 1 \) and satisfies following properties.

1. The polynomial \( Q^{l,m}_{D} \) is symmetric in \( x_{s_1}, \ldots, x_{s_n} \) and anti-symmetric in \( x_{s_1}, x_{s_2} \).
2. We have \( \deg_{x_{s_1}}(Q^{l,m}_{D}) = nm + l + 1 \). The leading coefficient of \( Q^{l,m}_{D} \) in \( x_{s_1} \) is \((-1)^{m+1}m! \prod_{i=0}^{m}(mn + l + 1 - s)\).
3. Let \( i \in \{1, \ldots, n\} \setminus \{1, 2\} \). We have \( \deg_{x_{s_i}}(Q^{l,m}_{D}) = m \). The leading coefficient of \( Q^{l,m}_{D} \) in \( x_{s_i} \) is equal to \((-1)^mQ^{l,m}_{D}\).

**Proof.** We show the case \( D = T(1, 2) \) since the proofs of other cases are similar. We set \( T = T(1, 2) \).
(1) It follows from the fact that \( t^l \prod_{i=1}^{n} (t - x_i)^m \) is symmetric in \( x_1, x_2, \ldots, x_n \).

(2) We show this statement by induction on \( m \).

When \( m = 0 \), the polynomial \( Q_{T_1}^{l,0} \) is \( (1/(l+1))(x_1^{l+1} - x_i^{l+1}) \). So, the statement holds.

When \( m \geq 1 \), assume that the statement holds for all numbers less than \( m \). In \( [2] \), the polynomial \( Q_{T_1}^{l,m} \) is expressed as:

\[
Q_{T_1}^{l,m} = \sum_{i=0}^{n} (-1)^i e_i Q_{T_1}^{n+l-i,m-1}.
\]

By the induction assumption on \( m \), we have \( \deg_{x_1} (Q_{T_1}^{n+l-i,m-1}) = nm + l - i + 1 \).

From (2.5), we have \( \deg_{x_1} (Q_{T_1}^{l,m}) = nm + l + 1 \) and the leading term is in \( e_0 Q_{T_1}^{n+l,m-1} - e_1 Q_{T_1}^{n+l-1,m-1} \). The leading coefficient of \( Q_{T_1}^{l,m} \) in \( x_1 \) is

\[
(-1)^m(m-1)! \prod_{s=0}^{m-1}(mn + l + 1 - s)
\]

and

\[
(-1)^m(m-1)! \prod_{s=0}^{m-1}(mn + l - s) = \prod_{s=0}^{m}(mn + l + 1 - s).
\]

(3) Expanding \( (t - x_i)^m \) in \( Q_{T_1}^{l,m} \), we have

\[
Q_{T_1}^{l,m} = \sum_{s=0}^{m} (-1)^s \binom{m}{s} Q_{T_1}^{l,m} x_i^s.
\]

Thus, the statement holds. \( \square \)

As a corollary of this proposition, we have \( Q_{D}^{l,m} \neq 0 \) when \( D \) is a tableau of shape \( \eta(n, 2) \).

3. A basis for the isotypic component of shape \( (n - k + 1, 1^k-1) \)

We give a basis for the \( \eta(n, k) \)-isotypic component. Let \( s_1, s_2, \ldots, s_n \) be mutually distinct positive integers. Throughout this section, we set \( D = D(s_1, \ldots, s_k, s_1, s_{k+1}, \ldots, s_n) \) and \( T = T(1, 2, \ldots, k) \).

DEFINITION 3.1. (1) Let \( p \) be a non-negative integer. For \( i, j \) such that \( 1 \leq i < j \leq k \), we define a polynomial \( R_{D, s_i, s_j}^{p,m} \) in \( Q[x_{s_1}, x_{s_2}, \ldots, x_{s_n}] \) as

\[
R_{D, s_i, s_j}^{p,m} = \int_{x_{s_i}}^{x_{s_j}} t^p \prod_{l=1}^{n}(t - x_{s_l})^{m} dt.
\]
(2) Let $k$ be an integer such that $k \geq 2$. Take a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_{k-1})$ such that $\mu_1 > \mu_2 > \cdots > \mu_{k-1} \geq 0$. We define a polynomial $Q_D^{\mu,m}$ in $Q[x_1, x_2, \ldots, x_n]$ as follows:

\[
Q_D^{\mu,m} = \begin{vmatrix}
R_{D;1,2}^{\mu_1;m} & R_{D;1,2}^{\mu_2;m} & \cdots & R_{D;1,2}^{\mu_{k-1};m} \\
R_{D;2,3}^{\mu_1;m} & R_{D;2,3}^{\mu_2;m} & \cdots & R_{D;2,3}^{\mu_{k-1};m} \\
\vdots & \vdots & \ddots & \vdots \\
R_{D;k-1,k}^{\mu_1;m} & R_{D;k-1,k}^{\mu_2;m} & \cdots & R_{D;k-1,k}^{\mu_{k-1};m}
\end{vmatrix}
\]

We denote the empty sequence by $\emptyset$. When $k = 1$, $\mu$ is the empty sequence $\emptyset$. We set $Q_D^{\emptyset,m} = 1$. We simply write $Q_D^m$ as $Q_D^{0,m}$.

**Remark 3.2.** Setting $D' = D(s_1, s_2; s_1, s_3, \ldots, s_n)$, we have $R_{D';s_1,s_2}^{\mu,m} = Q_{D'}^{\mu,m}$.

The polynomial $Q_D^{\mu,m}$ has the following properties, which we will use to show our main results.

**Proposition 3.3.** Let $s_1, s_2, \ldots, s_n$ be mutually distinct positive integers. We set $D = D(s_1, \ldots, s_k; s_1, s_{k+1}, \ldots, s_n)$. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_{k-1})$ be a partition such that $\mu_1 > \mu_2 > \cdots > \mu_{k-1} \geq 0$.

Then, the polynomial $Q_D^{\mu,m}$ satisfies the following.

1. The polynomial $Q_D^{\mu,m}$ is symmetric in $x_{s_{k+1}}, x_{s_{k+2}}, \ldots, x_s$ and anti-symmetric in $x_{s_1}, x_{s_2}, \ldots, x_{s_k}$. In particular, $Q_D^{\mu,m}$ is divisible by $V_{D,m+1}^{2m+1}$.
2. We have $\deg_{x_1}(Q_D^{\mu,m}) = (n + k - 2)m + \mu_1 + 1$. The leading coefficient of $Q_D^{\mu,m}$ in $x_1$ is

\[
(-1)^{(k-1)m+1}m! \prod_{s=0}^{m}(mn + \mu_1 + 1 - s) Q_D^{\mu_2, \ldots, \mu_{k-1};m}.
\]

In particular, we have $\deg(Q_D^{\mu,m}) = (k - 1)nm + |\mu| + k - 1$.
3. We have $\deg_{x_1}(Q_D^{\mu,m}) = (k - 1)m$. The leading coefficient of $Q_D^{\mu,m}$ in $x_{k+1}$ is

\[
(-1)^{(k-1)m} Q_D^{\mu_1, \ldots, \mu_{k-2}, m}_{D';s_{k+1}}.
\]
4. The polynomial $Q_D^{\mu,m}$ is invariant under $\gamma_D$.

Proof. We show the case $D = T$. The proofs of other cases are similar.

(1) From Proposition 2.8 (1), it follows that the polynomial $Q_T^{\mu,m}$ is symmetric in $x_{k+1}, x_{k+2}, \ldots, x_n$.

Adding the first row to the second row, we get

\[
Q_T^{\mu,m} = \begin{vmatrix}
R_{T;1,2}^{\mu_1;m} & R_{T;1,2}^{\mu_2;m} & \cdots & R_{T;1,2}^{\mu_{k-1};m} \\
R_{T;1,3}^{\mu_1;m} & R_{T;1,3}^{\mu_2;m} & \cdots & R_{T;1,3}^{\mu_{k-1};m} \\
\vdots & \vdots & \ddots & \vdots \\
R_{T;k-1,k}^{\mu_1;m} & R_{T;k-1,k}^{\mu_2;m} & \cdots & R_{T;k-1,k}^{\mu_{k-1};m}
\end{vmatrix}
\]
Repeating this process, we get

\[
Q_T^{\mu:m} = \begin{vmatrix}
R_{T:1,2}^{\mu:m} & R_{T:1,2}^{\mu:m} & \cdots & R_{T:1,2}^{\mu:m} \\
R_{T:1,3}^{\mu:m} & R_{T:1,3}^{\mu:m} & \cdots & R_{T:1,3}^{\mu:m} \\
& \vdots & \ddots & \vdots \\
R_{T:1,k}^{\mu:m} & R_{T:1,k}^{\mu:m} & \cdots & R_{T:1,k}^{\mu:m}
\end{vmatrix}.
\]

(3.3)

Thus, the polynomial \(Q_T^{\mu:m}\) is anti-symmetric in \(x_2, \ldots, x_k\). We can show that \(Q_T^{\mu:m}\) is anti-symmetric in \(x_1, x_3, \ldots, x_k\) and \(x_1, x_2, x_4 \cdots, x_k\) in similar ways. Thus the first statement holds.

From Remark 2.7 and (3.3), the polynomial \(Q_T^{\mu:m}\) is divisible by \(\prod_{i=2}^n (x_1 - x_i)^{2m+1}\).

Using this proposition (1), we see \(Q_T^{\mu:m}\) is also divisible by \(V_T^{2n+1}\).

(2) We see \(Q_T^{\mu:m}\) as a polynomial in \(x_1\). From Proposition 2.8 (2), (3), the leading term of \(Q_T^{\mu:m}\) in \(x_1\) is in \(R_{T:1,2}^{\mu:m} R_{T:1,3}^{\mu:m} \cdots R_{T:k-1,k}^{\mu:m}\). We use Proposition 2.8 (2), (3) again, and the statement holds.

(3) From Proposition 2.8 (3), the leading coefficient of \(Q_T^{\mu:m}\) in \(x_{k+1}\) is

\[
\begin{vmatrix}
(-1)^m R_{T^1:1,2}^{\mu:m} & (-1)^m R_{T^1:1,2}^{\mu:m} & \cdots & (-1)^m R_{T^1:1,2}^{\mu:m} \\
(-1)^m R_{T^2:1,2}^{\mu:m} & (-1)^m R_{T^2:1,2}^{\mu:m} & \cdots & (-1)^m R_{T^2:1,2}^{\mu:m} \\
& \vdots & \ddots & \vdots \\
(-1)^m R_{T^{k-1}:1,k-1}^{\mu:m} & (-1)^m R_{T^{k-1}:1,k-1}^{\mu:m} & \cdots & (-1)^m R_{T^{k-1}:1,k-1}^{\mu:m}
\end{vmatrix}.
\]

(3.4)

The polynomial (3.4) is equal to \((-1)^{k-1} m Q_T^{\mu:m}\).

(4) To prove (4), we define the following notation.

For positive integers \(i, j\) such that \(i \neq j\), we define a tableau \((i, j)D\) as follows. When \(i, j \notin mem(D)\), we define \((i, j)D = D\). When \(i \in mem(D)\) and \(j \notin mem(D)\), \((i, j)D\) is a tableau obtained by replacing the entry \(i\) in \(D\) with \(j\). When \(i, j \in mem(D)\), \((i, j)D\) is a tableau obtained by interchanging the entry \(i\) and \(j\) in \(D\).

Using Proposition 2.3, \(\gamma_T\) is equal to

\[
\frac{1}{n(n-k)!(k-1)!} \left\{ 1 - \sum_{s=2}^k (1, s) \right\} [S_{[2,3,\ldots,k]}^T] \left\{ 1 + \sum_{s=k+1}^n (1, s) \right\} [S_{[k+1,\ldots,n]}].
\]

From (1), we obtain

\[
\gamma_T(Q_T^{\mu:m}) = \frac{1}{n} \left\{ k Q_T^{\mu:m} + \sum_{s=k+1}^n \{ 1 - (1, 2) - \cdots - (1, k) \} Q_T^{\mu:m} \right\}.
\]
We consider the sum \( \sum_{s=k+1}^{n} [1 - (1, 2) - \cdots - (1, k)] Q_{(1,s)T}^{\mu:m} \). We have

\[
\sum_{s=k+1}^{n} [1 - (1, 2) - (1, 3) - \cdots - (1, k)] Q_{(1,s)T}^{\mu:m} = \sum_{s=k+1}^{n} \{ Q_{(1,s)T}^{\mu:m} + Q_{(2,s)T}^{\mu:m} + Q_{(3,s)T}^{\mu:m} + \cdots + Q_{(k,s)T}^{\mu:m} \}.
\]

Consider the sum \( Q_{(1,s)T}^{\mu:m} + Q_{(2,s)T}^{\mu:m} \). By definition, we have

\[
Q_{(1,s)T}^{\mu:m} + Q_{(2,s)T}^{\mu:m} = \begin{vmatrix} R_{T;2,3}^{\mu:m} & R_{T;2,3}^{\mu:m} & \cdots & R_{T;2,3}^{\mu:m-1} \\ R_{T;3,2}^{\mu:m} & R_{T;3,2}^{\mu:m} & \cdots & R_{T;3,2}^{\mu:m-1} \\ \vdots & \vdots & \ddots & \vdots \\ R_{T;k-1,k}^{\mu:m} & R_{T;k-1,k}^{\mu:m} & \cdots & R_{T;k-1,k}^{\mu:m-1} \end{vmatrix} + \begin{vmatrix} R_{T;1,3}^{\mu:m} & R_{T;1,3}^{\mu:m} & \cdots & R_{T;1,3}^{\mu:m-1} \\ R_{T;2,3}^{\mu:m} & R_{T;2,3}^{\mu:m} & \cdots & R_{T;2,3}^{\mu:m-1} \\ \vdots & \vdots & \ddots & \vdots \\ R_{T;k-1,k}^{\mu:m} & R_{T;k-1,k}^{\mu:m} & \cdots & R_{T;k-1,k}^{\mu:m-1} \end{vmatrix}.
\]

Adding the first row to the second row in the second determinant, we get

\[
Q_{(1,s)T}^{\mu:m} + Q_{(2,s)T}^{\mu:m} = \begin{vmatrix} R_{T;2,3}^{\mu:m} & R_{T;2,3}^{\mu:m} & \cdots & R_{T;2,3}^{\mu:m-1} \\ R_{T;3,2}^{\mu:m} & R_{T;3,2}^{\mu:m} & \cdots & R_{T;3,2}^{\mu:m-1} \\ \vdots & \vdots & \ddots & \vdots \\ R_{T;k-1,k}^{\mu:m} & R_{T;k-1,k}^{\mu:m} & \cdots & R_{T;k-1,k}^{\mu:m-1} \end{vmatrix} + \begin{vmatrix} R_{T;1,3}^{\mu:m} & R_{T;1,3}^{\mu:m} & \cdots & R_{T;1,3}^{\mu:m-1} \\ R_{T;2,3}^{\mu:m} & R_{T;2,3}^{\mu:m} & \cdots & R_{T;2,3}^{\mu:m-1} \\ \vdots & \vdots & \ddots & \vdots \\ R_{T;k-1,k}^{\mu:m} & R_{T;k-1,k}^{\mu:m} & \cdots & R_{T;k-1,k}^{\mu:m-1} \end{vmatrix}.
\]

Adding the two terms, we obtain

\[
Q_{(1,s)T}^{\mu:m} + Q_{(2,s)T}^{\mu:m} = \begin{vmatrix} R_{T;1,3}^{\mu:m} & R_{T;1,3}^{\mu:m} & \cdots & R_{T;1,3}^{\mu:m-1} \\ R_{T;2,3}^{\mu:m} & R_{T;2,3}^{\mu:m} & \cdots & R_{T;2,3}^{\mu:m-1} \\ \vdots & \vdots & \ddots & \vdots \\ R_{T;k-1,k}^{\mu:m} & R_{T;k-1,k}^{\mu:m} & \cdots & R_{T;k-1,k}^{\mu:m-1} \end{vmatrix}.
\]

Repeating this process, we get

\[
(1 - (1, 2) - (1, 3) - \cdots - (1, k)) Q_{(1,s)T}^{\mu:m} = Q_{T}^{\mu:m}.
\]

Thus, the statement holds.

As a corollary of this proposition, we have \( Q_{T}^{\mu:m} \in \gamma_T(QL_m) \) where \( T \in ST(\eta(n,k)) \). We introduce the following notations.
DEFINITION 3.4. Let \( s, t, u \) be non-negative integers. When \( u \geq 1 \), we set the subsets \( P(s; t; u) \), \( P(t; u) \) and \( Q(s; t; u) \) of the set of partitions as:

\[
P(s; t; u) = \{ \lambda \in \mathbb{Z}^n \mid |\lambda| = s, \ t \geq \lambda_1 > \lambda_2 > \cdots > \lambda_u \geq 0 \},
\]

\[
Q(s; t; u) = P(s; t; u) \setminus P(s; t - 1; u),
\]

\[
P(t; u) = \bigcup_{s \geq 0} P(s; t; u).
\]

When \( u = 0 \), we set

\[
P(0; t; 0) = \{ \emptyset \},
\]

\[
P(t; 0) = \{ \emptyset \}.
\]

Let \( l \) be a positive integer. We set \( P(l; t; 0) \) as empty set.

We define \( p(s; t; u) = \#P(s; t; u) \) and \( q(s; t; u) = \#Q(s; t; u) \).

REMARK 3.5. Let \( \mu \in P(n - 2; k - 1) \) (resp. \( \mu \in \bigcup_{s \geq 0} Q(s; n - 2; k - 1) \)). We have

\[
\frac{(k - 1)(k - 2)}{2} \leq |\mu| \leq (k - 1)(n - k) + \frac{(k - 1)(k - 2)}{2}
\]

(resp. \( n - 2 + (k - 2)(k - 3)/2 \leq |\mu| \leq (k - 1)(n - k) + (k - 1)(k - 2)/2 \)).

We have the following proposition.

**Proposition 3.6.** Let \( k \) be an integer such that \( k \geq 2 \).

1. Let \( l \) be an integer such that \( 0 \leq l \leq n - k - 1 \). Then, we have

\[
p\left(l + \frac{(k - 1)(k - 2)}{2}; n - 3; k - 1 \right) = p\left(l + \frac{(k - 1)(k - 2)}{2}; n - 2; k - 1 \right).
\]

2. Let \( l \) be an integer such that \( l \geq n - k \). Then, we have

\[
p\left(l + \frac{(k - 1)(k - 2)}{2}; n - 2; k - 1 \right)
\]

\[
= p\left(l + \frac{(k - 1)(k - 2)}{2}; n - 3; k - 1 \right)
\]

\[
+ p\left(l + k - n + \frac{(k - 2)(k - 3)}{2}; n - 3; k - 2 \right).
\]

3. Let \( l \) be an integer such that \( 0 \leq l \leq k - 2 \). Then, we have

\[
p\left((k - 1)(n - k) + \frac{(k - 1)(k - 2)}{2} - l; n - 2; k - 1 \right)
\]

\[
= p\left((k - 2)(n - k) + \frac{(k - 2)(k - 3)}{2} - l; n - 3; k - 2 \right).
\]
Proof. (1) By definition, we have
\[
q \left( l + \frac{(k-1)(k-2)}{2}; n-2; k-1 \right)
= p \left( l + \frac{(k-1)(k-2)}{2}; n-2; k-1 \right) - p \left( l + \frac{(k-1)(k-2)}{2}; n-3; k-1 \right).
\]
Therefore we show \( q(l + (k-1)(k-2)/2; n-2; k-1) = 0. \)

We have \( l + (k-1)(k-2)/2 \leq n-k-1 + (k-1)(k-2)/2 < n-2 + (k-2)(k-3)/2. \)
From Remark 3.5, we have \( Q(l+(k-1)(k-2)/2; n-2; k-1) = \emptyset. \) Thus, the proposition follows.

(2) To prove (2), we show
\[
q \left( l + \frac{(k-1)(k-2)}{2}; n-2; k-1 \right)
= p \left( l + k - n + \frac{(k-2)(k-3)}{2}; n-3; k-2 \right).
\]

Let \( \mu = (j, \mu_2, \ldots, \mu_k) \in Q(i; j; k). \) Then, we have \( (\mu_2, \ldots, \mu_k) \in Q(i-j; \mu_2; k-1). \)
So, we get \( Q(i; j; k) = \bigcup_{j=0}^{k-1} Q(i-j; s; k-1). \) Thus, we have
\[
q \left( l + \frac{(k-1)(k-2)}{2}; n-2; k-1 \right) = \sum_{s=0}^{n-3} q \left( l + \frac{(k-1)(k-2)}{2}; n+2; s; k-2 \right).
\]

We have \( l + (k-1)(k-2)/2 - n + 2 = l + k - n + (k-2)(k-3)/2. \) So, we get
\[
q \left( l + \frac{(k-1)(k-2)}{2}; n-2; k-1 \right)
= \sum_{s=0}^{n-3} q \left( l + k - n + \frac{(k-2)(k-3)}{2}; s; k-2 \right).
\]

By definition, we obtain
\[
\sum_{s=0}^{n-3} q \left( l + k - n + \frac{(k-2)(k-3)}{2}; s; k-2 \right)
= p \left( l + k - n + \frac{(k-2)(k-3)}{2}; n-3; k-2 \right).
\]

(3) By definition, we have
\[
p \left( (k-1)(n-k) + \frac{(k-1)(k-2)}{2}; n-2; k-1 \right)
= \sum_{s=0}^{n-2} q \left( (k-1)(n-k) + \frac{(k-1)(k-2)}{2}; l; s; k-1 \right).
\]
From Remark 3.5, we have $q((k - 1)(n - k) + (k - 1)(k - 2)/2 - l; s; k - 1) = 0$ when $s \leq n - 3$. Therefore, we obtain

$$
p(k - 1)(n - k) + (k - 1)(k - 2)/2 - l; n - 2; k - 1

$$

From (2), we have

$$
q(k - 1)(n - k) + (k - 1)(k - 2)/2 - l; n - 2; k - 1

$$

We next consider the Hilbert series of $\gamma_T(QI_m^s)$. To simplify notation, we write $p_{s,n-2,k-1} = p(s + (k - 1)(k - 2)/2; n - 2; k - 1)$.

Proposition 3.6 is rewritten as:

1. $p_{l,n-3,k-1} = p_{l,n-2,k-1}$.
2. $p_{l,n-2,k-1} = p_{l,n-3,k-1} + p_{l+k-n,n-3,k-2}$.
3. $p_{(k-1)(n-k)-1,n-2,k-1} = p_{(k-2)(n-k)-1,n-3,k-2}$.

**Lemma 3.7.** We have

$$(3.5) \quad H(\gamma_T(QI_m^s); t) = t^{(k-1)(n-1)-1} \sum_{s=0}^{(k-1)(n-k)} p_{s,n-2,k-1} t^s.$$ 

Proof. From (2.3), the Hilbert series $H(\gamma_T(QI_m^s); t)$ is equal to

$$
\prod_{(i,j)\in\lambda} \prod_{l=1}^{n} t^{m(l(i,j) - a(i,j)) + l(i,j)} \frac{1 - t^l}{1 - t^{h(i,j)}}.

$$

For $2 \leq i \leq n - k + 1$ and $2 \leq j \leq k$, we have

$$
a(1, 1) = n - k, \quad l(1, 1) = k - 1, \quad h(1, 1) = n,

a(1, i) = n - k + 1 - i, \quad l(1, i) = 0, \quad h(1, i) = n - k + 2 - i,

a(j, 1) = 0, \quad l(j, 1) = k - j, \quad h(j, 1) = k - j + 1.

$$
Thus, we have

\[ H(\gamma_T(\mathbf{Q}^s_m); t) = t^{(k-1)n + (k+1)/2} \prod_{s=1}^{k-1} \frac{(1 - t^{n-s})}{(1 - t^s)}. \]

Therefore, we must show

\[ \prod_{s=1}^{k-1} \frac{(1 - t^{n-s})}{(1 - t^s)} = \sum_{s=0}^{(k-1)(n-k)} P_{s,n-2,k-1} t^s. \]

We show this by induction on \( n \).

If \( n = k \), then both of l.h.s. and r.h.s. are equal to 1.

When \( n \geq k + 1 \), we assume that (3.6) holds with all numbers less than \( n \). We have the following identity:

\[
\prod_{s=1}^{k-1} \frac{(1 - t^{n-s})}{(1 - t^s)} = \prod_{s=1}^{k-1} \frac{(1 - t^{n-s-1})}{(1 - t^s)} + t^n \prod_{s=1}^{k-2} \frac{(1 - t^{n-s-1})}{(1 - t^s)}.
\]

By the induction assumption, we obtain

\[
\prod_{s=1}^{k-1} \frac{(1 - t^{n-s-1})}{(1 - t^s)} + t^n \prod_{s=1}^{k-2} \frac{(1 - t^{n-s-1})}{(1 - t^s)} = \sum_{s=0}^{(k-1)(n-k)} P_{s,n-3,k-1} t^s + t^n \sum_{s=0}^{(k-2)(n-k)} P_{s,n-3,k-2} t^s.
\]

We can rewrite this as

\[
\prod_{s=1}^{k-1} \frac{(1 - t^{n-s})}{(1 - t^s)} = \sum_{s=n-k}^{(k-1)(n-k)} (P_{s,n+k,n-3,k-2} + P_{s,n-3,k-1}) t^s + \sum_{s=(k-1)(n-k)-k+2}^{n-k+1} P_{s,n+k,n-3,k-2} t^s + \sum_{s=0}^{n-k} P_{s,n-3,k-1} t^s.
\]

Using Proposition 3.6 (2), we have

\[
\sum_{s=n-k}^{(k-1)(n-k)} (P_{s,n+k,n-3,k-2} + P_{s,n-3,k-1}) t^s = \sum_{s=n-k}^{(k-1)(n-k)} P_{s,n-2,k-1} t^s.
\]
From Proposition 3.6 (1) and (3), the lemma holds.

We state the main theorem in this paper.

**Theorem 3.8.** The set \( \{Q_T^{\mu,m}\}_{\mu \in P(n-2,k-1)} \) is a basis of \( \gamma_T(QI^s_m) \).

To simplify notation, we set

\[
P_{s,n-2,k-1} = P \left(s + \frac{(k-1)(k-2)}{2}; n-2; k-1\right),
\]

\[
P_{n-2,k-1} = P(n-2; k-1),
\]

\[
Q_{s,n-2,k-1} = Q \left(s + \frac{(k-1)(k-2)}{2}; n-2; k-1\right).
\]

We define following notations.

Let \( X = \{s_1, s_2, \ldots, s_n\} \) be the set of \( n \) positive integers. We recall that \( S_X \) is the symmetric group on \( X \) and \( S_X \) acts on \( \mathbb{Q}[x_{s_1}, x_{s_2}, \ldots, x_{s_n}] \) from the left.

We define \( \Lambda_X \) as the subspace of \( \mathbb{Q}[x_{s_1}, x_{s_2}, \ldots, x_{s_n}] \) spanned by all polynomials which is invariant under \( S_X \). We define \( \Lambda^d_X \) as the subspace of \( \Lambda_X \) spanned by homogeneous polynomials of degree \( d \). We define \( \Lambda^d_X = \{0\} \) if \( d < 0 \).

Theorem 3.8 follows from the following proposition.

**Proposition 3.9.** Let \( D \) be a tableau of shape \( \eta(n, k) \). If

\[
\sum_{\mu \in P(n-2,k-1)} f_{\mu} Q_{D}^{\mu,m} = 0
\]

where \( f_{\mu} \in \Lambda_{\text{mem}(D)} \), then all \( f_{\mu} \) is equal to 0.

Proof. We show this proposition by induction on the size \( n \) of tableau \( D \).

In the case \( k = 1 \), (3.7) is \( f Q_D^m = 0 \) where \( f \in \Lambda_{\text{mem}(D)} \). Therefore, the proposition holds when \( k = 1 \). We assume that \( k \geq 2 \).

We recall that \( n \geq k \). When \( n = 2 \), we have \( k = 2 \). Then l.h.s. of (3.7) is equal to \( f_0 Q_D^{0,m} \). Therefore, the lemma holds when \( n = 2 \).

Assume that (3.7) holds when the size of the tableau \( D \) is less than \( n \) for \( n \geq 3 \). We show the case \( D = T \) since the proofs of other cases are similar.

We recall that \( \Lambda_n \) is a graded ring. Therefore, we can decompose

\[
f_{\mu} = \sum_{l \geq 0} f_{\mu,l}
\]
where \( f_{\mu,l} \in \Lambda^l_n \). Thus, (3.7) is written as

\[
(3.8) \quad \sum_{\mu \in P(n-2,k-1)} \sum_{l \geq 0} f_{\mu,l} Q_T^{\mu;m} = 0
\]

where \( f_{\mu,l} \in \Lambda^l_n \). We have \( \deg(Q^{\mu;m}_T) = (k-1)nm + |\mu| + k - 1 \), and we obtain \( \deg(f_{\mu,l} Q^{\mu;m}_T) = (k-1)nm + |\mu| + d + k - 1 \).

Thus, (3.8) is written as

\[
(3.9) \quad \sum_{d \geq 0} \sum_{\mu \in P(n-2,k-1)} f_{\mu,d-(k-1)nm-|\mu|-k+1} Q_T^{\mu;m} = 0.
\]

Hence, for any \( d \) we obtain

\[
(3.10) \quad \sum_{\mu \in P(n-2,k-1)} f_{\mu,d-(k-1)nm-|\mu|-k+1} Q_T^{\mu;m} = 0.
\]

Fix \( d \). Recall that the set \( P_{s,n-2,k-1} \) is not the empty set if \( 0 \leq s \leq (k-1)(n-k) \).

Let \( s \) be an integer such that \( 0 \leq s \leq (k-1)(n-k) \) and take \( \mu \in P_{s,n-2,k-1} \). Then, we have \( \deg(Q^{\mu;m}_T) = (k-1)nm + k(k-1)/2 + s \). We set \( d' = d - (k-1)nm - k(k-1)/2 \). We express \( f_{\mu,d'-s} \) as

\[
\sum_{r=0}^{d'-s} \sum_{|y|=d'-s \atop \l(y)=r} a_{r,y}^\mu e_y.
\]

We recall that

\[
P_{s,n-2,k-1} = P\left(s + \frac{(k-1)(k-2)}{2}; n-2; k-1\right),
\]

\[
P_{n-2,k-1} = P(n-2; k-1),
\]

\[
Q_{s,n-2,k-1} = Q\left(s + \frac{(k-1)(k-2)}{2}; n-2; k-1\right).
\]

Therefore, (3.10) is written as

\[
(3.11) \quad \sum_{s=0}^{(k-1)(n-k)} \sum_{\mu \in P_{s,n-2,k-1}} \sum_{r=0}^{d'-s} \sum_{|y|=d'-s \atop \l(y)=r} a_{r,y}^\mu e_y Q_T^{\mu;m} = 0.
\]

We show \( a_{r,y}^\mu = 0 \) for \( r \geq 0 \). We show this by induction on \( r \). To prove this, we consider the leading terms in \( x_{k+1} \).

As a polynomial in \( x_{k+1} \), the degree of l.h.s. of (3.11) is \((k-1)m + d'\) and the leading term is in \( a_{d',(1^{d'})}^{(k-2,k-3,...,0)} e_{(1^{d'})} Q_T^{(k-2,k-3,...,0;m)}. \) Hence we have \( a_{d',(1^{d'})}^{(k-2,k-3,...,0)} = 0.\)

Using the following lemma, we complete the proof of Proposition 3.9.
Lemma 3.10. Let $k$ be an integer such that $k \geq 3$. We assume that for each integer $l$ such that $2 \leq l \leq n - 1$ and each tableau of shape \( \eta(n - 1, l) \), the statement of Proposition 3.9 holds.

Let $r$ be an integer such that $1 \leq r \leq d' - 1$. If we have the following equation:

\[
\sum_{s=0}^{(k-1)(n-k)} \sum_{\mu \in P_s, a_{s, l-1}} \sum_{i=0}^{r} \sum_{l(y)=i} a_{r, y}^\mu e_y Q_T^{\mu; m} = 0,
\]

then all constants $a_{r, y}^\mu$ are equal to 0.

Proof. We set

\[
I = \sum_{s=0}^{(k-1)(n-k)} \sum_{\mu \in P_s, a_{s, l-1}} \sum_{i=0}^{r} \sum_{l(y)=i} a_{r, y}^\mu e_y Q_T^{\mu; m}.
\]

From Proposition 3.3 (3), we have $\deg_{x_{k+1}} (I) = (k - 1)m + r$. The leading term of $I$ in $x_{k+1}$ is in

\[
\sum_{s=0}^{(k-1)(n-k)} \sum_{\mu \in P_s, a_{s, l-1}} \sum_{i=0}^{r} \sum_{l(y)=i} a_{r, y}^\mu e_y Q_T^{\mu; m}.
\]

Recall that we have $P_{s, n-2, k-1} = Q_{s, n-2, k-1} \cup P_{s, n-3, k-1}$ and this union is disjoint. Therefore, we can rewrite this as

\[
\sum_{s=n-k}^{(k-1)(n-k)} \sum_{\mu \in Q_{s, a_{s, l-1}}} \sum_{i=0}^{r} \sum_{l(y)=i} a_{r, y}^\mu e_y Q_T^{\mu; m} + \sum_{s=0}^{(k-1)(n-k-1)} \sum_{\mu \in P_{s, a_{s, l-1}}} \sum_{i=0}^{r} \sum_{l(y)=i} a_{r, y}^\mu e_y Q_T^{\mu; m}.
\]

We set

\[
I_1 = \sum_{s=n-k}^{(k-1)(n-k)} \sum_{\mu \in Q_{s, a_{s, l-1}}} \sum_{i=0}^{r} \sum_{l(y)=i} a_{r, y}^\mu e_y Q_T^{\mu; m},
\]

\[
I_2 = \sum_{s=0}^{(k-1)(n-k-1)} \sum_{\mu \in P_{s, a_{s, l-1}}} \sum_{i=0}^{r} \sum_{l(y)=i} a_{r, y}^\mu e_y Q_T^{\mu; m}.
\]
First, we show that the constants $d_{v,i}^\mu$ in $I_1$ are equal to 0.

If $r > d' - n + k$, we have $|\mu| < (k-1)(k-2)/2 + n - k$. On the other hand, if $\mu \in Q_{s,n-2,k-1}$, we have $|\mu| \geq (k-1)(k-2)/2 + n - k$. Therefore if $r > d' - n + k$, the sum in $I_1$ is empty. We only need to consider the case when $r \leq d' - n + k$.

We define the following notations. Let $X = \{s_1, s_2, \ldots, s_n\}$ be the set of $n$ positive integers. For a partition $\nu = (\nu_1, \nu_2, \ldots)$, we define

$$e_{X,i} = \sum_{1 \leq i_1 < \cdots < i_l \leq n} x_{i_1} \cdots x_{i_l},$$

$$e_{X,\nu} = \prod_i e_{X,i}^{\nu_i},$$

$$e_{X,i}^{(j)} = e_i(x_{s_1}, \ldots, x_{s_{j-1}}, x_{s_{j+1}}, \ldots, x_{s_n}),$$

$$e_{X,\nu}^{(j)} = \prod_{i} e_{X,i}^{(j)}.$$

In particular, if $X = \{1, 2, \ldots, n\}$, then we simply write $e_{X,i}^{(j)}$ as $e_i^{(j)}$ and $e_{X,\nu}^{(j)}$ as $e^{(j)}$.

When $r \leq d' - n + k$, the leading term of $I$ in $x_1$ is in $I_1$. For $\mu \in Q_{s,n-2,k-1}$, there exists $\mu' = (\mu'_1, \ldots, \mu'_{k-2}) \in P_{n-3,k-2}$ such that $\mu = (n-2, \mu'_1, \ldots, \mu'_{k-2})$. In particular, we have $\mu' \in P_{s+k-n,n-3,k-2}$. The leading coefficient of $I_1$ in $x_1$ is

$$b_{\mu',\mu}^{(1)} = \frac{(-1)^{(k-1)m+1}m!}{\prod_{s=0}^{m}(mn+n-1-s)}a_{r,\mu}^{(n-2,\mu'_{\ldots})}.$$ We can rewrite this as

$$b_{\mu',\mu}^{(1)} = \frac{(-1)^{(k-2)m+1}m!}{\prod_{s=0}^{m}(mn+n-1-s)}a_{r,\mu}^{(n-2,\mu'_{\ldots})}.$$
We next consider $I_2$. The leading coefficient of $I_2$ in $x_{k+1}$ is

\[
\sum_{s=0}^{(k-1)(n-k-1)} \sum_{\mu \in P_{s, n-3, k-1}} \sum_{l(\nu^{(2)l})=r} c^{(k+1)}_{\nu^{(2)}, T} Q^{\mu; m}_{T^{k+1}}
\]

where we set $c^{(k+1)}_{\nu^{(2)}, T} = (-1)^{(k-2)m} d^{(k+1)}_{\nu^{(2)}, T}$.

Since $e^{(k+1)}_{\nu^{(2)}, T} = e_{\mu}(T^{k+1}, \nu^{(2)}, T)$, we can rewrite (3.13) as

\[
\sum_{s=0}^{(k-1)(n-k-1)} \sum_{\mu \in P_{s, n-3, k-1}} \sum_{l(\nu^{(2)})=r} c^{(k+1)}_{\nu^{(2)}, T} Q^{\mu; m}_{T^{k+1}}
\]

The tableau $T^{k+1}$ has $n-1$ boxes. By the induction assumption on $n$, all $c^{(k+1)}_{\nu^{(2)}, T}$ are equal to 0. Thus, all $d^{(k+1)}_{\nu^{(2)}, T}$ are equal to 0.

Thus, the lemma follows. Therefore, the proposition also follows. \qed

From Theorem 3.8 and Proposition 3.9, we obtain the following corollary.

**Corollary 3.11.** Let $T \in ST(n, k)$. The space $\gamma_T(QI_m)$ is a free module over $\Lambda_n$ and the set $\{Q^{\mu; m}_T\}_{\mu \in P(n-2, k-1)}$ is a free basis.

**Proof.** In this proof, we simply write $Q^{\mu; m}_T$ as $Q^\mu$. Using Proposition 3.9, the set $\{Q^\mu\}$ is linearly independent over $\Lambda_n$.

Since $H(\gamma_T(QI_m^\mu); t) = t^{(k-1)nm+k(k-1)/2} \sum_{i=0}^{(k-1)(n-k)} P_{s, n-2, k-1} t^i$, we have

\[
\gamma_T(QI_m) = \bigoplus_{d \geq (k-1)nm + k(k-1)/2} \gamma_T(QI_m)[d].
\]

Let $d$ be a non-negative integer such that $d \geq (k-1)nm + k(k-1)/2$. We show that the subspace of $\gamma_T(QI_m)[d]$ is generated by $\{Q^\mu\}$ over $\Lambda_n$ by induction on $d$.

When $d = (k-1)nm + k(k-1)/2$, the coefficient of $t^{(k-1)nm+k(k-1)/2}$ in the polynomial $H(\gamma_T(QI_m^\mu); t)$ is equal to 1. Therefore, $\gamma_T(QI_m)[d]$ is a space spanned by $Q^{(k-2, k-1-\ldots, 0)}$. Thus the statement follows when $d = (k-1)nm + k(k-1)/2$.

When $d \geq (k-1)nm + k(k-1)/2 + 1$, we assume that the statement holds with all numbers less than $d$. We denote by $V$ the vector space over $Q$ spanned by $\{Q^\mu\}_{\mu \in P(n-2, k-1)}$.

Take $f \in \gamma_T(QI_m)[d]$. From Theorem 3.8, we can find $g \in V[d]$ such that $[f] = [g]$ in $\gamma_T(QI_m^\mu)$. Thus, we have $f - g \in I_m$. This is expressed as

\[
f - g = \sum_{s \geq 1} A_s u_s
\]
where $A_s \in \Lambda_n^s$ and $u_s \in \gamma_T(QI_m)$.

Since $\gamma_T(QI_m)$ is a graded space, we can decompose $u_s = \sum_{i \geq 0} u_{s,i}$ where $u_{s,i} \in \gamma_T(QI_m)[i]$. We have $\deg(A_s u_{s,i}) = s + i$. Thus, we have

$$f - g = \sum_{l \geq 0} \sum_{s+l=i} A_s u_{s,i}.$$ 

Since $f - g \in \gamma_T(QI_m)[d]$, we get $\sum_{l \neq d} \sum_{s+i=l} A_s u_{s,i} = 0$. Therefore, we have

$$f - g = \sum_{s \geq 1} A_s u_{s,d-s}.$$ 

The polynomial $A_s$ has the degree at least 1. So, the polynomial $u_{s,d-s}$ has the degree less than $d$. By the induction assumption, $u_{s,d-s}$ can be expressed as

$$u_{s,d-s} = \sum_l B_l v_l$$

where $B_l \in \Lambda_n$ and $v_l \in V$. Thus, the statement follows. }

\[ \square \]

4. The operator $L_m$

The operator $L_m$ is defined as

$$L_m = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} - 2m \sum_{1 \leq i < j \leq n} \frac{1}{x_i - x_j} \left( \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right).$$

This operator is discussed in [4] and [5]. It is related to the quasiinvariants. In [5], Feigin and Veselov showed that the operator $L_m$ preserves $QI_m$. We consider how $L_m$ acts on our polynomial $Q^m_{T(1,2)}$. In [2], for $T(1,2)$ Bandlow and Musiker showed the following formulas for the action of $L_m$.

**Theorem 4.1** ([2]). Let $k, m$ be non-negative integers.

Then, we have $L_m(Q^k_{T(1,2)} = k(k-1)Q^{k-2; m}_{T(1,2)}$ for $k \geq 2$ and $L_m(Q^k_{T(1,2)} = 0$ for $k = 0, 1$.

We extend these formulas. We set $T = T(1, 2, \ldots, k)$. To write formulas simply, we define the following polynomials.

**Definition 4.2.** Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{k-1}) \in \mathbb{Z}^{k-1}$. 

\[ \square \]
We define a polynomial $Q_{T}^{α,m}$ as follows:

\[
Q_{T}^{α,m} = \begin{vmatrix}
R_{T;1}^{α,1,m} & R_{T;1}^{α,2,m} & \cdots & R_{T;1}^{α,n,m} \\
R_{T;2}^{α,1,m} & R_{T;2}^{α,2,m} & \cdots & R_{T;2}^{α,n,m} \\
\vdots & \vdots & \ddots & \vdots \\
R_{T;k-1}^{α,1,m} & R_{T;k-1}^{α,2,m} & \cdots & R_{T;k-1}^{α,n,m}
\end{vmatrix}
\]

(4.1)

when $α_i \geq 0, i = 1, \ldots, k - 1$. Otherwise we define $Q_{T}^{α,m} = 0$.

**Remark 4.3.** If $α$ is a partition, $Q_{T}^{α,m}$ is equal to a polynomial defined in Definition 3.1 (2). If $α ∈ \mathbb{Z}_{≥0}^{k−1}$, $Q_{T}^{α,m}$ is equal to $Q_{T}^{μ,m}$ up to a sign where $μ$ is a partition sorted $α$.

We obtain the following formulas for the action of $L_m$. To write the formula simply, for $α = (α_1, α_2, \ldots, α_{k−1}) ∈ \mathbb{Z}^{k−1}$ we define

$$α^{(i,j)} = (α_1, \ldots, α_{i−1}, α_i - 1, α_{i+1}, \ldots, α_{j−1}, α_j - 1, α_{j+1}, \ldots, α_{k−1}).$$

**Theorem 4.4.** Let $α = (α_1, \ldots, α_{k−1}) ∈ \mathbb{Z}^{k−1}$ and take $T ∈ ST(η(n, k))$. Then we have

$$L_m(Q_{T}^{α,m}) = \sum_{i=1}^{n} α_i(α_i - 1)Q_{T}^{α_1,\ldots,α_{i−2},α_i−2,\ldots,α_k,m} + 2m \sum_{1 ≤ i < j ≤ k−1}(-α_jQ_{T}^{α^{(i,j)},m})$$

$$+ \sum_{s+t=α_i+α_j−2} (s−t)Q_{T}^{α_1,\ldots,α_{i−1},s,α_{i+1},\ldots,α_{j−1},t,α_{j+1},\ldots,α_k,m}.$$ 

This follows from following lemma. We define a polynomial $R_{T;1,2,3}^{α,m}$ as

$$R_{T;1,2,3}^{α,m} = \begin{vmatrix}
R_{T;1,2}^{α,m} & R_{T;1,2}^{α,m} \\
R_{T;2,3}^{α,m} & R_{T;2,3}^{α,m}
\end{vmatrix}.$$ 

**Lemma 4.5.** (1) We have

$$L_m(fg) = L_m(f)g + fL_m(g) + 2 \sum_{i=1}^{n} \left( \frac{∂}{∂x_i} f \right) \left( \frac{∂}{∂x_i} g \right).$$

(2) Let $k$ be a non-negative integer and $m$ be a positive integer. Then, we have

$$k \int_{s_1}^{x_1} t^{k−1} \prod_{s=1}^{n}(t−x_s)^m dt = -m \sum_{r=1}^{n} \int_{s_1}^{x_1} t^k(t−x_r)^{m−1} \prod_{s≠r}(t−x_s)^m dt.$$
(3) Let $k, l$ be non-negative integers such that $k > l$. Then we have

$$\sum_{i=1}^{n} \left( \frac{\partial}{\partial x_i} R_{T:1,2}^{k,m} \right) \left( \frac{\partial}{\partial x_i} R_{T:1,3}^{l,m} \right) - \left( \frac{\partial}{\partial x_i} R_{T:1,2}^{k,m} \right) \left( \frac{\partial}{\partial x_i} R_{T:1,3}^{l,m} \right)$$

(4.2)

$$= m \left( -l R_{T:1,2,3}^{k-1,l-1:m} + \sum_{s+i=k+l-2}^{k-2} (s-t) R_{T:1,2,3}^{s,m} \right).$$

Proof. (1) It follows from Leibniz’s rule.
(2) It follows from the following identity:

$$\int_{x_i}^{x_j} \frac{\partial}{\partial t} t^k \prod_{s=1}^{n} (t-x_s)^m \, dt = 0.$$

(3) When $m = 0$, it follows from $R_{T:1,2}^{k,m} = (x_2^{k+1} - x_1^{k+1})/(k + 1)$. We consider the case $m \geq 1$.

We show this formula by induction on $k - l$. We define $f(t, x) = \prod_{s=1}^{n} (t-x_s)^m$ and $f_i(t, x) = (t-x_i)^{m-1} \prod_{s \neq i} (t-x_s)^m$.

When $k - l = 1$, l.h.s. of (4.2) is equal to

$$m^2 \sum_{i=1}^{n} \int_{x_1}^{x_2} t^k f_i(t, x) \, dt \int_{x_1}^{x_3} u^{k-1} f_i(u, x) \, du$$

$$- m^2 \sum_{i=1}^{n} \int_{x_1}^{x_2} t^k f_i(t, x) \, dt \int_{x_1}^{x_3} u^{k-1} f_i(u, x) \, du.$$

So, this is equal to

$$m^2 \sum_{i=1}^{n} \int_{x_1}^{x_2} t^{k-1} (t-x_i) f_i(t, x) \, dt \int_{x_1}^{x_3} u^{k-1} f_i(u, x) \, du$$

$$- m^2 \sum_{i=1}^{n} \int_{x_1}^{x_2} t^{k-1} (t-x_i) f_i(t, x) \, dt \int_{x_1}^{x_3} u^{k-1} f_i(u, x) \, du$$

$$= m^2 \sum_{i=1}^{n} \int_{x_1}^{x_2} t^{k-1} f(t, x) \, dt \int_{x_1}^{x_3} u^{k-1} f_i(u, x) \, du$$

$$- m^2 \sum_{i=1}^{n} \int_{x_1}^{x_2} t^{k-1} f(t, x) \, dt \int_{x_1}^{x_3} u^{k-1} f_i(u, x) \, du.$$

Using (2), we have

l.h.s. of (4.2) $= -m(k - 1) R_{T:1,2,3}^{k-1,k-2:m}$. 
We consider the case \( k - l = 2 \). Calculating it in the same way, we have

\[
\text{l.h.s. of (4.2)} = -m(k - 2)R_{l+1,2,3}^{k-1,k-3,m}
\]
\[
+ m^2 \sum_{i=1}^{n} \int_{x_1}^{x_2} t^{k-1} f_i(t, x) \, dt \int_{x_1}^{x_3} x_i u^{k-2} f_i(u, x) \, du
\]
\[
- m^2 \sum_{i=1}^{n} \int_{x_1}^{x_3} t^{k-1} f_i(t, x) \, dt \int_{x_1}^{x_2} x_i u^{k-2} f_i(u, x) \, du.
\]

From \( x_i = u - (u - x_i) \), we get

\[
\text{l.h.s. of (4.2)} = -m(k - 2)R_{l+1,2,3}^{k-1,k-3,m}
\]
\[
+ m^2 \sum_{i=1}^{n} \int_{x_1}^{x_2} t^{k-1} f_i(t, x) \, dt \int_{x_1}^{x_3} [u - (u - x_i)] u^{k-2} f_i(u, x) \, du
\]
\[
- m^2 \sum_{i=1}^{n} \int_{x_1}^{x_3} t^{k-1} f_i(t, x) \, dt \int_{x_1}^{x_2} [u - (u - x_i)] u^{k-2} f_i(u, x) \, du.
\]

It is equal to \(-m(k - 2)R_{l+1,2,3}^{k-1,k-3,m}\). Thus the statement holds when \( k - l = 2 \).

When \( k - l \geq 3 \), we assume that the formula (4.2) holds with all numbers less than \( k - l \). Calculating l.h.s. of (4.2) in the same way, we have

\[
\text{l.h.s. of (4.2)} = -m(k - 2)R_{l+1,2,3}^{k-1,k-3,m}
\]
\[
+ m^2 \sum_{i=1}^{n} \left( \frac{\partial}{\partial x_i} R_{l+1,2,3}^{k-1,m} \right) \left( \frac{\partial}{\partial x_i} R_{l+1,2,3}^{k-1,m} \right) - \left( \frac{\partial}{\partial x_i} R_{l+1,2,3}^{k-1,m} \right) \left( \frac{\partial}{\partial x_i} R_{l+1,2,3}^{k-1,m} \right).
\]

Hence the formula (4.2) holds by the induction assumption, and the statement has been proved.

\[ \Box \]

Acknowledgments. I would like to thank Professor Etsuro Date for introducing me to this subject of the quasimvarients and for his many valuable advices. I would also like to thank Professor Misha Feigin for his interest and encouragement during preparation of this paper and for useful comments while he was fully occupied.

References


Graduate School of Information Science and Technology
Osaka University
Toyonaka, Osaka 560–0043
Japan
e-mail: t-tsuchida@ist.osaka-u.ac.jp