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AN APPROXIMATE POSITIVE PART OF A SELF-ADJOINT PSEUDO-DIFFERENTIAL OPERATOR I

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1. Introduction

Among many problems concerning pseudo-differential operators, one of the most interesting problem is "to what extent does the symbol function $p(x, \xi)$ describe the spectral properties of an operator $p(x, D)$?" Motivation of this paper comes from this problem.

Actually what we do in this note is the following: Assume that $P=p(x, D)$ is a self-adjoint pseudo-differential operator of class $L^0_{1,0}$ of Hörmander [4]. Then starting from its principal symbol, we explicitly construct self-adjoint operators P^+, P^-, R, F^+ and F^- with the following properties;

(i) $F^+ + F^- = Id$.

(ii) $P = P^+ - P^- + R$.

(iii) P^+, P^- and F^+, F^- are non-negative self-adjoint operators.

(iv) We have the following estimates;

$$|(P^+F^-u, F^\pm v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3},$$
$$|(P^-F^+u, F^\pm v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3},$$
$$|(Ru, v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3},$$

for any $u, v \in C_0^\infty(\boldsymbol{R}^n)$.

Theorem I gives more precise statement. Proof is found in §5 and §6.

If the principal symbol does not change sign, the problem has been settled. In fact strong Gårding inequality [3], [6] means that we can take $P^-=0$, $F^-=0$ and that R satisfies stronger inequality

$$|(Ru, v)| \leqq C\|u\|_{-1/2}\|v\|_{-1/2}.$$

However our result seems new if the principal symbol changes sign. Difficulty arises at the point of characteristics of the operator $p(x, D)$. The operator F^+ and F^- are closely related to location of characteristics of $p(x, D)$. This is discussed in §7.

1) As to general theory of pseudo-differential operators. See [1], [2], [5] and [7].

Our method is based on localization of Hörmander in [4]. His terminology will frequently be used.

2. Localization

We treat a pseudo-differential operator $p(x, D)$ defined by

$$(2.1)\qquad p(x, D)u(x) = (2\pi)^{-n}\iint_{R^n\times R^n} p(x, \xi)e^{i\langle x-y\rangle\cdot\xi}u(y)\,dy\,d\xi\,.$$

We assume that the symbol $p(x, \xi)$ is of the form

$$p(x, \xi) = p_0(x, \xi)+p_1(x, \xi)\,,$$

where $p_0(x, \xi)$ is homogeneous of degree 0 with respect to ξ for large $|\xi|$ and $p_1(x, \xi)$ is a function in $S^{-1}_{1,0}(\boldsymbol{R}^n)$ in the sense of Hörmander [4]. We further assume that the principal part $p_0(x, \xi)$ vanishes unless x lies in a bounded domain $\Omega\subset\boldsymbol{R}^n$. (See [4]). We use Hörmander's localization in [4]. Let $g_0=0$, $g_1, g_2, \cdots$, be the unit lattice points in $\boldsymbol{R}^n$. Then $\boldsymbol{R}^n$ is covered by open cubes of side 2 with center at these points. Let $\Theta(x)$ be a non-negative C_0^∞ function which equals 1 in $|x_j|\leqq 1$ and zero outside $|x_j|\leqq\frac{3}{2}$, $1\leqq i\leqq n$. We use

$$(2.2)\qquad \varphi_k(x) = \Theta(x-g_k)/(\sum_{k=0}^{\infty}\Theta(x-g_k)^2)^{1/2}\quad\text{and}$$

$$\dot{\varphi}_k(x) = \varphi_k\left(\frac{x-g_k}{2}k+g_k\right).$$

The following properties hold:

$$(2.3)\qquad \sum_k \varphi_k(x)^2 \equiv 1 \quad\text{and}$$

$$(2.4)\qquad \sum_k D^\alpha\varphi_k(x) \leqq C_\alpha\,,$$

where α is an arbitrary multi-index $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_n)$. D^α is the usual notation, *i.e.*, $D^\alpha=\left(-i\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(-i\frac{\partial}{\partial x_n}\right)^{\alpha_n}$.

$$(2.5)\qquad |x-y|\leqq 2\sqrt{n}\qquad\text{if}\quad x, y\in\operatorname{supp}\varphi_k\,.$$

Let

$$(2.6)\qquad \psi_k(\xi) = \varphi_k\left(\frac{\xi}{|\xi|^{2/3}}\right),\quad \dot{\psi}_k(\xi) = \dot{\varphi}_k\left(\frac{\xi}{|\xi|^{2/3}}\right).$$

Then

$$(2.7)\qquad \sum_k \psi_k(\xi)^2 = 1\,,$$

(2.8) $\quad |\xi|^{3/4|\alpha|}\sum_k |D^\alpha\psi_k(\xi)|^2 \leqq C_\alpha \quad$ for $\ ^\forall\alpha$.

(2.9) $\quad |\xi-\eta|\leqq C|\xi|^{2/3} \quad$ if $\ \xi, \eta\in\operatorname{supp}\psi_k$.

(2.10) $\quad \sum_k |\psi_k(\xi)-\psi_k(\eta)|^2 \leqq \dfrac{C|\xi-\eta|^2}{(1+|\xi|^{2/3})(1+|\eta|)^{2/3}} \quad$ for $\ ^\forall\xi, \eta\in \boldsymbol{R}^n$.

Functions $\dot\varphi_k$ and $\dot\psi_k$ are identically one in some neighbourhood of supp φ_k and supp ψ_k respectively. They also have properties (2.4)~(2.10) except (2.7). Note that $\delta_j^2 g_j$ belongs to supp ψ_j if $\delta_j=|g_j|$. We define operator $\psi_j(D)$ by

(2.11) $$\psi_j(D)u(x)=(2\pi)^{-n}\iint_{R^n\times R^n} e^{i(x-y)\cdot\xi}\psi_j(\xi)u(y)\,dy\,d\xi\,.$$

Obviously we have

(2.12) $$\sum_j \varphi_j(D)^2 = Id\,,$$

and

(2.13) $$C\|u\|_s^2 \leqq \sum_{j=0}^{\infty}\delta_j^{6s}\|\varphi_j(D)u\|_0^2 \leqq C^{-1}\|u\|_s^2\,,$$

where $\|u\|_s$ is Sobolev norm of u of order s in $\boldsymbol{R}^n$.
We set $\varphi_{jk}(x)=\varphi_j(\delta_k x)$ and $\phi_{jk}(x,\xi)=\varphi_{jk}(x)\psi_k(\xi)$. Note that for any multi-indices α, β, we have

(2.14) $$|D_x^\alpha D_\xi^\beta \phi_{jk}(x,\xi)| \leqq C_{\alpha\beta}\delta_k^{|\alpha|}|\xi|^{-2/3|\beta|} \leqq C_{\alpha\beta}|\xi|^{1/3|\alpha|-2/3|\beta|}\,.$$

This means that ϕ_{jk} belongs to class $S^0_{2/3,1/3}$ of Hörmander. It follows from (2.3) and (2.13) that

(2.15) $$C\|u\|_s^2 \leqq \sum_{jk}\delta_k^{6s}\|\phi_j(x,D)u\|_0^2 \leqq C^{-1}\|u\|_s^2\,,$$

and

(2.16) $$\sum_{jk}\phi_{jk}(x,D)^*\phi_{jk}(x,D) = Id\,.$$

For any pair (j, k) of integers we set

(2.17) $$P_{jk}(x,D) = p_0(x^{jk},\xi^k)+\sum_{\nu=1}^n p_{0(\nu)}(x^{jk},\xi^k)(x-x^{jk})_\nu + \sum_{\nu=1}^n p_0^{(\nu)}(x^{jk},\xi^k)(D-\xi^k)_\nu\,,$$

where ξ^k is a point in supp ψ_k and x^{jk} is a point in supp φ_{jk}. The following proposition is due to Hörmander.

Proposition 2.1. *For any $^\forall u, v\in D(\boldsymbol{R}^n)$, we have*

(2.18) $$|(p(x,D)u,v)-\sum_{jk}(p_{jk}(x,D)\phi_{jk}(x,D)u,\phi_{jk}(x,D)v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3}\,.$$

Proof is found in [4].

3. Spectral decomposition of localized operators

We shall call $P_{jk}(x, D)$ localized operator. $P_{jk}(x, D)$ is an operator of order 1. The spectral decomposition of $P_{jk}(x, D)$ is well known. In fact, after multiplication of $e^{ix\cdot\xi^k}$ and suitable change of coordinates, $P_{jk}(x, D)$ is unitarily transformed to an operator of the form

$$L = \alpha D_1 + b\cdot x\,,$$

where α is a real constant and $b\cdot x$ is Euclidean scalar product of two vectors

$$b = (b_1, b_2, \cdots, b_n) \quad \text{and} \quad x = (x_1, x_2, \cdots, x_n)\,.$$

Let

$$L = \int_{-\infty}^{\infty} \lambda\, dE(\lambda)$$

be spectral decomposition of L. Then the projection operator $E(\lambda)$ is the multiplication of function $Y(\lambda - b\cdot x)$ if $\alpha=0$. Here $Y(t)$, $t\in \boldsymbol{R}$, stands for Heaviside function, that is,

$$Y(t) = \begin{cases} 1 & t\geqq 0 \\ 0 & t<0\,. \end{cases}$$

If $\alpha \neq 0$, we set
$$L' = e^{-i\frac{b_1 x_1^2}{2\alpha_1}} L e^{i\frac{b_1 x_1^2}{2\alpha_1}}\,.$$

L' is an operator of the form

$$L' = \alpha D_1 + b'\cdot x'\,,$$

where $b' = (b_2, \cdots, b_n)$ and $x' = (x_2, \cdots, x_n)$.

Taking partial Fourier transform with respect to x_1, we have reduced to the case that $\alpha=0$.

We shall use the following notations:

$$(3.1) \qquad P_{jk}(x, D) = \int_{-\infty}^{\infty} \lambda\, dE_{jk}(\lambda)\,.$$

Here $E_{jk}(\lambda)$ is the spectral measure of P_{jk}.

We put $\quad E_{jk}^- = E_{jk}(0) \qquad E_{jk}^+ = I - E_{jk}^-$

$\qquad\qquad P_{jk}^+ = P_{jk}E_{jk}^+ \qquad P_{jk}^- = -P_{jk}E_{jk}^-\,.$

4. Statement of Theorem I

We put

$$P^+ = \sum_{jk} \phi_{jk}(x, D)^* P^+_{jk} \phi_{jk}(x, D)\,, \tag{4.1}$$

$$P^- = \sum_{jk} \phi_{jk}(x, D)^* P^-_{jk} \phi_{jk}(x, D)\,, \tag{4.2}$$

$$F^+ = \sum_{jk} \phi_{jk}(x, D)^* E^+_{jk} \phi_{jk}(x, D)\,, \tag{4.3}$$

$$F^- = \sum_{jk} \phi_{jk}(x, D)^* E^-_{jk} \phi_{jk}(x, D)\,. \tag{4.4}$$

Then we have

Theorem I. *Operators P^+, P^-, F^+ and F^- are self-adjoint and satisfy the following properties*:

(4.5) (i) $I = F^+ + F^-$.

(4.6) (ii) $(P^{\pm}u, u) \geqq 0$.

(4.7) (iii) $|(F^-P^+F^+u, v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3}$.

(4.8) $|(F^-P^+F^-u, v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3}$.

(4.9) $|(F^-P^-F^+u, v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3}$.

(4.10) $|(F^+P^-F^+u, v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3}$.

(4.11) (iv) $|([P, F^{\pm}]u, v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3}$.

(4.12) $|([P^{\pm}, F^{\pm}]u, v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3}$.

(v) *If we set $R = P - (P^+ - P^-)$ then*

(4.13) $|(Ru, v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3}$.

Corollary 4.2. *We have*

(4.14) $|(PF^+u, v) - (F^+P^+F^+u, v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3}$

(4.15) $|(PF^-u, v) + (F^-P^-F^-u, v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3}$.

(4.16) $|(P^+F^-u, v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3}$,

(4.17) $|(P^-F^+u, v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3}$,

(4.18) $P = P^+ - P^- + R$.

We shall prove Theorem I in §6.

5. Some lemmas about self-adjoint operators

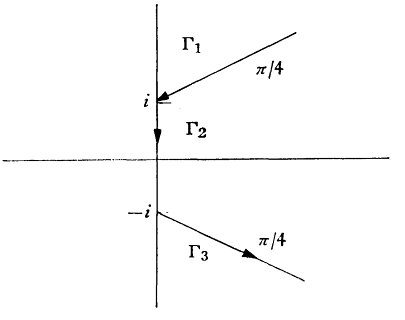
In this section X stands for an abstract Hilbert space.

Lemma 5.1. *Let A be a self-adjoint operator in X and A^+ be its positive part. Then*

$$A^+u = \frac{1}{2\pi i}\int_\Gamma \left(\lambda(\lambda - A)^{-1} - 1 - \frac{A}{\lambda+1}\right) u\, d\lambda$$

provided $u \in D(A^2)$*=domain of* A^2. Γ *is the complex contour as is shown in* fig 1.

fig 1.



Proof. Note that

$$\lambda(\lambda-\sigma)^{-1} - 1 - \frac{\sigma}{\lambda+1} = \frac{\sigma(\sigma+1)}{(\lambda-\sigma)(\lambda+1)}.$$

Integrate this with respect to λ on Γ then we have σ if $\sigma>0$ and 0 if $\sigma<0$. Therefore if we use spectral decomposition of A, then we can prove our Lemma.

Lemma 5.2. *Let A be a self-adjoint operator in X and let B be a bounded linear operator. We assume that operators AB and A^2B are densely defined. We further assume that the communtator $[A, B]$, $[A, [A, B]]$ are bounded.*
Then we have

(5.2) $$\|[A^\pm, B]\| \leqq C(\|B\| + \|[A, B]\| + \|[[A, B], A]\|).$$

Proof. Let $u \in D(A^2) \cap D(A^2B) \cap D(AB)$,

$$2\pi i[A^+, B]u = \int_\Gamma \left[\left(\lambda(\lambda-A)^{-1} - 1 - \frac{A}{\lambda+1}\right), B\right] u\, d\lambda.$$

We split Γ into theree parts $\Gamma_1+\Gamma_2+\Gamma_3$. (see fig. 1). Corresponding integrals are denoted by A_1, A_2 and A_3. Obviously $[A^+, B] = [A_1, B] + [A_2, B] + [A_3, B]$.

Since $$\int_{\Gamma_2} \frac{1}{\lambda+1} d\lambda = \log(1-i) - \log(1+i),$$

we have

(5.3) $$\|[A_2, B]\| \leqq 4(\|B\| + \|[A, B]\|).$$

Let us treat

$$2\pi i[A_2+A_3, B]=\int_{\Gamma_1+\Gamma_2}\left[\left[\lambda(\lambda-A)^{-1}-1-\frac{A}{\lambda+1}\right], B\right]d\lambda$$

$$=-[A, B]\int_{\Gamma_1+\Gamma_3}\frac{d\lambda}{\lambda+1}+\int_{\Gamma_1+\Gamma_3}\lambda(\lambda-A)^{-1}[A, B](\lambda-A)^{-1}d\lambda\,.$$

We know

$$\left|\int_{\Gamma_1+\Gamma_3}\frac{d\lambda}{\lambda+1}\right|\leqq \text{const.}$$

On the other hand we have

$$\int_{\Gamma_1+\Gamma_3}\lambda(\lambda-A)^{-1}[A, B](\lambda-A)^{-1}d\lambda$$

$$=\int_{\Gamma_1+\Gamma_3}\lambda(\lambda-A)^{-2}[A, B]d\lambda+\int_{\Gamma_1+\Gamma_3}\lambda(\lambda-A)^{-2}[A, [A, B]](\lambda-A)^{-1}d\lambda\,.$$

The last term is majorized by $C||[A, [A, B]]||$.

The first is

$$\int_{\Gamma_1+\Gamma_3}\lambda(\lambda-A)^{-2}d\lambda=\int_{\Gamma_1+\Gamma_3}d\lambda\int_{-\infty}^{\infty}\lambda(\lambda-\sigma)^{-2}dE(\sigma)$$

$$=\int_{\Gamma_1+\Gamma_3}d\lambda\int_{-\infty}^{\infty}\left(\frac{1}{\lambda-\sigma}+\frac{\sigma}{(\lambda-\sigma)^2}\right)dE(\sigma)\,.$$

Since $\left|\int_{\Gamma_1+\Gamma_3}\frac{d\lambda}{\lambda-\sigma}\right|\leqq \text{Const.}$ and $\left|\int\frac{\sigma}{(\lambda-\sigma)^2}d\lambda\right|\leqq 2\left|\frac{\sigma}{i+\sigma}\right|\leqq 2$,

we have

$$||\int_{\Gamma_1+\Gamma_3}\lambda(\lambda-A)^{-2}[A, B]\,d\lambda||\leqq C||[A, B]||\,.$$

We have thus proved our lemma.

Lemma 5.3. *Let A and B be two self-adjoint operators in X. If the commutator $[A, B]$ is bounded, then for any*

$$x\in D(A^2)\cap D(B^2)$$

we have

$$||(A^+-B^+)x||\leqq C(||[A, B]||\,||x||+\sum_{k=1}^{2}||(A-B)^k x||+||x||+||[A, B]]|\,||(A-B)x||)\,.$$

Proof. We have to majorize

$$(2\pi i)\,(A^+x-B^+x)=\int_\Gamma\left(\lambda(\lambda-A)^{-1}-\lambda(\lambda-B)^{-1}-\frac{A}{\lambda+1}+\frac{B}{\lambda+1}\right)xd\lambda\,.$$

We decompose Γ as we did in the proof of Lemma 5.2. The integral over Γ_2 is majorized by $C(||x||+||(A-B)x||)$.

Note that

$$\begin{aligned}
&\lambda(\lambda-A)^{-1}-\lambda(\lambda-B)^{-1}x \\
&\quad = -\lambda(\lambda-B)^{-1}(A-B)(\lambda-A)^{-1}x \\
&\quad = -\lambda(\lambda-B)^{-1}(\lambda-A)^{-1}(A-B)x \\
&\qquad -\lambda(\lambda-B)^{-1}(\lambda-A)^{-1}[A, B](\lambda-A)^{-1}x \\
&\quad = -\lambda(\lambda-B)^{-2}\{1+(A-B)(\lambda-A)^{-1}\}(A-B)x \\
&\qquad +\lambda(\lambda-B)^{-1}(\lambda-A)^{-1}[A, B](\lambda-A)^{-1}x \\
&\quad = -\lambda(\lambda-B)^{-2}\{1+(\lambda-A)^{-1}(A-B) \\
&\qquad +(\lambda-A)^{-1}[A, B](\lambda-A)^{-1}\}(A-B)x \\
&\qquad +\lambda(\lambda-B)^{-1}(\lambda-A)^{-1}[A, B](\lambda-A)^{-1}x\,.
\end{aligned}$$

From this we can majorize the integral over $\Gamma_1+\Gamma_3$ by

$$C(||(A-B)x||+||(A-B)^2x||+||[A, B]||\ ||x||+||[A, B]||\ ||(A-B)x||)\,.$$

We have thus proved our lemma.

6. Proof of Theorem

We start with the propositions which simplify discussions later.

Proposition 6.1. *Let $u\in C_0^\infty(\boldsymbol{R}^n)$ be arbitrary and (j, k) be a pair of indices. Then there is a point $\bar{x}$ satisfying*

$$(6.1)\qquad |\bar{x}-x^{jk}|\leq\alpha\delta_k\,,$$

$$(6.2)\qquad \int(x_\nu-\bar{x}_\nu)|\phi_{jk}(x, D)u(x)|^2dx = 0$$

for $\nu=1, 2, 3, \cdots, n$. Here α is a positive constant independent of u and (j, k).

Proof is found in [3], page 171.
The point $\bar{x}$ can be chosen in supp φ_{jk}.

Proposition 6.2. *There exists a bounded sequence $\{\phi'_{jk}(x, \xi)\}_{jk}$ of symbols in $S^0_{2/3,1/3}$ such that we have*

$$(6.3)\qquad \text{(i)}\quad ||(D_\nu-\xi^k_\nu)\phi_{jk}(x, D)u||^2\leq C\delta_k^4||\phi'_{jk}(x, D)u||^2$$

and

$$(6.4)\qquad \text{(ii)}\quad \text{supp }\phi'_{jk}\subset\text{supp }\phi_{jk}$$

for $\nu=1, 2, 3, \cdots, n$.

Proof. We have

$$(6.5)\qquad (D_\nu-\xi_\nu^k)\phi_{jk}(x, D)u=\delta_k^2\phi'_{jk}(x, D)u\,,$$

where

$$(6.6)\qquad \phi'_{jk}(x, \xi)=\delta_k^{-2}\Big(-i\frac{\partial}{\partial x_\nu}\varphi_{jk}(x)\psi_k(\xi)+\varphi_{jk}(x)\psi_k(\xi)(\xi_\nu-\xi_\nu^k)\Big)\,.$$

The sequence $\{\phi'_{jk}(x, \xi)\}_{j,k}$ is bounded in $S^0_{2/3,1/3}$ because of (2.9) and

$$-i\frac{\partial}{\partial x_\nu}\varphi_{jk}(x)=-i\delta_k\Big(\frac{\partial}{\partial x_\nu}\varphi_j\Big)(\delta_k x)\,.$$

Proposition 6.3. *Let $\{(\dot{x}^{jk}, \xi^{jk})\}_{jk}$ be another sequence of points. Let $\dot{P}_{jk}$, $\dot{P}^\pm$ and $\dot{F}^\pm$ be operators defined by* (2.17), (4.1), (4.2), (4.3) *and* (4.4) *where (x^{jk}, ξ^k) is replaced by $(\dot{x}^{jk}, \xi^{jk})$. If there exists a constant $C>0$ satisfying*

$$(6.7)\qquad |x^{jk}-\dot{x}^{jk}|\leqslant C\delta_k^{-1}\quad \text{and}\quad |\xi^k-\xi^{jk}|\leqslant C\delta_k^2\,,$$

then we have

$$(6.8)\qquad \|(\dot{P}_{jk}-P_{jk})\phi_{jk}(x, D)u\|^2\leqslant C\delta_k^{-4}\|\phi_{jk}^{(1)}(x, D)u\|^2\,,$$

$$(6.9)\qquad \|(\dot{P}_{jk}-P_{jk})^2\phi_{jk})(x, D)u\|^2\leqslant C\delta_k^{-8}\|\phi_{jk}^{(2)}(x, D)u\|^2\,,$$

$$(6.10)\qquad \|[P_{jk}, \dot{P}_{jk}]\|\leqslant C\delta_k^{-4}\,,$$

$$(6.11)\qquad \|(\dot{P}_{jk}^\pm-P_{jk}^\pm)\phi_{jk}(x, D)u\|\leqslant C\delta_k^{-2}\|\phi_{jk}^{(3)}(x, D)u\|\,,$$

$$(6.12)\qquad |((\dot{P}^\pm-P^\pm)u, u)|\leq C\|u\|_{-1/3}\|v\|_{-1/3}\,.$$

Here, $\{\phi_{jk}^{(l)}\}_{jk}$, $l=1, 2, 3$, are bounded sequences of symbols in $S^0_{2/3,1/3}$ with the property that supp $\phi_{jk}^{(l)}\subset$ supp ϕ_{jk}.

REMARK 6.4. We require that the point (x^{jk}, ξ^k) lies in supp ϕ_{jk} but we don't require that $(\dot{x}^{jk}, \xi^{jk})$ lies in supp ϕ_{jk}.

Proof. It follows from Taylor's formula that

$$(6.13)\qquad P_0(\dot{x}^{jk}, \xi^{jk})=P_0(x^{jk}, \xi^k)+\sum_\nu(\dot{x}^{jk}-x^{jk})P_{0(\nu)}(x^{jk}, \xi^k)$$
$$+\sum_\nu(\xi^{jk}-\xi^k)_\nu P_0^{(\nu)}(x^{jk}, \xi^k)+R_1\,,$$

$$(6.14)\qquad P_{0(\nu)}(\dot{x}^{jk}, \xi^{jk})=P_{0(\nu)}(x^{jk}, \xi^k)+R_{2,(\nu)}\quad \text{and}$$

$$(6.15)\qquad P_0^{(\nu)}(\dot{x}^{jk}, \xi^{jk})=P_0^{(\nu)}(x^{jk}, \xi^k)+R_3^{(\nu)}\,.$$

By (6.7) the remainder terms are majorized as

$$(6.16)\qquad |R_1|\leqq C\delta_k^{-2},\ |R_{2(\nu)}|\leqq C\delta_k^{-1},\ |R_3^{(\nu)}|\leqq C\delta_k^{-4}.$$

We have

$$(6.17)\qquad \dot{P}_{jk}(x, D)-P_{jk}(x, D) = R_1+\sum_\nu (x-x^{jk})_\nu R_{2(\nu)}+\sum_\nu (D-\xi^{jk})_\nu R\,.$$

This implies that

$$\begin{aligned}&\|(\dot{P}_{jk}(x, D)-P_{jk}(x, D)\phi_{jk}(x, D)u\|^2\\ &\leqq C(\delta_k^{-4}\|\phi_{jk}(x, D)u\|^2+\delta_k^{-2}\sum_\nu \|(x-x^{jk})_\nu \phi_{jk}(x, D)u\|^2\\ &+\delta_k^{-8}\sum_\nu \|(D-\xi^{jk})_\nu \phi_{jk}(x, D)u\|^2)\\ &\leqq C\delta_k^{-4}\|\phi_{jk}^{(1)}(x, D)u\|^2\,.\end{aligned}$$

This is (6.8).
Similarly

$$(6.18)\qquad \begin{aligned}&\|(\dot{P}_{jk}(x, D)-P_{jk}(x, D))^2\phi_{jk}(x, D)u\|^2\\ &\leqq C\delta_k^{-8}\|\phi_{jk}^{(2)}(x, D)u\|^2\,.\end{aligned}$$

Now

$$(6.19)\qquad \begin{aligned}[P_{jk}, \dot{P}_{jk}] &= [P_{jk}, \dot{P}_{jk}-P_{jk}]\\ &= -[i\sum_\nu R_{2(\nu)}P_0^{(\nu)}(x^{jk}, \xi^k)-\sum_\nu R_3^{(\nu)}P_{0(\nu)}(x^{jk}, \xi^k)]\,.\end{aligned}$$

This proves (6.10).

We apply Lemma 5.3 to operators $A=\delta_k^2P_{jk}$, and $B=\delta_k^2\dot{P}_{jk}$. Then we have

$$(6.20)\qquad \|(A^+-B^+)\phi_{jk}(x, D)u\|\leqq C\|\phi_{jk}^{(3)}(x, D)u\|\,.$$

This proves that

$$(6.21)\qquad \|(P_{jk}^+-\dot{P}_{jk}^+)\phi_{jk}(x, D)u\|\leqq C\delta_k^{-2}\|\phi_{jk}^{(3)}(x, D)u\|\,.$$

Let v be in $C_0^\infty(\boldsymbol{R}^n)$. Then

$$\begin{aligned}|((P^+-\dot{P}^+)u, v)| &\leqq \sum_{jk} |((P_{jk}^+-\dot{P}_{jk}^+)\phi_{jk}(x, D)u, \phi_{jk}(x, D)v)\\ &\leqq C\sum_{jk} \delta_k^{-2}\|\phi_{jk}^{(3)}(x, D)u\|\,\|\phi_{jk}(x, D)v\|\,.\end{aligned}$$

Take arbitrary positive $t>0$. Then

$$\begin{aligned}|((P^+-\dot{P}^+)u, v)| &\leqq C\sum_{jk} \frac{t^2}{2}\delta_k^{-2}\|\phi_{jk}^{(3)}(x, D)u\|^2+\frac{t^{-2}}{2}\delta_k^{-2}\|\phi_{jk}(x, D)v\|^2\\ &\leqq C\left(\frac{t^2}{2}\|u\|_{-1/3}^2+\frac{t^{-2}}{2}\|v\|_{-1/3}^2\right).\end{aligned}$$

Taking the minimum of this with respect to t, we have

$$|((P^+-\dot{P}^+)u, v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3}.$$

Next we need bounds for commutators

$$[P_{jk}^{\pm}, \phi_{lm}(x, D)], \quad [E_{jk}^{\pm}, \phi_{lm}(x, D)] \quad \text{etc.}$$

These are needed only when supp $\phi_{jk} \cap$ supp $\phi_{lm} \neq \phi$.
We introduce notation

$$I(j, k) = \{(l, m) | \text{supp } \phi_{jk} \cap \text{supp } \phi_{lm} \neq \phi\}.$$

It is obvious that there is a constant $C>0$ such that

$$C^{-1} \leqq \frac{\delta_m}{\delta_k} \leqq C. \qquad \text{if} \quad (l, m) \in I(j, k).$$

The number of indices (l, m) in $I(j, k)$ is bounded.

Proposition 6.5. *We have the following estimates for commutators: If $(l, m) \in I(j, k)$, then*

(6.22) $\|[P_{jk}, \phi_{lm}]\| \leqq C\delta_k^{-2}$,

(6.23) $\|[[P_{jk}, \phi_{lm}], P_{jk}]\| \leqq C\delta_k^{-4}$,

(6.24) $\|[[P_{jk}, \phi_{lm}], \phi_{lm}]\| \leqq C\delta_k^{-2}$,

(6.25) $\|[[P_{jk}, \phi_{lm}{}^*], \phi_{lm}]\| \leqq C\delta_k^{-2}$,

(6.26) $\|[P_{jk}^{\pm}, \phi_{lm}]\| \leqq C\delta_k^{-2}$,

(6.27) $\|[P_{jk}, [E_{jk}^{\pm}, \phi_{lm}]]\| \leqq C\delta_k^{-2}$.

Proof.
$$\begin{aligned}[P_{jk}, \phi_{lm}] &= [P_{jk}, \varphi_{lm}(x)\psi_m(D)] \\ &= [P_{jk}, \varphi_{lm}]\varphi_m(D) + \varphi_{lm}[P_{jk}, \psi_m(D)] \\ &= \delta_k \sum_\nu P_0^{(\nu)}(x^{jk}, \xi^k) D_\nu \varphi_{lm}(x)\psi_m(D) \\ &\quad -\varphi_{lm}\delta_k^{-2} \sum_\nu D_\nu \psi_m(D) P_{0(\nu)}(x^{jk}, \xi^k).\end{aligned}$$

This proves that

$$\|[P_{jk}, \phi_{lm}]\| \leqq C\delta_k^{-2}.$$

More precisely, $\{\delta_k^2[P_{jk}, \phi_{lm}]\}_{jk}$ is bounded sequence of operators in $L^0_{2/3,1/3}$ of Hörmander. By just the same argument we can prove (6.23). (6.24) and (6.25) are consequences of the fact that

$$\{\delta_k[P_{jk}, \phi_{lm}]\}_{jk}$$

is a bounded set in $L^0_{2/3,1/3}$.

We set $A = \delta_k^2 P_{jk}$, $B = \delta_k^{-2}\phi_{lm}$ and apply Lemma 5.2.

Then we have

$$||[P^{\pm}_{jk}, \phi_{lm}]|| \leqq C\delta_k^{-2}\,.$$

Since

$$(6.29) \qquad P_{jk}[E^{\pm}_{jk}, \phi_{lm}] = [P^{\pm}_{jk}, \phi_{lm}] - E^{\pm}_{jk}[P_{jk}, \phi_{lm}]\,,$$

(6.27) is a consequence of (6.26).
Now we are ready for proving our Theorem I.

Proof of (iii). Let (j, k) and (j', k') be two pairs of indices. Then we put

$$I(jk, j'k') = \{(l, m) \quad | \operatorname{supp} \phi_{lm} \cap \operatorname{supp} \phi_{jk} \neq \phi \\ \operatorname{supp} \phi_{lm} \cap \operatorname{supp} \phi_{j'k'} \neq \phi\}\,.$$

By definition of P^+, F^+ and F^-, we have

$$(6.30) \qquad (F^-P^-F^+u, v) = \sum_{jk} \sum_{j'k'} (P^-\phi^*_{jk}E^+_{jk}\phi_{jk}u, \phi^*_{j'k'}E^-_{j'k'}\phi_{j'k'}v)\,.$$

If $\operatorname{supp} \phi_{lm} \cap \operatorname{supp} \phi_{jk} = \phi$ and $\operatorname{supp} \varphi_k \cap \operatorname{supp} \psi_m \neq 0$, then

$$(6.31) \qquad ||\phi_{lm}(x, D)\phi_{jk}(x, D)^* w|| \leqq C\delta_k^{-N}||w||$$

for any $N>0$. If $\operatorname{supp} \psi_k \cap \operatorname{supp} \psi_m = \phi$, then $\phi_{lm}(x, D)\phi_{jk}(x, D)^* u = 0$.

Thus we have

$$(6.32) \qquad \sum_{(l,m)\notin I(j,k)} ||\phi^*_{lm}P^-_{lm}\phi_{lm}\phi^*_{jk}E^+_{jk}\phi_{jk}u|| \\ \leqq C|\Omega|\delta_k^n\delta_k^{-N}||E^+_{jk}\phi_{jk}u|| \\ \leqq C|\Omega|\delta_k^{n-N}||\phi_{jk}u||\,,$$

where $|\Omega|$ is the volume of the domain Ω.
Similarly

$$(6.33) \qquad \sum_{(l,m)\notin I(j',k')} ||\phi^*_{lm}P^-_{lm}\phi_{lm}\phi^*_{j'k'}E^*_{j'k'}\phi_{j'k'}v||^2 \leqq C|\Omega|\delta_k^{n-N}||\phi_{j'k'}v||^2.$$

(6.32) and (6.33) imply that

$$(6.34) \qquad (F^-P^-F^+u, v) - \sum_{jk}\sum_{j'} \sum_{(l,m)I(jk,j'k')} (\phi^*_{lm}P^-\phi_{lm}\phi^*_{jk}E^+_{jk}\phi_{jk}u\phi_{j'k'}E'_{jk}\phi_{jk'}v) \\ \leqq C|\Omega|(\sum_{jk,j'k'} \delta_k^{n-N}|||\phi_{jk}u||\,||\phi_{j'k'}v||) \\ \leqq C|\Omega|\,||u||_{-1/3}||v||_{-1/3}\,.$$

We have

$$(6.35)\qquad \sum_{(lm)\in I(jk,j'k')} \phi_{lm}^{*}P_{lm}^{-}\phi_{lm}\phi_{jk}^{*}E_{jk}^{+}\phi_{jk}u = \sum_{(lm)\in I(jk,j'k')} \phi_{lm}^{*}P_{jk}^{-}\phi_{lm}\phi_{jk}^{*}E_{jk}^{+}\phi_{jk}u + \sum_{(lm)\in I(jk,j'k')} \phi_{lm}^{*}(P_{lm}^{-}-P_{jk}^{-})\phi_{lm}\phi_{jk}^{*}E_{jk}^{+}\phi_{jk}u\,.$$

We apply Proposition 6.3 and have

$$(6.36)\qquad \|\sum_{lm}\phi_{lm}^{*}(P_{lm}^{-}-P_{jk}^{-})\phi_{lm}\phi_{jk}^{*}E_{jk}^{+}\phi_{jk}u\| \leqq C\delta_k^{-2}\|\phi_{jk}u\|\,.$$

On the other hand,

$$(6.37)\qquad \sum_{(l,m)\in I(jk,j'k')} \phi_{lm}^{*}P_{jk}^{-}\phi_{lm}\phi_{jk}^{*}E_{jk}^{+}\phi_{jk}u = \sum_{lm}\{\phi_{lm}^{*}[P_{jk}^{-}\phi_{lm}]\phi_{jk}^{*}E_{jk}^{+}\phi_{jk}u+\phi_{lm}^{*}\phi_{lm}[P_{jk}^{-},\phi_{jk}^{*}]E_{jk}^{+}\phi_{jk}u\}\,.$$

By proposition 6.5, we have

$$(6.38)\qquad \|\phi_{lm}^{*}[P_{jk}^{-},\phi_{lm}^{*}]\phi_{jk}^{*}E_{jk}^{+}\phi_{jk}\|u \leqq C\delta_k^{-2}\|\phi_{jk}u\|$$

and

$$(6.39)\qquad \|\phi_{lm}^{*}\phi_{lm}[P_{jk}^{-},\phi_{jk}^{*}]E_{jk}^{+}\phi_{jk}u\| \leqq C\delta_k^{-2}\|\phi_{jk}u\|\,.$$

(6.37), (6.38) and (6.39) imply that

$$(6.40)\qquad \|\sum_{(lm)\in I(jk,j'k')} \phi_{lm}^{*}P_{jk}^{-}\phi_{lm}\phi_{jk}^{*}E_{jk}^{+}\phi_{jk}u\| \leqq C\delta_k^{-2}\|\phi_{jk}u\|\,.$$

As a consequence of (6.34) and (6.40), we have

$$(6.41)\qquad |(F^{-}P^{-}F^{+}u,v)| \leqq \sum_{(jk)(j'k')} C\delta_k^{-2}\|\phi_{jk}u\|\,\|\phi_{j'k'}v\| \leqq C\|u\|_{-1/3}\|v\|_{-1/3}\,,$$

where the summation ranges over those (jk) and (j', k') that $I(jk, j'k')\neq\phi$. This proved (iii). Proof of remaining part of Theorem I is the same.

7. The role of characteristics

So far the choice of sequence $\{(x^{jk}, \xi^{k})\}$ is not specified. In the following we shall make use of special choice of it in order to simplify operators $P_{jk}^{\pm}$ and $E_{jk}^{\pm}$.

The set

$$(7.1)\qquad \Sigma^{0}(P) = \{(x,\xi)\in \boldsymbol{R}^{2n}\,|\,\xi\neq 0,\ P_0(x,\xi)=0\}$$

is called the characteristics of the operator P. We also use the following notations;

$$(7.2)\qquad \sum^{+}(P) = \{(x, \xi)\in \boldsymbol{R}^{2n} \mid \xi \neq 0,\ P_0(x, \xi)>0\}\ ,$$

$$(7.3)\qquad \sum^{-}(P) = \{(x, \xi)\in \boldsymbol{R}^{2n} \mid \xi \neq 0,\ P_0(x, \xi)<0\}\ .$$

Proposition 7.1. *Assume that* $(x^{jk}, \xi^k)\in\sum^{+}(P)\cup\sum^{0}(P)$ *and that* $P(x, \xi)\geqq 0$ *for any* $x\in\operatorname{supp}\phi_{jk}$ *and* ξ *with* $|\xi-\xi^k|<\alpha\delta_k^2$, *where* α *is the constant appeared in Proposition* 6.1. *Then we can replace* E_{jk}^{+} *by the identity operator without altering results in Theorem* I.

Proof of Proposition 7.1.

We put $L_k=\{j\mid (x^{jk}, \xi^k)$ satisfies the assumption of Proposition 7.1$\}$

$$(7.4)\qquad Q_k = \sum_{j\in L_k}\phi_{jk}^{*}(x, D)P_{jk}^{-}\phi_{jk}(x, D)$$

and

$$(7.5)\qquad G_k = \sum_{j\in L_k}\phi_{jk}^{*}(x, D)E_{jk}^{-}\phi_{jk}(x, D)\ .$$

We claim that there exists a constant $C>0$ such that

$$(7.6)\qquad \|Q_k u\|\leqq C\delta_k^{-2}\|\psi_k(D)u\|\ .$$

We admit this for a moment. Replacing E_j^{+} $(j\in L_k,\ k=0, 1, 2, \cdots)$ in (4.1)$\sim$(4.4) with the identity, we obtain operators $Q^{\pm}$ and $G^{\pm}$.
Differences between old and new operators are

$$(7.7)\qquad Q^{\pm}-P^{\pm} = \sum_k Q_k\ ,$$

$$(7.8)\qquad G^{\pm}-F^{\pm} = \sum_k G_k\ .$$

These relations imply that

$$(7.9)\qquad \begin{aligned}(G^{-}Q^{+}G^{+}u, v) = {} & \sum_k (G^{-}Q_kG^{+}u, v)-\sum_k (G_kP^{+}F^{+}u, v)\\ & +\sum_k (F^{-}P^{+}G_ku, v)-\sum_{k,l}(G_kP^{+}G_lu, v)\\ & +(F^{-}P^{+}F^{+}u, v)\ .\end{aligned}$$

We know by Theorem I that

$$(7.10)\qquad |(F^{-}P^{+}F^{+}u, v)|\leqq C\|u\|_{-1/3}\|v\|_{-1/3}\ .$$

On the other hand we can use (7.6) and prove the following inequalities in the same way as the proof of (6.34):

$$
(7.11) \qquad \begin{aligned}
&\sum_k |(G_k P^+ F^+ u, v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3}, \\
&\sum_k |(F^- P^+ G_k u, v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3}, \\
&\sum_{k,l} |(G_k P^+ G_l u, v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3}, \\
&\sum_k |(G^- Q_k G^+ u, v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3}.
\end{aligned}
$$

These prove

$$
(7.12) \qquad |(G^- Q^+ G^+ u, v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3}
$$

which corresponds to (4.7). Other inequalities can be proved in the same manner.

Now we must prove our claim (7.6). We choose $\bar{x}$ as in Proposition 6.1. Let

$$
(7.13) \qquad \begin{aligned}
Q_{jk}(x, D) = p_0(\bar{x}, \xi^k) &+ \sum_{\nu=1}^{n} p_{0(\nu)}(\bar{x}, \xi^k)(x-\bar{x})_\nu \\
&+ \sum_{\nu=1}^{n} p_0^{(\nu)}(\bar{x}, \xi^k)(D-\xi^k)_\nu .
\end{aligned}
$$

Then

$$
(7.14) \qquad \begin{aligned}
&(Q_{jk}(x, D)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u) \\
&= ((p_0(\bar{x}, \xi^k) + \sum_\nu p_0^{(\nu)}(\bar{x}, \xi^k)(D-\xi^k)_\nu)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u) \\
&= p_0(\bar{x}, \xi^k)((1-\dot{\psi}(D))\phi_{jk}(x, D)u, \phi_{jk}(x, D)u) \\
&\quad + ((p_0(\bar{x}, \xi^k) + \sum_\nu p_0^{(\nu)}(\bar{x}, \xi^k)(D-\xi^k)_\nu)\dot{\psi}_k(D)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u) \\
&\quad + \sum_\nu p_0^{(\nu)}(\bar{x}, \xi^k)((D-\xi^k)_\nu(1-\dot{\psi}_k(D))\phi_{jk}(x, D)u, \phi_{jk}(x, D)u)
\end{aligned}
$$

because of (6.2).

Since α is large, we may assume that $p_0(\bar{x}, \xi) \geqq 0$ if $\xi \in \operatorname{Supp} \dot{\psi}_{k0}$. Taylor's expansion of $p_0(\bar{x}, \xi)$ at $\xi = \xi^k$ imply that there exists a constant $C>0$ such that

$$
\begin{aligned}
&((p_0(\bar{x}, \xi^k) + \sum_\nu p_0^{(\nu)}(\bar{x}, \xi^k)(D-\xi^k)_\nu)\dot{\psi}_k(D)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u) \\
&\geqq -C\delta_k^{-2}\|\phi_{jk}(x, D)u\|^2 .
\end{aligned}
$$

We know that

$$
(D-\xi^k)_\nu(1-\dot{\psi}_k(D))\phi_{jk}(x, D)u = (D-\xi^k)_\nu(1-\dot{\psi}_k(D))\varphi_{jk}(x)\psi_k(D)\dot{\psi}_k(D)u
$$

and that the sequence of double symbols $\{(\xi-\xi^k)(1-\dot{\psi}_k(\xi))\varphi_{jk}(x)\psi_k(\eta)\}_{j,k}$ is bounded in $S^{-\infty}$. Therefore we have estimate for any $N>0$,

$$
\|(D-\xi^k)(1-\dot{\psi}_k(D))\phi_{jk}(x, D)u\|^2 \leqq C\delta_k^{-N}\|\psi_k(D)u\|^2 .
$$

This implies that

$$(Q_{jk}(x, D)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u)+C\delta_k^{-2}\|\phi_{jk}(x, D)u\|^2+C\delta_k^{-N}\|\psi_k(D)u\|^2\geqq 0 .$$

This and Proposition 6.3 prove that

$$(P_{jk}(x, D)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u)+C(\delta_k^{-2}\|\phi'_{jk}(x, D)u\|^2+\delta_k^{-N}\|\psi_k(D)u\|^2)\geqq 0 ,$$

where $\{\phi'_{jk}(x, \xi)\}$ is a bounded sequence in $S^0_{2/3,1/3}$ as of Proposition 6.3. Taking sum of these with respect to $j\in L_k$, we have

$$\sum_{j\in L_k}(P_{jk}(x, D)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u)+C\delta_k^{-2}\|\psi_k(D)u\|^2\geqq 0 .$$

Our claim is an immediate consequence of this inequality.

REMARK. Result similar to Proposition 7.1 holds for E^-_{jk}.

Next we discuss the case that $P_0(x, \xi)$ changes sign in the neighbourhood of supp ϕ_{jk}. In this case we compare $P_{jk}(x, D)$ with the operator $\dot{P}_{jk}(x, D)$ which is determined at a characteristic point.

Proposition 7.2. *Assume that $P_0(x, \xi)$ changes sign at some point $(\dot{x}, \dot{\xi})$ with*

$$|X^{jk}-\dot{x}|<\alpha\delta_k^{-1} , \quad |\xi^k-\dot{\xi}|<\alpha\delta_k^2 . \tag{7.6}$$

Then we can replace $P_{jk}(x, D)$ by

$$\bar{P}_{jk}(x, D) = \sum_\nu P_{0(\nu)}(\dot{x}, \dot{\xi})(x-\dot{x})_\nu + \sum_\nu P_0^{(\nu)}(\dot{x}, \dot{\xi})(D-\dot{\xi})_\nu \tag{7.7}$$

without altering results in Theorem I.

Proof. This proposition is contained in Proposition 6.3.
Finally we discuss the case where the operator $E^\pm_{jk}$ can be arbitrarily chosen.

Proposition 7.3. *Assume that we have*

$$P_0(\dot{x}, \dot{\xi}) = 0 \quad \mathrm{grad}_{x,\xi}P_0(\dot{x}, \dot{\xi}) = 0$$

at some point $(\dot{x}, \dot{\xi})$ with $|\dot{x}-x^{jk}|<\alpha\delta_k^{-1}$, $|\dot{\xi}-\xi^k|<\alpha\delta_k^2$. Then we can replace $P_{jk}(x, D)$ by zero operator 0 *without altering Theorem* I.

Proof. This is because of Proposition 6.3.

REMARK 7.4. In this case, the operator $E^\pm_{jk}$ does not matter. We can put $E^+_{jk}=Id$ or 0 at our disposal. From Proposition 7.1, 7.2 and 7.3, we can see F^+ and F^- depend only on location of sets $\sum^+(P)$, $\sum^-(P)$ and $\sum^0(P)$. An interesting consequence comes out when one compare two pseudo-differential operators whose characteristics are the same. Let Q be another self-adjoint pseudo-differential operator of class $L^0_{1,0}$. We assume Q has homogeneous principal symbol $q_0(x, \xi)$ and $Q-q_0(x, D)\in L^{-1}_{1,0}$. Just as we did for the operator $P(x, D)$ we can consider operators Q^+, Q^-, F^+_Q, F^-_Q and sets $\sum^0(Q)$, $\sum^+(Q)$, $\sum^-(Q)$.

Theorem II. If $\sum^+(Q)\cup\sum^0(Q)\supset\sum^+(P)\cup\sum^0(P)$ and $\sum^-(Q)\cup\sum^0(Q)\supset\sum^-(P)\cup\sum^0(P)$, then we can take $F^+=F^+_Q$ and $F^-=F^-_Q$.

Proof. If Proposition 7.1 applies to (x^{jk}, ξ^k) and operator P, then the same applies to the operator Q. If Proposition 7.2 applies to (x^{jk}, ξ^k) and P, then we have $(\mathring{x}, \xi)\in\sum^0(P)\subset\sum^0(Q)$. If Proposition 7.2 does not apply to (x^{jk}, ξ^k) and Q, then $(\mathring{x}, \xi)$ satisfies $q_0(\mathring{x}, \xi)=0$, $\text{grad}_{x,\xi}q_0(\mathring{x}, \xi)=0$. Proposition 7.3 can be applied to this case and we come to the conclusion that we may take $Q_{jk}=0$ and the operator $E^{\pm}_{jk}$ does not matter so far as Q is concerned.

Acknowledgement:

The original manuscript of the author fallaciously asserted that operators $\phi'_{jk}(x, D)$, $\phi^{(l)}_{jk}(x, D)$, $l=1, 2, 3$, in Propositions 6.2 and 6.3 could be replaced by $\phi_{jk}(x, D)$ itself. This error was pointed out by the editors. The author expresses his hearty thanks to the editors.

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