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AN APPROXIMATE POSITIVE PART OF A SELF-ADJOINT PSEUDO-DIFFERENTIAL OPERATOR I

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1. Introduction

Among many problems concerning pseudo-differential operators, one of the most interesting problem is "to what extent does the symbol function $p(x, \xi)$ describe the spectral properties of an operator $p(x, D)$?" Motivation of this paper comes from this problem.

Actually what we do in this note is the following: Assume that $P=p(x, D)$ is a self-adjoint pseudo-differential operator of class $L_{1,0}^0$ of Hörmander [4]. Then starting from its principal symbol, we explicitly construct self-adjoint operators P^+ , P^- , R , F^+ and F^- with the following properties;

- (i) $F^+ + F^- = Id.$
- (ii) $P = P^+ - P^- + R.$
- (iii) P^+ , P^- and F^+ , F^- are non-negative self-adjoint operators.
- (iv) We have the following estimates;

$$|(P^+ F^- u, F^\pm v)| \leq C \|u\|_{-1/3} \|v\|_{-1/3},$$

$$|(P^- F^+ u, F^\pm v)| \leq C \|u\|_{-1/3} \|v\|_{-1/3},$$

$$|(Ru, v)| \leq C \|u\|_{-1/3} \|v\|_{-1/3},$$

for any $u, v \in C_0^\infty(\mathbf{R}^n).$

Theorem I gives more precise statement. Proof is found in §5 and §6.

If the principal symbol does not change sign, the problem has been settled. In fact strong Gårding inequality [3], [6] means that we can take $P^- = 0$, $F^- = 0$ and that R satisfies stronger inequality

$$|(Ru, v)| \leq C \|u\|_{-1/2} \|v\|_{-1/2}.$$

However our result seems new if the principal symbol changes sign. Difficulty arises at the point of characteristics of the operator $p(x, D)$. The operator F^+ and F^- are closely related to location of characteristics of $p(x, D)$. This is discussed in §7.

1) As to general theory of pseudo-differential operators. See [1], [2], [5] and [7].

Our method is based on localization of Hörmander in [4]. His terminology will frequently be used.

2. Localization

We treat a pseudo-differential operator $p(x, D)$ defined by

$$(2.1) \quad p(x, D)u(x) = (2\pi)^{-n} \iint_{\mathbf{R}^n \times \mathbf{R}^n} p(x, \xi) e^{i(x-y) \cdot \xi} u(y) dy d\xi.$$

We assume that the symbol $p(x, \xi)$ is of the form

$$p(x, \xi) = p_0(x, \xi) + p_1(x, \xi),$$

where $p_0(x, \xi)$ is homogeneous of degree 0 with respect to ξ for large $|\xi|$ and $p_1(x, \xi)$ is a function in $S_{1,0}^{-1}(\mathbf{R}^n)$ in the sense of Hörmander [4]. We further assume that the principal part $p_0(x, \xi)$ vanishes unless x lies in a bounded domain $\Omega \subset \mathbf{R}^n$. (See [4]). We use Hörmander's localization in [4]. Let $g_0=0, g_1, g_2, \dots$, be the unit lattice points in \mathbf{R}^n . Then \mathbf{R}^n is covered by open cubes of side 2 with center at these points. Let $\Theta(x)$ be a non-negative C_∞^∞ function which equals 1 in $|x_j| \leq 1$ and zero outside $|x_j| \leq \frac{3}{2}, 1 \leq j \leq n$. We use

$$(2.2) \quad \varphi_k(x) = \Theta(x-g_k) / \left(\sum_{k=0}^{\infty} \Theta(x-g_k)^2 \right)^{1/2} \quad \text{and} \\ \dot{\varphi}_k(x) = \varphi_k \left(\frac{x-g_k}{2} + g_k \right).$$

The following properties hold:

$$(2.3) \quad \sum_k \varphi_k(x)^2 \equiv 1 \quad \text{and}$$

$$(2.4) \quad \sum_k D^\alpha \varphi_k(x) \leq C_\alpha,$$

where α is an arbitrary multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$. D^α is the usual notation, i.e., $D^\alpha = \left(-i \frac{\partial}{\partial x_1} \right)^{\alpha_1} \dots \left(-i \frac{\partial}{\partial x_n} \right)^{\alpha_n}$.

$$(2.5) \quad |x-y| \leq 2\sqrt{n} \quad \text{if } x, y \in \text{supp } \varphi_k.$$

Let

$$(2.6) \quad \psi_k(\xi) = \varphi_k \left(\frac{\xi}{|\xi|^{2/3}} \right), \quad \dot{\psi}_k(\xi) = \dot{\varphi}_k \left(\frac{\xi}{|\xi|^{2/3}} \right).$$

Then

$$(2.7) \quad \sum_k \psi_k(\xi)^2 = 1,$$

$$(2.8) \quad |\xi|^{3/4|\alpha|} \sum_k |D^\alpha \psi_k(\xi)|^2 \leq C_\alpha \quad \text{for } \forall \alpha.$$

$$(2.9) \quad |\xi - \eta| \leq C |\xi|^{2/3} \quad \text{if } \xi, \eta \in \text{supp } \psi_k.$$

$$(2.10) \quad \sum_k |\psi_k(\xi) - \psi_k(\eta)|^2 \leq \frac{C |\xi - \eta|^2}{(1 + |\xi|^{2/3})(1 + |\eta|^{2/3})} \quad \text{for } \forall \xi, \eta \in \mathbf{R}^n.$$

Functions $\dot{\varphi}_k$ and $\dot{\psi}_k$ are identically one in some neighbourhood of $\text{supp } \varphi_k$ and $\text{supp } \psi_k$ respectively. They also have properties (2.4)~(2.10) except (2.7). Note that $\delta_j^2 g_j$ belongs to $\text{supp } \psi_j$ if $\delta_j = |g_j|$. We define operator $\psi_j(D)$ by

$$(2.11) \quad \psi_j(D)u(x) = (2\pi)^{-n} \iint_{\mathbf{R}^n \times \mathbf{R}^n} e^{i(x-y) \cdot \xi} \psi_j(\xi) u(y) dy d\xi.$$

Obviously we have

$$(2.12) \quad \sum_j \varphi_j(D)^2 = Id,$$

and

$$(2.13) \quad C \|u\|_s^2 \leq \sum_{j=0}^\infty \delta_j^{6s} \|\varphi_j(D)u\|_0^2 \leq C^{-1} \|u\|_s^2,$$

where $\|u\|_s$ is Sobolev norm of u of order s in \mathbf{R}^n .

We set $\varphi_{jk}(x) = \varphi_j(\delta_k x)$ and $\phi_{jk}(x, \xi) = \varphi_{jk}(x) \psi_k(\xi)$. Note that for any multi-indices α, β , we have

$$(2.14) \quad |D_\alpha^\alpha D_\xi^\beta \phi_{jk}(x, \xi)| \leq C_{\alpha\beta} \delta_k^{|\alpha|} |\xi|^{-2/3|\beta|} \leq C_{\alpha\beta} |\xi|^{1/3|\alpha| - 2/3|\beta|}.$$

This means that ϕ_{jk} belongs to class $S_{2/3, 1/3}^0$ of Hörmander. It follows from (2.3) and (2.13) that

$$(2.15) \quad C \|u\|_s^2 \leq \sum_{jk} \delta_k^{6s} \|\phi_{jk}(x, D)u\|_0^2 \leq C^{-1} \|u\|_s^2,$$

and

$$(2.16) \quad \sum_{jk} \phi_{jk}(x, D)^* \phi_{jk}(x, D) = Id.$$

For any pair (j, k) of integers we set

$$(2.17) \quad P_{jk}(x, D) = p_0(x^{jk}, \xi^k) + \sum_{\nu=1}^n p_{0(\nu)}(x^{jk}, \xi^k)(x - x^{jk})_\nu + \sum_{\nu=1}^n p_\nu(x^{jk}, \xi^k)(D - \xi^k)_\nu,$$

where ξ^k is a point in $\text{supp } \psi_k$ and x^{jk} is a point in $\text{supp } \varphi_{jk}$. The following proposition is due to Hörmander.

Proposition 2.1. *For any $\forall u, v \in D(\mathbf{R}^n)$, we have*

$$(2.18) \quad |(p(x, D)u, v) - \sum_{jk} (p_{jk}(x, D)\phi_{jk}(x, D)u, \phi_{jk}(x, D)v)| \leq C \|u\|_{-1/3} \|v\|_{-1/3}.$$

Proof is found in [4].

3. Spectral decomposition of localized operators

We shall call $P_{jk}(x, D)$ localized operator. $P_{jk}(x, D)$ is an operator of order 1. The spectral decomposition of $P_{jk}(x, D)$ is well known. In fact, after multiplication of $e^{ix \cdot \xi^k}$ and suitable change of coordinates, $P_{jk}(x, D)$ is unitarily transformed to an operator of the form

$$L = \alpha D_1 + b \cdot x,$$

where α is a real constant and $b \cdot x$ is Euclidean scalar product of two vectors

$$b = (b_1, b_2, \dots, b_n) \quad \text{and} \quad x = (x_1, x_2, \dots, x_n).$$

Let

$$L = \int_{-\infty}^{\infty} \lambda dE(\lambda)$$

be spectral decomposition of L . Then the projection operator $E(\lambda)$ is the multiplication of function $Y(\lambda - b \cdot x)$ if $\alpha = 0$. Here $Y(t)$, $t \in \mathbf{R}$, stands for Heaviside function, that is,

$$Y(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0. \end{cases}$$

If $\alpha \neq 0$, we set

$$L' = e^{-i \frac{b_1 x_1^2}{2\alpha_1}} L e^{i \frac{b_1 x_1^2}{2\alpha_1}}.$$

L' is an operator of the form

$$L' = \alpha D_1 + b' \cdot x',$$

where $b' = (b_2, \dots, b_n)$ and $x' = (x_2, \dots, x_n)$.

Taking partial Fourier transform with respect to x_1 , we have reduced to the case that $\alpha = 0$.

We shall use the following notations:

$$(3.1) \quad P_{jk}(x, D) = \int_{-\infty}^{\infty} \lambda dE_{jk}(\lambda).$$

Here $E_{jk}(\lambda)$ is the spectral measure of P_{jk} .

$$\begin{aligned} \text{We put} \quad E_{jk}^- &= E_{jk}(0) & E_{jk}^+ &= I - E_{jk}^- \\ P_{jk}^+ &= P_{jk} E_{jk}^+ & P_{jk}^- &= -P_{jk} E_{jk}^-. \end{aligned}$$

4. Statement of Theorem I

We put

$$(4.1) \quad P^+ = \sum_{jk} \phi_{jk}(x, D)^* P_{jk}^+ \phi_{jk}(x, D),$$

$$(4.2) \quad P^- = \sum_{jk} \phi_{jk}(x, D)^* P_{jk}^- \phi_{jk}(x, D),$$

$$(4.3) \quad F^+ = \sum_{jk} \phi_{jk}(x, D)^* E_{jk}^+ \phi_{jk}(x, D),$$

$$(4.4) \quad F^- = \sum_{jk} \phi_{jk}(x, D)^* E_{jk}^- \phi_{jk}(x, D).$$

Then we have

Theorem I. *Operators P^+ , P^- , F^+ and F^- are self-adjoint and satisfy the following properties:*

$$(4.5) \quad (i) \quad I = F^+ + F^-.$$

$$(4.6) \quad (ii) \quad (P^\pm u, u) \geq 0.$$

$$(4.7) \quad (iii) \quad |(F^- P^+ F^+ u, v)| \leq C \|u\|_{-1/3} \|v\|_{-1/3}.$$

$$(4.8) \quad |(F^- P^+ F^- u, v)| \leq C \|u\|_{-1/3} \|v\|_{-1/3}.$$

$$(4.9) \quad |(F^- P^- F^+ u, v)| \leq C \|u\|_{-1/3} \|v\|_{-1/3}.$$

$$(4.10) \quad |(F^+ P^- F^+ u, v)| \leq C \|u\|_{-1/3} \|v\|_{-1/3}.$$

$$(4.11) \quad (iv) \quad |([P, F^\pm] u, v)| \leq C \|u\|_{-1/3} \|v\|_{-1/3}.$$

$$(4.12) \quad |([P^\pm, F^\pm] u, v)| \leq C \|u\|_{-1/3} \|v\|_{-1/3}.$$

(v) *If we set $R = P - (P^+ - P^-)$ then*

$$(4.13) \quad |(Ru, v)| \leq C \|u\|_{-1/3} \|v\|_{-1/3}.$$

Corollary 4.2. *We have*

$$(4.14) \quad |(PF^+ u, v) - (F^+ P^+ F^+ u, v)| \leq C \|u\|_{-1/3} \|v\|_{-1/3}$$

$$(4.15) \quad |(PF^- u, v) + (F^- P^- F^- u, v)| \leq C \|u\|_{-1/3} \|v\|_{-1/3}.$$

$$(4.16) \quad |(P^+ F^- u, v)| \leq C \|u\|_{-1/3} \|v\|_{-1/3},$$

$$(4.17) \quad |(P^- F^+ u, v)| \leq C \|u\|_{-1/3} \|v\|_{-1/3},$$

$$(4.18) \quad P = P^+ - P^- + R.$$

We shall prove Theorem I in §6.

5. Some lemmas about self-adjoint operators

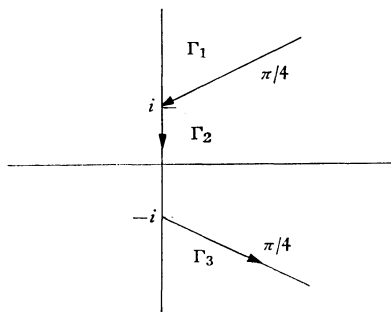
In this section X stands for an abstract Hilbert space.

Lemma 5.1. *Let A be a self-adjoint operator in X and A^+ be its positive part. Then*

$$A^+u = \frac{1}{2\pi i} \int_{\Gamma} \left(\lambda(\lambda - A)^{-1} - 1 - \frac{A}{\lambda + 1} \right) u d\lambda$$

provided $u \in D(A^2) = \text{domain of } A^2$. Γ is the complex contour as is shown in fig 1.

fig 1.



Proof. Note that

$$\lambda(\lambda - \sigma)^{-1} - 1 - \frac{\sigma}{\lambda + 1} = \frac{\sigma(\sigma + 1)}{(\lambda - \sigma)(\lambda + 1)}.$$

Integrate this with respect to λ on Γ then we have σ if $\sigma > 0$ and 0 if $\sigma < 0$. Therefore if we use spectral decomposition of A , then we can prove our Lemma.

Lemma 5.2. *Let A be a self-adjoint operator in X and let B be a bounded linear operator. We assume that operators AB and A^2B are densely defined. We further assume that the commutator $[A, B]$, $[A, [A, B]]$ are bounded.*

Then we have

$$(5.2) \quad \|[A^\pm, B]\| \leq C(\|B\| + \|[A, B]\| + \|[A, [A, B]]\|).$$

Proof. Let $u \in D(A^2) \cap D(A^2B) \cap D(AB)$,

$$2\pi i [A^+, B]u = \int_{\Gamma} \left[\left(\lambda(\lambda - A)^{-1} - 1 - \frac{A}{\lambda + 1} \right), B \right] u d\lambda.$$

We split Γ into three parts $\Gamma_1 + \Gamma_2 + \Gamma_3$. (see fig. 1). Corresponding integrals are denoted by A_1, A_2 and A_3 . Obviously $[A^+, B] = [A_1, B] + [A_2, B] + [A_3, B]$.

Since
$$\int_{\Gamma_2} \frac{1}{\lambda + 1} d\lambda = \log(1 - i) - \log(1 + i),$$

we have

$$(5.3) \quad \|[A_2, B]\| \leq 4(\|B\| + \|[A, B]\|).$$

Let us treat

$$\begin{aligned}
 2\pi i[A_2 + A_3, B] &= \int_{\Gamma_1 + \Gamma_2} \left[\left[\lambda(\lambda - A)^{-1} - 1 - \frac{A}{\lambda + 1} \right], B \right] d\lambda \\
 &= -[A, B] \int_{\Gamma_1 + \Gamma_3} \frac{d\lambda}{\lambda + 1} + \int_{\Gamma_1 + \Gamma_3} \lambda(\lambda - A)^{-1} [A, B] (\lambda - A)^{-1} d\lambda.
 \end{aligned}$$

We know

$$\left| \int_{\Gamma_1 + \Gamma_3} \frac{d\lambda}{\lambda + 1} \right| \leq \text{const.}$$

On the other hand we have

$$\begin{aligned}
 &\int_{\Gamma_1 + \Gamma_3} \lambda(\lambda - A)^{-1} [A, B] (\lambda - A)^{-1} d\lambda \\
 &= \int_{\Gamma_1 + \Gamma_3} \lambda(\lambda - A)^{-2} [A, B] d\lambda + \int_{\Gamma_1 + \Gamma_3} \lambda(\lambda - A)^{-2} [A, [A, B]] (\lambda - A)^{-1} d\lambda.
 \end{aligned}$$

The last term is majorized by $C \|[A, [A, B]]\|$.

The first is

$$\begin{aligned}
 \int_{\Gamma_1 + \Gamma_3} \lambda(\lambda - A)^{-2} d\lambda &= \int_{\Gamma_1 + \Gamma_3} d\lambda \int_{-\infty}^{\infty} \lambda(\lambda - \sigma)^{-2} dE(\sigma) \\
 &= \int_{\Gamma_1 + \Gamma_3} d\lambda \int_{-\infty}^{\infty} \left(\frac{1}{\lambda - \sigma} + \frac{\sigma}{(\lambda - \sigma)^2} \right) dE(\sigma).
 \end{aligned}$$

Since $\left| \int_{\Gamma_1 + \Gamma_3} \frac{d\lambda}{\lambda - \sigma} \right| \leq \text{Const.}$ and $\left| \int \frac{\sigma}{(\lambda - \sigma)^2} d\lambda \right| \leq 2 \left| \frac{\sigma}{i + \sigma} \right| \leq 2,$

we have

$$\left\| \int_{\Gamma_1 + \Gamma_3} \lambda(\lambda - A)^{-2} [A, B] d\lambda \right\| \leq C \|[A, B]\|.$$

We have thus proved our lemma.

Lemma 5.3. *Let A and B be two self-adjoint operators in X . If the commutator $[A, B]$ is bounded, then for any*

$$x \in D(A^2) \cap D(B^2)$$

we have

$$\|(A^+ - B^+)x\| \leq C(\|[A, B]\| \|x\| + \sum_{i=1}^2 \|(A - B)^i x\| + \|x\| + \|[A, B]\| \|(A - B)x\|).$$

Proof. We have to majorize

$$(2\pi i)(A^+x - B^+x) = \int_{\Gamma} \left(\lambda(\lambda - A)^{-1} - \lambda(\lambda - B)^{-1} - \frac{A}{\lambda + 1} + \frac{B}{\lambda + 1} \right) x d\lambda.$$

We decompose Γ as we did in the proof of Lemma 5.2. The integral over Γ_2 is majorized by $C(\|x\| + \|(A-B)x\|)$.

Note that

$$\begin{aligned} & \lambda(\lambda-A)^{-1} - \lambda(\lambda-B)^{-1}x \\ &= -\lambda(\lambda-B)^{-1}(A-B)(\lambda-A)^{-1}x \\ &= -\lambda(\lambda-B)^{-1}(\lambda-A)^{-1}(A-B)x \\ &\quad -\lambda(\lambda-B)^{-1}(\lambda-A)^{-1}[A, B](\lambda-A)^{-1}x \\ &= -\lambda(\lambda-B)^{-2}\{1+(A-B)(\lambda-A)^{-1}\}(A-B)x \\ &\quad +\lambda(\lambda-B)^{-1}(\lambda-A)^{-1}[A, B](\lambda-A)^{-1}x \\ &= -\lambda(\lambda-B)^{-2}\{1+(\lambda-A)^{-1}(A-B) \\ &\quad +(\lambda-A)^{-1}[A, B](\lambda-A)^{-1}\}(A-B)x \\ &\quad +\lambda(\lambda-B)^{-1}(\lambda-A)^{-1}[A, B](\lambda-A)^{-1}x. \end{aligned}$$

From this we can majorize the integral over $\Gamma_1 + \Gamma_3$ by

$$C(\|(A-B)x\| + \|(A-B)^2x\| + \|[A, B]\| \|x\| + \|[A, B]\| \|(A-B)x\|).$$

We have thus proved our lemma.

6. Proof of Theorem

We start with the propositions which simplify discussions later.

Proposition 6.1. *Let $u \in C_0^\infty(\mathbf{R}^n)$ be arbitrary and (j, k) be a pair of indices. Then there is a point \bar{x} satisfying*

$$(6.1) \quad |x - x^{jk}| \leq \alpha \delta_k,$$

$$(6.2) \quad \int (x_\nu - \bar{x}_\nu) |\phi_{jk}(x, D)u(x)|^2 dx = 0$$

for $\nu=1, 2, 3, \dots, n$. Here α is a positive constant independent of u and (j, k) .

Proof is found in [3], page 171.

The point \bar{x} can be chosen in $\text{supp } \phi_{jk}$.

Proposition 6.2. *There exists a bounded sequence $\{\phi'_{jk}(x, \xi)\}_{jk}$ of symbols in $S_{2/3, 1/3}^0$ such that we have*

$$(6.3) \quad (i) \quad \|(D_\nu - \xi_\nu^k) \phi_{jk}(x, D)u\|^2 \leq C \delta_k^4 \|\phi'_{jk}(x, D)u\|^2$$

and

$$(6.4) \quad (ii) \quad \text{supp } \phi'_{jk} \subset \text{supp } \phi_{jk}$$

for $\nu=1, 2, 3, \dots, n$.

Proof. We have

$$(6.5) \quad (D_\nu - \xi_\nu^k) \phi_{jk}(x, D)u = \delta_k^2 \phi'_{jk}(x, D)u,$$

where

$$(6.6) \quad \phi'_{jk}(x, \xi) = \delta_k^{-2} \left(-i \frac{\partial}{\partial x_\nu} \varphi_{jk}(x) \psi_k(\xi) + \varphi_{jk}(x) \psi_k(\xi) (\xi_\nu - \xi_\nu^k) \right).$$

The sequence $\{\phi'_{jk}(x, \xi)\}_{j,k}$ is bounded in $S_{2/3, 1/3}^0$ because of (2.9) and

$$-i \frac{\partial}{\partial x_\nu} \varphi_{jk}(x) = -i \delta_k \left(\frac{\partial}{\partial x_\nu} \varphi_j \right) (\delta_k x).$$

Proposition 6.3. *Let $\{(\hat{x}^{jk}, \xi^{jk})\}_{j,k}$ be another sequence of points. Let \hat{P}_{jk}^\pm and \hat{F}^\pm be operators defined by (2.17), (4.1), (4.2), (4.3) and (4.4) where (x^{jk}, ξ^k) is replaced by (\hat{x}^{jk}, ξ^{jk}) . If there exists a constant $C > 0$ satisfying*

$$(6.7) \quad |x^{jk} - \hat{x}^{jk}| \leq C \delta_k^{-1} \quad \text{and} \quad |\xi^k - \xi^{jk}| \leq C \delta_k^2,$$

then we have

$$(6.8) \quad \|(\hat{P}_{jk} - P_{jk}) \phi_{jk}(x, D)u\|^2 \leq C \delta_k^{-4} \|\phi_{jk}^{(1)}(x, D)u\|^2,$$

$$(6.9) \quad \|(\hat{P}_{jk} - P_{jk})^2 \phi_{jk}(x, D)u\|^2 \leq C \delta_k^{-8} \|\phi_{jk}^{(2)}(x, D)u\|^2,$$

$$(6.10) \quad \|[P_{jk}, \hat{P}_{jk}]\| \leq C \delta_k^{-4},$$

$$(6.11) \quad \|(\hat{P}_{jk}^\pm - P_{jk}^\pm) \phi_{jk}(x, D)u\| \leq C \delta_k^{-2} \|\phi_{jk}^{(3)}(x, D)u\|,$$

$$(6.12) \quad |((\hat{P}^\pm - P^\pm)u, u)| \leq C \|u\|_{-1/3} \|v\|_{-1/3}.$$

Here, $\{\phi_{jk}^{(l)}\}_{j,k}$, $l=1, 2, 3$, are bounded sequences of symbols in $S_{2/3, 1/3}^0$ with the property that $\text{supp } \phi_{jk}^{(l)} \subset \text{supp } \phi_{jk}$.

REMARK 6.4. We require that the point (x^{jk}, ξ^k) lies in $\text{supp } \phi_{jk}$ but we don't require that (\hat{x}^{jk}, ξ^{jk}) lies in $\text{supp } \phi_{jk}$.

Proof. It follows from Taylor's formula that

$$(6.13) \quad P_0(\hat{x}^{jk}, \xi^{jk}) = P_0(x^{jk}, \xi^k) + \sum_\nu (\hat{x}^{jk} - x^{jk}) P_{0(\nu)}(x^{jk}, \xi^k) + \sum_\nu (\xi^{jk} - \xi^k)_\nu P_0^{(\nu)}(x^{jk}, \xi^k) + R_1,$$

$$(6.14) \quad P_{0(\nu)}(\hat{x}^{jk}, \xi^{jk}) = P_{0(\nu)}(x^{jk}, \xi^k) + R_{2(\nu)} \quad \text{and}$$

$$(6.15) \quad P_0^{(\nu)}(\hat{x}^{jk}, \xi^{jk}) = P_0^{(\nu)}(x^{jk}, \xi^k) + R_3^{(\nu)}.$$

By (6.7) the remainder terms are majorized as

$$(6.16) \quad |R_1| \leq C \delta_k^{-2}, \quad |R_{2(\nu)}| \leq C \delta_k^{-1}, \quad |R_3^{(\nu)}| \leq C \delta_k^{-4}.$$

We have

$$(6.17) \quad \begin{aligned} & \dot{P}_{jk}(x, D) - P_{jk}(x, D) \\ &= R_1 + \sum_{\nu} (x - x^{jk})_{\nu} R_{2(\nu)} + \sum_{\nu} (D - \xi^{jk})_{\nu} R. \end{aligned}$$

This implies that

$$\begin{aligned} & \|(\dot{P}_{jk}(x, D) - P_{jk}(x, D)) \phi_{jk}(x, D) u\|^2 \\ & \leq C(\delta_k^{-4} \|\phi_{jk}(x, D) u\|^2 + \delta_k^{-2} \sum_{\nu} \|(x - x^{jk})_{\nu} \phi_{jk}(x, D) u\|^2 \\ & \quad + \delta_k^{-8} \sum_{\nu} \|(D - \xi^{jk})_{\nu} \phi_{jk}(x, D) u\|^2) \\ & \leq C\delta_k^{-4} \|\phi_{jk}^{(1)}(x, D) u\|^2. \end{aligned}$$

This is (6.8).

Similarly

$$(6.18) \quad \begin{aligned} & \|(\dot{P}_{jk}(x, D) - P_{jk}(x, D))^2 \phi_{jk}(x, D) u\|^2 \\ & \leq C\delta_k^{-8} \|\phi_{jk}^{(2)}(x, D) u\|^2. \end{aligned}$$

Now

$$(6.19) \quad \begin{aligned} [P_{jk}, \dot{P}_{jk}] &= [P_{jk}, \dot{P}_{jk} - P_{jk}] \\ &= -[i \sum_{\nu} R_{2(\nu)} P_0^{(\nu)}(x^{jk}, \xi^k) - \sum_{\nu} R_3^{(\nu)} P_{0(\nu)}(x^{jk}, \xi^k)]. \end{aligned}$$

This proves (6.10).

We apply Lemma 5.3 to operators $A = \delta_k^2 P_{jk}$, and $B = \delta_k^2 \dot{P}_{jk}$. Then we have

$$(6.20) \quad \|(A^+ - B^+) \phi_{jk}(x, D) u\| \leq C \|\phi_{jk}^{(3)}(x, D) u\|.$$

This proves that

$$(6.21) \quad \|(P_{jk}^+ - \dot{P}_{jk}^+) \phi_{jk}(x, D) u\| \leq C\delta_k^{-2} \|\phi_{jk}^{(3)}(x, D) u\|.$$

Let v be in $C_0^\infty(\mathbf{R}^n)$. Then

$$\begin{aligned} |((P^+ - \dot{P}^+) u, v)| &\leq \sum_{jk} |((P_{jk}^+ - \dot{P}_{jk}^+) \phi_{jk}(x, D) u, \phi_{jk}(x, D) v)| \\ &\leq C \sum_{jk} \delta_k^{-2} \|\phi_{jk}^{(3)}(x, D) u\| \|\phi_{jk}(x, D) v\|. \end{aligned}$$

Take arbitrary positive $t > 0$. Then

$$\begin{aligned} |((P^+ - \dot{P}^+) u, v)| &\leq C \sum_{jk} \frac{t^2}{2} \delta_k^{-2} \|\phi_{jk}^{(3)}(x, D) u\|^2 + \frac{t^{-2}}{2} \delta_k^{-2} \|\phi_{jk}(x, D) v\|^2 \\ &\leq C \left(\frac{t^2}{2} \|u\|_{-1/3}^2 + \frac{t^{-2}}{2} \|v\|_{-1/3}^2 \right). \end{aligned}$$

Taking the minimum of this with respect to t , we have

$$|((P^+ - \dot{P}^+)u, v)| \leq C \|u\|_{-1/3} \|v\|_{-1/3}.$$

Next we need bounds for commutators

$$[P_{jk}^\pm, \phi_{lm}(x, D)], [E_{jk}^\pm, \phi_{lm}(x, D)] \text{ etc.}$$

These are needed only when $\text{supp } \phi_{jk} \cap \text{supp } \phi_{lm} \neq \emptyset$.

We introduce notation

$$I(j, k) = \{(l, m) | \text{supp } \phi_{jk} \cap \text{supp } \phi_{lm} \neq \emptyset\}.$$

It is obvious that there is a constant $C > 0$ such that

$$C^{-1} \leq \frac{\delta_m}{\delta_k} \leq C \quad \text{if } (l, m) \in I(j, k).$$

The number of indices (l, m) in $I(j, k)$ is bounded.

Proposition 6.5. *We have the following estimates for commutators: If $(l, m) \in I(j, k)$, then*

$$(6.22) \quad \|[P_{jk}, \phi_{lm}]\| \leq C\delta_k^{-2},$$

$$(6.23) \quad \|[[P_{jk}, \phi_{lm}], P_{jk}]\| \leq C\delta_k^{-4},$$

$$(6.24) \quad \|[[[P_{jk}, \phi_{lm}], \phi_{lm}]\| \leq C\delta_k^{-2},$$

$$(6.25) \quad \|[[[P_{jk}, \phi_{lm}^*], \phi_{lm}]\| \leq C\delta_k^{-2},$$

$$(6.26) \quad \|[P_{jk}^\pm, \phi_{lm}]\| \leq C\delta_k^{-2},$$

$$(6.27) \quad \|[[P_{jk}, [E_{jk}^\pm, \phi_{lm}]]\| \leq C\delta_k^{-2}.$$

Proof. $[P_{jk}, \phi_{lm}] = [P_{jk}, \varphi_{lm}(x)\psi_m(D)]$
 $= [P_{jk}, \varphi_{lm}]\varphi_m(D) + \varphi_{lm}[P_{jk}, \psi_m(D)]$
 $= \delta_k \sum_{\nu} P_0^{(\nu)}(x^{jk}, \xi^k) D_\nu \varphi_{lm}(x) \psi_m(D)$
 $- \varphi_{lm} \delta_k^{-2} \sum_{\nu} D_\nu \psi_m(D) P_{0(\nu)}(x^{jk}, \xi^k).$

This proves that

$$\|[[P_{jk}, \phi_{lm}]\| \leq C\delta_k^{-2}.$$

More precisely, $\{\delta_k^2 [P_{jk}, \phi_{lm}]\}_{jk}$ is bounded sequence of operators in $L_{2/3, 1/3}^0$ of Hörmander. By just the same argument we can prove (6.23), (6.24) and (6.25) are consequences of the fact that

$$\{\delta_k [P_{jk}, \phi_{lm}]\}_{jk}$$

is a bounded set in $L_{2/3, 1/3}^0$.

We set $A = \delta_k^2 P_{jk}$, $B = \delta_k^{-2} \phi_{lm}$ and apply Lemma 5.2.

Then we have

$$\| [P_{jk}^\pm, \phi_{lm}] \| \leq C\delta_k^{-2}.$$

Since

$$(6.29) \quad P_{jk}[E_{jk}^\pm, \phi_{lm}] = [P_{jk}^\pm, \phi_{lm}] - E_{jk}^\pm[P_{jk}, \phi_{lm}],$$

(6.27) is a consequence of (6.26).

Now we are ready for proving our Theorem I.

Proof of (iii). Let (j, k) and (j', k') be two pairs of indices. Then we put

$$I(jk, j'k') = \{(l, m) \mid \text{supp } \phi_{lm} \cap \text{supp } \phi_{jk} \neq \emptyset, \text{supp } \phi_{lm} \cap \text{supp } \phi_{j'k'} \neq \emptyset\}.$$

By definition of P^+ , F^+ and F^- , we have

$$(6.30) \quad (F^-P^-F^+u, v) = \sum_{jk} \sum_{j'k'} (P^- \phi_{jk}^* E_{jk}^+ \phi_{jk} u, \phi_{j'k'}^* E_{j'k'}^- \phi_{j'k'} v).$$

If $\text{supp } \phi_{lm} \cap \text{supp } \phi_{jk} = \emptyset$ and $\text{supp } \phi_{lm} \cap \text{supp } \phi_{j'k'} = \emptyset$, then

$$(6.31) \quad \|\phi_{lm}(x, D)\phi_{jk}(x, D)^*v\| \leq C\delta_k^{-N}\|v\|$$

for any $N > 0$. If $\text{supp } \phi_{lm} \cap \text{supp } \phi_{j'k'} = \emptyset$, then $\phi_{lm}(x, D)\phi_{j'k'}(x, D)^*u = 0$.

Thus we have

$$(6.32) \quad \begin{aligned} & \sum_{(l,m) \in I(j,k)} \|\phi_{lm}^* P_{lm}^- \phi_{lm} \phi_{jk}^* E_{jk}^+ \phi_{jk} u\| \\ & \leq C |\Omega| \delta_k^n \delta_k^{-N} \|E_{jk}^+ \phi_{jk} u\| \\ & \leq C |\Omega| \delta_k^{n-N} \|\phi_{jk} u\|, \end{aligned}$$

where $|\Omega|$ is the volume of the domain Ω .

Similarly

$$(6.33) \quad \sum_{(l,m) \in I(j',k')} \|\phi_{lm}^* P_{lm}^- \phi_{lm} \phi_{j'k'}^* E_{j'k'}^+ \phi_{j'k'} v\|^2 \leq C |\Omega| \delta_k^{n-N} \|\phi_{j'k'} v\|^2.$$

(6.32) and (6.33) imply that

$$(6.34) \quad \begin{aligned} & (F^-P^-F^+u, v) - \sum_{jk} \sum_{j'k'} \sum_{(l,m) \in I(jk, j'k')} (\phi_{lm}^* P_{lm}^- \phi_{lm} \phi_{jk}^* E_{jk}^+ \phi_{jk} u \phi_{j'k'}^* E_{j'k'}^- \phi_{j'k'} v) \\ & \leq C |\Omega| \left(\sum_{jk, j'k'} \delta_k^{n-N} \|\phi_{jk} u\| \|\phi_{j'k'} v\| \right) \\ & \leq C |\Omega| \|u\|_{-1/3} \|v\|_{-1/3}. \end{aligned}$$

We have

$$\begin{aligned}
 (6.35) \quad & \sum_{(l,m) \in I(jk, j'k')} \phi_{lm}^* P_{lm}^- \phi_{lm} \phi_{jk}^* E_{jk}^+ \phi_{jk} u \\
 &= \sum_{(l,m) \in I(jk, j'k')} \phi_{lm}^* P_{jk}^- \phi_{lm} \phi_{jk}^* E_{jk}^+ \phi_{jk} u \\
 &+ \sum_{(l,m) \in I(jk, j'k')} \phi_{lm}^* (P_{lm}^- - P_{jk}^-) \phi_{lm} \phi_{jk}^* E_{jk}^+ \phi_{jk} u.
 \end{aligned}$$

We apply Proposition 6.3 and have

$$(6.36) \quad \left\| \sum_{(l,m)} \phi_{lm}^* (P_{lm}^- - P_{jk}^-) \phi_{lm} \phi_{jk}^* E_{jk}^+ \phi_{jk} u \right\| \leq C \delta_k^{-2} \|\phi_{jk} u\|.$$

On the other hand,

$$\begin{aligned}
 (6.37) \quad & \sum_{(l,m) \in I(jk, j'k')} \phi_{lm}^* P_{jk}^- \phi_{lm} \phi_{jk}^* E_{jk}^+ \phi_{jk} u \\
 &= \sum_{lm} \{ \phi_{lm}^* [P_{jk}^- \phi_{lm}] \phi_{jk}^* E_{jk}^+ \phi_{jk} u + \phi_{lm}^* \phi_{lm} [P_{jk}^-, \phi_{jk}^*] E_{jk}^+ \phi_{jk} u \}.
 \end{aligned}$$

By proposition 6.5, we have

$$(6.38) \quad \|\phi_{lm}^* [P_{jk}^-, \phi_{lm}^*] \phi_{jk}^* E_{jk}^+ \phi_{jk} u\| \leq C \delta_k^{-2} \|\phi_{jk} u\|$$

and

$$(6.39) \quad \|\phi_{lm}^* \phi_{lm} [P_{jk}^-, \phi_{jk}^*] E_{jk}^+ \phi_{jk} u\| \leq C \delta_k^{-2} \|\phi_{jk} u\|.$$

(6.37), (6.38) and (6.39) imply that

$$(6.40) \quad \left\| \sum_{(l,m) \in I(jk, j'k')} \phi_{lm}^* P_{jk}^- \phi_{lm} \phi_{jk}^* E_{jk}^+ \phi_{jk} u \right\| \leq C \delta_k^{-2} \|\phi_{jk} u\|.$$

As a consequence of (6.34) and (6.40), we have

$$\begin{aligned}
 (6.41) \quad |(F^- P^- F^+ u, v)| &\leq \sum_{(jk) \subset (j'k')} C \delta_k^{-2} \|\phi_{jk} u\| \|\phi_{j'k'} v\| \\
 &\leq C \|u\|_{-1/3} \|v\|_{-1/3},
 \end{aligned}$$

where the summation ranges over those (jk) and (j', k') that $I(jk, j'k') \neq \emptyset$. This proved (iii). Proof of remaining part of Theorem I is the same.

7. The role of characteristics

So far the choice of sequence $\{(x^{jk}, \xi^k)\}$ is not specified. In the following we shall make use of special choice of it in order to simplify operators P_{jk}^\pm and E_{jk}^\pm .

The set

$$(7.1) \quad \Sigma^0(P) = \{(x, \xi) \in \mathbf{R}^{2n} \mid \xi \neq 0, P_0(x, \xi) = 0\}$$

is called the characteristics of the operator P . We also use the following notations;

$$(7.2) \quad \Sigma^+(P) = \{(x, \xi) \in \mathbf{R}^{2n} \mid \xi \neq 0, P_0(x, \xi) > 0\},$$

$$(7.3) \quad \Sigma^-(P) = \{(x, \xi) \in \mathbf{R}^{2n} \mid \xi \neq 0, P_0(x, \xi) < 0\}.$$

Proposition 7.1. *Assume that $(x^{jk}, \xi^k) \in \Sigma^+(P) \cup \Sigma^0(P)$ and that $P(x, \xi) \geq 0$ for any $x \in \text{supp } \phi_{jk}$ and ξ with $|\xi - \xi^k| < \alpha \delta_k^2$, where α is the constant appeared in Proposition 6.1. Then we can replace E_{jk}^+ by the identity operator without altering results in Theorem I.*

Proof of Proposition 7.1.

We put $L_k = \{j \mid (x^{jk}, \xi^k) \text{ satisfies the assumption of Proposition 7.1}\}$

$$(7.4) \quad Q_k = \sum_{j \in L_k} \phi_{jk}^*(x, D) P_{jk}^- \phi_{jk}(x, D)$$

and

$$(7.5) \quad G_k = \sum_{j \in L_k} \phi_{jk}^*(x, D) E_{jk}^- \phi_{jk}(x, D).$$

We claim that there exists a constant $C > 0$ such that

$$(7.6) \quad \|Q_k u\| \leq C \delta_k^{-2} \|\psi_k(D)u\|.$$

We admit this for a moment. Replacing E_j^+ ($j \in L_k, k=0, 1, 2, \dots$) in (4.1)~(4.4) with the identity, we obtain operators Q^\pm and G^\pm .

Differences between old and new operators are

$$(7.7) \quad Q^\pm - P^\pm = \sum_k Q_k,$$

$$(7.8) \quad G^\pm - F^\pm = \sum_k G_k.$$

These relations imply that

$$(7.9) \quad \begin{aligned} (G^- Q^+ G^+ u, v) &= \sum_k (G^- Q_k G^+ u, v) - \sum_k (G_k P^+ F^+ u, v) \\ &\quad + \sum_k (F^- P^+ G_k u, v) - \sum_{k,l} (G_k P^+ G_l u, v) \\ &\quad + (F^- P^+ F^+ u, v). \end{aligned}$$

We know by Theorem I that

$$(7.10) \quad |(F^- P^+ F^+ u, v)| \leq C \|u\|_{-1/3} \|v\|_{-1/3}.$$

On the other hand we can use (7.6) and prove the following inequalities in the same way as the proof of (6.34):

$$(7.11) \quad \begin{aligned} \sum_k |(G_k P^+ F^+ u, v)| &\leq C \|u\|_{-1/3} \|v\|_{-1/3}, \\ \sum_k |(F^- P^+ G_k u, v)| &\leq C \|u\|_{-1/3} \|v\|_{-1/3}, \\ \sum_{k,l} |(G_k P^+ G_l u, v)| &\leq C \|u\|_{-1/3} \|v\|_{-1/3}, \\ \sum_k |(G^- Q_k G^+ u, v)| &\leq C \|u\|_{-1/3} \|v\|_{-1/3}. \end{aligned}$$

These prove

$$(7.12) \quad |(G^- Q^+ G^+ u, v)| \leq C \|u\|_{-1/3} \|v\|_{-1/3}$$

which corresponds to (4.7). Other inequalities can be proved in the same manner.

Now we must prove our claim (7.6). We choose \bar{x} as in Proposition 6.1.

Let

$$(7.13) \quad \begin{aligned} Q_{jk}(x, D) &= p_0(\bar{x}, \xi^k) + \sum_{\nu=1}^n p_{0(\nu)}(\bar{x}, \xi^k)(x - \bar{x})_\nu \\ &\quad + \sum_{\nu=1}^n p_0^{(\nu)}(\bar{x}, \xi^k)(D - \xi^k)_\nu. \end{aligned}$$

Then

$$(7.14) \quad \begin{aligned} &(Q_{jk}(x, D)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u) \\ &= ((p_0(\bar{x}, \xi^k) + \sum_{\nu} p_0^{(\nu)}(\bar{x}, \xi^k)(D - \xi^k)_\nu)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u) \\ &= p_0(\bar{x}, \xi^k)((1 - \dot{\psi}_k(D))\phi_{jk}(x, D)u, \phi_{jk}(x, D)u) \\ &\quad + ((p_0(\bar{x}, \xi^k) + \sum_{\nu} p_0^{(\nu)}(\bar{x}, \xi^k)(D - \xi^k)_\nu)\dot{\psi}_k(D)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u) \\ &\quad + \sum_{\nu} p_0^{(\nu)}(\bar{x}, \xi^k)(D - \xi^k)_\nu(1 - \dot{\psi}_k(D))\phi_{jk}(x, D)u, \phi_{jk}(x, D)u) \end{aligned}$$

because of (6.2).

Since α is large, we may assume that $p_0(\bar{x}, \xi) \geq 0$ if $\xi \in \text{Supp } \dot{\psi}_{k_0}$. Taylor's expansion of $p_0(\bar{x}, \xi)$ at $\xi = \xi^k$ imply that there exists a constant $C > 0$ such that

$$\begin{aligned} &((p_0(\bar{x}, \xi^k) + \sum_{\nu} p_0^{(\nu)}(\bar{x}, \xi^k)(D - \xi^k)_\nu)\dot{\psi}_k(D)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u) \\ &\geq -C\delta_k^{-2} \|\phi_{jk}(x, D)u\|^2. \end{aligned}$$

We know that

$$(D - \xi^k)_\nu(1 - \dot{\psi}_k(D))\phi_{jk}(x, D)u = (D - \xi^k)_\nu(1 - \dot{\psi}_k(D))\varphi_{jk}(x)\psi_k(D)\dot{\psi}_k(D)u$$

and that the sequence of double symbols

$\{(\xi - \xi^k)(1 - \dot{\psi}_k(\xi))\varphi_{jk}(x)\psi_k(\eta)\}_{j,k}$ is bounded in $S^{-\infty}$. Therefore we have estimate for any $N > 0$,

$$\|(D - \xi^k)(1 - \dot{\psi}_k(D))\phi_{jk}(x, D)u\|^2 \leq C\delta_k^{-N} \|\psi_k(D)u\|^2.$$

This implies that

$$(Q_{jk}(x, D)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u) + C\delta_k^{-2}\|\phi_{jk}(x, D)u\|^2 + C\delta_k^{-N}\|\psi_{jk}(D)u\|^2 \geq 0.$$

This and Proposition 6.3 prove that

$$(P_{jk}(x, D)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u) + C(\delta_k^{-2}\|\phi'_{jk}(x, D)u\|^2 + \delta_k^{-N}\|\psi_{jk}(D)u\|^2) \geq 0,$$

where $\{\phi'_{jk}(x, \xi)\}$ is a bounded sequence in $S_{2/3, 1/3}^0$ as of Proposition 6.3. Taking sum of these with respect to $j \in L_k$, we have

$$\sum_{j \in L_k} (P_{jk}(x, D)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u) + C\delta_k^{-2}\|\psi_{jk}(D)u\|^2 \geq 0.$$

Our claim is an immediate consequence of this inequality.

REMARK. Result similar to Proposition 7.1 holds for E_{jk}^- .

Next we discuss the case that $P_0(x, \xi)$ changes sign in the neighbourhood of $\text{supp } \phi_{jk}$. In this case we compare $P_{jk}(x, D)$ with the operator $\bar{P}_{jk}(x, D)$ which is determined at a characteristic point.

Proposition 7.2. *Assume that $P_0(x, \xi)$ changes sign at some point $(\dot{x}, \dot{\xi})$ with*

$$(7.6) \quad |X^{jk} - \dot{x}| < \alpha\delta_k^{-1}, \quad |\xi^{jk} - \dot{\xi}| < \alpha\delta_k^2.$$

Then we can replace $P_{jk}(x, D)$ by

$$(7.7) \quad \bar{P}_{jk}(x, D) = \sum_{\nu} P_{0(\nu)}(\dot{x}, \dot{\xi})(x - \dot{x})_{\nu} + \sum_{\nu} P_0^{(\nu)}(\dot{x}, \dot{\xi})(D - \dot{\xi})_{\nu}$$

without altering results in Theorem I.

Proof. This proposition is contained in Proposition 6.3.

Finally we discuss the case where the operator E_{jk}^{\pm} can be arbitrarily chosen.

Proposition 7.3. *Assume that we have*

$$P_0(\dot{x}, \dot{\xi}) = 0 \quad \text{grad}_{x, \xi} P_0(\dot{x}, \dot{\xi}) = 0$$

at some point $(\dot{x}, \dot{\xi})$ with $|\dot{x} - x^{jk}| < \alpha\delta_k^{-1}$, $|\dot{\xi} - \xi^{jk}| < \alpha\delta_k^2$. Then we can replace $P_{jk}(x, D)$ by zero operator 0 without altering Theorem I.

Proof. This is because of Proposition 6.3.

REMARK 7.4. In this case, the operator E_{jk}^{\pm} does not matter. We can put $E_{jk}^+ = Id$ or 0 at our disposal. From Proposition 7.1, 7.2 and 7.3, we can see F^+ and F^- depend only on location of sets $\Sigma^+(P)$, $\Sigma^-(P)$ and $\Sigma^0(P)$. An interesting consequence comes out when one compare two pseudo-differential operators whose characteristics are the same. Let Q be another self-adjoint pseudo-differential operator of class $L_{1,0}^0$. We assume Q has homogeneous

principal symbol $q_0(x, \xi)$ and $Q - q_0(x, D) \in L_{1,0}^{-1}$. Just as we did for the operator $P(x, D)$ we can consider operators Q^+, Q^-, F_q^+, F_q^- and sets $\Sigma^0(Q), \Sigma^+(Q), \Sigma^-(Q)$.

Theorem II. If $\Sigma^+(Q) \cup \Sigma^0(Q) \supset \Sigma^+(P) \cup \Sigma^0(P)$ and $\Sigma^-(Q) \cup \Sigma^0(Q) \supset \Sigma^-(P) \cup \Sigma^0(P)$, then we can take $F^+ = F_q^+$ and $F^- = F_q^-$.

Proof. If Proposition 7.1 applies to (x^{jk}, ξ^k) and operator P , then the same applies to the operator Q . If Proposition 7.2 applies to (x^{jk}, ξ^k) and P , then we have $(\hat{x}, \xi) \in \Sigma^0(P) \subset \Sigma^0(Q)$. If Proposition 7.2 does not apply to (x^{jk}, ξ^k) and Q , then (\hat{x}, ξ) satisfies $q_0(\hat{x}, \xi) = 0, \text{grad}_{x,\xi} q_0(\hat{x}, \xi) = 0$. Proposition 7.3 can be applied to this case and we come to the conclusion that we may take $Q_{jk} = 0$ and the operator E_{jk}^\pm does not matter so far as Q is concerned.

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The original manuscript of the author fallaciously asserted that operators $\phi'_{jk}(x, D), \phi^{(l)}_{jk}(x, D), l=1, 2, 3$, in Propositions 6.2 and 6.3 could be replaced by $\phi_{jk}(x, D)$ itself. This error was pointed out by the editors. The author expresses his hearty thanks to the editors.

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