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## CONSTRUCTION OF AFFINE PLANE CURVES WITH ONE PLACE AT INFINITY

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### 1. Introduction

Let  $C$  be an irreducible algebraic curve in complex affine plane  $\mathbf{C}^2$ . We say that  $C$  has *one place at infinity*, if the closure of  $C$  intersects with the  $\infty$ -line in  $\mathbf{P}^2$  at only one point  $P$  and  $C$  is locally irreducible at that point  $P$ .

The problem of finding the canonical models of curves with one place at infinity under the polynomial transformations of the coordinates of  $\mathbf{C}^2$  has been studied by many mathematicians since Suzuki [17] and Abhyankar-Moh [2] proved independently that the canonical model of  $C$  is a line when  $C$  is non-singular and simply connected. Zaidenberg-Lin [19] proved that  $C$  has the canonical model of type  $y^q = x^p$ , where  $p$  and  $q$  are coprime integers  $> 1$ , when  $C$  is singular and simply connected. A'Campo-Oka [5] studied the case of genus  $g \leq 3$  as an application of a resolution tower of toric modifications. For the case  $g \leq 4$  Neumann [12] studied from the viewpoint of the link at infinity, and Miyanishi [9] studied from the algebrico-geometric viewpoint. Nakazawa-Oka [11] gave the classifications of all the canonical models for the case  $g \leq 7$  using the result of A'Campo-Oka, and gave the classifications for the case  $g \leq 16$  without proof. Jaworski [8] studied normal forms of irreducible germs of functions of two variables with given Puiseux pairs. Oka [14, 15] gave the normal form of plane curves which are locally irreducible at the origin and with a given sequence of weight vectors corresponding to the Tschirnhausen-good resolution tower, and showed that the moduli space of such curves is of the form  $(\mathbf{C}^*)^a \times \mathbf{C}^b$ . Furthermore, Oka translated this result to the case of affine curves with one place at infinity.

Also, Abhyankar-Moh [1, 3, 4] investigated properties of  $\delta$ -sequences which are sequences of pole orders of *approximate roots* of  $C$ . This result is called Abhyankar-Moh's semigroup theorem. Sathaye-Stenerson [16] proved that if a sequence  $S$  of natural numbers satisfies Abhyankar-Moh's condition then there exists a curve with one place at infinity of the  $\delta$ -sequence  $S$ . Suzuki [18] made it clear the relationship between the  $\delta$ -sequence and the dual graph of the minimal resolution of the singularity of the curve  $C$  at infinity, and gave an algebrico-geometric proof of semigroup theorem and its inverse theorem due to Sathaye-Stenerson.

In this paper, we develop Suzuki's result and give an algebrico-geometric proof of Oka's result (Theorem 7 and Corollary 1). We shall also give an algorithm to compute

the normal form and the moduli space of the curve with one place at infinity from a given  $\delta$ -sequence<sup>1</sup>.

Our construction method of normal forms is different from [8, 14, 15] in the following respects. First, this method uses  $\delta$ -sequences generating semigroups of affine plane curves with one place at infinity. Second, this method directly generates defining polynomials at the origin of curves with one place at infinity.

## 2. Preparations

In this section, we introduce some definitions and facts which is needed to describe our theorem.

Let  $C$  be a curve with one place at infinity defined by a polynomial equation  $f(x, y) = 0$  in the complex affine plane  $\mathbf{C}^2$ . Assume that  $\deg_x f = m$ ,  $\deg_y f = n$  and  $d = \gcd(m, n)$ . By the consideration of the Newton boundary, we can get

$$f(x, y) = (ux^p + vy^q)^d + \sum_{q\alpha + p\beta < pqd} c_{\alpha\beta} x^\alpha y^\beta,$$

where  $u, v \in \mathbf{C}^*$ ,  $m = pd$  and  $n = qd$ . By a finitely many times of the coordinate transformations of the form

$$\begin{cases} x_1 = x \\ y_1 = y + cx^p \end{cases}$$

and the exchange of the coordinates  $x$  and  $y$ , we can reduce the polynomial  $f$  into one of the following two types:

(A)  $m = 1, n = 0$

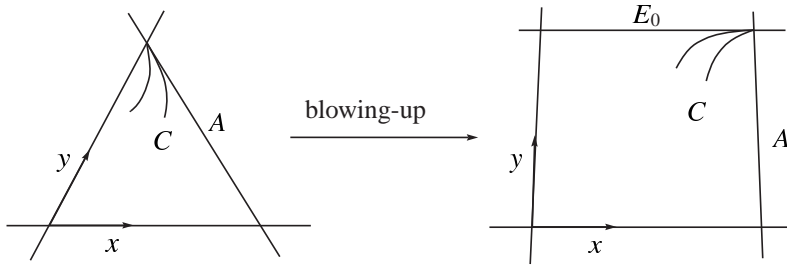
(B)  $m = pd, n = qd, \gcd(p, q) = 1, p > q > 1$ .

A curve of type (A) is a line. We call the curve of type (B) *non-linearizable*. In this paper, we shall consider only the curves of the type (B) from now on. The closure  $\overline{C}$  of  $C$  in the projective plane  $\mathbf{P}^2$  passes through the intersection point  $O$  of the  $\infty$ -line  $A$  and the line  $x = 0$  by the assumption  $p > q$ .

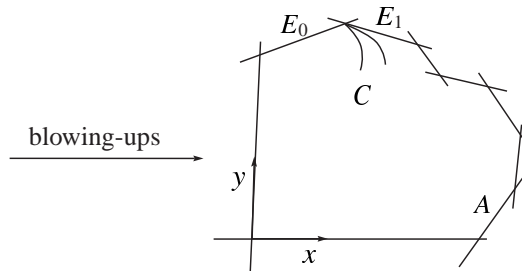
Let us denote by  $E_0$  the  $(-1)$ -curve appeared by the blowing-up of the point  $O$ , and continue to denote the proper transform of  $A$  by the same character  $A$ . Let  $a$  be the natural number satisfying  $aq < p < (a + 1)q$ . If  $a = 1$ , then the proper transform of  $\overline{C}$  is tangent to  $A$ , or else is tangent to  $E_0$ .

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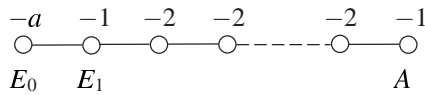
<sup>1</sup>The computer calculation by our algorithm verified the result of Nakazawa-Oka [11].



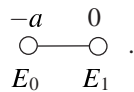
In case  $a > 1$ , after further  $a - 1$  times of the blowing-ups of the point at infinity of the curve  $C$ , the proper transform of  $\bar{C}$  is tangent to the  $(-1)$ -curve  $E_1$  obtained by the last blowing-up. (In case  $a = 1$ , we set  $E_1 = A$ .)



Thus we get a compactification of  $\mathbf{C}^2$  with the boundary curve of which the dual graph is of the following form:



By  $a - 1$  times of the blowing-downs of the  $(-1)$ -curve on the right hand side from  $A$  of the above dual graph, we get the following dual graph:

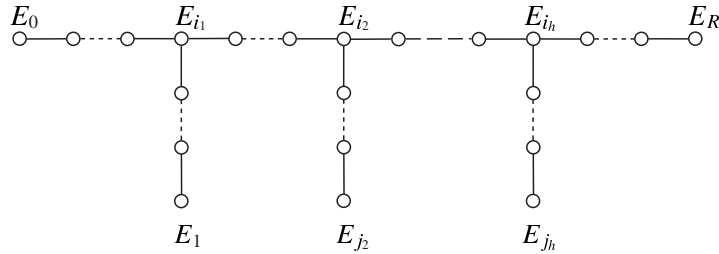


Let  $(M_1, E_0 \cup E_1)$  be the compactification of  $\mathbf{C}^2$  thus obtained.

The intersection point of  $E_0$  and  $E_1$  is the indetermination point of  $f$ . Now, we blow up from the surface  $M_1$  the indetermination points of  $f$  successively, until the indetermination points of  $f$  disappear. Let  $M_f$  be the surface thus obtained. We denote the proper transform in  $M_f$  of  $E_0$  (resp.  $E_1$ ) by the same character  $E_0$  (resp.  $E_1$ ). Let  $E_i (2 \leq i \leq R)$  be the proper transform in  $M_f$  of the  $(-1)$ -curve obtained by the  $(i - 1)$ -th blowing-up. Furthermore, we set  $E_f = E_0 \cup E_1 \cup \dots \cup E_R$ .

The following theorem about the compactification  $(M_f, E_f)$  of  $\mathbf{C}^2$  is very important for the classification problem of the curves with one place at infinity.

**Theorem 1** ([18]). (i) *The dual graph  $\Gamma(E_f)$  of  $E_f$  has the following form:*



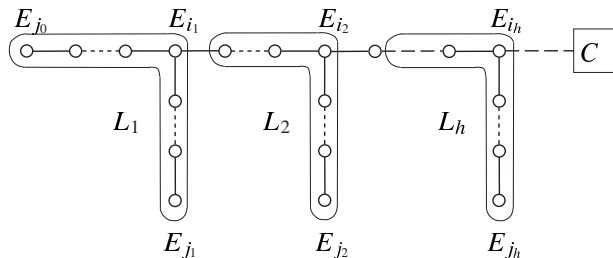
- (ii)  $f$  is non-constant only on  $E_R$  and has the pole on  $E_f - E_R$ .
- (iii) The degree of  $f$  on  $E_R$  is 1.
- (iv)  $E_R$  is the unique  $(-1)$ -curve in  $E_f$ .

**Note.** There is a small gap in the proof of (i) described in [18]. Let  $Z$  (resp.  $P$ ,  $S$ ) be the union of the components of  $E_f$  on which  $f = 0$  (resp.  $f = \infty$ ,  $f = \text{non-constant}$ ). Let  $T$  be the union of the other components of  $E_f$ . From the proof of (i) described in [18], we know that  $Z$  and  $P$  are both connected and  $S = E_R$ . Here, since  $f$  is non-zero constant on  $T$ ,  $T$  does not intersect  $Z$  and  $P$ . If  $T \neq \emptyset$ , then  $T$  intersects only  $S$ . But since  $S (= E_R)$  is the last  $(-1)$ -curve on  $M_f$ , the relations of intersection among  $Z$ ,  $P$ ,  $S$  and  $T$  is one of the following two types:

$$(I) P-S-Z \quad (II) P-S-T.$$

If  $Z \neq \emptyset$ , then we get the contradiction as it is described in [18]. The similar argument applies to the case of  $T \neq \emptyset$ . Thus we get  $Z = \emptyset$  and  $T = \emptyset$ . As a consequence,  $\Gamma(E_f)$  has the above form.

In  $\Gamma(E_f)$ , let  $i_1, i_2, \dots, i_h$  (resp.  $j_0, j_1, \dots, j_h$ ) be the indices of the branch vertices (resp. the terminal vertices) from the left hand side, where  $j_0 = 0$  and  $j_1 = 1$ . Let  $M_C$  be the surface obtained by the blowing-down of  $E_R, E_{R-1}, \dots, E_{i_h+1}$  from  $M_f$ . For  $i$  ( $0 \leq i \leq i_h$ ), we shall continue to denote by  $E_i$  the proper transform of  $E_i$  in  $M_C$ . Further, we set  $E_C = E_0 \cup E_1 \cup \dots \cup E_{i_h}$ . We shall call the pair  $(M_C, E_C)$  the compactification of  $\mathbf{C}^2$  obtained by the *minimal resolution* of the singularity of  $C$  at infinity. We set  $L_k = \bigcup_{i_{k-1} < i \leq i_k} E_i$  for each  $k$  ( $1 \leq k \leq h$ ) like the following figure, where  $i_0 = -1$ .

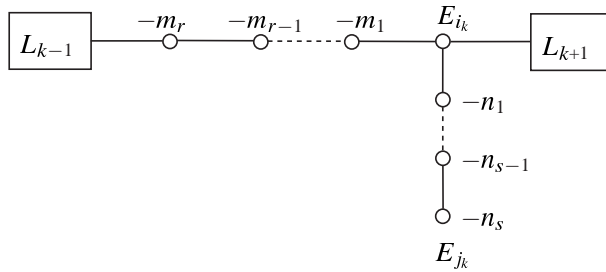


DEFINITION 1 ( $\delta$ -sequence). Let  $\delta_k (0 \leq k \leq h)$  be the order of the pole of  $f$  on  $E_{j_k}$ . We shall call the sequence  $\{\delta_0, \delta_1, \dots, \delta_h\}$  the  $\delta$ -sequence of  $C$  (or of  $f$ ).

We have the following fact since  $\deg_x f = m$  and  $\deg_y f = n$ .

**Fact 1.**  $\delta_0 = n, \delta_1 = m$ .

DEFINITION 2 ( $(p, q)$ -sequence). Now, we assume that the weights of  $L_k$  is of the following form:



We define the natural numbers  $p_k, a_k, q_k, b_k$  satisfying

$$(p_k, a_k) = 1, (q_k, b_k) = 1, 0 < a_k < p_k, 0 < b_k < q_k,$$

$$\frac{p_k}{a_k} = m_1 - \frac{1}{m_2 - \frac{1}{m_3 - \dots - \frac{1}{m_r}}}$$

and

$$\frac{q_k}{b_k} = n_1 - \frac{1}{n_2 - \frac{1}{n_3 - \dots - \frac{1}{n_s}}}$$

We shall call the sequence  $\{(p_1, q_1), (p_2, q_2), \dots, (p_h, q_h)\}$  the  $(p, q)$ -sequence of  $C$  (or of  $f$ ).

We shall assume that  $f(x, y)$  is monic in  $y$ . We define approximate roots by Abhyankar's definition.

DEFINITION 3 (approximate roots). Let  $f(x, y)$  be the defining polynomial, monic in  $y$ , of a curve with one place at infinity. Let  $\{\delta_0, \delta_1, \dots, \delta_h\}$  be the  $\delta$ -sequence of  $f$ . We set  $n = \deg_y f, d_k = \gcd\{\delta_0, \delta_1, \dots, \delta_{k-1}\}$  and  $n_k = n/d_k (1 \leq k \leq h+1)$ . Then, for each  $k (1 \leq k \leq h+1)$ , a pair of polynomials  $(g_k(x, y), \psi_k(x, y))$  satisfying the following conditions is uniquely determined:

- (i)  $g_k$  is monic in  $y$  and  $\deg_y g_k = n_k$ ,
- (ii)  $\deg_y \psi_k < n - n_k$ ,
- (iii)  $f = g_k^{d_k} + \psi_k$ .

We call this  $g_k$  the  $k$ -th approximate root of  $f$ .

We can easily get the following fact from the definition of approximate roots.

**Fact 2.** *We have*

$$g_1 = y + \sum_{j=0}^{\lfloor p/q \rfloor} c_j x^j, \quad g_{h+1} = f$$

where  $c_k \in \mathbf{C}$ ,  $p = \deg_x f/d$ ,  $q = \deg_y f/d$ ,  $d = \gcd\{\deg_x f, \deg_y f\}$  and  $\lfloor p/q \rfloor$  is the maximal integer  $l$  such that  $l \leq p/q$ .

**DEFINITION 4** (*g-sequence*). The sequence of polynomials  $g_0 := x, g_1, \dots, g_{h+1}$  is called the *g-sequence* of  $f$ .

Here, we denote by  $C_k$  the curve defined by  $g_k(x, y) = 0$  in  $\mathbf{C}^2$ . The following theorem about  $C_k$  plays a vital role in the main theorem.

**Theorem 2.** *For each  $k$  ( $0 \leq k \leq h$ ),  $C_k$  is also with one place at infinity. Further, its closure  $\overline{C}_k$  in  $M_C$  intersects transversely  $E_{j_k}$ , and does not intersect other irreducible components of  $E_C$ .*

Suzuki [18] gave the algebrico-geometric proof of this theorem. We get the following theorem as a corollary of the above theorem.

**Theorem 3.** *For each  $k$  ( $0 \leq k \leq h$ ),  $g_k$  has the pole of order  $\delta_k$  on  $E_{i_h}$ .*

The following lemma about approximate roots will be used in Theorem 6.

**Lemma 1.** *Let  $f$  be the defining polynomial, monic in  $y$ , of a curve with one place at infinity. Let  $\{\delta_0, \delta_1, \dots, \delta_h\}$  be the  $\delta$ -sequence of  $f$ , and  $g_0, g_1, \dots, g_h, g_{h+1}$  be the  $g$ -sequence of  $f$ . Then,  $g_k$  ( $0 \leq k \leq h-1$ ) is also the  $k$ -th approximate root of  $g_j$  for any  $j$  with  $k < j < h+1$ .*

Proof. For example, see Proposition 2.2 in [5]. □

### 3. Intersection matrix and successive blow-up

Let  $M$  be a non-singular projective algebraic surface over complex number field, and  $E$  be an algebraic curve on  $M$ . We shall assume that  $E_1, E_2, \dots, E_s$  are irreducible components of  $E$ , and denote by  $I_E$  the intersection matrix  $((E_i \cdot E_j))_{i,j=1,\dots,s}$  of  $E$ . The following lemma about the intersection matrix is well-known by Mumford.

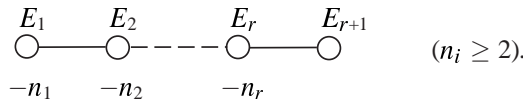
**Lemma 2.**  *$E$  is an exceptional set if and only if  $I_E$  is negative definite.*

Let  $E'_1$  be the  $(-1)$ -curve appeared by blowing-up at a point  $P_0$  on a surface  $M$ , and let  $P_1$  be a point on  $E'_1$ . For  $i (\geq 1)$ , let  $E'_{i+1}$  be the  $(-1)$ -curve appeared by blowing-up at a point  $P_i$ , and let  $P_{i+1}$  be the point on  $E'_{i+1}$ . We get  $\{P_i\}_{i=0,\dots,r}$  and  $\{E'_i\}_{i=1,\dots,r}$  by the above finite operations. In this paper we call this finite sequence of blowing-ups a *successive blow-up from  $P_0$* . Let  $M'$  be the surface obtained by a successive blow-up from  $P_0$ . For  $i (1 \leq i \leq r)$ , we shall continue to denote by  $E'_i$  the proper transform of  $E'_i$  in  $M'$ . Further, we set  $E' = \bigcup_{i=1}^r E'_i$  and  $\Delta_{E'} = \det(-I_{E'})$ . We have the following fact since  $\Delta_{E'}$  is invariant under the successive blow-up.

**Fact 3.**  $\Delta_{E'} = 1$ .

The following lemma is Lemma 1 in [18]. Here, we describe it because it is used many times in the next section.

**Lemma 3.** *Let  $E_1, E_2, \dots, E_r, E_{r+1}$  be the irreducible components of  $E$  and assume that the dual graph  $\Gamma(E)$  is of the following linear type:*



Assume further that there exists a holomorphic function  $f$  on a neighborhood  $U$  of  $\bigcup_{i=1}^r E_i$  such that the zero divisor  $(f)$  of  $f$  on  $U$  is written in the following form:

$$\sum_{i=1}^r m_i E_i + m_{r+1} E_{r+1} \cap U.$$

Let  $(p_i, p_{i+1})$  be the coprime integers defined by the following continued fraction:

$$\frac{p_{i+1}}{p_i} = n_i - \frac{1}{n_{i-1} - \dots - \frac{1}{n_1}} \quad (1 \leq i \leq r).$$

Then,  $m_i = m_1 p_i (1 \leq i \leq r + 1)$ .

Now, consider a pair of natural numbers  $(p, q)$  with  $\gcd(p, q) = 1, p > q > 0$ . We can easily show that there exists a unique pair of natural numbers  $(a, b)$  with  $pq - aq - bp = 1, 0 < a < p, 0 < b < q$ .



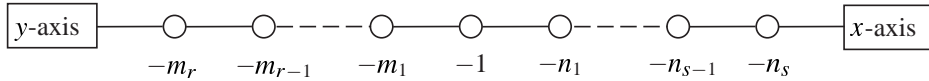
We consider the following continued fractions for the above mentioned  $p, q, a, b$ :

$$\frac{p}{a} = m_1 - \frac{1}{m_2 - \frac{1}{m_3 - \dots - \frac{1}{m_r}}}, \quad \frac{q}{b} = n_1 - \frac{1}{n_2 - \frac{1}{n_3 - \dots - \frac{1}{n_s}}},$$

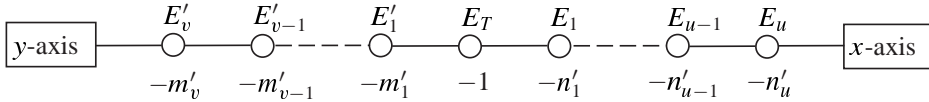
where  $m_i \geq 2$  and  $n_j \geq 2$ .

Let  $(x, y)$  be the local coordinate for the neighborhood of a point  $P$  on  $M$  which has  $P$  as the origin. Then,

**Lemma 4.** *we can construct a exceptional curve with the following weights by a successive blow-up from  $P$ .*



Proof. We consider the curve  $C$  defined by  $x^p + y^q = 0$ . The resolution graph at origin of  $C$  is as follows:



Let  $I_E$  be the intersection matrix of the exceptional curve  $E$  corresponding to the above dual graph. Here, we set

$$\frac{p'}{a'} = m'_1 - \frac{1}{m'_2 - \dots - \frac{1}{m'_u}}, \quad \frac{q'}{b'} = n'_1 - \frac{1}{n'_2 - \dots - \frac{1}{n'_v}}.$$

We get  $\det(-I_E) = p'q' - a'q' - b'p'$ . On the other hand,  $E$  is the exceptional curve obtained by a successive blow-up from origin. Therefore, we get  $\det(-I_E) = 1$  by Fact 3. Thus  $p'q' - a'q' - b'p' = 1$ .

As the above dual graph, let  $E_i (1 \leq i \leq u)$ ,  $E_T, E'_j (1 \leq j \leq v)$  be the irreducible components of  $E$ . We denote by  $\mu_i (1 \leq i \leq u)$  the zero order of the function  $x$  on  $E_i$  and by  $\mu_T$  the zero order of the function  $x$  on  $E_T$ . Also, we denote by  $\nu_j (1 \leq j \leq v)$  the zero order of the function  $y$  on  $E'_j$  and by  $\nu_T$  the zero order of the function  $y$  on  $E_T$ . Since  $q = \mu_T$  and  $\mu_u = 1$ , we get  $q' = \mu_T/\mu_u = q$  by Lemma 3. As the same way, we get  $p = p'$ . Thus  $pq - a'q - b'p = 1$ . Further, it must be  $a = a', b = b'$ , since  $0 < a' < p$  and  $0 < b' < q$ . Therefore, we get  $v = r, m'_i = m_i (1 \leq i \leq r), u = s,$

$n'_j = n_j$  ( $1 \leq j \leq s$ ) by the uniqueness of the expansion into continued fraction. As a result, the assertion was proved.  $\square$

**4. Construction of a curve with one place at infinity**

We set  $\mathbf{N} = \{n \in \mathbf{Z} \mid n \geq 0\}$  and  $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$ . The following theorem about  $\delta$ -sequence and  $(p, q)$ -sequence is called Abhyankar-Moh's Semigroup Theorem.

**Theorem 4** (Abhyankar-Moh). *Let  $C$  be a non-linearizable affine plane curve with one place at infinity. Let  $\{\delta_0, \delta_1, \dots, \delta_h\}$  be the  $\delta$ -sequence of  $C$  and  $\{(p_1, q_1), \dots, (p_h, q_h)\}$  be the  $(p, q)$ -sequence of  $C$ . We set  $d_k = \gcd\{\delta_0, \delta_1, \dots, \delta_{k-1}\}$  ( $1 \leq k \leq h + 1$ ). We have then,*

- (i)  $q_k = d_k/d_{k+1}$ ,  $d_{h+1} = 1$  ( $1 \leq k \leq h$ ),
- (ii)  $d_{k+1}p_k = \begin{cases} \delta_1 & (k = 1) \\ q_{k-1}\delta_{k-1} - \delta_k & (2 \leq k \leq h) \end{cases}$ ,
- (iii)  $q_k\delta_k \in \mathbf{N}\delta_0 + \mathbf{N}\delta_1 + \dots + \mathbf{N}\delta_{k-1}$  ( $1 \leq k \leq h$ ).

The following theorem gives the converse of the above theorem.

**Theorem 5** (Sathaye-Stenerson [16]). *Let  $\{\delta_0, \delta_1, \dots, \delta_h\}$  ( $h \geq 1$ ) be the sequence of  $h + 1$  natural numbers. We set  $d_k = \gcd\{\delta_0, \delta_1, \dots, \delta_{k-1}\}$  ( $1 \leq k \leq h + 1$ ) and  $q_k = d_k/d_{k+1}$  ( $1 \leq k \leq h$ ). Furthermore, suppose that the following conditions are satisfied:*

- (1)  $\delta_0 < \delta_1$ ,
- (2)  $q_k \geq 2$  ( $1 \leq k \leq h$ ),
- (3)  $d_{h+1} = 1$ ,
- (4)  $\delta_k < q_{k-1}\delta_{k-1}$  ( $2 \leq k \leq h$ ),
- (5)  $q_k\delta_k \in \mathbf{N}\delta_0 + \mathbf{N}\delta_1 + \dots + \mathbf{N}\delta_{k-1}$  ( $1 \leq k \leq h$ ).

*Then, there exists a curve with one place at infinity of the  $\delta$ -sequence  $\{\delta_0, \delta_1, \dots, \delta_h\}$ .*

Suzuki [18] gave an algebrico-geometric proof of the above two theorem by the consideration of the resolution graph at infinity.

**DEFINITION 5** (Abhyankar-Moh's condition). We shall call the conditions (1)–(5) concerning  $\{\delta_0, \delta_1, \dots, \delta_h\}$  in Theorem 5 *Abhyankar-Moh's condition*.

**Theorem 6.** *Let  $\{\delta_0, \delta_1, \dots, \delta_h\}$  ( $h \geq 1$ ) be the sequence of  $h + 1$  natural numbers satisfying Abhyankar-Moh's condition. Set  $d_k = \gcd\{\delta_0, \delta_1, \dots, \delta_{k-1}\}$  ( $1 \leq k \leq h + 1$ ) and  $q_k = d_k/d_{k+1}$  ( $1 \leq k \leq h$ ). Then,*

- (i) *the defining polynomial  $f$ , monic in  $y$ , of a curve with one place at infinity of*

the  $\delta$ -sequence  $\{\delta_0, \delta_1, \dots, \delta_h\}$  has the following form using the approximate roots  $g_0, g_1, \dots, g_h$  of  $f$ :

$$f = g_h^{q_h} + a_{\bar{\alpha}_0 \bar{\alpha}_1 \dots \bar{\alpha}_{h-1}} g_0^{\bar{\alpha}_0} g_1^{\bar{\alpha}_1} \dots g_{h-1}^{\bar{\alpha}_{h-1}} + \sum_{(\alpha_0, \alpha_1, \dots, \alpha_h) \in \Lambda} c_{\alpha_0 \alpha_1 \dots \alpha_h} g_0^{\alpha_0} g_1^{\alpha_1} \dots g_h^{\alpha_h}$$

where  $a_{\bar{\alpha}_0 \bar{\alpha}_1 \dots \bar{\alpha}_{h-1}} \in \mathbf{C}^*$ ,  $c_{\alpha_0 \alpha_1 \dots \alpha_h} \in \mathbf{C}$ ,  $(\bar{\alpha}_0, \bar{\alpha}_1, \dots, \bar{\alpha}_{h-1})$  is the sequence of  $h$  non-negative integers satisfying

$$\sum_{i=0}^{h-1} \bar{\alpha}_i \delta_i = q_h \delta_h, \quad \bar{\alpha}_i < q_i \quad (0 < i < h)$$

and

$$\Lambda = \left\{ (\alpha_0, \alpha_1, \dots, \alpha_h) \in \mathbf{N}^{h+1} \left| \alpha_i < q_i \quad (0 < i < h), \alpha_h < q_h - 1, \sum_{i=0}^h \alpha_i \delta_i < q_h \delta_h \right. \right\}.$$

(ii) Conversely, let  $g_h$  be the defining polynomial, monic in  $y$ , of a curve with one place at infinity of the  $\delta$ -sequence  $\{\delta_0/q_h, \delta_1/q_h, \dots, \delta_{h-1}/q_h\}$ , and  $g_0, g_1, \dots, g_{h-1}$  be the approximate roots of  $g_h$ . For any non-zero complex number  $a_{\bar{\alpha}_0 \bar{\alpha}_1 \dots \bar{\alpha}_{h-1}}$  corresponding to the sequence of  $h$  non-negative integers  $(\bar{\alpha}_0, \bar{\alpha}_1, \dots, \bar{\alpha}_{h-1})$  satisfying

$$\sum_{i=0}^{h-1} \bar{\alpha}_i \delta_i = q_h \delta_h, \quad \bar{\alpha}_i < q_i \quad (0 < i < h)$$

and any complex numbers  $c_{\alpha_0 \alpha_1 \dots \alpha_h}$  corresponding to the sequences of  $h + 1$  non-negative integers  $(\alpha_0, \alpha_1, \dots, \alpha_h)$  satisfying

$$\sum_{i=0}^h \alpha_i \delta_i < q_h \delta_h, \quad \alpha_i < q_i \quad (0 < i < h), \quad \alpha_h < q_h - 1,$$

we consider

$$f = g_h^{q_h} + a_{\bar{\alpha}_0 \bar{\alpha}_1 \dots \bar{\alpha}_{h-1}} g_0^{\bar{\alpha}_0} g_1^{\bar{\alpha}_1} \dots g_{h-1}^{\bar{\alpha}_{h-1}} + \sum_{(\alpha_0, \alpha_1, \dots, \alpha_h) \in \Lambda} c_{\alpha_0 \alpha_1 \dots \alpha_h} g_0^{\alpha_0} g_1^{\alpha_1} \dots g_h^{\alpha_h}$$

where

$$\Lambda = \left\{ (\alpha_0, \alpha_1, \dots, \alpha_h) \in \mathbf{N}^{h+1} \left| \alpha_i < q_i \quad (0 < i < h), \alpha_h < q_h - 1, \sum_{i=0}^h \alpha_i \delta_i < q_h \delta_h \right. \right\}.$$

Then, the curve defined by  $f = 0$  is a curve with one place at infinity of the  $\delta$ -sequence  $\{\delta_0, \delta_1, \dots, \delta_h\}$ , and has the approximate roots  $g_0, g_1, \dots, g_h$ .

Proof of Theorem 6. We shall prove (i). By the procedure described in the proof of Proposition 10 in [18], using the approximate roots  $g_0, g_1, \dots, g_h$  of  $f$  and the set of  $h + 1$  non-negative integers  $(\alpha_0, \alpha_1, \dots, \alpha_h)$  with  $\max\{\sum_{i=0}^h \alpha_i \delta_i\} = q_h \delta_h$ , we can write  $f$  as follows:

$$f = \sum_{\alpha_i < q_i (1 \leq i \leq h)} c_{\alpha_0 \alpha_1 \dots \alpha_h} g_0^{\alpha_0} g_1^{\alpha_1} \dots g_h^{\alpha_h} + g_h^{q_h}, \quad c_{\alpha_0 \alpha_1 \dots \alpha_h} \in \mathbf{C}.$$

Here, we suppose  $f = g_h^{q_h} + g_h^{q_h-1}$ . We have  $\deg_y g_h^{q_h-1} = n_h(q_h - 1) = \deg_y f - n_h = n - n_h$ . But this is a contradiction, since  $g_h$  is  $h$ -th approximate root of  $f$ . Thus we get  $\alpha_h < q_h - 1$ . By Theorem 4(iii) and the uniqueness of  $\{\alpha_i\}_{i=0, \dots, h}$  (e.g., Lemma 7 in [18]), we have  $\{\tilde{\alpha}_i\}_{i=0, \dots, h-1}$  with  $\sum_{i=0}^{h-1} \tilde{\alpha}_i \delta_i = q_h \delta_h$ . As a result, (i) was proved.

We shall prove (ii).

CASE  $h = 1$ . Set  $\delta_0 = q$  and  $\delta_1 = p$ . We can write  $f$  as follows:

$$f = y^q + ax^p + \sum_{q\alpha + p\beta < pq} c_{\alpha\beta} x^\alpha y^\beta, \quad a \in \mathbf{C}^*, \quad c_{\alpha\beta} \in \mathbf{C}.$$

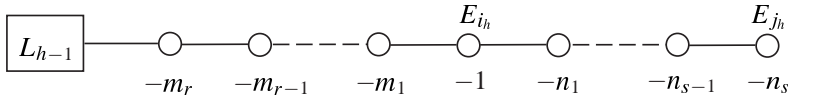
The curve defined by  $f = 0$  has one place at infinity of the  $\delta$ -sequence  $\{q, p\}$  by the consideration of Newton boundary.

CASE  $h \geq 2$ . Set  $\delta_i/q_h = \tilde{\delta}_i (0 \leq i \leq h - 1)$ . We denote by  $C_k$  the curve defined by  $g_k = 0$  for each  $k$  with  $0 \leq k \leq h$ . Further, we shall denote by  $(\tilde{M}, \tilde{E})$  the compactification of  $\mathbf{C}^2$  obtained by the minimal resolution of  $C_h$  at infinity. Let  $\tilde{C}_k$  be the proper transform of  $C_k$  on  $\tilde{M}$  and  $\tilde{E}_i$  be the irreducible components of  $\tilde{E}$ . (The way of numbering about indices is same as Section 2.) By Theorem 2,  $\tilde{C}_k$  has one place at infinity and intersects transversely  $\tilde{E}_{j_k} (0 \leq k \leq h - 1)$ .

Let  $Q$  be the intersection point of  $\tilde{C}_h$  and  $\tilde{E}_{i_{h-1}}$ . Set  $p_h = q_{h-1} \delta_{h-1} - \delta_h$ . ( $p_h > 0$  since Abhyankar-Moh's condition (4).) We have  $\gcd(p_h, q_h) = 1$  from  $\gcd(q_h, \delta_h) = d_{h+1} = 1$  and get a unique pair of natural numbers  $(a_h, b_h)$  with  $p_h q_h - a_h q_h - b_h p_h = 1, 0 < a_h < p_h, 0 < b_h < q_h$ . We define  $\{m_i\}_{i=1, \dots, r}, \{n_j\}_{j=1, \dots, s}$  using the following expansion into continued fractions by  $p_h, a_h, q_h, b_h$ :

$$\frac{p_h}{a_h} = m_1 - \frac{1}{m_2 - \frac{1}{m_3 - \dots - \frac{1}{m_r}}}, \quad \frac{q_h}{b_h} = n_1 - \frac{1}{n_2 - \frac{1}{n_3 - \dots - \frac{1}{n_s}}}.$$

By Lemma 4, we can obtain the following branch  $L_h$  such that  $C_h$  intersects transversely  $E_{j_h}$  using the successive blow-up from  $Q$ :



Let  $M$  be the surface thus obtained,  $E$  be the total transform of  $\tilde{E}$  on  $M$ . We denote by  $E_i$  (resp.  $\bar{C}_k$ ) the proper transform of  $\tilde{E}_i$  (resp.  $\tilde{C}_k$ ).

Set  $d_k = \gcd\{\delta_0, \delta_1, \dots, \delta_{k-1}\}$  ( $1 \leq k \leq h + 1$ ) and  $q_k = d_k/d_{k+1}$  ( $1 \leq k \leq h$ ). By Theorem 3,  $g_k$  has the pole of order  $\tilde{\delta}_k$  on  $E_{i_{h-1}}$  for each  $k$  ( $0 \leq k \leq h - 1$ ). Thus  $g_k$  has the pole of order  $\tilde{\delta}_k$  on  $E_{j_h}$  and of order  $q_h \tilde{\delta}_k (= \delta_k)$  on  $E_{i_h}$ . On the other hand,  $g_h$  has the pole of order  $\delta_h$  on  $E_{i_h}$ . In fact, we can write  $g_h$  on a neighborhood of  $Q$  as follows:

$$g_h = \frac{v}{u^{q_{h-1} \tilde{\delta}_{h-1}}} \times (\text{non-const}).$$

Hence  $g_h$  has the pole of order  $q_h(q_{h-1} \tilde{\delta}_{h-1}) - p_h$  on  $E_{i_h}$ . This value is equal to  $\delta_h$  by the assumption of  $p_h$ .

Now, we consider the curve  $C$  defined by  $f = 0$ . Set  $\phi = f - g_h^{q_h}$  and  $\Phi = \phi/g_h^{q_h}$ . Since the both of  $g_h^{q_h}$  and  $\phi$  has the pole of order  $q_h \delta_h$  on  $E_{i_h}$ ,  $\Phi$  is non-constant or constant ( $\neq 0$ ) on  $E_{i_h}$ .

Let  $A$  (resp.  $B$ ) be the closure of the connected component of  $E - E_{i_h}$  which contains  $E_0$  (resp.  $E_{j_h}$ ). Let  $P_{g_h}$  be the pole divisor of  $g_h$  on  $M$ , and  $D$  be its restriction to  $A$ . Here, let  $F_1$  be the irreducible component of  $A$  intersecting  $E_{i_h}$ . Since  $g_h$  has the pole of order  $\delta_h$  on  $E_{i_h}$ , we have  $(D \cdot F_1) < 0$ . Also, since  $(D \cdot E_i) = 0$  for any  $E_i$  with  $E_i \neq F_1$ , using Proposition 2 in [6], the intersection matrix of  $A$  is negative definite. Thus it follows that  $A$  is exceptional set.  $\Phi$  is holomorphic on  $A$  since  $A \cap \bar{C}_h = \emptyset$ . On the other hand,

$$\begin{aligned} \deg_y g_0^{\alpha_0} g_1^{\alpha_1} \cdots g_h^{\alpha_h} &= \sum_{i=0}^h \alpha_i n_i = \sum_{i=1}^h \alpha_i n_i \\ &= \alpha_1 + \alpha_2 q_1 + \alpha_3 q_2 q_1 + \cdots + \alpha_h q_{h-1} \cdots q_1 \\ &< (q_1 - 1) + (q_2 - 1)q_1 + (q_3 - 1)q_2 q_1 + \cdots + (q_h - 1)q_{h-1} \cdots q_1 \\ &= q_h q_{h-1} \cdots q_1 - 1 \\ &< q_h q_{h-1} \cdots q_1 = q_h n_h = \deg_y g_h^{q_h}. \end{aligned}$$

Therefore, we get  $\deg_y \phi < \deg_y g_h^{q_h}$ . Hence,  $\Phi = 0$  on  $E_0$ . Further,  $\Phi = 0$  on  $A$ , since  $A$  is compact. As a result, it must be that  $\Phi$  is non-constant on  $E_{i_h}$ .

Let  $P_\Phi$  be the pole divisor of  $\Phi$  on  $M$ . We denote by  $B_1, B_2, \dots, B_s$  the irreducible components of  $B$  in order from the component intersecting  $E_{i_h}$ . Since  $\Phi$  has the pole on  $B_1$  and  $\bar{C}_h$ , the support of  $P_\Phi$  is  $B \cup \bar{C}_h$  and we can write  $P_\Phi =$

$q_h \bar{C}_h + \sum_{i=1}^s \mu_i B_i (\mu_i > 0)$ . By

$$n_s - \frac{1}{n_{s-1} - \dots - \frac{1}{n_1}} = \frac{q_h}{b'}$$

and Lemma 3, we get  $\mu_1 q_h = q_h$ , where  $\mu_1$  is the pole order of  $\Phi$  on  $B_1$ . Hence,  $\mu_1 = 1$ . This implies that  $\Phi$  is a rational function of degree 1 on  $E_{i_h}$ . Therefore, the curve defined by  $\Phi = -1$  intersects transversely  $E_{i_h}$  at only one point. Since the curve  $\Phi = -1$  coincides with  $\bar{C}$ , we get

$$(\bar{C} \cdot E_i) = \begin{cases} 1 & (i = i_h) \\ 0 & (i \neq i_h) \end{cases}.$$

As a result,  $C$  has one place at infinity.

We have  $f = g_h^{q_h}$  on  $A$ , since  $\Phi = 0$  on  $A$ . Hence,  $f$  has the pole of the same order as  $g_h^{q_h}$  on each irreducible component of  $A$ . In particular,  $f$  has the pole of order  $q_h \delta_k = \delta_k$  on each  $E_{i_k} (0 \leq k \leq h - 1)$ . Since  $\Phi$  is non-constant on  $E_{i_h}$ ,  $f$  has the pole of the same order as  $g_h^{q_h}$  on  $E_{i_h}$ . Since the value of its pole order is  $q_h \delta_h$ , using Lemma 3, it follows that  $f$  has the pole of order  $\delta_h$  on  $E_{j_h}$ . Consequently,  $\{\delta_0, \delta_1, \dots, \delta_h\}$  is the  $\delta$ -sequence of  $f$ .

Finally, we show that  $g_0, g_1, \dots, g_h$  are the approximate roots of  $f$ . By

$$\begin{aligned} \deg_y g_0^{\alpha_0} g_1^{\alpha_1} \dots g_h^{\alpha_h} &= n_0 \alpha_0 + n_1 \alpha_1 + \dots + n_h \alpha_h \\ &\leq n_1(q_1 - 1) + n_2(q_2 - 1) + \dots + n_{h-1}(q_{h-1} - 1) + n_h(q_h - 2) \\ &= -n_1 + n_h q_h - n_h < n - n_h, \end{aligned}$$

$g_h$  is  $h$ -th approximate root of  $f$ . Therefore, by Lemma 1,  $g_0, g_1, \dots, g_h$  are the approximate roots of  $f$ . □

The following theorem is the main theorem in this paper, and is obtained by using Theorem 6 inductively.

**Theorem 7.** *Let  $\{\delta_0, \delta_1, \dots, \delta_h\} (h \geq 1)$  be a sequence of natural numbers satisfying Abhyankar-Moh's condition (see Definition 5). Set  $d_k = \gcd\{\delta_0, \delta_1, \dots, \delta_{k-1}\} (1 \leq k \leq h + 1)$  and  $q_k = d_k/d_{k+1} (1 \leq k \leq h)$ .*

(1) We define  $g_k$  ( $0 \leq k \leq h + 1$ ) as follows:

$$\left\{ \begin{array}{l} g_0 = x, \\ g_1 = y + \sum_{j=0}^{\lfloor p/q \rfloor} c_j x^j, \quad c_j \in \mathbf{C}, \quad p = \frac{\delta_1}{d_2}, \quad q = \frac{\delta_0}{d_2}, \\ g_{i+1} = g_i^{q_i} + a_{\bar{\alpha}_0 \bar{\alpha}_1 \dots \bar{\alpha}_{i-1}} g_0^{\bar{\alpha}_0} g_1^{\bar{\alpha}_1} \dots g_{i-1}^{\bar{\alpha}_{i-1}} \\ \quad + \sum_{(\alpha_0, \alpha_1, \dots, \alpha_i) \in \Lambda_i} c_{\alpha_0 \alpha_1 \dots \alpha_i} g_0^{\alpha_0} g_1^{\alpha_1} \dots g_i^{\alpha_i}, \\ a_{\bar{\alpha}_0 \bar{\alpha}_1 \dots \bar{\alpha}_{i-1}} \in \mathbf{C}^*, \quad c_{\alpha_0 \alpha_1 \dots \alpha_i} \in \mathbf{C} \quad (1 \leq i \leq h), \end{array} \right.$$

where  $(\bar{\alpha}_0, \bar{\alpha}_1, \dots, \bar{\alpha}_{i-1})$  is the sequence of  $i$  non-negative integers satisfying

$$\sum_{j=0}^{i-1} \bar{\alpha}_j \delta_j = q_i \delta_i, \quad \bar{\alpha}_j < q_j \quad (0 < j < i)$$

and

$$\Lambda_i = \left\{ (\alpha_0, \alpha_1, \dots, \alpha_i) \in \mathbf{N}^{i+1} \mid \alpha_j < q_j \quad (0 < j < i), \alpha_i < q_i - 1, \sum_{j=0}^i \alpha_j \delta_j < q_i \delta_i \right\}.$$

Then,  $g_0, g_1, \dots, g_h$  are approximate roots of  $f(=g_{h+1})$ , and  $f$  is the defining polynomial, monic in  $y$ , of a curve with one place at infinity of the  $\delta$ -sequence  $\{\delta_0, \delta_1, \dots, \delta_h\}$ .

(2) The defining polynomial  $f$ , monic in  $y$ , of a curve with one place at infinity of the  $\delta$ -sequence  $\{\delta_0, \delta_1, \dots, \delta_h\}$  is obtained by the procedure of (1), and the values of parameters  $\{a_{\bar{\alpha}_0 \bar{\alpha}_1 \dots \bar{\alpha}_{i-1}}\}_{1 \leq i \leq h}$  and  $\{c_{\alpha_0 \alpha_1 \dots \alpha_i}\}_{0 \leq i \leq h}$  are uniquely determined for  $f$ .

The above theorem gives normal forms of defining polynomials of curves with one place at infinity and the method of construction of their defining polynomials.

**Corollary 1.** Let  $\{\delta_0, \delta_1, \dots, \delta_h\}$  ( $h \geq 1$ ) be a sequence of natural numbers satisfying Abhyankar-Moh's condition. The moduli space of the curve  $C$  with one place at infinity of the  $\delta$ -sequence  $\{\delta_0, \delta_1, \dots, \delta_h\}$  is isomorphic to

$$(\mathbf{C}^*)^h \times \mathbf{C}^b,$$

where  $b$  is the total number of parameters  $\{c_{\alpha_0 \alpha_1 \dots \alpha_i}\}_{0 \leq i \leq h}$  appeared in the defining polynomial, monic in  $y$ , of  $C$  obtained in Theorem 7.

Proof. We consider the defining polynomial  $f$ , monic in  $y$ , of the curve  $C$  with one place at infinity of the  $\delta$ -sequence  $\{\delta_0, \delta_1, \dots, \delta_h\}$ . We denote by  $a$  the number

of non-zero parameters in  $f$  and by  $b$  the number of others. By Theorem 7, the moduli space of  $C$  is  $(\mathbf{C}^*)^a \times \mathbf{C}^b$ .  $f$  has  $h + 2$  polynomials  $g_0, g_1, \dots, g_{h+1}$ . Here, both of  $g_0$  and  $g_1$  do not have non-zero parameter. Also,  $g_{i+1}$  ( $1 \leq i \leq h$ ) has exactly one non-zero parameter because the sequence of  $i+1$  non-negative integers  $(\alpha_0, \alpha_1, \dots, \alpha_i)$  with  $\sum_{j=0}^i \alpha_j \tilde{\delta}_j = q_i \tilde{\delta}_i$  is determined uniquely. As a result, we get  $a = h$ .  $\square$

By the above results, we can easily get an algorithm generating the defining polynomial and computing the moduli space from a  $\delta$ -sequence. We will introduce them in the next section.

### 5. Algorithms

Using Theorem 7, the following algorithm generating the defining polynomial of the curve with one place at infinity from a  $\delta$ -sequence is obtained.

**Algorithm 1:** generating polynomial

**Input:**  $\delta$ -sequence  $\{\delta_0, \delta_1, \dots, \delta_h\}$

**Output:** the defining polynomial  $f(x, y)$  of the curve with one place at infinity of the  $\delta$ -sequence  $\{\delta_0, \delta_1, \dots, \delta_h\}$

```

 $D \leftarrow [\delta_h, \delta_{h-1}, \dots, \delta_0]$ 
 $d_k \leftarrow \gcd\{\delta_0, \delta_1, \dots, \delta_{k-1}\} (1 \leq k \leq h + 1)$ 
 $Q \leftarrow [q_h, \dots, q_1]$  where  $q_k = d_k/d_{k+1} (1 \leq k \leq h)$ 
 $DL \leftarrow \text{cons}(D, [ ])$ 
 $QL \leftarrow \text{cons}(Q, [ ])$ 
 $m \leftarrow h + 1$ 
while  $m \neq 2$  do
     $T \leftarrow \text{reverse}(\text{cdr}(D))$ 
     $D \leftarrow [ ]$ 
    while  $T \neq [ ]$  do
         $D \leftarrow \text{cons}(\text{car}(T)/\text{car}(Q), D)$ 
         $T \leftarrow \text{cdr}(T)$ 
    end
     $DL \leftarrow \text{cons}(D, DL)$ 
     $Q \leftarrow \text{cdr}(Q)$ 
     $QL \leftarrow \text{cons}(Q, QL)$ 
     $m \leftarrow \text{length}(D)$ 
end
 $AL \leftarrow [x]$ 
 $D \leftarrow \text{car}(DL)$ 
 $l \leftarrow \lfloor \text{car}(D)/\text{car}(\text{cdr}(D)) \rfloor$ 
 $g_1 \leftarrow y + \sum_{j=0}^l c_j x^j$ 

```



```

AL ← cons(g1, AL)
while DL ≠ [ ] do
  D ← car(DL)
  Q ← car(QL)
  q0 ← ⌊car(Q) × car(D)/car(reverse(D))⌋ + 1
  L ← append(Q, [q0])
  k ← length(D) − 1, i.e., D = [δ̄k, …, δ̄0], L = [qk, …, q0].
  (ᾱ0, ᾱ1, …, ᾱk−1) ← the sequence of non-negative integers with
    ∑i=0k−1 ᾱiδ̄i = δ̄kqk, ᾱi < qi (0 ≤ i ≤ k − 1), δ̄i ∈ D and qi ∈ L
  {(α0, α1, …, αk)} ← the set of sequences of non-negative integers with
    ∑i=0k αiδ̄i < δ̄kqk, αi < qi (0 ≤ i < k), αk < qk − 1, δ̄i ∈ D and qi ∈ L
  gk+1 ← gkqk + aᾱ0, ᾱ1, …, ᾱk−1 ∏i=0k−1 giᾱi + ∑ cα0, α1, …, αk ∏i=0k giαi
  AL ← cons(gk+1, AL)
  DL ← cdr(DL)
  QL ← cdr(QL)
end
return car(AL)

```

SUPPLEMENTATION:

- [ . . . ] := A list. (This is a data structure with ordered elements.)
- ⌊p⌋ := The maximal integer  $n$  such that  $n \leq p$ .
- car(L) := The first element of a given non-null list  $L$ .
- cdr(L) := The list obtained by removing the first element of a given non-null list  $L$ .
- cons(A, L) := The list obtained by adding an element  $A$  to the top of a given list  $L$ .
- reverse(L) := The reversed list of a given list  $L$ .
- append(L<sub>1</sub>, L<sub>2</sub>) := The list obtained by adding all elements in a list  $L_2$  according to the order as it is to the last element in a list  $L_1$ .
- length(L) := The number of elements of a given list  $L$ .
- a<sub>\*,\*,\*,\*,\*</sub> is a parameter in  $\mathbf{C}^*$ .
- c<sub>\*,\*,\*,\*,\*</sub> is a parameter in  $\mathbf{C}$ .

The moduli space of  $f$  is obtained by counting the numbers of  $\{a_{*,*,*,*,*}\}$  and  $\{c_{*,*,*,*,*}\}$  in  $f$  which the above algorithm outputted. But we can compute the moduli space from a  $\delta$ -sequence without generating the defining polynomial. The following algorithm directly compute the moduli space from a  $\delta$ -sequence.

**Algorithm 2:** computation of moduli space

**Input:**  $\delta$ -sequence  $\{\delta_0, \delta_1, \dots, \delta_h\}$

**Output:**  $[M, N]$  (This means the moduli space  $(\mathbf{C}^*)^M \times \mathbf{C}^N$ .)

$D \leftarrow [\delta_h, \delta_{h-1}, \dots, \delta_0]$

```

 $d_k \leftarrow \gcd\{\delta_0, \delta_1, \dots, \delta_{k-1}\} \ (1 \leq k \leq h+1)$ 
 $Q \leftarrow [q_h, \dots, q_1]$  where  $q_k = d_k/d_{k+1} \ (1 \leq k \leq h)$ 
 $Q \leftarrow \text{cons}(1, Q)$ 
 $M \leftarrow h$ 
 $N \leftarrow 0$ 
while true do
     $k \leftarrow \text{length}(D) - 1$ , i.e.,  $D = [\bar{\delta}_k, \dots, \bar{\delta}_0]$ 
     $D \leftarrow [\bar{\delta}_k/\text{car}(Q), \bar{\delta}_{k-1}/\text{car}(Q), \dots, \bar{\delta}_0/\text{car}(Q)]$ 
     $Q \leftarrow \text{cdr}(Q)$ 
     $q_0 \leftarrow \lfloor \text{car}(Q) \times \text{car}(D) / \text{car}(\text{reverse}(D)) \rfloor + 1$ 
     $L \leftarrow \text{append}(Q, [q_0])$ , i.e.,  $L = [q_k, \dots, q_0]$ 
     $n \leftarrow$  the number of  $(\alpha_0, \alpha_1, \dots, \alpha_k)$  with  $\sum_{i=0}^k \alpha_i \bar{\delta}_i < \text{car}(Q) \times \text{car}(D)$ ,
         $\alpha_i < q_i \ (0 \leq i \leq k-1)$ ,  $\alpha_k < q_k - 1$ ,  $\bar{\delta}_i \in D$  and  $q_i \in L$ 
     $N \leftarrow N + n$ 
    if  $\text{length}(D) = 2$  then break
     $D \leftarrow \text{cdr}(D)$ 
end
 $N \leftarrow N + \lfloor p/q \rfloor + 1$ 
return  $[M, N]$ 

```

## 6. Polynomial curve

**6.1. Abhyankar's question.** In this section, we will introduce Abhyankar's question.

**DEFINITION 6** (planar semigroup). Let  $\{\delta_0, \delta_1, \dots, \delta_h\} \ (h \geq 1)$  be a sequence of natural numbers satisfying Abhyankar-Moh's condition. A semigroup generated by  $\{\delta_0, \delta_1, \dots, \delta_h\}$  is said to be a *planar semigroup*.

**DEFINITION 7** (polynomial curve). Let  $C$  be an algebraic curve defined by  $f(x, y) = 0$ , where  $f(x, y)$  is an irreducible polynomial in  $\mathbf{C}[x, y]$ . We call  $C$  a *polynomial curve*, if  $C$  has a parametrisation  $x = x(t)$ ,  $y = y(t)$ , where  $x(t)$  and  $y(t)$  are polynomials in  $\mathbf{C}[t]$ .

**Abhyankar's Question.** Let  $\Omega$  be a planar semigroup. Is there a polynomial curve with  $\delta$ -sequence generating  $\Omega$ ?

This question is still open. Moh [10] showed that there is no polynomial curve with  $\delta$ -sequence  $\{6, 8, 3\}$ . But there is a polynomial curve  $(x, y) = (t^3, t^8)$  with  $\delta$ -sequence  $\{3, 8\}$  which generates the same semigroup as above. Sathaye-Stenerson [16] proved that the semigroup generated by  $\{6, 22, 17\}$  has no other  $\delta$ -sequence generating the same semigroup, and proposed the following conjecture for

this question.

**Sathaye-Stenerson’s Conjecture.** There is no polynomial curve having the  $\delta$ -sequence  $\{6, 22, 17\}$ .

By Algorithm 1, the defining polynomial of the curve with one place at infinity of the  $\delta$ -sequence  $\{6, 22, 17\}$  as follows:

$$f = (g_2^2 + a_{2,1}x^2g_1) + c_{5,0,0}x^5 + c_{4,0,0}x^4 + c_{3,0,0}x^3 + c_{2,0,0}x^2 + c_{1,1,0}xg_1 + c_{1,0,0}x + c_{0,1,0}g_1 + c_{0,0,0}$$

where

$$\begin{aligned} g_1 &= y + c_3x^3 + c_2x^2 + c_1x + c_0, \\ g_2 &= (g_1^3 + a_{11}x^{11}) + c_{10,0}x^{10} + c_{9,0}x^9 + c_{8,0}x^8 + (c_{7,1}g_1 + c_{7,0})x^7 \\ &\quad + (c_{6,1}g_1 + c_{6,0})x^6 + (c_{5,1}g_1 + c_{5,0})x^5 + (c_{4,1}g_1 + c_{4,0})x^4 \\ &\quad + (c_{3,1}g_1 + c_{3,0})x^3 + (c_{2,1}g_1 + c_{2,0})x^2 + (c_{1,1}g_1 + c_{1,0})x + c_{0,1}g_1 + c_{0,0}. \end{aligned}$$

This result gives us a new approach to investigate the curve with one place at infinity of the  $\delta$ -sequence  $\{6, 22, 17\}$  using a computer algebra system.

**6.2. Computation of moduli space.** Suzuki gave an algorithm generating the list of  $\delta$ -sequences of curves with one place at infinity, and implemented on a computer. From the list of  $\delta$ -sequences obtained by Suzuki, we could get normal forms and moduli spaces of curves with one place at infinity of genus  $\leq 100$  by using the algorithm introduced in previous section. As a result, we could verify the result of Nakazawa-Oka [11].

The following is the list of moduli spaces of curves with one place at infinity for the cases genus  $\leq 30$ .

EXAMPLE 1. The case

$$[7, [4, 6, 11], [2, 15]]$$

means that the moduli space of the curve with one place at infinity of genus 7 and the  $\delta$ -sequence  $\{4, 6, 11\}$  is isomorphic to  $(\mathbf{C}^*)^2 \times \mathbf{C}^{15}$ .

[1, [2, 3], [1, 5]],	[5, [2, 11], [1, 17]],	[7, [3, 8], [1, 17]],
[2, [2, 5], [1, 8]],	[5, [4, 6, 7], [2, 11]],	[7, [4, 6, 11], [2, 15]],
[3, [2, 7], [1, 11]],	[6, [2, 13], [1, 20]],	[7, [6, 9, 5], [2, 12]],
[3, [3, 4], [1, 9]],	[6, [3, 7], [1, 15]],	[7, [8, 12, 3], [2, 10]],
[3, [4, 6, 3], [2, 7]],	[6, [4, 5], [1, 14]],	[7, [10, 15, 2], [2, 9]],
[4, [2, 9], [1, 14]],	[6, [4, 6, 9], [2, 13]],	[7, [4, 10, 7], [2, 13]],
[4, [3, 5], [1, 11]],	[6, [6, 9, 4], [2, 10]],	[7, [6, 15, 2], [2, 10]],
[4, [4, 6, 5], [2, 9]],	[6, [4, 10, 5], [2, 11]],	[7, [6, 8, 3], [2, 10]],
[4, [6, 9, 2], [2, 7]],	[7, [2, 15], [1, 23]],	[7, [8, 12, 6, 3], [3, 8]],

[8, [2, 17], [1, 26]], [13, [18, 27, 6, 2], [3, 9]], [16, [8, 20, 10, 9], [3, 15]],  
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