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ON THE SCHUR INDICES OF CHARACTERS OF FINITE REDUCTIVE GROUPS IN BAD CHARACTERISTIC CASES

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1.

Let ${}^2F_4(q^2)$ be the finite Ree group of type (F_4), where $q^2 = 2^{2n+1}$. One of the original motivation of writing this paper is to get informations about the Schur indices of the complex irreducible characters of ${}^2F_4(q^2)$.

Let G^F be a finite reductive group. That is, G is a connected, reductive linear algebraic group over an algebraic closure K of the prime field F_p of characteristic p, F is a surjective endomorphism of G such that some power F^d of F is the Frobenius endomorphism of G relative to a rational structure on G over a finite subfield of K, and G^F is the group of F-fixed points of G (cf. Carter [3, p. 31]). Then we say that a complex irreducible character χ of G^F is regular if it is an irreducible component of a Gel'fand-Graev character of G^F and that χ is semisimple if it is the dual character of a regular character of G^F (up to ± 1) in the sense of Curtis and Kawanaka ([4, 11]). In [15, 16], we obtained some results on the Schur indices of the regular characters of G^F and, under the assumption that p is a good prime for G, of the semisimple characters of G^F. The first purpose of this paper is to drop out this assumption. Thus, in particular, we see that any semisimple character of ${}^2F_4(q^2)$ has the Schur index 1. (It is clear that any regular character of ${}^2F_4(q^2)$ has the Schur index 1.)

Our second purpose is to give a proof of the following theorem when $p = 2$.

Theorem (cf. M.J.J. Barry [1]). *Any complex irreducible character of the Steinberg's triality group* ${}^3D_4(q^3)$ *has the Schur index* 1.

We note that the theorem is proved by Barry when $p \neq 2$ ([1]) and that when $p = 2$ R. Gow has determined the Schur indices of the regular characters and the semisimple characters of ${}^3D_4(q^3)$. But the latter results can be also obtained from the first results of this paper.

NOTATION. If χ is an absolutely irreducible character of a finite group over an algebraically closed field of characteristic 0 and if k is a field of characteristic 0, then $m_k(\chi)$ denotes the Schur index of χ with respect to k, where we consider χ as a character over an algebraically closed extension of k. If l is a prime number, then $\overline{Q}_l$ denotes an algebraic closure of the l-adic number field Q_l.

2.

Let K be an algebraic closure of the prime field F_p of characteristic p, G a connected, reductive linear algebraic group over K, F a surjective endomorphism of G such that some power F^d of F is the Frobenius endomorphism of G relative to a rational structure on G over a finite subfield of K, and G^F the group of F-fixed points of G Let B^* be an F-stable Borel subgroup of G and T^* an F-stable maximal torus of G contained in B^*. Let U^* be the unipotent radical of B^*. Let R be the root system of G with respect to T^*, and, for $\alpha \in R$, let U^*_α denote the root subgroup of G corresponding to α. Let $R^+ = \{\alpha \in R \mid U^*_\alpha \subset B^*\}$ be the set of positive roots determined by B^*, and let S be the set of corresponding simple roots. Let ρ be the permutation on R given by $F(U^*_\alpha) = U^*_{\rho\alpha}$; we have $\rho(R^+) = R^+$ and $\rho(S) = S$. Let I be the set of orbits of ρ an S. Let $U^*_.$ be the subgroup of U^* generated by the root subgroups U^*_α corresponding to the non-simple positive roots α. Then we have $U^*/U^*_. = \prod_{\alpha\in S} U^*_\alpha = \prod_{i\in I} U^*_i$, where, for $i \in I$, $U^*_i = \prod_{\alpha\in i} U^*_\alpha$, and $U^{*F}/U^{*F}_. = (U^*/U^*_.)^F = \prod_{i\in I} U^{*F}_i$. Let Λ be the set of all complex linear characters λ of U^{*F} such that $\lambda | U^{*F}_. = 1$, and let Λ_0 be the set of all linear characters λ in Λ such that $\lambda | U^{*F}_i \neq 1$ for all $i \in I$. For $\lambda \in \Lambda_0$, let $\Gamma_\lambda = \lambda^{G^F} = \mathrm{Ind}_{U^{*F}}^{G^F}(\lambda)$, which we call a Gel'fand-Graev character of G^F. It is well known that any Gel'fand-Graev character of G^F is multiplicity-free (Gel'fand-Graev, Yokonuma. Steinberg; see Deligne and Lusztig [5, Theorem 10.7] and Carter [3, Theorem 8.1.3]). We say that a complex irreducible character of G^F is regular if it is an irreducible component of a Gel'fand-Graev character of G^F and that a complex irreducible character of G^F is semisimple if it is the dual character of a regular character of G^F (up to ± 1) in the sense of Curtis ([4]) and Kawanaka ([11]) (see Carter [3, §8.2]).

Assume that the centre Z of G is connected. Then $\Gamma_G = \Gamma_\lambda$ is independent of $\lambda \in \Lambda_0$ and any regular or semisimple character of G^F is expressed as a Q-linear combination of the Deligne-Lusztig virtual characters R^θ_T (Deligne and Lusztig [5, Theorem 10.7]; see also Carter [3, §8.4]). The degree of any semisimple character of G^F is coprime to p and when p is a good prime for G a complex irreducible character of G^F is semisimple if and only if its degree is coprime to p (see Carter [3, p. 280]).

Let us consider the case that Z is not necessarily connected. Then we still have:

Lemma 1. *Assume that G is defined over a finite subfield of K and F is the corresponding Frobenius endomorphism of G . Let χ be a complex irreducible character of G^F. Then, if χ is semisimple, its degree is coprime to p. When p is a good prime for G, χ is semisimple if and only if its degree is coprime to p.*

Proof. We embed G in a connected, reductive group G_1 with connected centre and the same derived group (cf. Deligne and Lusztig [5, 5.18]). Let $\chi_1, \ldots, \chi_t$ be the ${G_1}^F$-conjugates of χ. Then, by Clifford theory, we see that there is a complex irreducible character θ of ${G_1}^F$ and a positive integer e such that $\theta | G^F = e(\chi_1 + \cdots + \chi_t)$. According to a result of Lusztig ([14, Proposition 10]), we have $e = 1$. Assume that χ is semisimple. Then by a result of Digne, Lehrer and Michel [7, (3.15.3)], we see that one can assume that θ is a semisimple character of ${G_1}^F$. Since the centre of G_1 is connected, the degree of θ is coprime to p. Hence the degree of χ must be coprime to p. Assume that p is a good prime for G and that the degree of χ is coprime to p. Then the degree of θ is also coprime to p so that it must be semisimple. Hence, by [loc. cit.], χ must be semisimple. □

Let J be any subset of I. Let $P(J) = \langle B^*, U^*_{-\alpha} \mid \alpha \in i, i \in J \rangle$. and $L(J) = \langle T^*, U^*_{\alpha}, U^*_{-\alpha} \mid \alpha \in i, i \in J \rangle$. Let $U(J)$ be the unipotent radical of $P(J)$. For a character $\lambda \in \Lambda_0$, let $\lambda(J) = (\lambda \mid (U^{*F} \cap L(J)^F)) \times 1_{U(J)^F}$, a linear character in Λ.

Let $\lambda \in \Lambda_0$, and let Δ_λ be the dual (generalized) character of Γ_λ. Then by [7, (2.12.2)], we have

$$\Delta_\lambda = \sum_{J \subset I} (-1)^{|J|} \lambda(J)^{G^F}, \tag{2.1}$$

where the sum is taken over all the subsets J of I. (In [7], it is assumed that G is defined over a finite subfield of K and F is the corresponding Frobenius endomorphism of G. But (2.12.2) in [7] still holds in our case.) Since Γ_λ is multiplicity-free, by a result of Curtis, Alvis and Kawanaka (See Carter [3, Theorem 8.2.1]), we must have

$$\Delta_\lambda = \varepsilon_1 \chi_1 + \cdots + \varepsilon_m \chi_m, \tag{2.2}$$

where $m = (\Gamma_\lambda, \Gamma_\lambda)_{G^F}$, $\varepsilon_i = \pm 1$ $(1 \leqq i \leqq m)$ and $\chi_1, \ldots, \chi_m$ are distinct irreducible (semisimple) characters of G^F.

Let H be a finite group, k a field of characteristic 0 and C an algebraically closed extension of k. Let ξ be a generalized character of H over C. Then we say that ξ is virtually realizable in k if it can be written as $a_1\xi_1 + \cdots + a_n\xi_n$, where $a_1, \ldots, a_n$ are rational integers and $\xi_1, \ldots, \xi_n$, are proper characters of H which are realizable in k. In this case, if χ is an absolutely irreducible character of H over C, then, by a property of the Schur index, we see that $m_k(\chi)$ divides each multiplicity $(\xi_i, \chi)_H$ $(1 \leqq i \leqq n)$, so that $m_k(\chi)$ divides the inner product $(\xi, \chi)_H$.

Suppose that k is a field of characteristic 0 such that for any $\lambda \in \Lambda$, λ^{G^F} is realizable in k. Then, by (2.1), we see that, for any $\lambda \in \Lambda_0$, Δ_λ is virtually realizable in k, so that, by (2.2), we have $m_k(\chi) = 1$ for any semisimple character χ of G^F.

Lemma 2 (cf. [15, 16]). *Let $\lambda \in \Lambda$. Then we have the following*:
(i) *If $p = 2$, then λ^{G^F} is realizable in Q. Assume that $p \neq 2$,*

(ii) *Let* $k = Q\left(\sqrt{(-1)^{(p-1)/2}p}\right)$. *Then, if* $p \equiv -1 \pmod 4$, λ^{G^F} *is realizable in* k, *and if* $p \equiv 1 \pmod 4$, *for any finite place* v *of* k. λ^{G^F} *is realizable in the completion* k_v *of* k *at* v.
(iii) *Assume that* G *is defined over a finite subfield with* q *elements of* K *where* q *is an even power of* p *and* F *is the corresponding Frobenius endomorphism of* G. *Then, for each prime number* $l \neq p$, λ^{G^F} *is realizable in* Q_l.

Assume that Z *is connected.*
(iv) *For each prime number* $l \neq p$, λ^{G^F} *is realizable in* Q_l.
(v) *Assume that* Z^F *is trivial or that* G *is defined and split over a finite subfield of* K *and* F *is the corresponding Frobenius endomorphism of* G. *Then* λ^{G^F} *is realizable in* Q.

Since $U^{*F}/U_{\cdot}^{*F}$ is an elementary abelian p-group, λ is realizable in $Q(\zeta_p)$, where ζ_p is a primitive p-th root of unity. Thus, if $p = 2$, λ is realizable in Q, hence λ^{G^F} is realizable in $Q((\mathrm{i}))$. Assume that $p \neq 2$. Then (iii) is proved in [16] and (iv), (v) are proved in [15]. (ii) is proved in [16] when G is defined over a finite subfield of K and F is the corresponding Frobenius endomorphism of G. Therefore it remains to prove

Lemma 3 (cf. [16, Lemma 2]). *Assume that* $p \neq 2$. *Let* ν *be a generator of the cyclic group* ${F_p}^{\times}$. *Then there is an element* t *in* T^{*F} *such that* $t^{P-1} = 1$ *(possibly* $t^{(p-1)/2} = 1$*) and* $\alpha(t) = \nu^2$ *for all simple roots* α.

Proof. We repeat the proof of Lemma 2 in [16].

Firstly, we observe that it suffices to prove the lemma for the derived group G' of G. Let $\pi\colon \tilde{G} \to G'$ be the simply-connected covering of G'. Then, by [20, 9.16], we see that there exists a unique isogeney $\tilde{F}\colon \tilde{G} \to \tilde{G}$ such that $\pi \circ \tilde{F} = F \circ \pi$. We see that if F^d is the Frobenius endomorphism of G' corresponding to a rational structure on G' over a finite subfield F_q of K, then $\tilde{F}^d$ is the Frobenius endomorphism of G corresponding to a rational structure on $\tilde{G}$ over F_q (cf. Satake [18, Remark 5, p. 63]). Then, by the argument in the proof of Lemma 2 in [16], we can be reduced to the case that G is a simply connected simple algebraic group. If G is defined over a finite subfield of K and F is the corresponding Frobenius endomorphism of G, then Lemma 3 is just Lemma 2 in [16]. Therefore, since $p \neq 2$, it remains to treat the case where $p = 3$, $G = G_2$ and F is an exceptional isogeney such that F^2 is the Frobenius endomorphism of G corresponding to a rational structure on G over a finite subfield of K with 3^{2n+1} elements (i.e. $G^F = {}^2G_2(q^2)$). But, in this ease, G is an adjoint group, so the assertion is proved in [15] (this case is also implicit in Gow [10, Thearem 9]). □

By Lemma 2, we get

Theorem 1 (cf. [15, 16]). *Let χ be a complex irreducible character of G^F such that $(\lambda^{G^F}, \chi)_{G^F} = 1$ for some $\lambda \in \Lambda$ (e.g. χ is regular) or that χ is semisimple. Then we have the following*:

(i) *If $p = 2$, then we have $m_Q(\chi) = 1$.*

(ii) *Let $k = Q\big(\sqrt{(-1)^{(p-1)/2}p}\big)$. Then, if $p \equiv -1 \pmod 4$, we have $m_k(\chi) = 1$, and If $p \equiv 1 \pmod 4$, for any finite place v of k, we have $m_{k_v}(\chi) = 1$. Thus we have $m_Q(\chi) \leqq 2$.*

(iii) *Assume that G is defined over a subfield with q elements of K where q is an even power of p and F is the corresponding Frobenius endomorphism of G. Then, for each prime number $l \neq p$, we have $m_{Q_l}(\chi) = 1$.*

Assume that Z is connected.

(iv) *For each prime number $l \neq p$, we have $m_{Q_l}(\chi) = 1$.*

(v) *Assume that Z^F is trivial or that G is defined and split over a finite subfield of K and F is the corresponding Frobenius endomorphism of G. Then we have $m_Q(\chi) = 1$.*

REMARK. Let χ be a semisimple character of G^F. Then, in [15, 16], Theorem 1 is proved by a different method under the assumption that p is a good prime for G.

EXAMPLE. By Theorem 1, we see that any regular or semisimple character of the Ree group ${}^2F_4(q^2)$ of type (F_4) has the Schur index 1. We can also determine the local Schur indices of any unipotent character of ${}^2F_4(q^2)$. There is just one unipotent character χ of ${}^2F_4(q^2)$ such that $m_R(\chi) = m_{Q_2}(\chi) = 2$ and $m_{Q_l}(\chi) = 1$ for each prime number $l \neq 2$. This character has the property that it occurs with even multiplicity in each Deligne-Lusztig virtual character R_T^1 (cf. [13]). Other unipotent characters of ${}^2F_4(q^2)$ have the Schur index 1.

By the proof of Lemma 2 in [15, 16] and by Schur's lemma, we get

Proposition 1. *Assume that $p \neq 2$. Let χ be as in* Theorem 1 *and assume that χ is trivial on Z^F. Let $k = Q\big(\sqrt{(-1)^{(p-1)/2}p}\big)$. Then we have $m_k(\chi) = 1$. If Z is connected or if G is defined over a finite with q elements of K where q is an even power of p and F is the corresponding Frobenius endomorphism of G, then we have $m_Q(\chi) = 1$.*

By Lemma 4 of [16], we get

Theorem 2. *Assume that $p \neq 2$ and let χ be as in* Theorem 1. *Let G be such that G/Z is a simple algebraic group of any one of the following types*: *A_r with $2|r$ or $\mathrm{ord}_2(r+1) > \mathrm{ord}_2(p-1)$; ${}^2A_r$ with $2|r$; B_r with $4|r(r+1)$; D_r with either* (a) *$4|r(r-1)$ or* (b) *$\mathrm{ord}_2(r-1) = 1$ and $p \equiv -1 \pmod 4$; ${}^2D_r$ with $4|r(r-1)$; ${}^3D_4$; E_6; ${}^2E_6$. Then we have $m_Q(\chi) = 1$.*

3.

In this section we shall give a proof of the following theorem when $p = 2$.

Theorem 3 (cf. Barry [1] for $p \neq 2$). *Any complex irreducible character of* ${}^3D_4(q^3)$ *has the Schur index* 1 *over* Q.

Let q be a power of any fixed prime number p. Let G be a connected, reductive algebraic group, defined over the subfield F_q with q elements of K (an algebraic closure of F_p), with Frobenius endomorphism F such that G/Z is a simple algebraic group of type (${}^3D_4$).

Firstly, by Theorems 1, 2, we see that any regular or semisimple character of G^F has the Schur index 1 over Q (in the case where $G^F = {}^3D_4(q^3)$ with q even, the rationality of the semisimple characters of G^F has been already observed by Gow; see below).

Next, we treat the unipotent characters of G^F. Let B^*, T^* be as in §2. Let $W = N_G(T^*)/T^*$ be the Weyl group of G, where $N_G(T^*)$ is the normalizer of T^* in G. Let X be the variety of all Borel subgroups of G. Let l be any fixed prime number different from p. Let we $w \in W$, and let $\dot{w}$ be an element of $N_G(T^*)$ such that $\dot{w}T^* = w$ in W. Then, for B, $B' \in X$, we write $B \overset{W}{\to} B'$ if there is an element g of G such that $B = gB^*g^{-1}$ and $B' = g\dot{w}B^*\dot{w}^{-1}g^{-1}$. Let $X(w)$ be the subvariety of X which consists of all $B \in X$ such that $B \overset{w}{\to} F(B)$. Then $X(w)$ is smooth and purely of dimension $l(w)$, where $l(\ )$ denotes the length function on W with respect to the simple reflections determined by B^* (see Deligne and Lusztig [5, 1.4]). G^F acts on $X(w)$ by conjugation, so G^F acts on each i-th l-adic cohomology group with compact support $H_c^i(X(w), Q_l)$ of $X(w)$ $(0 \leqq i \leqq 2l(w))$. For $0 \leqq i \leqq 2l(w)$, let $H_c^i(X(w)) = H_c^i(X(w), \overline{Q_l}) = H_c^i(X(w), Q_l) \bigotimes_{Q_l} \overline{Q_l}$, and let

$$R^1(w) = \sum_{i=0}^{2l(w)} (-1)^i H_c^i(X(w))$$

(an element of the Grothendieck group of representations of G^F over $\overline{Q_l}$). Then the character of $R^1(w)$ has rational integeral values, independent of l ([5, (3.3)]). So we can regard $R^1(w)$ as a generalized complex character of G^F. We say that a complex irreducible character χ of G^F is unipotent if $(R^1(w), \chi)_{G^F} \neq 0$ for some $w \in W$ ([5, 7.8]).

Let $w \in W$, and, for $0 \leqq i \leqq 2l(w)$, let $\xi_i(w)$ be the character of the G^F-module $H_c^i(X(w))$. Then $\xi_i(w)$ is clearly realizable in Q_l so that the generalized character $R^1(w)$ is virtually realizable in Q_l. Therefore, for any unipotent character χ of G^F, $m_{Q_l}(\chi)$ divides $(R^1(w), \chi)_{G^F}$.

By [5, Proposition 7.10], we see that in order to investigate the rationality properties of the unipotent characters of G^F we may assume that G is a simple adjoint group.

Assume therefore that G is a simple adjoint group. Then G^F is isomorphic to ${}^3D_4(q^3)$. Following the notation of Spaltenstein [19], the unipotent characters of G^F are $[1] = 1_{G^F}$, $[\varepsilon_1]$, $[\varepsilon_2]$, $[\varepsilon] = St_G$, $[\rho_1]$, $[\rho_2]$, ${}^3D_4[-1]$ and ${}^3D_4[1]$. The first six characters are the irreducible components of ${1_{B^{*F}}}^{G^F}$, so that, by a result of Benson and Curtis [2], we see that they are realizable in Q. By a result of Lusztig ([12, (7.6)], we see that the character ${}^3D_4[-1]$ is also realizable in Q. And, by an argument similar to that in the proof of the theorem in [17], we can prove that the character ${}^3D_4[1]$ is realizable in Q.

In the case where $p = 2$, we can also argue as follows. Assume that $p = 2$ and $G^F = {}^3D_4(q^3)$. Then G^F contains exactly $q^{16} + q^{12} - q^4 - 1$ involutions and this number is equal to the sum of the degrees of the irreducible characters of G^F minus 1 (Gow's observation). Thus all irreducible characters of G^F are real-valued and have the Schur index 1 over R (a theorem of Frobenius and Schur [8]). Let χ be any unipotent character of G^F. Then we see from [19] that χ is of rational-valued and that there is some $w \in W$ such that $(R^1(w), \chi)_{G^F} = \pm 1$. Therefore we have $m_{Q_l}(\chi) = 1$ for any prime number $l \neq 2$ and $m_R(\chi) = 1$. Therefore, by Hasse's sum formula, we must have $m_{Q_2}(\chi) = 1$. Hence $m_Q(\chi) = 1$. We also note that, since all irreducible characters of G^F are real, by the Baruer-Speiser theorem, we see that they have the Schur indices at most two over Q, so that, since any semisimple character of G^F has add degree, we see that it has the Schur index 1 over Q.

Assume that $p = 2$ and $G^F = {}^3D_4(q^3)$. Then, in view of the table an page 53 of Deriziotis and Michler [6], we find that the remaining characters are $\chi_{4,qs}$ and $\chi_{9,qs'}$.

We use the notation of [19] freely. Let $A = \{x_8(t)x_9(t^q)x_{10}(t^{q^2}) \mid t \in F_{q^3}\}$, an elementary abelian 2-subgroup of G^F, of order q^3. For $t \neq 0$, the element $x_8(t)x_9(t^q)x_{10}(t^{q^2})$ belongs to the class $3A_1$. Let μ be any non-principal complex linear character of A. Then μ^{G^F} is clearly realizable in Q. We have

$$\begin{aligned}\left(\mu^{G^F}, \chi_{4,qs}\right)_{G^F} &= (\mu, \chi_{4,qs} \mid A)_A \\ &= \frac{1}{q^3}\{\chi_{4,qs}(1) - \chi_{4,qs}(3A_1)\} \\ &= q^8 - q^6 + 2q^5 + q^4 - 2q^3 + q^2 + q - 1 \\ &\not\equiv 0 \qquad (\text{mod } 2)\end{aligned}$$

and

$$\left(\mu^{G^F}, \chi_{9,qs'}\right)_{G^F} = q^8 - q^6 - 2q^5 + q^4 + 2q^3 + q^2 - q - 1 \not\equiv 0 \qquad (\text{mod } 2).$$

Therefore, by the property of the Schur index, we find that $m_Q(\chi_{4,qs})$ and $m_Q(\chi_{9,qs'})$ are relatively prime to 2. On the other hand, since these two series of characters are real valued, they have the Schur indices at most two over Q. Therefore we conclude that $m_Q(\chi_{4,qs}) = m_Q(\chi_{9,qs'}) = 1$.

This completes the proof of Theorem 3 when q is even.

REMARK. There is an alternative proof of Theorem 3 when q is odd. Assume that $p \neq 2$ and that G is an adjoint simple algebraic group, defined over F_q, of type $({}^3D_4)$ and F is the corresponding Frobenius endomorphism of G. Then we see from results of Geck [9], that, for any complex irreducible character χ of G^F, the greatest common divisor of the multiplicities of χ in the generalized Gel'fand-Graev characters of G^F is equal to one. On the other hand, we can prove that each generalized Gel'fand-Graev character of G^F is realizable in Q. Therefore, by the property of the Schur index, we can conclude that $m_Q(\chi) = 1$ for any complex irreducible character χ of G^F.

By the same argument, we can prove that any complex irreducible character of $GL_n(F_q)$ (q is a power of any prime number p) has the Schur index 1 over Q (this is a well known result of Zelevinsky [21]).

Added in the proof (26 Aug. 2003): After this paper had been accepted for publication, I knew the existence of the following paper:

M. Geck: *Character values, Schur indicates and character sheaves*, Representation Theory **7** (2003), 19–55, An Electronic Journal of the American Mathematical Society (Print form in 2001).

In it, it is established the existence of the unipotent representation of ${}^2F_4(q^2)$ of the Schur index 2.

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