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ON THE SCHUR INDICES OF CHARACTERS OF FINITE REDUCTIVE GROUPS IN BAD CHARACTERISTIC CASES

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1.

Let ${}^2F_4(q^2)$ be the finite Ree group of type (F_4) , where $q^2 = 2^{2n+1}$. One of the original motivation of writing this paper is to get informations about the Schur indices of the complex irreducible characters of ${}^2F_4(q^2)$.

Let G^F be a finite reductive group. That is, G is a connected, reductive linear algebraic group over an algebraic closure K of the prime field F_p of characteristic p , F is a surjective endomorphism of G such that some power F^d of F is the Frobenius endomorphism of G relative to a rational structure on G over a finite subfield of K , and G^F is the group of F -fixed points of G (cf. Carter [3, p. 31]). Then we say that a complex irreducible character χ of G^F is regular if it is an irreducible component of a Gel'fand-Graev character of G^F and that χ is semisimple if it is the dual character of a regular character of G^F (up to ± 1) in the sense of Curtis and Kawanaka ([4, 11]). In [15, 16], we obtained some results on the Schur indices of the regular characters of G^F and, under the assumption that p is a good prime for G , of the semisimple characters of G^F . The first purpose of this paper is to drop out this assumption. Thus, in particular, we see that any semisimple character of ${}^2F_4(q^2)$ has the Schur index 1. (It is clear that any regular character of ${}^2F_4(q^2)$ has the Schur index 1.)

Our second purpose is to give a proof of the following theorem when $p = 2$.

Theorem (cf. M.J.J. Barry [1]). *Any complex irreducible character of the Steinberg's triality group ${}^3D_4(q^3)$ has the Schur index 1.*

We note that the theorem is proved by Barry when $p \neq 2$ ([1]) and that when $p = 2$ R. Gow has determined the Schur indices of the regular characters and the semisimple characters of ${}^3D_4(q^3)$. But the latter results can be also obtained from the first results of this paper.

NOTATION. If χ is an absolutely irreducible character of a finite group over an algebraically closed field of characteristic 0 and if k is a field of characteristic 0, then $m_k(\chi)$ denotes the Schur index of χ with respect to k , where we consider χ as a character over an algebraically closed extension of k . If l is a prime number, then $\overline{\mathbb{Q}}_l$ de-

notes an algebraic closure of the l -adic number field Q_l .

2.

Let K be an algebraic closure of the prime field F_p of characteristic p , G a connected, reductive linear algebraic group over K , F a surjective endomorphism of G such that some power F^d of F is the Frobenius endomorphism of G relative to a rational structure on G over a finite subfield of K , and G^F the group of F -fixed points of G . Let B^* be an F -stable Borel subgroup of G and T^* an F -stable maximal torus of G contained in B^* . Let U^* be the unipotent radical of B^* . Let R be the root system of G with respect to T^* , and, for $\alpha \in R$, let U_α^* denote the root subgroup of G corresponding to α . Let $R^+ = \{\alpha \in R \mid U_\alpha^* \subset B^*\}$ be the set of positive roots determined by B^* , and let S be the set of corresponding simple roots. Let ρ be the permutation on R given by $F(U_\alpha^*) = U_{\rho\alpha}^*$; we have $\rho(R^+) = R^+$ and $\rho(S) = S$. Let I be the set of orbits of ρ on S . Let U_\cdot^* be the subgroup of U^* generated by the root subgroups U_α^* corresponding to the non-simple positive roots α . Then we have $U^*/U_\cdot^* = \prod_{\alpha \in S} U_\alpha^* = \prod_{i \in I} U_i^*$, where, for $i \in I$, $U_i^* = \prod_{\alpha \in i} U_\alpha^*$, and $U^{*F}/U_\cdot^{*F} = (U^*/U_\cdot^*)^F = \prod_{i \in I} U_i^{*F}$. Let Λ be the set of all complex linear characters λ of U^{*F} such that $\lambda|_{U_\cdot^{*F}} = 1$, and let Λ_0 be the set of all linear characters λ in Λ such that $\lambda|_{U_i^{*F}} \neq 1$ for all $i \in I$. For $\lambda \in \Lambda_0$, let $\Gamma_\lambda = \lambda^{G^F} = \text{Ind}_{U^{*F}}^{G^F}(\lambda)$, which we call a Gel'fand-Graev character of G^F . It is well known that any Gel'fand-Graev character of G^F is multiplicity-free (Gel'fand-Graev, Yokonuma, Steinberg; see Deligne and Lusztig [5, Theorem 10.7] and Carter [3, Theorem 8.1.3]). We say that a complex irreducible character of G^F is regular if it is an irreducible component of a Gel'fand-Graev character of G^F and that a complex irreducible character of G^F is semisimple if it is the dual character of a regular character of G^F (up to ± 1) in the sense of Curtis ([4]) and Kawanaka ([11]) (see Carter [3, §8.2]).

Assume that the centre Z of G is connected. Then $\Gamma_G = \Gamma_\lambda$ is independent of $\lambda \in \Lambda_0$ and any regular or semisimple character of G^F is expressed as a Q -linear combination of the Deligne-Lusztig virtual characters R_T^θ (Deligne and Lusztig [5, Theorem 10.7]; see also Carter [3, §8.4]). The degree of any semisimple character of G^F is coprime to p and when p is a good prime for G a complex irreducible character of G^F is semisimple if and only if its degree is coprime to p (see Carter [3, p. 280]).

Let us consider the case that Z is not necessarily connected. Then we still have:

Lemma 1. *Assume that G is defined over a finite subfield of K and F is the corresponding Frobenius endomorphism of G . Let χ be a complex irreducible character of G^F . Then, if χ is semisimple, its degree is coprime to p . When p is a good prime for G , χ is semisimple if and only if its degree is coprime to p .*

Proof. We embed G in a connected, reductive group G_1 with connected centre and the same derived group (cf. Deligne and Lusztig [5, 5.18]). Let χ_1, \dots, χ_t be the G_1^F -conjugates of χ . Then, by Clifford theory, we see that there is a complex irreducible character θ of G_1^F and a positive integer e such that $\theta|_{G^F} = e(\chi_1 + \dots + \chi_t)$. According to a result of Lusztig ([14, Proposition 10]), we have $e = 1$. Assume that χ is semisimple. Then by a result of Digne, Lehrer and Michel [7, (3.15.3)], we see that one can assume that θ is a semisimple character of G_1^F . Since the centre of G_1 is connected, the degree of θ is coprime to p . Hence the degree of χ must be coprime to p . Assume that p is a good prime for G and that the degree of χ is coprime to p . Then the degree of θ is also coprime to p so that it must be semisimple. Hence, by [loc. cit.], χ must be semisimple. \square

Let J be any subset of I . Let $P(J) = \langle B^*, U_{-\alpha}^* \mid \alpha \in i, i \in J \rangle$, and $L(J) = \langle T^*, U_{\alpha}^*, U_{-\alpha}^* \mid \alpha \in i, i \in J \rangle$. Let $U(J)$ be the unipotent radical of $P(J)$. For a character $\lambda \in \Lambda_0$, let $\lambda(J) = (\lambda \mid (U^{*F} \cap L(J)^F)) \times 1_{U(J)^F}$, a linear character in Λ .

Let $\lambda \in \Lambda_0$, and let Δ_λ be the dual (generalized) character of Γ_λ . Then by [7, (2.12.2)], we have

$$(2.1) \quad \Delta_\lambda = \sum_{J \subset I} (-1)^{|J|} \lambda(J)^{G^F},$$

where the sum is taken over all the subsets J of I . (In [7], it is assumed that G is defined over a finite subfield of K and F is the corresponding Frobenius endomorphism of G . But (2.12.2) in [7] still holds in our case.) Since Γ_λ is multiplicity-free, by a result of Curtis, Alvis and Kawanaka (See Carter [3, Theorem 8.2.1]), we must have

$$(2.2) \quad \Delta_\lambda = \varepsilon_1 \chi_1 + \dots + \varepsilon_m \chi_m,$$

where $m = (\Gamma_\lambda, \Gamma_\lambda)_{G^F}$, $\varepsilon_i = \pm 1$ ($1 \leq i \leq m$) and χ_1, \dots, χ_m are distinct irreducible (semisimple) characters of G^F .

Let H be a finite group, k a field of characteristic 0 and C an algebraically closed extension of k . Let ξ be a generalized character of H over C . Then we say that ξ is virtually realizable in k if it can be written as $a_1 \xi_1 + \dots + a_n \xi_n$, where a_1, \dots, a_n are rational integers and ξ_1, \dots, ξ_n are proper characters of H which are realizable in k . In this case, if χ is an absolutely irreducible character of H over C , then, by a property of the Schur index, we see that $m_k(\chi)$ divides each multiplicity $(\xi_i, \chi)_H$ ($1 \leq i \leq n$), so that $m_k(\chi)$ divides the inner product $(\xi, \chi)_H$.

Suppose that k is a field of characteristic 0 such that for any $\lambda \in \Lambda$, λ^{G^F} is realizable in k . Then, by (2.1), we see that, for any $\lambda \in \Lambda_0$, Δ_λ is virtually realizable in k , so that, by (2.2), we have $m_k(\chi) = 1$ for any semisimple character χ of G^F .

Lemma 2 (cf. [15, 16]). *Let $\lambda \in \Lambda$. Then we have the following:*

- (i) *If $p = 2$, then λ^{G^F} is realizable in \mathbb{Q} . Assume that $p \neq 2$,*

(ii) Let $k = Q(\sqrt{(-1)^{(p-1)/2}p})$. Then, if $p \equiv -1 \pmod{4}$, λ^{G^F} is realizable in k , and if $p \equiv 1 \pmod{4}$, for any finite place v of k , λ^{G^F} is realizable in the completion k_v of k at v .

(iii) Assume that G is defined over a finite subfield with q elements of K where q is an even power of p and F is the corresponding Frobenius endomorphism of G . Then, for each prime number $l \neq p$, λ^{G^F} is realizable in Q_l .

Assume that Z is connected.

(iv) For each prime number $l \neq p$, λ^{G^F} is realizable in Q_l .

(v) Assume that Z^F is trivial or that G is defined and split over a finite subfield of K and F is the corresponding Frobenius endomorphism of G . Then λ^{G^F} is realizable in Q .

Since U^{*F}/U^{*F} is an elementary abelian p -group, λ is realizable in $Q(\zeta_p)$, where ζ_p is a primitive p -th root of unity. Thus, if $p = 2$, λ is realizable in Q , hence λ^{G^F} is realizable in Q (i). Assume that $p \neq 2$. Then (iii) is proved in [16] and (iv), (v) are proved in [15]. (ii) is proved in [16] when G is defined over a finite subfield of K and F is the corresponding Frobenius endomorphism of G . Therefore it remains to prove

Lemma 3 (cf. [16, Lemma 2]). *Assume that $p \neq 2$. Let ν be a generator of the cyclic group F_p^\times . Then there is an element t in T^{*F} such that $t^{p-1} = 1$ (possibly $t^{(p-1)/2} = 1$) and $\alpha(t) = \nu^2$ for all simple roots α .*

Proof. We repeat the proof of Lemma 2 in [16].

Firstly, we observe that it suffices to prove the lemma for the derived group G' of G . Let $\pi: \tilde{G} \rightarrow G'$ be the simply-connected covering of G' . Then, by [20, 9.16], we see that there exists a unique isogeny $\tilde{F}: \tilde{G} \rightarrow \tilde{G}$ such that $\pi \circ \tilde{F} = F \circ \pi$. We see that if F^d is the Frobenius endomorphism of G' corresponding to a rational structure on G' over a finite subfield F_q of K , then \tilde{F}^d is the Frobenius endomorphism of G corresponding to a rational structure on \tilde{G} over F_q (cf. Satake [18, Remark 5, p. 63]). Then, by the argument in the proof of Lemma 2 in [16], we can be reduced to the case that G is a simply connected simple algebraic group. If G is defined over a finite subfield of K and F is the corresponding Frobenius endomorphism of G , then Lemma 3 is just Lemma 2 in [16]. Therefore, since $p \neq 2$, it remains to treat the case where $p = 3$, $G = G_2$ and F is an exceptional isogeny such that F^2 is the Frobenius endomorphism of G corresponding to a rational structure on G over a finite subfield of K with 3^{2n+1} elements (i.e. $G^F = {}^2G_2(q^2)$). But, in this case, G is an adjoint group, so the assertion is proved in [15] (this case is also implicit in Gow [10, Theorem 9]). □

By Lemma 2, we get

Theorem 1 (cf. [15, 16]). *Let χ be a complex irreducible character of G^F such that $(\lambda^{G^F}, \chi)_{G^F} = 1$ for some $\lambda \in \Lambda$ (e.g. χ is regular) or that χ is semisimple. Then we have the following:*

- (i) *If $p = 2$, then we have $m_Q(\chi) = 1$.*
- (ii) *Let $k = Q(\sqrt{(-1)^{(p-1)/2}p})$. Then, if $p \equiv -1 \pmod{4}$, we have $m_k(\chi) = 1$, and if $p \equiv 1 \pmod{4}$, for any finite place v of k , we have $m_{k_v}(\chi) = 1$. Thus we have $m_Q(\chi) \leq 2$.*
- (iii) *Assume that G is defined over a subfield with q elements of K where q is an even power of p and F is the corresponding Frobenius endomorphism of G . Then, for each prime number $l \neq p$, we have $m_{Q_l}(\chi) = 1$.*
Assume that Z is connected.
- (iv) *For each prime number $l \neq p$, we have $m_{Q_l}(\chi) = 1$.*
- (v) *Assume that Z^F is trivial or that G is defined and split over a finite subfield of K and F is the corresponding Frobenius endomorphism of G . Then we have $m_Q(\chi) = 1$.*

REMARK. Let χ be a semisimple character of G^F . Then, in [15, 16], Theorem 1 is proved by a different method under the assumption that p is a good prime for G .

EXAMPLE. By Theorem 1, we see that any regular or semisimple character of the Ree group ${}^2F_4(q^2)$ of type (F_4) has the Schur index 1. We can also determine the local Schur indices of any unipotent character of ${}^2F_4(q^2)$. There is just one unipotent character χ of ${}^2F_4(q^2)$ such that $m_R(\chi) = m_{Q_2}(\chi) = 2$ and $m_{Q_l}(\chi) = 1$ for each prime number $l \neq 2$. This character has the property that it occurs with even multiplicity in each Deligne-Lusztig virtual character R_T^1 (cf. [13]). Other unipotent characters of ${}^2F_4(q^2)$ have the Schur index 1.

By the proof of Lemma 2 in [15, 16] and by Schur's lemma, we get

Proposition 1. *Assume that $p \neq 2$. Let χ be as in Theorem 1 and assume that χ is trivial on Z^F . Let $k = Q(\sqrt{(-1)^{(p-1)/2}p})$. Then we have $m_k(\chi) = 1$. If Z is connected or if G is defined over a finite with q elements of K where q is an even power of p and F is the corresponding Frobenius endomorphism of G , then we have $m_Q(\chi) = 1$.*

By Lemma 4 of [16], we get

Theorem 2. *Assume that $p \neq 2$ and let χ be as in Theorem 1. Let G be such that G/Z is a simple algebraic group of any one of the following types: A_r with $2|r$ or $\text{ord}_2(r+1) > \text{ord}_2(p-1)$; 2A_r with $2|r$; B_r with $4|r(r+1)$; D_r with either (a) $4|r(r-1)$ or (b) $\text{ord}_2(r-1) = 1$ and $p \equiv -1 \pmod{4}$; 2D_r with $4|r(r-1)$; 3D_4 ; E_6 ; 2E_6 . Then we have $m_Q(\chi) = 1$.*

3.

In this section we shall give a proof of the following theorem when $p = 2$.

Theorem 3 (cf. Barry [1] for $p \neq 2$). *Any complex irreducible character of ${}^3D_4(q^3)$ has the Schur index 1 over Q .*

Let q be a power of any fixed prime number p . Let G be a connected, reductive algebraic group, defined over the subfield F_q with q elements of K (an algebraic closure of F_p), with Frobenius endomorphism F such that G/Z is a simple algebraic group of type $({}^3D_4)$.

Firstly, by Theorems 1, 2, we see that any regular or semisimple character of G^F has the Schur index 1 over Q (in the case where $G^F = {}^3D_4(q^3)$ with q even, the rationality of the semisimple characters of G^F has been already observed by Gow; see below).

Next, we treat the unipotent characters of G^F . Let B^*, T^* be as in §2. Let $W = N_G(T^*)/T^*$ be the Weyl group of G , where $N_G(T^*)$ is the normalizer of T^* in G . Let X be the variety of all Borel subgroups of G . Let l be any fixed prime number different from p . Let $w \in W$, and let \dot{w} be an element of $N_G(T^*)$ such that $\dot{w}T^* = w$ in W . Then, for $B, B' \in X$, we write $B \xrightarrow{w} B'$ if there is an element g of G such that $B = gB^*g^{-1}$ and $B' = g\dot{w}B^*\dot{w}^{-1}g^{-1}$. Let $X(w)$ be the subvariety of X which consists of all $B \in X$ such that $B \xrightarrow{w} F(B)$. Then $X(w)$ is smooth and purely of dimension $l(w)$, where $l(\)$ denotes the length function on W with respect to the simple reflections determined by B^* (see Deligne and Lusztig [5, 1.4]). G^F acts on $X(w)$ by conjugation, so G^F acts on each i -th l -adic cohomology group with compact support $H_c^i(X(w), Q_l)$ of $X(w)$ ($0 \leq i \leq 2l(w)$). For $0 \leq i \leq 2l(w)$, let $H_c^i(X(w)) = H_c^i(X(w), \overline{Q}_l) = H_c^i(X(w), Q_l) \otimes_{Q_l} \overline{Q}_l$, and let

$$R^1(w) = \sum_{i=0}^{2l(w)} (-1)^i H_c^i(X(w))$$

(an element of the Grothendieck group of representations of G^F over \overline{Q}_l). Then the character of $R^1(w)$ has rational integral values, independent of l ([5, (3.3)]). So we can regard $R^1(w)$ as a generalized complex character of G^F . We say that a complex irreducible character χ of G^F is unipotent if $(R^1(w), \chi)_{G^F} \neq 0$ for some $w \in W$ ([5, 7.8]).

Let $w \in W$, and, for $0 \leq i \leq 2l(w)$, let $\xi_i(w)$ be the character of the G^F -module $H_c^i(X(w))$. Then $\xi_i(w)$ is clearly realizable in Q_l so that the generalized character $R^1(w)$ is virtually realizable in Q_l . Therefore, for any unipotent character χ of G^F , $m_{Q_l}(\chi)$ divides $(R^1(w), \chi)_{G^F}$.

By [5, Proposition 7.10], we see that in order to investigate the rationality properties of the unipotent characters of G^F we may assume that G is a simple adjoint

group.

Assume therefore that G is a simple adjoint group. Then G^F is isomorphic to ${}^3D_4(q^3)$. Following the notation of Spaltenstein [19], the unipotent characters of G^F are $[1] = 1_{G^F}$, $[\varepsilon_1]$, $[\varepsilon_2]$, $[\varepsilon] = St_G$, $[\rho_1]$, $[\rho_2]$, ${}^3D_4[-1]$ and ${}^3D_4[1]$. The first six characters are the irreducible components of $1_{B^{*F}G^F}$, so that, by a result of Benson and Curtis [2], we see that they are realizable in \mathcal{Q} . By a result of Lusztig ([12, (7.6)]), we see that the character ${}^3D_4[-1]$ is also realizable in \mathcal{Q} . And, by an argument similar to that in the proof of the theorem in [17], we can prove that the character ${}^3D_4[1]$ is realizable in \mathcal{Q} .

In the case where $p = 2$, we can also argue as follows. Assume that $p = 2$ and $G^F = {}^3D_4(q^3)$. Then G^F contains exactly $q^{16} + q^{12} - q^4 - 1$ involutions and this number is equal to the sum of the degrees of the irreducible characters of G^F minus 1 (Gow's observation). Thus all irreducible characters of G^F are real-valued and have the Schur index 1 over R (a theorem of Frobenius and Schur [8]). Let χ be any unipotent character of G^F . Then we see from [19] that χ is of rational-valued and that there is some $w \in W$ such that $(R^1(w), \chi)_{G^F} = \pm 1$. Therefore we have $m_{Q_l}(\chi) = 1$ for any prime number $l \neq 2$ and $m_R(\chi) = 1$. Therefore, by Hasse's sum formula, we must have $m_{Q_2}(\chi) = 1$. Hence $m_{\mathcal{Q}}(\chi) = 1$. We also note that, since all irreducible characters of G^F are real, by the Baruer-Speiser theorem, we see that they have the Schur indices at most two over \mathcal{Q} , so that, since any semisimple character of G^F has add degree, we see that it has the Schur index 1 over \mathcal{Q} .

Assume that $p = 2$ and $G^F = {}^3D_4(q^3)$. Then, in view of the table an page 53 of Deriziotis and Michler [6], we find that the remaining characters are $\chi_{4,qs}$ and $\chi_{9,qs'}$.

We use the notation of [19] freely. Let $A = \{x_8(t)x_9(t^q)x_{10}(t^{q^2}) \mid t \in F_{q^3}\}$, an elementary abelian 2-subgroup of G^F , of order q^3 . For $t \neq 0$, the element $x_8(t)x_9(t^q)x_{10}(t^{q^2})$ belongs to the class $3A_1$. Let μ be any non-principal complex linear character of A . Then μ^{G^F} is clearly realizable in \mathcal{Q} . We have

$$\begin{aligned} \left(\mu^{G^F}, \chi_{4,qs}\right)_{G^F} &= (\mu, \chi_{4,qs} \mid A)_A \\ &= \frac{1}{q^3} \{ \chi_{4,qs}(1) - \chi_{4,qs}(3A_1) \} \\ &= q^8 - q^6 + 2q^5 + q^4 - 2q^3 + q^2 + q - 1 \\ &\not\equiv 0 \pmod{2} \end{aligned}$$

and

$$\left(\mu^{G^F}, \chi_{9,qs'}\right)_{G^F} = q^8 - q^6 - 2q^5 + q^4 + 2q^3 + q^2 - q - 1 \not\equiv 0 \pmod{2}.$$

Therefore, by the property of the Schur index, we find that $m_{\mathcal{Q}}(\chi_{4,qs})$ and $m_{\mathcal{Q}}(\chi_{9,qs'})$ are relatively prime to 2. On the other hand, since these two series of characters are real valued, they have the Schur indices at most two over \mathcal{Q} . Therefore we conclude

that $m_Q(\chi_{4,qs}) = m_Q(\chi_{9,qs'}) = 1$.

This completes the proof of Theorem 3 when q is even.

REMARK. There is an alternative proof of Theorem 3 when q is odd. Assume that $p \neq 2$ and that G is an adjoint simple algebraic group, defined over F_q , of type $({}^3D_4)$ and F is the corresponding Frobenius endomorphism of G . Then we see from results of Geck [9], that, for any complex irreducible character χ of G^F , the greatest common divisor of the multiplicities of χ in the generalized Gel'fand-Graev characters of G^F is equal to one. On the other hand, we can prove that each generalized Gel'fand-Graev character of G^F is realizable in Q . Therefore, by the property of the Schur index, we can conclude that $m_Q(\chi) = 1$ for any complex irreducible character χ of G^F .

By the same argument, we can prove that any complex irreducible character of $GL_n(F_q)$ (q is a power of any prime number p) has the Schur index 1 over Q (this is a well known result of Zelevinsky [21]).

Added in the proof (26 Aug. 2003): After this paper had been accepted for publication, I knew the existence of the following paper:

M. Geck: *Character values, Schur indices and character sheaves*, Representation Theory **7** (2003), 19–55, An Electronic Journal of the American Mathematical Society (Print form in 2001).

In it, it is established the existence of the unipotent representation of ${}^2F_4(q^2)$ of the Schur index 2.

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