Conformally invariant quantum field theories in two dimensions

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Abstract

The general framework to analyze the structure of two-dimensional quantum field theories with the conformal and the Kac-Moody invariance is discussed. It is shown that the stress-energy tensor naturally exhibits the Sugawara form irrespective of models. The generalization of the Knizhnik-Zamolodchikov equations for arbitrary symmetry groups is presented. As applications of the present method, the solutions of two models are studied. One is the conformally invariant SU(N) Thirring model. All multipoint correlation functions of the theory are explicitly obtained by solving the corresponding Knizhnik-Zamolodchikov equations. The other is the SU(N)×U(1) Wess-Zumino model. This model is studied in connection with the fermion theory mentioned above. As a result, the bosonization prescription of the conformally invariant SU(N) Thirring model is obtained.

CONTENTS

§1. Introduction

One of the most interesting problem in quantum field theories consists in the study of solvable models. The investigation into the exact solutions of these models have presented valuable insights on the structure of quantum field theories and the exact knowledge on the behavior of each model. Many efforts in this direction have been devoted, and some two-dimensional models were exactly solved in practice : o(4)-σ model [1], Thirring model [2,3] etc. In particular, the significance of the study of conformally invariant field theories in two dimensions [4,5] has been recently recognized due to their connections to unified theories and statistical physics.

Recently, much attention has been paid to (super)string theories [6,7] as the candidates for the unified theory of elementary forces which contain gravity. These theories can be described as a conformally invariant field theories on the world-sheet. One can also find some other applications of conformal theories in statistical physics. Many two-dimensional statistical systems (Ising model, three-state Potts model etc) at critical temperature can also be interpreted as the special kind of conformal theories (see, for example, [8]).

With the increasing of the physical interests, the method in the analysis of conformal theories have been developed from a mathematical aspect. Friedan, Qiu and Shenker [9] analyzed these theories as the representation theories of the Virasoro algebra. They showed that the conformal invariance and the unitarity severely limit the possible theories in two dimensions, and they consequently classified the unitary theories.

Furthermore, in 1984, it has been shown by Belavin, Polyakov and Zamolodchikov (B.P.Z.) [10] that the bootstrap approach based on the operator algebra hypothesis is powerful to solve conformally invariant field theories in two dimensions. Several authors applied this approach to some models and obtained successful results: for instance, to string theories [11,12], and to statistical systems [13-16]. Among others, Knizhnik and Zamolodchikov [17] investigated the Wess-Zumino model [18,19] which has a certain internal symmetry (Kac-Moody invariance) besides the conformal symmetry [20]. They found the anomalous dimensions of the Wess-Zumino fields and showed that the multipoint correlation functions satisfy special first order linear differential equations (K-Z equations) (see also [21,22]). However their derivations seem to be model-dependent and the groups of the internal symmetry are limited to simple ones.

Main purpose of this thesis is to give the general framework for the investigation of two-dimensional quantum field theories which have the conformal and the Kac-Moody invariance. We also study the conformally invariant SU(N) Thirring model and the SU(N)×U(1) Wess-Zumino model as applications.

This thesis is organized as follows. In section 2, as preliminary for the present work, we briefly review the bootstrap approach to conformally invariant field theories in two dimensions. This section is devoted to the explanation of the basic tools and concepts. In section 3, we generally study two-dimensional quantum field theories which are invariant with respect to the action of the conformal and current algebras (Kac-Moody algebras) based on the bootstrap approach. It is shown that the stress-energy tensor exhibits the current-current form (Sugawara form [23-26]) irrespective of models. The genelization of the K-Z equations for arbitrary symmetry groups is presented. In section 4, we investigate two dynamical models by applying the general framework developed in section 3. One is the N-component Dirac theories [5] which possess the conformal and the SU(N)×U(1) Kac-Moody invariance. All multipoint correlation functions are explicitly given as the solutions of the generalized K-Z equations. The other is the SU(N)×U(1) Wess-Zumino model which is another realization of the conformal and the SU(N)×U(1) Kac-Moody invariance. Our attention will be mainly devoted to studying the relation between this model and the N-component Dirac theories. As a result, we find the bosonization rules [20,27-30] of the conformally invariant SU(N) Thirring model. The last section is devoted to summary.

§2. Conformally invariant field theories

2-1) Operator algebras

In this section we review, according to the paper [10] by Belavin, Polyakov and Zamolodchikov, the bootstrap approach [31] to the conformally invariant field theories in two dimensions (see also [32]).

The bootstrap approach is based on the operator algebra hypothesis : an infinite set of operators of local fields $\{A_i\}$ which contains the identity operator I forms a closed and associative algebra :

$$A_i(\xi)A_j(0) = \sum_k C^k_{ij}(\xi)A_k(0),$$

where ξ denotes the coordinates (ξ_1,ξ_2) and the structure constants $C^k_{ij}(\xi)$ are single-valued functions (see also [33]). The above relations are understood as an expansion of the (n+2)-point correlation functions with n arbitrary :

$$\langle A_i(\xi)A_j(0)A_{l_1}(\xi_1)\dots A_{l_n}(\xi_n)\rangle$$

$$= \sum_k C^k_{ij}(\xi)\langle A_k(0)A_{l_1}(\xi_1)\dots A_{l_n}(\xi_n)\rangle.$$

2-2) The Conformal invariance and the stress-energy tensor

In quantum field theories, the conformal symmetry take place provided that the stress-energy tensor is traceless.

The conformal transformations of the coordinates ξ_μ (μ=1,...,D with D being the dimension of the space) are substitutions

$$\xi_\mu \rightarrow f_\mu(\xi), \tag{2.1}$$

under the conditions

$$\partial_\mu f_\nu(\xi) + \partial_\nu f_\mu(\xi) - \frac{2}{D}\,\delta_{\mu\nu}\partial_\rho f_\rho(\xi) = 0. \tag{2.2}$$

In two dimension (D=2), since (2.2) are regarded as the Cauthy-Riemann equations, conformal transformations are arbitrary analytic substitutions

$$z \rightarrow f(z), \qquad \bar{z} \rightarrow f(\bar{z}), \tag{2.3}$$

where z and $\bar{z}$ are complex coordinates

$$z = \xi_1 + i\xi_2, \quad \bar{z} = \xi_1 - i\xi_2, \tag{2.4}$$

and $f(\bar{f})$ is an analytic function of the single variable $z(\bar{z})$.

Hereaftrer we work in the complex space $\mathbb{C}^2$. Hence we treat the complex coordinates (2.4) as two independent complex variables, and also f and $\bar{f}$ as two unrelated analytic functions.

The infinitesimal forms of the transformations (2.3) are

$$z \rightarrow z + \varepsilon(z), \quad \bar{z} \rightarrow \bar{z} + \bar{\varepsilon}(\bar{z}), \tag{2.5}$$

where $\varepsilon(z)$ and $\bar{\varepsilon}(\bar{z})$ are infinitesimal analytic functions.

The variation of a certain local field $A_i(\xi)$ under the infinitesimal transformation (2.5) is a linear combination of the function $\varepsilon(z)$ and a finite number of its derivatives taken at the point z :

$$\delta_\varepsilon A_i(z,\bar{z}) = \sum_{k=0}^{\nu_i} B_i^{(k-1)}(z,\bar{z}) \frac{d^k}{dz^k} \varepsilon(z), \tag{2.6}$$

where $B_i^{(k-1)}$ are local fields belonging to the set $\{A_i\}$ and ν_i is a certain integer (similarly for $\bar{z}$; the treatment of the left and right variable z and $\bar{z}$ is completely analogous, and we will usually present only the left part of equations). Remembering the transformation properties of local fields under infinitesimal translations and dilatations [34] , one finds

$$B_i^{(-1)}(z,\bar{z}) = \frac{\partial}{\partial z} A_i(z,\bar{z}), \tag{2.7a}$$

$$B_i^{(0)}(z,\bar{z}) = \Delta_i A_i(z,\bar{z}), \tag{2.7b}$$

where Δ_i is called the left conformal dimension of the field A_i (we use $\bar{\Delta}_i$ for the right conformal dimension).

In the conformally invariant field theories, the transformations (2.6) are generated by the symmetric and traceless stress-energy tensor $T(z)$ ($=T_{11}-iT_{12}$) ($\bar{T}(\bar{z})$ ($=T_{11}+iT_{12}$) for right transformations) of the theories.

Consider a correlation function of some local fields,

$$\langle X\rangle = \langle A_1(z_1,\bar{z}_1)\ldots A_n(z_n,\bar{z}_n)\rangle. \tag{2.8}$$

Then the conformal Ward identities [35,36] of the theory are written as

$$\langle\delta_\varepsilon X\rangle = \frac{1}{2\pi i}\int_C d\xi\ \varepsilon(\xi)\langle T(\xi)X\rangle, \tag{2.9}$$

where the contour C encloses all the points $z_1,\ldots,z_n$. For a single field we can then write the transformation law as

$$\delta_\varepsilon A_i(z,\bar{z}) = \frac{1}{2\pi i}\int_C d\xi\ \varepsilon(\xi)T(\xi)A(z,\bar{z}). \tag{2.10}$$

Taking account of the tensorial property of the field $T(z)$, one can write down the most general expression [4] for the variations $\delta_\varepsilon T$ and $\delta_{\bar{\varepsilon}}T$:

$$\delta_\varepsilon T(z) = \varepsilon(z)T'(z) + 2\varepsilon'(z)T(z) + \frac{c}{12}\varepsilon'''(z), \tag{2.11a}$$

$$\delta_{\bar{\varepsilon}}T(z) = 0. \tag{2.11b}$$

Let us introduce the operators L_n and $\bar{L}_n$ ($n = 0,\pm1,\pm2,\ldots$) as the coefficients of the Laurant expansions (see also Appendix A)

$$T(z) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}} , \qquad (2.12a)$$

$$\bar{T}(\bar{z}) = \sum_{n=-\infty}^{\infty} \frac{\bar{L}_n}{\bar{z}^{n+2}} . \qquad (2.12b)$$

It follows from (2.11a) that the operator L_n satisfy the commutation relations (*Virasoro algebra* [37,38])

$$[L_n , L_m] = (n - m)L_{n+m} + \frac{c}{12} (n^3 - n)\delta_{n+m,0} . \qquad (2.13)$$

The same relations are satisfied by $\bar{L}_n$'s, with the operators L_n and $\bar{L}_m$ being commutative.

2-3) Conformal families

Let us recall the equation (2.6)

$$\delta_\varepsilon A_i(z,\bar{z}) = \sum_{k=0}^{\nu_i} B_i^{(k-1)}(z,\bar{z}) \frac{d^k}{dz^k} \varepsilon(z). \qquad (2.6)$$

It is evident that the dimension of the field $B_i^{(k-1)}$ is equal to

$$\Delta_{i,(k-1)} = \Delta_i + 1 - k , \quad k = 0,1,\dots,\nu_i . \qquad (2.14)$$

In physically stable theories the dimensions of all the fields A_i should satisfy the inequality

$$\Delta_i \geq 0. \tag{2.15}$$

In what follows we assume that the only field with zero dimensions $\Delta=\bar{\Delta}=0$ is the identity operator I. From the condition (2.15), we see that the sum in (2.6) contains a finite number of terms :

$$\nu_i \leq \Delta_i + 1. \tag{2.16}$$

The spectrum of dimensions $\{\Delta_i\}$ in any two-dimensional conformal quantum field theory consists of infinite series

$$\Delta_p^{(k)} = \Delta_p + k , \qquad k = 0,1,\ldots , \tag{2.17}$$

where Δ_p denotes the minimal dimension of each series, whereas the index p labels the series (the same is valid for the dimensions $\bar{\Delta}_i$).

Let Φ_p be the field with the dimensions Δ_p and $\bar{\Delta}_p$. The variation of this field has the form

$$\delta_\varepsilon \Phi_p(z,\bar{z}) = \varepsilon(z) \frac{\partial}{\partial z} \Phi_p(z,\bar{z}) + \Delta_p \varepsilon'(z) \Phi_p(z,\bar{z}). \tag{2.18}$$

This formula is equivalent to the commutation relation

$$[L_m , \Phi_p(z,\bar{z})] = z^{m+1} \frac{\partial}{\partial z} \Phi_p(z,\bar{z}) + \Delta_p (m+1) z^m \Phi_p(z). \tag{2.19}$$

We shall call the operators Φ_p having the transformation property (2.18) *primary fields*.

The conformal Ward identities for the primary fields are

$$\langle T(z)\Phi_1(z_1)\ldots\Phi_n(z_n)\rangle$$

$$= \sum_{i=1}^{n} \{ \frac{\Delta_i}{(z-z_i)^2} + \frac{1}{z-z_i}\frac{\partial}{\partial z} \}\langle\Phi_1(z_1)\ldots\Phi_n(z_n)\rangle, \quad (2.20)$$

where $\Delta_1,\ldots,\Delta_n$ are the left conformal dimensions of the fields $\Phi_1,\ldots,\Phi_n$ respectively.

The primary fields themselves cannot form a closed operator algebra. From (2.17), there are infinite many other fields associated with each of the primary fields Φ_p . We shall refer to these fields as to the *secondary fields* with respect to the primary fields Φ_p .

The secondary fields associated with the primary field Φ_p are defined as

$$\Phi_p^{\{k\}\{\bar{k}\}}(z,\bar{z}) = L_{-k_1}(z)\ldots L_{-k_n}(z)L_{-\bar{k}_1}(\bar{z})L_{-\bar{k}_m}(\bar{z})\Phi_p(z,\bar{z}), \quad (2.21)$$

for

$$\{k\} = (k_1,\ldots,k_n) , \qquad \{\bar{k}\} = (\bar{k}_1,\ldots,\bar{k}_m), \quad (2.22)$$

where $k_i,\bar{k}_j \geq 1$ ($i=1,\ldots,n$, $j=1,\ldots,m$, with n and m taking all natural numbers), and

$$L_{-k_i}(z) = \int_C d\xi_i \; T(\xi_i)(\xi_i - z)^{-k+1}, \tag{2.23a}$$

$$\bar{L}_{-\bar{k}_j}(\bar{z}) = \int_C d\bar{\xi}_j \; \bar{T}(\bar{\xi}_j)(\bar{\xi}_j - \bar{z})^{-k+1}. \tag{2.23b}$$

The integration contours C associated with each of the operators L_{-k_i} ($\bar{L}_{-\bar{k}_j}$) in (2.23) enclose the point z ($\bar{z}$) as well as the points ξ_{i+1} ($\bar{\xi}_{j+1}$),...,ξ_n ($\bar{\xi}_m$). The operators L_n introduced in (2.12) are no other than $L_n(0)$. The dimensions of the fields (2.21) are

$$\Delta_p^{\{k\}} = \Delta_p + k_1 + \dots + k_n , \tag{2.24a}$$

$$\bar{\Delta}_p^{\{\bar{k}\}} = \bar{\Delta}_p + \bar{k}_1 + \dots + \bar{k}_m . \tag{2.24b}$$

An infinite set of the fields (2.21) constitutes the *conformal family* $[\Phi_p]$.

2-4) Conformal properties of operator algebras

Consider the product of two primary fields $\Phi_n(\xi)\Phi_m(0)$. The operator product expansion can be represented as

$$\Phi_n(z,\bar{z})\Phi_m(0,0) = \sum_p \sum_{\{k\}\{\bar{k}\}} C_{nm}^{p\{k\}\{\bar{k}\}}$$

$$\times\, z^{\Delta_p-\Delta_n-\Delta_m+\sum k_i}\bar{z}^{\bar{\Delta}_p-\bar{\Delta}_n-\bar{\Delta}_m+\sum\bar{k}_i}\Phi_p^{\{k\}\{\bar{k}\}}(0,0). \tag{2.25}$$

Both sides of (2.25) should exhibit the same conformal properties. The transformation law of the left-hand side is determined by (2.18), and that of the right-hand side can be derived from (2.21). The requirement of the conformal invariance of (2.25) leads to the relations among the numerical constants $C_{nm}^{p\{k\}\{\bar{k}\}}$ with different {k}'s but with the same indeces p, n and m. These relations can be solved in the factorized form

$$C_{nm}^{p\{k\}\{\bar{k}\}} = C_{nm}^{p}\beta_{nm}^{p\{k\}}\bar{\beta}_{nm}^{p\{\bar{k}\}}, \tag{2.26}$$

where the conditions $\beta_{nm}^{p\{o\}} = \beta_{nm}^{p\{0\}} = 1$ is implied (see Appendix B). Thus the expression (2.25) can be written as

$$\Phi_n(z,\bar{z})\Phi_m(0,0) = \sum_p C_{nm}^{p} z^{\Delta_p-\Delta_n-\Delta_m}\bar{z}^{\bar{\Delta}_p-\bar{\Delta}_n-\bar{\Delta}_m}\,\Psi_p(z,\bar{z}|0,0), \tag{2.27}$$

where

$$\Psi_p(z,\bar{z}|0,0) = \sum_{\{k\}\{\bar{k}\}} \beta_{nm}^{p\{k\}}\bar{\beta}_{nm}^{p\{\bar{k}\}} z^{\sum k_i}\bar{z}^{\sum\bar{k}_i}\Phi_p^{\{k\}\{\bar{k}\}}(0,0). \tag{2.28}$$

§3. Quantum field theories with the conformal and the Kac-Moody invariance [39]

3-1) The conformal and the Kac-Moody invariance

Knizhnik and Zamolodchikov [17] investigated the Wess-Zumino model which has a certain internal symmetry (Kac-Moody invariance) besides the conformal symmetry. They found the anomalous dimensions of the Wess-Zumino fields and showed that the multipoint correlation functions satisfy some special first order linear differential equations (*K-Z equations*). However their derivations seem to be model-dependent and the groups of internal symmetries are limited to simple ones.

In this section, we generally study two-dimensional quantum field theories which is invariant with respect to the action of the conformal and also arbitrary current algebras (*Kac-Moody algebras*), irrespective of models. Our investigations are based on the bootstrap approach developed by B.P.Z. (reviewed in sec.2). The generalization of the current-current form (*Sugawara form*) of the stress-energy tensor and K-Z equations are found for arbitrary groups which may not be simple.

Recalling the equation (2.27) in the previous section, operator algebras in conformally invariant theories are expressed in terms of conformal families $[\Phi_n]$ as

$$\Phi_n(z,\bar{z})\Phi_m(0,0) = \sum_p C^p_{nm} z^{\Delta_p-\Delta_n-\Delta_m} \bar{z}^{\bar{\Delta}_p-\bar{\Delta}_n-\bar{\Delta}_m}$$

$$\times \sum_{\{k\}\{\bar{k}\}} \beta_{nm}^{p\{k\}} \bar{\beta}_{nm}^{p\{\bar{k}\}} z^{\sum k_i} \bar{z}^{\sum \bar{k}_i} \Phi_p^{\{k\}\{\bar{k}\}}(0,0). \tag{3.1}$$

Left conformal transformations for the primary field Φ_n are generated by the traceless stress-energy tensor $T(z)$ ($\bar{T}(\bar{z})$ for right transformations) :

$$\delta_\varepsilon \Phi_n(z,\bar{z}) = \frac{1}{2\pi i} \int_C d\xi \ T(\xi)\varepsilon(\xi)\Phi_n(z,\bar{z})$$

$$= \varepsilon(z)\partial_z \Phi_n(z,\bar{z}) + \Delta_n \varepsilon'(z)\Phi_n(z,\bar{z}). \tag{3.2}$$

and internal transformations are generated by the conserved currents J_α^a :

$$\delta_{\omega_\alpha} \Phi_n(z,\bar{z}) = \int_C d\xi \ J_\alpha^a(\xi)\omega_\alpha^a(\xi)\Phi \ (z,\bar{z})$$

$$= t_{\alpha n}^a \omega_\alpha^a(z)\Phi_n(z,\bar{z}) \quad \text{for } \alpha = 1,\ldots,M, \tag{3.3}$$

where we consider a group $G=G_1\times\ldots\times G_M$, and $t_{\alpha n}^a$ are the left representation matrices of the algebra of G_α for Φ_n (we use $\bar{t}_{\alpha n}^a$ as the right representation matrices). The transformation prorperties of the generators themselves for the conformal and the internal group have the following forms :
for conformal transformations,

$$\delta_\varepsilon T(z) = \varepsilon(z)T'(z) + 2\varepsilon'(z)T(z) + \frac{c}{12}\varepsilon'''(z), \tag{3.4}$$

$$\delta_\varepsilon J_\alpha(z) = \varepsilon(z) J^{a\,\prime}_\alpha(z) + \varepsilon'(z) J^a_\alpha(z), \tag{3.5}$$

and for internal transformations,

$$\delta_{\omega_\alpha} T(z) = \omega^{a\,\prime}_\alpha(z) J^a_\alpha(z), \tag{3.6}$$

$$\delta_{\omega_\alpha} J^a_\alpha(z) = f^{abc}_\alpha \omega^b_\alpha(z) J^c_\alpha(z) - \frac{k_\alpha}{2} \omega_\alpha(z), \tag{3.7}$$

where f^{abc}_α are structure constants of the group G_α , and k_α is called the central charge of the G_α Kac-Moody algebra (3.7). The variations $\delta_{\bar\varepsilon}$ and $\delta_{\bar\omega}$ of the fields $\bar T$ and $\bar J$ are given by the same equations, whereas the variations $\delta_{\bar\varepsilon}$ (δ_ε) and $\delta_{\bar\omega}$ (δ_ω)of T ($\bar T$) and J ($\bar J$) vanish. The equations (3.2)~(3.7) can be rewritten in the form of operator product expansions :

$$T(z')\Phi_n(z,\bar z) = \frac{\Delta_n}{(z'-z)^2}\Phi_n(z,\bar z) + \frac{1}{z'-z}\partial_z\Phi_n(z,\bar z) + \text{regular terms}, \tag{3.8}$$

$$J^a_\alpha(z')\Phi_n(z,\bar z) = \frac{t^a_{\alpha n}}{z'-z}\Phi_n(z,\bar z) + \text{regular terms}, \tag{3,9}$$

$$T(z')T(z) = \frac{c}{2(z'-z)^4} + \frac{2}{(z'-z)^2} T(z) + \frac{1}{z'-z} T'(Z) + \text{regular terms}, \tag{3.10}$$

$$T(z')J_\alpha^a(z) = \frac{1}{(z'-z)^2} J_\alpha^a(z) + \frac{1}{z'-z} J_\alpha^{a\prime}(z)$$
$$+ \text{regular terms}, \qquad (3.11)$$

$$J_\alpha^a(z')T(z) = \frac{1}{(z'-z)^2} J_\alpha^a(z) + \text{regular terms}, \qquad (3.12)$$

$$J^a(z')J^b(z) = -\frac{k_\alpha \delta^{ab}}{2(z'-z)^2} - \frac{f_\alpha^{abc}}{z'-z} J_\alpha^c(z)$$
$$+ \text{regular terms}. \qquad (3.13)$$

3-2) The Sugawara form of the stress-energy tensor

We first derive the Sugawara form of the stress-energy tensor for the group $G=G_1\times\ldots\times G_M$. Consider a linear combination of the operator expansions (3.13)

$$J_1^a(z')J_1^a(z) + x_2 J_2^a(z')J_2^a(z) + \ldots + x_M J_M^a(z')J_M^a(z)$$

$$= -\frac{k_1 D_1 + x_2 k_2 D_2 + \ldots + x_M k_M D_M}{2(z'-z)^2} + \text{regular terms}, \qquad (3.14)$$

where the constants $x_2,\ldots,x_M$ are determined soon later, and D_α is the dimension of the group G_α . Since the equation (3.14) is regarded as an operator algebra of (3.1), one can know that the zeroth order's in (z'-z) of the regular part contains T(z) which belongs to the conformal family of the identity operator. In fact, by applying the method explained in Appendix B , we obtain the equations

$$\beta_{11}^{0\{1\}} = \beta_{11}^{0\{1,1\}} = 0 \ , \qquad \beta_{11}^{0\{2\}} = \frac{2}{c} \ , \tag{3.15}$$

which mean

$$J_1^a(z')J_1^a(z) + x_2 J_2^a(z')J_2^a(z) + \dots + x_M J_M^a(z')J_M^a(z)$$

$$= - \frac{k_1 D_1 + x_2 k_2 D_2 + \dots + x_M k_M D_M}{2(z'-z)^2}$$

$$- \frac{k_1 D_1 + x_2 k_2 D_2 + \dots + x_M k_M D_M}{c} T(z) + \text{regular terms.} \tag{3.16}$$

Here we show that one can consitently determine the constants $x_2,\dots,x_M$ and the central charge c setting other zeroth order's of the regular part zero. Comparison of the transformation properties of both sides of (3.16) yields some algebraic equations for the constants $x_2,\dots,x_M$ as well as the central charge c :

$$c_{V1} + k_1 = \frac{k_1 D_1 + x_2 k_2 D_2 + \dots + x_M k_M D_M}{c} \ ,$$

$$x_\beta (c_{V\beta} + k_\beta) = \frac{k_1 D_1 + x_2 k_2 D_2 + \dots + x_M k_M D_M}{c} \ ,$$

$$\text{for } \beta = 2, \dots, M \ , \tag{3.17}$$

where $c_{V\alpha}$ ($\alpha = 1,\dots,M$)is defined as

$$f_\alpha^{abc} f_\alpha^{bcd} = c_{V\alpha} \delta^{ab} \ . \tag{3.18}$$

The solution of these equations is easily obtained :

$$x_\beta = \frac{c_{V1} + k_1}{c_{V\beta} + k_\beta} \quad , \qquad \text{for } \beta = 2, \ \dots \ ,M \ , \tag{3.19}$$

$$c = \sum_{\alpha=1}^{M} \frac{k_\alpha D_\alpha}{c_{V\alpha} + k_\alpha} \ . \tag{3.20}$$

Consequently we find the Sugawara type formula in the form of operator algebra :

$$T(z) = - \sum_{\alpha=1}^{M} \frac{1}{c_{V\alpha} + k_\alpha} J^a(z')J^a(z) - \frac{c}{2(z'-z)^2} - O(z'-z), \tag{3.21}$$

where c is given by the equation (3.20). This equation can be expressed as

$$T(z) = - \sum_{\alpha=1}^{M} \frac{1}{c_{V\alpha} + k_\alpha} : J^a(z)J^a(z) :$$

$$= - \sum_{\alpha=1}^{M} \frac{1}{c_{V\alpha} + k_\alpha} \lim_{z'\to z} [\ J_\alpha^a(z')J_\alpha^a(z) + \frac{k_\alpha D_\alpha}{2(z'-z)^2} \ ]. \tag{3.22}$$

3-3) The generalization of Knizhnik-Zamolodchikov equations

Next, in the same way, taking account of the conformal properties of the operator algebras (3.9), one can consistently set the following expression :

$$t^a_{1n}J^a_1(z')\Phi_n(z,\bar{z}) + y_2 t^a_{2n}J^a_2(z')\Phi_n(z,\bar{z}) + \dots + y_M t^a_{M2}J^a_M(z')\Phi_n(z,\bar{z})$$

$$= - \frac{c_{1n} + y_2 c_{2n} + \dots + y_M c_{Mn}}{z'-z}\Phi_n(z,\bar{z})$$

$$- \frac{c_{1n} + y_2 c_{2n} + \dots + y_M c_{Mn}}{2\Delta_n}\partial_z\Phi_n(z,\bar{z}) + O(z'-z), \quad (3.23)$$

where $c_{\alpha n}$ ($\alpha=1,\dots,M$)is defined as

$$t^a_{\alpha n}t^a_{\alpha n} = - c_{\alpha n} I . \quad (3.24)$$

In fact, comparison of the transformation properties of both sides of (3.23) yields algebraic equations for the constants $y_2,\dots,y_M$ as well as the conformal dimension Δ_n of the primary field Φ_n :

$$c_{V1} + k_1 = \frac{c_{1n} + y_2 c_{2n} + \dots + y_M c_{Mn}}{\Delta_n} ,$$

$$y_\beta(c_{V\beta} + k_\beta) = \frac{c_{1n} + y_2 c_{2n} + \dots + y_M c_{Mn}}{\Delta_n} ,$$

$$\text{for } \beta = 2, \dots ,M . \quad (3.25)$$

The solution for these equations is given by the formulas

$$y_\beta = \frac{c_{V1} + k_1}{c_{V\beta} + k_\beta}, \qquad \text{for} \quad \beta = 2, \ldots, M, \tag{3.26}$$

$$\Delta_n = \sum_{\alpha=1}^{M} \frac{c_{\alpha n}}{c_{V\alpha} + k_\alpha} \quad . \tag{3.27}$$

Therefore the left and the right operator equations for the primary field Φ_n are

$$\partial_z \Phi_n(z,\bar{z}) = -2 \sum_{\alpha=1}^{M} \frac{t^a_{\alpha n}}{c_{V\alpha} + k_\alpha} J^a_\alpha(z')\Phi_n(z,\bar{z})$$

$$- 2\Delta_n \frac{1}{z'-z} \Phi_n(z,\bar{z}) + O(z'-z), \tag{3.28a}$$

$$\partial_{\bar{z}} \Phi_n(z,\bar{z}) = -2 \sum_{\alpha=1}^{M} \frac{\bar{t}^a_{\alpha n}}{c_{V\alpha} + k_\alpha} \bar{J}^a_\alpha(\bar{z}')\Phi_n(z,\bar{z})$$

$$- 2\bar{\Delta}_n \frac{1}{\bar{z}'-\bar{z}} \Phi_n(z,\bar{z}) + O(\bar{z}'-\bar{z}). \tag{3.28b}$$

The equations (3.28) can be expressed in the forms :

$$\partial_z \Phi_n(z,\bar{z}) = -2 \sum_{\alpha=1}^{M} \frac{t^a_{\alpha n}}{c_{V\alpha} + k_\alpha} : J^a_\alpha(z)\Phi_n(z,\bar{z}) :$$

$$= -2 \sum_{\alpha=1}^{M} \frac{t^a_{\alpha n}}{c_{V\alpha} + k_\alpha} \lim_{z'\to z} [\, J^a_\alpha(z') - \frac{t^a_{\alpha n}}{z'-z} \,] \, \Phi_n(z,\bar{z}). \tag{3.29a}$$

$$\partial_{\bar{z}}\Phi_n(z,\bar{z}) = -2\sum_{\alpha=1}^{M} \frac{\bar{t}^a_{\alpha n}}{c_{V\alpha} + k_\alpha} : \bar{J}^a_\alpha(\bar{z})\Phi_n(z,\bar{z}) :$$

$$= -2\sum_{\alpha=1}^{M} \frac{\bar{t}^a_{\alpha n}}{c_{V\alpha} + k_\alpha} \lim_{z'\to z} [\ \bar{J}^a_\alpha(\bar{z}') - \frac{\bar{t}^a_{\alpha n}}{z'-z}\ ]\ \Phi_n(z,\bar{z}). \qquad (3.29b)$$

Substitute the above equation (3.28a) into the Ward identity for the internal symmetries

$$\langle J^a_\alpha(z)\Phi_1(z_1,\bar{z}_1)\ldots\Phi_N(z_N,\bar{z}_N)\rangle$$

$$= \sum_{i=1}^{N} \frac{t^a_{\alpha i}}{z'-z} \langle\Phi_1(z_1,\bar{z}_1)\ldots\Phi_N(z_N,\bar{z}_N)\rangle, \qquad (3.30)$$

and one obtains the generalization of K-Z equations

$$\{\ \frac{1}{2}\ \partial_{z_i} + \sum_{j\neq i}\sum_{\alpha=1}^{M} \frac{t^a_{\alpha i}t^a_{\alpha j}}{c_{V\alpha} + k_\alpha} \frac{1}{z_i - z_j}\ \}$$

$$\times \langle\Phi_1(z_1,\bar{z}_1)\ldots\Phi_N(z_N,\bar{z}_N)\rangle = 0\ . \qquad (3.31)$$

Of course the left part corresponding to the equation (3.31) has the same form.

§4. Solutions to some models in two dimensions [42]

4-1) The Conformally invanant SU(N) Thirring model

Dashen and Frishman showed in their pioneering work [5] that N-component Dirac theories with SU(N) and U(1) current-current interactions are conformally invariant in two dimensions only for a special value of SU(N) coupling constant (the value of U(1) coupling constant remains arbitrary). They derive this result by imposing the locality requirement, which is explained later, on the conformally invariant four-point correlation functions.

In this subsection, we study conformally invariant N-component Dirac theories according to the general framework developed in the previous section. Above result of Dashen and Frishman appears, when we apply the equations (3.29) to Dirac Fermions. The anomalous dimensions of Dirac fields are found simply applying the formula (3.27) and all multipoint correlation functions are explicitly obtained as the solutions of the generalized K-Z equations.

Let us start with the definition of the theories.

The basic fields are N-component Dirac fermions :

$$\Psi_\alpha = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}_\alpha , \qquad \text{for} \quad \alpha = 1, \ldots, N . \qquad (4.1)$$

These fields Ψ_1 and Ψ_2 are assumed to be primary fields with conformal dimensions $(\Delta_1, \bar{\Delta}_1)$ and $(\Delta_2, \bar{\Delta}_2)$ respectively. These conformal dimensions will be determined later.

We now consider the model which exactly maintain the left and right SU(N)×U(1) Kac-Moody invariance as well as the conformal invariance. Therefore there are left and right conserved SU(N) (U(1)) currents which are denoted here by $J^a(z)$ ($J(z)$) and $\bar{J}^a(\bar{z})$ ($\bar{J}(\bar{z})$) respectively.

The Jacobi identities severely constrains the SU(N) transformation properties of fields. Especially, in a case that the basic spinor field Ψ transforms as the fundamental representation of SU(N), only two types of SU(N) transformation properties of Ψ_1 and Ψ_2 are compatible with the Jacobi identity :

$$\delta_\omega \Psi_1(z,\bar{z}) = \frac{1}{2}(1+\varepsilon)t^a\omega^a(z)\Psi_1(z,\bar{z}), \tag{4.2a}$$

$$\delta_{\bar{\omega}} \Psi_1(z,\bar{z}) = \frac{1}{2}(1-\varepsilon)t^a\bar{\omega}^a(\bar{z})\Psi_1(z,\bar{z}), \tag{4.2b}$$

$$\delta_\omega \Psi_2(z,\bar{z}) = \frac{1}{2}(1-\varepsilon)t^a\omega^a(z)\Psi_2(z,\bar{z}), \tag{4.2c}$$

$$\delta_{\bar{\omega}} \Psi_2(z,\bar{z}) = \frac{1}{2}(1+\varepsilon)t^a\bar{\omega}^a(\bar{z})\Psi_2(z,\bar{z}), \tag{4.2d}$$

where $\varepsilon=\pm1$ and t^a are the N × N antihermitian matrices representing the algebra of the group SU(N).

The left and the right U(1) transformations of the fields are generally written as

$$\delta_\omega \Psi_1(z,\bar{z}) = q\,\omega(z)\Psi_1(z,\bar{z}), \tag{4.3a}$$

$$\delta_{\bar{\omega}}\Psi_1(z,\bar{z}) = \bar{q}\ \bar{\omega}(\bar{z})\Psi_1(z,\bar{z}), \qquad (4.3b)$$

$$\delta_{\omega}\Psi_2(z,\bar{z}) = \bar{q}\ \omega(z)\Psi_2(z,\bar{z}), \qquad (4.3c)$$

$$\delta_{\bar{\omega}}\Psi_2(z,\bar{z}) = q\ \bar{\omega}(\bar{z})\Psi_2(z,\bar{z}), \qquad (4.3d)$$

where q and $\bar{q}$ are the left (right) and the right (left) pure imaginary U(1) charges of Ψ_1 (Ψ_2) respectively.

However we do not go into the explicit structure of these currents in terms of the basic fields. For the purpose of solving models exactly, it is not appropriate (except the free theory) to define the currents as the normal ordering of fermion bilinear forms. These definitions are relevant for perturbation study.

Kac-Moody algebras of the currents are

$$\delta_{\omega}J^a(z) = f^{abc}\omega^b(z)J^c(z) - \frac{k}{2}\ \omega^{a\prime}(z), \qquad (4.4a)$$

$$\delta_{\bar{\omega}}\bar{J}^a(\bar{z}) = f^{abc}\bar{\omega}^b(\bar{z})\bar{J}^c(\bar{z}) - \frac{k}{2}\ \bar{\omega}^{a\prime}(\bar{z}), \qquad (4.4b)$$

$$\delta_{\omega}J(z) = -\ \frac{k'}{2}\ \omega'(z), \qquad (4.4c)$$

$$\delta_{\bar{\omega}}\bar{J}(\bar{z}) = -\ \frac{k'}{2}\ \bar{\omega}'(\bar{z}), \qquad (4.4d)$$

where f^{abc} are structure constants of the group SU(N), and k and k' are the central charges for SU(N) and U(1) respectively. At

this stage we have no idea of determining the value of k, since we do not specify the explicit form of the currents.
Dashen and Frishman showed that the locality requirement of the four-point correlation function fix the value of k to be unity both in the free and interacting cases. Afterward we will return to this point. On the other hand, the U(1) central charge k' itself remains arbitrary, since it just serves as the normalizations of the currents J and $\bar{J}$.

Various results come from the applications of the general formulas, developed in the previous section, to the case with G=SU(N)×U(1). The only thing one has to do is to substitute the charge assignments (4.2) and (4.3) into the formulas.

First of all, from the equations (3.29), we obtain

$$\partial_{\bar{z}}\Psi_1(z,\bar{z}) = -(1-\varepsilon)\frac{t^a}{N+k} : \bar{J}^a(\bar{z})\Psi\ (z,\bar{z}) :$$

$$- 2\frac{\bar{q}}{k'} : \bar{J}(\bar{z})\Psi_1(z,\bar{z}) :, \qquad (4.5a)$$

$$\partial_z\Psi_2(z,\bar{z}) = -(1-\varepsilon)\frac{t^a}{N+k} : J^a(z)\Psi_2(z,z) :$$

$$- 2\frac{\bar{q}}{k'} : J(z)\Psi_2(z,z) :, \qquad (4.5b)$$

$$\partial_z\Psi_1(z,\bar{z}) = -(1+\varepsilon)\frac{t^a}{N+k} : J^a(z)\Psi_1(z,\bar{z}) :$$

$$- 2\frac{q}{k'} : J(z)\Psi_1(z,\bar{z}) :, \qquad (4.6a)$$

$$\partial_{\bar{z}}\Psi_2(z,\bar{z}) = -(1+\varepsilon)\frac{t^a}{N+k} : \bar{J}^a(\bar{z})\Psi_2(z,\bar{z}) :$$

$$- 2\frac{\bar{q}}{k'} : \bar{J}(\bar{z})\Psi_2(z,\bar{z}) :. \qquad (4.6b)$$

The equations (4.5) are regarded as the field equations for the basic fields, while the equations (4.6) are not. That is the reason why the case with $\varepsilon=1$ and $\bar{q}=0$ corresponds to the free theory, and the case with $\varepsilon=-1$ corresponds to the SU(N) Thirring model. Hereafter we concentrate on the case with $\bar{q}=0$ for simplicity, which means vanishing of the U(1) coupling constant.

The conformal dimensions of the fields Ψ_1 and Ψ_2 are given from the formula (3.27) and its right-handed version as

$$\Delta_1 = \frac{1+\varepsilon}{2}\frac{N^2-1}{2N(N+k)} + \frac{|q|^2}{k'}, \quad \bar{\Delta}_1 = \frac{1-\varepsilon}{2}\frac{N^2-1}{2N(N+k)}, \qquad (4.7a)$$

$$\Delta_2 = \frac{1-\varepsilon}{2}\frac{N^2-1}{2N(N+k)}, \quad \bar{\Delta}_2 = \frac{1+\varepsilon}{2}\frac{N^2-1}{2N(N+k)} + \frac{|q|^2}{k'}, \qquad (4,7b)$$

The combinations $d=\Delta_1+\bar{\Delta}_1=\Delta_2+\bar{\Delta}_2$ and $s=\Delta_1-\bar{\Delta}_1=-(\Delta_2-\bar{\Delta}_2)$ are the anomalous dimension and the spin of the basic spinor fields Ψ respectively. Setting the natural condition $s=1/2$, the ratio $|q|^2/k'$ and therefore the value of Δ's are fixed :

for $\varepsilon = 1$,

$$\frac{|q|^2}{k'} = \frac{Nk+1}{2N(N+k)}, \qquad (4.8a)$$

$$\Delta_1 = \frac{1}{2}, \quad \bar{\Delta}_1 = 0, \qquad (4.8b)$$

$$\Delta_2 = 0 \ , \quad \bar{\Delta}_2 = \frac{1}{2} \ , \tag{4.8c}$$

and for $\varepsilon = -1$,

$$\frac{|q|^2}{k'} = \frac{2N^2+kN-1}{2N(N+k)} \ , \tag{4.8d}$$

$$\Delta_1 = \frac{2N^2+kN-1}{2N(N+k)} \ , \quad \bar{\Delta}_1 = \frac{N^2-1}{2N(N+k)} \ , \tag{4.8e}$$

$$\Delta_2 = \frac{N^2-1}{2N(N+k)} \ , \quad \bar{\Delta}_2 = \frac{2N^2+kN-1}{2N(N+k)} \ . \tag{4.8f}$$

Any multipoint correlation function for arbitrary primary fields $\Phi_n(z,\bar{z})$ satisfies the following first order linear differential equations :

$$\{ \partial_{z_i} + 2 \sum_{j\neq i} [\frac{T_i^a T_j^a}{N+k} + \frac{Q_i Q_j}{k'}] \frac{1}{z_i - z_j} \}$$

$$\times \langle \Phi_1(z_1,\bar{z}_1)\ldots\Phi_n(z_n,\bar{z}_n)\rangle = 0 \ , \tag{4.9a}$$

$$\{ \partial_{\bar{z}_i} + 2 \sum_{j\neq i} [\frac{\bar{T}_i^a \bar{T}_j^a}{N+k} + \frac{\bar{Q}_i \bar{Q}_j}{k'}] \frac{1}{\bar{z}_i - \bar{z}_j} \}$$

$$\times \langle \Phi_1(z_1,\bar{z}_1)\ldots\Phi_n(z_n,\bar{z}_n)\rangle = 0 \ , \tag{4.9b}$$

where T_i^a ($\bar{T}_i^a$) are the left (right) representation matrices of the SU(N) algebra for Φ_i and, Q_i ($\bar{Q}_i$) is the left (right)

U(1) charge of Φ_i . The propagators of the basic fields Ψ_1 and Ψ_2 are easily obtained from the equations (4.9) and the charge assignments (4.2) and (4.3) :

for $\varepsilon = 1$,

$$\langle \Psi_{1\alpha}(z)\Psi_1^{\dagger\beta}(z')\rangle = A\ \delta_\alpha^\beta\ (z-z')^{-1}, \tag{4.10a}$$

$$\langle \Psi_{2\alpha}(\bar z)\Psi_2^{\dagger\beta}(\bar z')\rangle = A\ \delta_\alpha^\beta\ (\bar z-\bar z')^{-1}, \tag{4.10b}$$

and for $\varepsilon = -1$,

$$\langle \Psi_{1\alpha}(z,\bar z)\Psi_1^{\dagger\beta}(z',\bar z')\rangle = B\ \delta_\alpha^\beta\ (z-z')^{-2\Delta_1}(\bar z-\bar z')^{-2\bar\Delta_1}, \tag{4.11a}$$

$$\langle \Psi_{2\alpha}(z,\bar z)\Psi_2^{\dagger\beta}(z',\bar z')\rangle = B\ \delta_\alpha^\beta\ (z-z')^{-2\Delta_2}(\bar z-\bar z')^{-2\bar\Delta_2}, \tag{4.11b}$$

where A and B are normalizations. Any other two-point functions are vanishing.

In general, the 2(n+m)-point correlation functions

$$\langle \Psi_{1\alpha_1}(x_1,\bar x_1)\Psi_1^{\dagger\beta_1}(y_1,\bar y_1)\ldots\Psi_{1\alpha_n}(x_n,\bar x_n)\Psi_1^{\dagger\beta_n}(y_n,\bar y_n)$$

$$\times\ \Psi_{2\gamma_1}(u_1,\bar u_1)\Psi_2^{\dagger\delta_1}(v_1,\bar v_1)\ldots\Psi_{2\gamma_m}(u_m,\bar u_m)\Psi_2^{\dagger\delta_m}(v_m,v_m)\rangle \tag{4.12}$$

are decomposed into the forms

$$\langle \Psi_{1\alpha_1}(x_1,\bar x_1)\ldots\Psi_2^{\dagger\beta_m}(v_m,\bar v_m)\rangle$$

$$= \langle \Psi_{1\alpha_1}(x_1,\bar{x}_1)\dots\Psi_1^{\dagger\beta_n}(y_n,\bar{y}_n)\rangle\langle\Psi_{2\gamma_1}(u_1,\bar{u}_1)\dots\Psi_2^{\dagger\delta_m}(v_m,\bar{v}_m)\rangle . \tag{4.13}$$

Let us solve the generalized K-Z equations (4.9) for the four-point function

$$\langle\Psi_{1\alpha_1}(z_1,\bar{z}_1)\Psi_1^{\dagger\beta_1}(z_2,\bar{z}_2)\Psi_{1\alpha_2}(z_3,\bar{z}_3)\Psi_1^{\dagger\beta_2}(z_4,\bar{z}_4)\rangle, \tag{4.14}$$

which is decomposed into the form

$$\langle\Psi_{1\alpha_1}(z_1,\bar{z}_1)\dots\Psi_1^{\dagger\beta_2}(z_4,\bar{z}_4)\rangle = \delta_{\alpha_1}^{\beta_1}\delta_{\alpha_2}^{\beta_2}F_1(z,\bar{z}) + \delta_{\alpha_1}^{\beta_2}\delta_{\alpha_2}^{\beta_1}F_2(z,\bar{z})$$

$$= \delta_{\alpha_1}^{\beta_1}\delta_{\alpha_2}^{\beta_2}G_1(z)H_1(\bar{z}) + \delta_{\alpha_1}^{\beta_2}\delta_{\alpha_2}^{\beta_1}G_2(z)H_2(\bar{z}). \tag{4.15}$$

Substituting (4.15) into (4.9a) and (4.9b) with i=1, we obtain the following equations :

for $\varepsilon = 1$,

$$\partial_{\bar{z}_1} H_1 = 0, \tag{4.17a}$$

$$\partial_{\bar{z}_1} H_2 = 0, \tag{4.17b}$$

$$\partial_{z_1} F_1 = -\frac{1}{z_1-z_2}\left(F_1 + \frac{1}{N+k} F_2 \right)$$

$$+ \frac{1}{z_1-z_3}\left(\frac{1}{N+k} F_2 + \frac{k}{N+k} F_1 \right) - \frac{1}{z_1-z_4}\frac{k}{N+k} F_1, \tag{4.18a}$$

$$\partial_{z_1} F_2 = -\frac{1}{z_1-z_2}\frac{k}{N+k}F_2 + \frac{1}{z_1-z_3}\left(\frac{1}{N+k}F_1 + \frac{k}{N+k}F_2\right)$$

$$-\frac{1}{z_1-z_4}\left(F_2 + \frac{1}{N+k}F_1\right), \qquad (4.18b)$$

and for $\varepsilon = -1$,

$$\partial_{z_1} G_1 = -2\Delta_1\left(\frac{1}{z_1-z_2} - \frac{1}{z_1-z_3} + \frac{1}{z_1-z_4}\right)G_1, \qquad (4.19a)$$

$$\partial_{z_1} G_2 = +2\Delta_1\left(\frac{1}{z_1-z_2} - \frac{1}{z_1-z_3} + \frac{1}{z_1-z_4}\right)G_2, \qquad (4.19b)$$

$$\partial_{\bar{z}_1} F_1 = -\frac{1}{\bar{z}_1-\bar{z}_2}\left(2\bar{\Delta}_1F_1 + \frac{1}{N+k}F_2\right)$$

$$+\frac{1}{\bar{z}_1-\bar{z}_3}\left(\frac{1}{N+k}F_2 - \frac{1}{N(N+k)}F_1\right) + \frac{1}{\bar{z}_1-\bar{z}_4}\frac{1}{N(N+k)}F_1,$$

(4.20a)

$$\partial_{\bar{z}_1} F_2 = \frac{1}{\bar{z}_1-\bar{z}_2}\frac{1}{N(N+k)}F_2 + \frac{1}{\bar{z}_1-\bar{z}_3}\left(\frac{1}{N+k}F_1 - \frac{1}{N(N+k)}F_2\right)$$

$$-\frac{1}{\bar{z}_1-\bar{z}_4}\left(2\bar{\Delta}_1F_2 + \frac{1}{N+k}F_1\right). \qquad (4.20b)$$

Other equations with i=2,3 and 4 are given by the following change of the arguments in the equations (4.17)~(4.20) : for i=2 $z_1(\bar{z}_1)\to z_2(\bar{z}_2)$ and $z_3(\bar{z}_3)\to z_4(\bar{z}_4)$, for i=3 $z_1(\bar{z}_1)\to z_3(\bar{z}_3)$ and $z_2(\bar{z}_2)\to z_4(\bar{z}_4)$, and for i=4 $z_1(\bar{z}_1)\to z_4(\bar{z}_4)$ and $z_2(\bar{z}_2)\to z_3(\bar{z}_3)$.

Equations (4.17) and (4.19) can be easily solved

$$H_1 = H_2 = A^2, \qquad \text{for } \varepsilon = 1 , \tag{4.21}$$

$$G_1 = G_2 = B^2[(z_1-z_2)(z_1-z_4)(z_2-z_3)(z_3-z_4)]^{-2\Delta_1}$$

$$\times [(z_1-z_3)(z_2-z_4)]^{2\Delta_1}, \quad \text{for } \varepsilon = -1, \tag{4.22}$$

where constants A and B are same ones appeared in two-point functions (4.10) and (4.11). These equations (4.21) and (4.22) combined with (4.18) and (4.20) yield the following equations:

for $\varepsilon = 1$,

$$\partial_{z_1} G_1 = - \frac{1}{z_1-z_2} (G_1 + \frac{1}{N+k} G_2)$$

$$+ \frac{1}{z_1-z_3} (\frac{1}{N+k} G_2 + \frac{1}{N+k} G_1) - \frac{1}{z_1-z_4} \frac{1}{N+k} G_1, \tag{4.23a}$$

$$\partial_{z_1} G_2 = - \frac{1}{z_1-z_2} \frac{k}{N+k} G_2 + \frac{1}{z_1-z_3} (\frac{1}{N+k} G_1 + \frac{k}{N+k} G_2)$$

$$- \frac{1}{z_1-z_4} (G_2 + \frac{1}{N+k} G_1), \tag{4.23b}$$

and for $\varepsilon = -1$,

$$\partial_{\bar{z}_1} H_1 = - \frac{1}{\bar{z}_1-\bar{z}_2} (2\bar{\Delta}_1 H_1 + \frac{1}{N+k} H_2)$$

$$+ \frac{1}{\bar{z}_1-\bar{z}_3} (\frac{1}{N+k} H_2 - \frac{1}{N(N+k)} H_1) + \frac{1}{\bar{z}_1-\bar{z}_4} \frac{1}{N(N+k)} H_1, \tag{4.24a}$$

$$\partial_{\bar{z}_1} H_2 = \frac{1}{\bar{z}_1-\bar{z}_2} \frac{1}{N(N+k)} H_2 + \frac{1}{\bar{z}_1-\bar{z}_3} \left(\frac{1}{N+k} H_1 - \frac{1}{N(N+k)} H_2 \right)$$

$$- \frac{1}{\bar{z}_1-\bar{z}_4} \left(2\bar{\Delta}_1 H_2 + \frac{1}{N+k} H_1 \right). \qquad (4.24b)$$

For the free theory ($\varepsilon=1$), we know the four-point function:

$$\langle \Psi_{1\alpha_1}(z_1,\bar{z}_1)\ldots\Psi_1^{\dagger\beta_2}(z_4,\bar{z}_4)\rangle$$

$$= A^2\delta_{\alpha_1}^{\beta_1}\delta_{\alpha_2}^{\beta_2}(z_1-z_2)^{-1}(z_3-z_4)^{-1} + A^2\delta_{\alpha_1}^{\beta_2}\delta_{\alpha_2}^{\beta_1}(z_1-z_4)^{-1}(z_2-z_3)^{-1}. \qquad (4.25)$$

This equation is consistent with the equations (4.23) if and only if we set k=1. Also for the interacting theory ($\varepsilon=-1$), the locality demands that as $z_1 \to z_2$ and $z_3 \to z_4$ the four-point function tends to $\delta_{\alpha_1}^{\beta_1}\delta_{\alpha_2}^{\beta_2}$ times a product of two-point functions in (z_1-z_2) and (z_3-z_4), and as $z_1 \to z_4$ and $z_2 \to z_3$ to $\delta_{\alpha_1}^{\beta_2}\delta_{\alpha_2}^{\beta_1}$ times a product of two-point functions in (z_1-z_4) and (z_2-z_3). This requirement is compatible with the equations (4.24) only in the case with k=1 (see ref.[5] for detail).

In the case with k=1, demanding the boundary conditions which are indicated by the locality requirement, the solution of the equations (4.24) is given as

$$H_1 = (\bar{z}_1-\bar{z}_2)^{-2\bar{\Delta}_1}(\bar{z}_1-\bar{z}_3)^{-\frac{1}{N}}(\bar{z}_1-\bar{z}_4)^{\frac{1}{N}}$$

$$\times (\bar{z}_2-\bar{z}_3)^{\frac{1}{N}}(\bar{z}_2-\bar{z}_4)^{-\frac{1}{N}}(\bar{z}_3-\bar{z}_4)^{-2\bar{\Delta}_1} , \qquad (4.26a)$$

$$H_2 = (\bar{z}_1-\bar{z}_2)^{\frac{1}{N}}(\bar{z}_1-\bar{z}_3)^{-\frac{1}{N}}(\bar{z}_1-\bar{z}_4)^{-2\bar{\Delta}_1}$$

$$\times (\bar{z}_2-\bar{z}_3)^{-2\bar{\Delta}_1}(\bar{z}_2-\bar{z}_3)^{-\frac{1}{N}}(\bar{z}_3-\bar{z}_4)^{\frac{1}{N}}. \qquad (4.26b)$$

One can easily covert the solution for Ψ_1 into the four-point function for Ψ_2 by the change $\Delta_1(\bar{\Delta}_1)\rightarrow\Delta_2(\bar{\Delta}_2)$ and $z\rightarrow\bar{z}$.

Now let us give the explicit forms of general $2(n+m)$-point functions (4.12) for $\varepsilon=-1$. For the purpose, it is sufficient to solve the differential equations (4.9) for the 2n-point function for Ψ_1 :

$$\langle\Psi_{1\alpha_1}(x_1,\bar{x}_1)\Psi_1^{\dagger\beta_1}(y_1,\bar{y}_1)\ldots\Psi_{1\alpha_n}(x_n,\bar{x}_n)\Psi_1^{\dagger\beta_n}(y_n,\bar{y}_n)\rangle. \qquad (4.27)$$

This function can be decomposed into the form

$$\langle\Psi_{1\alpha_1}(x_1,\bar{x}_1)\ldots\Psi_1^{\dagger\beta_n}(y_n,\bar{y}_n)\rangle = \sum_P \delta_{\alpha_1}^{\beta_{i_1}}\ldots\delta_{\alpha_n}^{\beta_{i_n}} F_{i_1\ldots i_n}$$

$$= \sum_P \delta_{\alpha_1}^{\beta_{i_1}}\ldots\delta_{\alpha_n}^{\beta_{i_n}} G_{i_1\ldots i_n}(x,y)H_{i_1\ldots i_n}(\bar{x},\bar{y}), \qquad (4.28)$$

where $(i_1,\ldots,i_n)$ denote permutations $P(1,\ldots,n)$, and the sum is taken over the all permutations. Substituting (4.28) into (4.9a) and (4.9b), one obtains

$$\partial_{x_i} G_{i_1\ldots i_n} = -2\Delta_1 \{ \sum_{j=1}^{n} \frac{1}{x_i - x_j} - \sum_{j\neq i} \frac{1}{x_i - x_j} \} G_{i_1\ldots i_n}, \quad (4.29a)$$

$$\partial_{y_i} G_{i_1\ldots i_n} = -2\Delta_1 \{ \sum_{j=1}^{n} \frac{1}{y_i - x_j} - \sum_{j\neq i} \frac{1}{y_i - y_j} \} G_{i_1\ldots i_n}, \quad (4.29b)$$

$$\partial_{\bar{x}_k} F_{i_1\ldots i_n} = -\frac{1}{\bar{x}_k - \bar{y}_{i_k}} \{ 2\bar{\Delta}_1 F_{i_1\ldots i_n}$$

$$+ \frac{1}{N+1} \sum_{l\neq k} F_{i_1\ldots i_{k-1} i_l i_{k+1}\ldots i_{l-1} i_k i_{l+1}\ldots i_n} \}$$

$$+ \frac{1}{N+1} \sum_{l\neq k} \frac{1}{\bar{x}_k - \bar{x}_l} F_{i_1\ldots i_{k-1} i_l i_{k+1}\ldots i_{l-1} i_k i_{l+1}\ldots i_n}$$

$$+ \frac{1}{N(N+1)} \{ \sum_{l\neq k} \frac{1}{\bar{x}_k - \bar{y}_{i_l}} - \sum_{l\neq k} \frac{1}{\bar{x}_k - \bar{x}_l} \} F_{i_1\ldots i_n}, \quad (4.30a)$$

$$\partial_{\bar{y}_k} F_{i_1\ldots i_n} = -\frac{1}{\bar{y}_k - \bar{x}_{i_k}} \{ 2\bar{\Delta}_1 F_{i_1\ldots i_n}$$

$$+ \frac{1}{N+1} \sum_{l\neq k} F_{i_1\ldots i_{k-1} i_l i_{k+1}\ldots i_{l-1} i_k i_{l+1}\ldots i_n} \}$$

$$+ \frac{1}{N+1} \sum_{l\neq k} \frac{1}{\bar{y}_k - \bar{y}_l} F_{i_1\ldots i_{k-1} i_l i_{k+1}\ldots i_{l-1} i_k i_{l+1}\ldots i_n}$$

$$+ \frac{1}{N(N+1)} \{ \sum_{l \neq k} \frac{1}{\bar{y}_k - \bar{x}_{i_l}} - \sum_{l \neq k} \frac{1}{\bar{y}_k - \bar{y}_l} \} F_{i_1 \ldots i_n}, \qquad (4.30b)$$

The equations for arbitrary $F_{i_1 \ldots i_n}$ are obtained from the equations for $F_{1 \ldots n}$: change the indices of F and the arguments y(x) according to the permutation $P(1,\ldots,n)=(i_1,\ldots,i_n)$ in both sides of the equations (4.30a) ((4.30b)) for $F_{1 \ldots n}$. Like (4.22) for the four-point function, with a conventional normlization, the solution of (4.29) is

$$G_{i_1 \ldots i_n} = B^n \prod_{i,j} :x_i - y_j:^{-2\Delta_1} \prod_{i<j} [(x_i - x_j)(y_i - y_j)]^{2\Delta_1}, \qquad (4.31a)$$

where

$$:x_i - y_j: = x_i - y_j, \quad \text{if} \quad i \leq j,$$

$$:x_i - y_j: = y_j - x_i, \quad \text{if} \quad i > j. \qquad (4.31b)$$

Using this solution and the definition of H (4.28), the equations (4.30) reduce to the equations for $H_{i_1 \ldots i_n}$

$$\partial_{\bar{x}_k} H_{i_1 \ldots i_n} = - \frac{1}{\bar{x}_k - \bar{y}_{i_k}} \{ 2\bar{\Delta}_1 H_{i_1 \ldots i_n}$$

$$+ \frac{1}{N+1} \sum_{l \neq k} H_{i_1 \ldots i_{k-1} i_l i_{k+1} \ldots i_{l-1} i_k i_{l+1} \ldots i_n} \}$$

$$+ \frac{1}{N+1} \sum_{l \neq k} \frac{1}{\bar{x}_k - \bar{x}_l} H_{i_1 \cdots i_{k-1} i_l i_{k+1} \cdots i_{l-1} i_k i_{l+1} \cdots i_n}$$

$$+ \frac{1}{N(N+1)} \{ \sum_{l \neq k} \frac{1}{\bar{x}_k - \bar{y}_{i_l}} - \sum_{l \neq k} \frac{1}{\bar{x}_k - \bar{x}_l} \} H_{i_1 \cdots i_n}, \qquad (4.32a)$$

$$\partial_{\bar{y}_k} H_{i_1 \cdots i_n} = - \frac{1}{\bar{y}_k - \bar{x}_{i_k}} \{ 2\bar{\Delta}_1 H_{i_1 \cdots i_n}$$

$$+ \frac{1}{N+1} \sum_{l \neq k} H_{i_1 \cdots i_{k-1} i_l i_{k+1} \cdots i_{l-1} i_k i_{l+1} \cdots i_n} \}$$

$$+ \frac{1}{N+1} \sum_{l \neq k} \frac{1}{\bar{y}_k - \bar{y}_l} H_{i_1 \cdots i_{k-1} i_l i_{k+1} \cdots i_{l-1} i_k i_{l+1} \cdots i_n}$$

$$+ \frac{1}{N(N+1)} \{ \sum_{l \neq k} \frac{1}{\bar{y}_k - \bar{x}_{i_l}} - \sum_{l \neq k} \frac{1}{\bar{y}_k - \bar{y}_l} \} H_{i_1 \cdots i_n}, \qquad (4.32b)$$

There are $2n \times n!$ independent first order linear differential equations for the n! functions $H_{i_1 \cdots i_n}$. Thus, if we impose the locality requirement which yields n! boundary conditions for $H_{i_1 \cdots i_n}$, the solution of (4.32) must be uniquely determined.

Here we prove that the functions

$$H_{i_1 \cdots i_n} = \sigma_P \eta \prod_{k=1}^{n} :\bar{x}_k - \bar{y}_{i_k}:^{-2\bar{\Delta}_1} \prod_{k<l} (\bar{x}_k - \bar{x}_l)^{-\frac{1}{N}} (\bar{y}_k - \bar{y}_l)^{-\frac{1}{N}}$$

$$\times \prod_{k \neq l} :\bar{x}_k - \bar{y}_{i_l}:^{\frac{1}{N}} \qquad (4.33)$$

are the solution of (4.32). Here the symbol σ_P is the signiture of the permutation $(1,\ldots,n) \to (i_1,\ldots,i_n)$, and η is taken to be

$$\begin{aligned} \eta &= -1, \text{ if cardinal numbers of the set} \\ &\qquad \{ (k, i_k) \mid k > i_k \ , \ k = 1,\ldots,n \} \text{ is odd,} \\ \eta &= +1, \qquad \text{otherwise.} \end{aligned} \qquad (4.34)$$

It is important to note that the functions (4.33) satisfy the following relations :

for $k \neq l$,

$$(\bar{x}_k - \bar{y}_{i_k})(\bar{x}_l - \bar{y}_{i_l}) H_{i_1 \cdots i_n} = -(\bar{x}_k - \bar{y}_{i_l})(\bar{x}_l - \bar{y}_{i_k})$$

$$\times H_{i_1 \cdots i_{k-1} i_l i_{k+1} \cdots i_{l-1} i_k i_{l+1} \cdots i_n}, \qquad (4.35)$$

which simplify the equations (4.32):

$$\partial_{\bar{x}_k} H_{i_1 \ldots i_n} = - \frac{1}{\bar{x}_k - \bar{y}_{i_k}} 2\bar{\Delta}_1 H_{i_1 \ldots i_n}$$

$$+ \frac{1}{N} \{ \sum_{l \neq k} \frac{1}{\bar{x}_k - \bar{y}_{i_l}} - \sum_{l \neq k} \frac{1}{\bar{x}_k - \bar{x}_l} \} H_{i_1 \ldots i_n}, \qquad (4.36a)$$

$$\partial_{\bar{y}_k} H_{i_1 \ldots i_n} = - \frac{1}{\bar{y}_k - \bar{x}_{i_k}} 2\bar{\Delta}_1 H_{i_1 \ldots i_n}$$

$$+ \frac{1}{N} \{ \sum_{l \neq k} \frac{1}{\bar{y}_k - \bar{x}_{i_l}} - \sum_{l \neq k} \frac{1}{\bar{y}_k - \bar{y}_l} \} H_{i_1 \ldots i_n}, \qquad (4.36b)$$

These equations combined with the locality requirement indeed have the solution (4.33).

Finally let us consider the operator algebras

$$\Psi_{1\alpha}(z,\bar{z})\Psi_2^{\dagger\beta}(z',\bar{z}') = \text{const.}(z-z')^{\kappa}(\bar{z}'-\bar{z}')^{\lambda}[\ \Phi_\alpha^\beta(z',\bar{z}') + \ldots\ ] \qquad (4.37a)$$

$$\Psi_{2\beta}(z,\bar{z})\Psi_1^{\dagger\alpha}(z',\bar{z}') = \text{const.}(z-z')^{\tilde{\kappa}}(\bar{z}'-\bar{z}')^{\tilde{\lambda}}[\ \tilde{\Phi}_\beta^\alpha(z',\bar{z}') + \ldots\ ] \qquad (4.37b)$$

as the preliminary for the next subsection. We will derive the fusion rules for the products (4.37) by the method used by B.P.Z. for degenerate primary fields [10]. Substituting each side of (4.37a) into the correlation functions of (4.9) and then comparing the most singular terms at $z \to z'$ and $\bar{z} \to \bar{z}'$, one obtains

$$\kappa = \lambda = 0. \tag{4.38}$$

In the same way, one also obtains

$$\tilde{\kappa} = \tilde{\lambda} = 0. \tag{4.39}$$

Therefore we conclude

$$\Phi_\alpha^\beta(z,\bar{z}) \sim :\Psi_{1\alpha}(z,\bar{z})\Psi_2^{\dagger\beta}(z,\bar{z}): = \lim_{z'\to z,\bar{z}'\to\bar{z}} \Psi_{1\alpha}(z,\bar{z})\Psi_2^{\dagger\beta}(z',\bar{z}'), \tag{4.40a}$$

$$\tilde{\Phi}_\beta^\alpha(z,\bar{z}) \sim :\Psi_{2\beta}(z,\bar{z})\Psi_1^{\dagger\alpha}(z,\bar{z}): = \lim_{z'\to z,\bar{z}'\to\bar{z}} \Psi_{2\beta}(z,\bar{z})\Psi_1^{\dagger\alpha}(z',\bar{z}'), \tag{4.40b}$$

We will return to these results in the next subsection in order to discuss non-abelian bosonizations.

4-2) The SU(N) × U(1) Wess-Zumino model and non-abelian bosonizations

In this subsection we study the SU(N)×U(1) Wess-Zumino model which is another realization of the conformal and the SU(N) × U(1) Kac-Moody algebras.

The basic fields are the U(1)-valued field g_1 and the SU(N)-valued matrix field g_2. The dynamics of these fields is governed by the action

$$S_{\alpha,k}(g_1,g_2) = S_\alpha(g_1) + S_k(g_2). \qquad (4.41)$$

Here S_α is defined as

$$S_\alpha(g) = \frac{\alpha}{16\pi} \int d^2\xi \; \partial_\mu g^{-1} \partial_\mu g \;, \qquad (4.42)$$

and S_k is the Wess-Zumino action

$$S_k(g) = k\{ \frac{1}{16\pi} \int d^2\xi \; tr.(\partial_\mu g^{-1} \partial_\mu g) \quad + \quad \Gamma(g) \}, \qquad (4.43)$$

where the Wess-Zumino term $\Gamma(g)$ is defined by the integral over the three-dimensional ball with two-dimensional space being the boundary :

$$\Gamma(g) = \frac{1}{24\pi} \int d^3X \; \varepsilon^{\alpha\beta\gamma} tr.(g^{-1}\partial_\alpha g \; g^{-1}\partial_\beta g \; g^{-1}\partial_\gamma g). \qquad (4.44)$$

The action (4.41) has the coformal invariance, and also is invariant under the transformations

$$g_1(\xi) \rightarrow \Omega_1(z) g_1(\xi) \bar{\Omega}_1^{-1}(\bar{z}), \qquad (4.45)$$

$$g_2(\xi) \rightarrow \Omega_2(z) g_2(\xi) \bar{\Omega}_2^{-1}(\bar{z}), \qquad (4.46)$$

where $\Omega_1(z)$ ($\bar{\Omega}_1(\bar{z})$) and $\Omega_2(z)$ ($\bar{\Omega}_2(\bar{z})$) are arbitrary S(N) and U(1)-valued functions respectively, and depend only on $z=\xi_1+i\xi_2$ ($\bar{z}=\xi_1-i\xi_2$). The corresponding conserved currents are the following ones : for U(1) transformations

$$J(z) = i\alpha\, \partial_z g_1 g_1^{-1}, \tag{4.47a}$$

$$\bar{J}(\bar{z}) = i\alpha\, \partial_{\bar{z}} g_1^{-1} g_1 , \tag{4.47b}$$

and for SU(N) transformations,

$$J^a(z) = k\ \mathrm{tr.}(t^a \partial_z g_2 g_2^{-1}), \tag{4.48a}$$

$$\bar{J}^a(\bar{z}) = k\ \mathrm{tr.}(t^a \partial_{\bar{z}} g_2^{-1} g_2). \tag{4.48b}$$

These currents satisfy the following relations [20] :

$$\{ J(z), J(z') \} = 2\pi\alpha\delta'(z-z'), \tag{4.49a}$$

$$\{ \bar{J}(\bar{z}), \bar{J}(\bar{z}') \} = 2\pi\alpha\delta'(\bar{z}-\bar{z}'), \tag{4.49b}$$

$$\{ J^a(z), J^b(z') \} = -2\pi f^{abc} J^c(z)\delta(z-z') + k\pi\delta^{ab}\delta'(z-z'), \tag{4.50a}$$

$$\{ \bar{J}^a(\bar{z}), \bar{J}^b(\bar{z}') \} = -2\pi f^{abc} \bar{J}^c(\bar{z})\delta(\bar{z}-\bar{z}') + k\pi\delta^{ab}\delta'(\bar{z}-\bar{z}'), \tag{4.50b}$$

and

$$\{ J(z), g_1(z',\bar{z}') \} = 2\pi i g_1(z,\bar{z})\delta(z-z'), \tag{4.51a}$$

$$\{ \bar{J}(\bar{z}), g_1(z',\bar{z}') \} = -2\pi i g_1(z,\bar{z})\delta(\bar{z}-\bar{z}'), \tag{4.51b}$$

$$\{ J^a(z), g_2(z',\bar{z}') \} = 2\pi t^a g_2(z,\bar{z})\delta(z-z'), \tag{4.52a}$$

$$\{ \bar{J}^a(\bar{z}), g_2(z',\bar{z}') \} = -2\pi g_2(z,\bar{z}) t^a \delta(\bar{z}-\bar{z}'), \tag{4.52a}$$

where { } denote Poisson brackets.

We postulate that the theory maintain the conformal and the Kac-Moody invariance at full quantum level. At the same time, Poisson brackets (4.49)~(4.52) must be read in the operator language as

$$\delta_\omega J(z) = -\alpha\ \omega'(z), \tag{4.53a}$$

$$\delta_{\bar{\omega}} \bar{J}(\bar{z}) = -\alpha\ \bar{\omega}'(\bar{z}), \tag{4.53b}$$

$$\delta_\omega J^a(z) = f^{abc}\omega^b(z) J^c(z) - \frac{k}{2}\ \omega^{a\prime}(z), \tag{4.54a}$$

$$\delta_{\bar{\omega}} \bar{J}^a(\bar{z}) = f^{abc}\bar{\omega}^b(\bar{z}) \bar{J}^c(\bar{z}) - \frac{k}{2}\ \bar{\omega}^{a\prime}(\bar{z}), \tag{4.54b}$$

and

$$\delta_\omega g_1(z,\bar{z}) = i\omega(z) g_1(z,\bar{z}), \tag{4.55a}$$

$$\delta_{\bar{\omega}} g_1(z,\bar{z}) = -i\bar{\omega}(\bar{z}) g_1(z,\bar{z}), \tag{4.55b}$$

$$\delta_\omega g_2(z,\bar{z}) = t^a \omega^a(z) g_2(z,\bar{z}), \tag{4.56a}$$

$$\delta_{\bar{\omega}} g_2(z,\bar{z}) = - \bar{\omega}^a(\bar{z}) g_2(z,\bar{z}) t^a. \tag{4.56b}$$

Here we will devote our attention to studying the relation between the SU(N)×U(1) Wess-Zumino model and N-component Dirac theories studied in the previous subsection rather than to obtaining the general forms of the multipoint correlation functions for arbitrary k and α.

If we set

$$k = 1, \tag{4.57}$$

and

$$\alpha = \frac{k'}{2} = N, \qquad \text{for} \quad \varepsilon = 1, \tag{4.58a}$$

$$\alpha = \frac{k'}{2} = \frac{N}{2N-1}, \qquad \text{for} \quad \varepsilon = -1, \tag{4.58b}$$

(see (4.8a) and (4.8d)), the current algebras (4.53) and (4.54) equal to ones for N-component Dirac theories. Therefore the K-Z equations are given by (4.9).

Let us consider the operator algebras

$$g_1^{\pm 1}(z,\bar{z})g_{2\alpha}^{\ \beta}(z',\bar{z}',) = \text{const.}(z-z')^{\sigma}(\bar{z}-\bar{z}')^{\rho}[\ g_{\pm\alpha}^{\ \beta}\ (z',\bar{z}') + \ldots\ ] \qquad (4,59a)$$

$$g_2^{-1\alpha}{}_{\beta}(z,\bar{z})g_1^{\mp 1}(z',\bar{z}',) = \text{const.}(z-z')^{\tilde{\sigma}}(\bar{z}-\bar{z}')^{\tilde{\rho}}[\ g_{\pm}^{-1\alpha}{}_{\beta}(z',\bar{z}') + \ldots\ ] \qquad (4.59b)$$

Substituting each side of (4.59a) into the correlation function of (4.9) and comparing the most singular term at $z \to z'$ and $\bar{z} \to \bar{z}'$, one obtains

$$\sigma = \rho = 0. \qquad (4.60)$$

In the same way, one also obtains

$$\tilde{\sigma} = \tilde{\rho} = 0. \qquad (4.61)$$

Therefore we conclude

$$g_{\pm\alpha}^{\ \beta}(z,\bar{z}) \sim\ :g_1^{\pm 1}(z,\bar{z})g_{2\alpha}^{\ \beta}(z,\bar{z}): = \lim_{z'\to z,\bar{z}'\to\bar{z}} g_1^{\pm 1}(z,\bar{z})g_{2\alpha}^{\ \beta}(z',\bar{z}') \qquad (4.62a)$$

$$g_{\pm}^{-1\alpha}{}_{\beta}(z,\bar{z}) \sim\ :g_2^{-1\alpha}{}_{\beta}(z,\bar{z})g_1^{\mp 1}(z,\bar{z}): = \lim_{z'\to z,\bar{z}'\to\bar{z}} g_2^{-1\alpha}{}_{\beta}(z,\bar{z})g_1^{\mp 1}(z',\bar{z}') \qquad (4.62b)$$

Both sides of the following equations have the same transformation properties under $SU(N)\times U(1)$ transformations

$$g_{+\alpha}^{\ \beta} \sim \Phi_\alpha^\beta \ , \quad g_{+\ \beta}^{-1\alpha} \sim \Phi_\beta^\alpha \ , \quad \text{for} \quad \varepsilon = 1, \tag{4.63a}$$

$$g_{-\alpha}^{\ \beta} \sim \tilde{\Phi}_\alpha^\beta \ , \quad g_{-\ \beta}^{-1\alpha} \sim \tilde{\Phi}_\beta^\alpha \ , \quad \text{for} \quad \varepsilon = -1, \tag{4.63b}$$

where Φ_α^β and $\tilde{\Phi}_\alpha^\beta$ are defined as (4.40). It is clear that the composite fields in both sides of (4.63) have the same correlation functions. These facts visualize Witten's non-abelian bosonizations both of the free N-component Dirac theory and the conformally invariant SU(N) Thirring model.

§5. Summary

In this thesis, the general framework to analyze the structure of two-dimensional quantum field theories with the conformal and the Kac-Moody invariance was discussed based on the bootstrap approach. It was shown that the stress-energy tensor naturally exhibits the Sugawara form (3.22) irrespective of models. The generalization of the K-Z equations (3.31) for arbitrary symmetry groups was presented. As applications of the present method, the solutions of two models have been studied. One is the conformally invariant SU(N) Thirring model. All multipoint correlation functions of the theory were explicitly obtained as (4.31) and (4.33) by solving the corresponding K-Z equations. The other is the SU(N)×U(1) Wess-Zumino model. We studied this model from the point of view of non-abelian bosonizations. The non-abelian bosonization prescriptions of two-dimensional free fermion theories were originally presented by Witten in ref.[20]. He pointed out the significance of the investigation in view of the representation theory of the Kac-Moody algebra. In this thesis, we visualized this concept of bosonizations from the entirely physical aspect. As a result, we extended the bosonization rules to an interacting theory, the conformally invariant SU(N) Thirring model (see (4.63)).

Some problems are left unsolved. The systematic analysis of the operator algebra, on which the bootstrap apprach depends, will be valuable for the more detailed investigation of the bosonization problem. In this regard, the introduction of flavor groups into the theories mentioned above is also important to solve the open question whether the Wess-Zumino fields can be regarded as the Goldstone bosons of a certain fermion theory [1,43-45].

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Appedix A

The Virasoro algebra contains a subgroup $sl(2,\mathbb{C})$, generated by the operators $L_{-1}(\bar{L}_{-1})$, $L_0(\bar{L}_0)$ and $L_1(\bar{L}_1)$. In particular, the operator $L_{-1}(\bar{L}_{-1})$ generates the translations whereas $L_0(\bar{L}_0)$ generates the dilatations of the coordinates $z(\bar{z})$.

It is convenient to introduce the coordinates σ and τ by

$$z = \exp(\ \tau+i\sigma\ ),\quad \bar{z} = \exp(\ \tau-i\sigma\ ), \tag{A1}$$

and to regard τ as "time" while regarding σ as "space". The operator

$$H = L_0 + \bar{L}_0 \tag{A2}$$

is the generator of "time" shifts, and it play the role of the hamiltonian. Note that the "infinite past" $\tau\to-\infty$ and the "infinite future" $\tau\to\infty$ correspond to the points $z=0$ and $z=\infty$ respectively.

Let the vacuum be the ground state of the hamiltonian (A2). For the manifestation of the conformal invariance, the in-vacuum must satisfies the equations

$$L_n\ |0\rangle = 0,\qquad \text{if } n \geq -1. \tag{A3}$$

Appendix B

Here we demonstrate that the coefficients $\beta_{nm}^{p\{k\}}$ and $\bar{\beta}_{nm}^{p\{\bar{k}\}}$

in (2.28) are completely determined by the requirement of the conformally covariant properties of the expansions (2.25).

Applying both sides of (2.27) to the vacuum state, one obtains the equation

$$\Phi_n(z,\bar{z})|m\rangle = \sum_p C^p_{nm}\, z^{\Delta_p-\Delta_n-\Delta_m}\bar{z}^{\bar{\Delta}_p-\bar{\Delta}_n-\bar{\Delta}_m}\, \varphi(z)\bar{\varphi}(\bar{z})|p\rangle, \quad (B1)$$

where $|m(p)\rangle=\Phi_{m(p)}(0,0)|0\rangle$, and the operator $\varphi(z)$ is given by the series

$$\varphi(z) = \sum_{\{k\}} z^{\sum k_i}\beta^{p\{k\}}_{nm} L_{-k_1}\ldots L_{-k_N}. \quad (B2)$$

The same formula with the substitutions $z\to\bar{z}$, $\beta\to\bar{\beta}$ and $L\to\bar{L}$ holds for $\bar{\varphi}(\bar{z})$.

Let us consider the state

$$|z,p\rangle = \varphi(z)|p\rangle. \quad (B3)$$

It can be represented as the power series

$$|z,p\rangle = \sum_{N=0}^{\infty} z^N|N,p\rangle, \quad (B4)$$

where the vectors $|N,p\rangle$ satisfy the equations

$$L_0|N,p\rangle = (\Delta_p + N)|N,p\rangle. \quad (B5)$$

To compute these vectors, let us apply the operators L_r ($r \geq 1$) to both sides of (B1). This leads to the equations

$$[\, z^{r+1} \frac{d}{dz} + (\Delta_p + r\Delta_n - \Delta_m) z^r \,] |z,p\rangle = L_r |z,p\rangle. \qquad (B6)$$

Substituting the power series (B4), one obtains

$$L_r |N,p\rangle = 0, \qquad \text{if} \quad 0 \leq N < r, \qquad (B7)$$

and

$$L_r |N+r,p\rangle = [\, N + \Delta_p + r\Delta_n - \Delta_m \,] |N,p\rangle. \qquad (B8)$$

Solving these equations, one can compute the power series (B4) order by order.

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