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# ON QUASI-HOMOGENEOUS FOURFOLDS OF $SL(3)$

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## Introduction

We recall that a quasi-homogeneous variety of an algebraic group $G$ is an algebraic variety with a regular $G$-action which has an open dense orbit. A general theory of quasi-homogeneous varieties has been presented in Luna-Vust [5], and in particular, quasi-homogeneous varieties of $\boldsymbol{SL}(2)$ have been studied by Popov [9], Jauslin-Moser [2]. On the other hand, the geometry of smooth projective quasi-homogeneous threefolds of $\boldsymbol{SL}(2)$ has been thoroughly studied in Mukai-Umemura [7] and Nakano [8] by means of Mori theory.

In this note, we shall study and classify the smooth irreducible complete quasi-homogeneous fourfolds of $\boldsymbol{SL}(3)$. The motivation for this research comes from Mabuchi's work [6], in which the smooth complete $n$-folds with a non-trivial $\boldsymbol{SL}(n)$-action have been completely classified. Since $\boldsymbol{SL}(n)$-varieties of dimension less than $n$ are obvious ones, we are interested in $\boldsymbol{SL}(n)$-varieties of dimension $n+1$. Let $X$ be a smooth complete $\boldsymbol{SL}(n)$-variety of dimension $n+1$, and let $d$ be the maximum of the dimensions of all orbits of $X$. It turns out that, if $d\leq n-1$, then $\boldsymbol{SL}(n)$-actions on $X$ are easy, and essential problems occur when (1) $d=n+1$ (quasi-homogeneous case) and (2) $d=n$ (codimension 1 case). We hope that the investigation of the case (1) for $n=3$ in this note will be a good example toward the understanding of the structure of $\boldsymbol{SL}(n)$-varieties of dimension $n+1$.

Our main result is the classification theorem 11 of smooth complete quasi-homogeneous 4-folds of $\boldsymbol{SL}(3)$, which turns out extermely simple compared to the $\boldsymbol{SL}(2)$-case. Indeed, all the varieties appearing in the classification are rational 4-folds of very simple type.

This note is organized as follows. First in §1, we classify the closed subgroups of $\boldsymbol{SL}(3)$ of codimension 4. The author is indebted to Prof. Ariki for Proposition 1. In §2, examples of quasi-homogeneous 4-folds of $\boldsymbol{SL}(3)$ are constructed by rather ad-hok methods. Finally, in §3, the classification will be done.

In this note, algebraic varieties, algebraic groups and Lie algebras are all defined over a fixed algebraically closed field $k$ of characteristic 0. An algebraic variety is always assumed to be reduced and irreducible, and an (algebraic)

$n$-fold is an algebraic variety of dimension $n$. The symbol $*$ in a matrix stands for any element in $k$, or some element in $k$ which we do not need to specify.

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## 1. Classification of closed algebraic subgroups of $SL(3)$ of codimension 4

This section is devoted to the proof of the following proposition due to Ariki. We denote by $\boldsymbol{SL}(3)$ the special linear group of degree 3 defined over $k$.

**Proposition 1.** *Let $G\subset \boldsymbol{SL}(3)$ be a closed algebraic subgroup of codimension 4. Then $G$ is one of the following subgroups up to conjugation.*

$$G_0=\{\begin{bmatrix} A & 0\\ 0 & b\end{bmatrix} \mid A\in \boldsymbol{GL}(2),\ b\in k^{\times},\ \det(A)\cdot b=1\}$$

$$G_1=\{\begin{bmatrix} x & * & *\\ 0 & y & 0\\ 0 & 0 & z\end{bmatrix} \mid xyz=1\}$$

$$N(G_1)=G_1\cdot\langle\begin{bmatrix} -1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}\rangle$$

$$G_2=\{\begin{bmatrix} x & 0 & *\\ 0 & y & *\\ 0 & 0 & z\end{bmatrix} \mid xyz=1\}$$

$$N(G_2)=G_2\cdot\langle\begin{bmatrix} 0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & -1\end{bmatrix}\rangle$$

$$G_{p,q}=\{\begin{bmatrix} x & * & *\\ 0 & y & *\\ 0 & 0 & 1/(xy)\end{bmatrix} \mid x^p y^q=1\} \text{ for } p,q\in \boldsymbol{Z},\ q\geq 0,$$

$(p,q)\neq(0,0)$.

Proof. (1) Let $\mathfrak{sl}(3)$ be the Lie algebra of $\boldsymbol{SL}(3)$. We first determine the Lie subalgebras of $\mathfrak{sl}(3)$ of dimension 4 and the corresponding connected closed subgroup of $\boldsymbol{SL}(3)$. Let $\mathfrak{g}\subset\mathfrak{sl}(3)$ be a Lie subalgebra of dimension 4. Then $\mathfrak{g}=\mathfrak{s}\oplus\mathfrak{r}$ (semi-direct sum), where $\mathfrak{s}$ is a semi-simple Lie subalgebra and $\mathfrak{r}$ is the maximal solvable ideal of $\mathfrak{g}$, by Levi-Malcev's theorem. Since the rank of $\mathfrak{s}\leq 2$, we have $\mathfrak{s}\simeq\mathfrak{sl}(2)$ or 0. In fact, if the rank of $\mathfrak{s}=2$, then $\mathfrak{s}\simeq A_1\oplus A_1$, $A_2$, $B_2$ or $G_2$ and hence $\dim_k \mathfrak{s}\geq 5$, which is impossible.

(a) First, we assume $\mathfrak{s}=\mathfrak{sl}(2)$. Consider the faithful representation of $\mathfrak{s}$ on $k^3$ which is the restriction of the natural representation of $\mathfrak{sl}(3)$ on $k^3$. We decompose this representation into irreducible ones and may asume that $\mathfrak{s}$ is one of the following two fomrs up to conjugation.

$$\mathfrak{s}=k\cdot\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}\oplus k\cdot\begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix}\oplus k\cdot\begin{bmatrix}1&0&0\\0&-1&0\\0&0&0\end{bmatrix}\qquad\text{(type 1)}$$

or

$$\mathfrak{s}=k\cdot\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}\oplus k\cdot\begin{bmatrix}0&0&0\\2&0&0\\0&2&0\end{bmatrix}\oplus k\cdot\begin{bmatrix}2&0&0\\0&0&0\\0&0&-2\end{bmatrix}\qquad\text{(type 2)}.$$

Consider the adjoint representation of $\mathfrak{s}$ on $\mathfrak{r}$: $(\mathfrak{r}, ad|_{\mathfrak{s}})$. Since $\dim\mathfrak{r}=1$, this is trivial and we find that $\mathfrak{r}=k\cdot R$, where $R$ commutes with any element of $\mathfrak{s}$. Assume that $\mathfrak{s}$ is of type 1. Then a simple calculation shows that $R=\begin{bmatrix}1&0&0\\0&1&0\\0&0&-2\end{bmatrix}$ up to scalar multiplication. The corresponding connected closed subgroup is

$$G_0=\left\{\begin{bmatrix}g&&0\\&&0\\0&0&1\end{bmatrix}\;\middle|\; g\in \boldsymbol{SL}(2)\right\}\cdot\left\{\begin{bmatrix}x&0&0\\0&x&0\\0&0&x^{-2}\end{bmatrix}\;\middle|\; x\in k^{\times}\right\}$$

$$=\left\{\begin{bmatrix}g&&0\\&&0\\0&0&1/\det g\end{bmatrix}\;\middle|\; g\in \boldsymbol{GL}(2)\right\}.$$

Assume that $\mathfrak{s}$ is of type 2. Then a simple calcualtion shows that there is no nonzero $R$ which commutes with every element of $\mathfrak{s}$. Hence the type 2 never occurs.

(b) Second, we assume that $\mathfrak{s}=\{0\}$. Since $\mathfrak{g}$ is solvable, $\mathfrak{g}=\mathfrak{t}\oplus\mathfrak{n}$, where $\mathfrak{t}$ is a maximal abelian subalgebra consisting of semi-simple elements and $\mathfrak{n}$ is the ideal of all nilpotent elements in $\mathfrak{g}$. We set

$$\mathfrak{b}:=\left\{\begin{bmatrix}*&*&*\\0&*&*\\0&0&*\end{bmatrix}\right\}\text{ and }\mathfrak{h}:=\left\{\begin{bmatrix}0&*&*\\0&0&*\\0&0&0\end{bmatrix}\right\}.$$

Then we may assume $\mathfrak{g}\subset\mathfrak{b}$ and $\mathfrak{n}=\mathfrak{g}\cap\mathfrak{h}$ by Lie's theorem.

If $\dim\mathfrak{n}=3$, then $\mathfrak{g}\supset\mathfrak{h}=\mathfrak{n}$. Then we have

$$\mathfrak{g}=\mathfrak{h}\oplus k\cdot\begin{bmatrix}a&0&0\\0&b&0\\0&0&-a-b\end{bmatrix}\text{ for some } a, b\in k.$$

The corresponding algebraic subgroup $G$ is of the form

$$G=\left\{\begin{bmatrix} x^a & * & * \\ 0 & x^b & * \\ 0 & 0 & x^{-a-b} \end{bmatrix} \mid x\in k^\times\right\} \text{ for } a, b\in \boldsymbol{Z}.$$

Since $G$ is connected, we conclude that $G=G_{b,a}$ for coprime $a, b\in\boldsymbol{Z}$ in this case.

If $\dim \mathfrak{n}=2$, then $\dim \mathfrak{t}=2$ and $\mathfrak{g}$ is full-rank in $\mathfrak{sl}(3)$. Hence we may assume that $\mathfrak{t}=\left\{\begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}\right\}$, and then,

$$\mathfrak{n}=\left\{\begin{bmatrix} 0 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\} \text{ or } \left\{\begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{bmatrix}\right\}$$

by root-decomposition of $\mathfrak{n}$ with respect to $\mathfrak{t}$. The corresponding connected subgroup is

$$G_1:=\left\{\begin{bmatrix} * & * & * \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}\right\} \text{ or } G_2:=\left\{\begin{bmatrix} * & 0 & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}\right\}.$$

If $\dim \mathfrak{n}\leq 1$, then $\dim \mathfrak{t}\geq 3$ which is impossible.

(2) Let $G$ be a connected closed subgroup of codimension 4 determined in (1). In order to determine not necessarily connected such subgroups, we calculate $N_{SL(3)}(G)/G$, where $N_{SL(3)}(G)$ is the normalizer of $G$ in $\boldsymbol{SL}(3)$. In the following, we set $N:=N_{SL(3)}(G)$.

(a) Suppose $G=G_0$. We consider the linear $N$-action on $k^3$ induced by the natural $\boldsymbol{SL}(3)$-action on $k^3$. Let $[x, y, z]$ be the coordinates of $k^3$, and set $P=[0, 0, 0]$, $l=\{x=y=0\}$ and $S=\{z=0\}$. Then the orbit decomposition of $k^3$ with respect to the $G$-action is given by

$$k^3=\{P\}\cup\{l-P\}\cup\{S-P\}\cup\{k^3-(l\cup S)\}.$$

For any $g\in N$, $g\circ l$ and $g\circ S$ are $G$-stable. Since $l$ (resp. $S$) is the unique $G$-stable line (resp. plane), $g\circ l=l$ and $g\circ S=S$. It follows that $g\in G$ and hence $N=G$.

(b) Suppose $G=G_1$. We set $l=\{y=z=0\}$, $S_1=\{z=0\}$ and $S_2=\{y=0\}$. Then the orbit decomposition of $k^3$ with respect to the $G$-action is given by

$$k^3=\{P\}\cup\{l-P\}\cup\{S_1-l\}\cup\{S_2-l\}\cup\{k^3-(S_1\cup S_2)\}.$$

For any $g\in N$, $g\circ l$ and $g\circ S_1$ is $G$-stable, and hence we have $g\circ l=l$, $g\circ S_1=S_1$ or $S_2$. Therefore we may assume that $g$ is of the following 2 types modulo $G$:

$$g_1=\begin{bmatrix}1&*&*\\0&1&*\\0&0&1\end{bmatrix} \text{ or } g_2=\begin{bmatrix}-1&*&*\\0&0&1\\0&1&*\end{bmatrix}.$$

Since $g_1Gg_1^{-1}\subset G$, a direct computation shows that $g_1\in G$ in this case. Similarly, $g_2=\begin{bmatrix}-1&0&0\\0&0&1\\0&1&0\end{bmatrix}$ modulo $G$. Hence we conclude that $N/G=\left\langle\begin{bmatrix}-1&0&0\\0&0&1\\0&1&0\end{bmatrix}\right\rangle\cong\boldsymbol{Z}_2$, and $N(G_1):=G_1\cdot\left\langle\begin{bmatrix}-1&0&0\\0&0&1\\0&1&0\end{bmatrix}\right\rangle$ is the only non-connected closed subgroup whose connected component containing the identity is $G_1$.

(c) Suppose $G=G_2$. Similar calculations as in (b) show that $N(G_2):=G_2\cdot\left\langle\begin{bmatrix}0&1&0\\1&0&0\\0&0&-1\end{bmatrix}\right\rangle$ is the only non-connected closed subgroup which has $G_2$ as the identity component.

(d) Suppose $G=G_{p,q}$ ($p$, $q$ are coprime). Then $N=B:=$ the Borel subgroup of all the upper triangular matrices. In fact, $N\supset B$ is obvious. Conversely, if $g\in N$, then $g\in N_{SL(3)}(U)=B$, where $U$ is the unipotent radical of $B$. Hence we find $N/G\simeq B/G_{p,q}$. Now, let $\varphi\colon B\to k^\times$ be the character of $B$ defined by $\varphi\left(\begin{bmatrix}x&*&*\\0&y&*\\0&0&z\end{bmatrix}\right)=x^py^q$. Then $\mathrm{Ker}(\varphi)=G_{p,q}$, and we have $B/G_{p,q}\simeq k^\times$. Since any finite subgroup of $k^\times$ is a group of roots of unity, we conclude that

$$G_{np,nq}=\left\{\begin{bmatrix}x&*&*\\0&y&*\\0&0&z\end{bmatrix}\;\middle|\;(x^py^q)^n=1,\ xyz=1\right\}\qquad(n\in\boldsymbol{N})$$

are the subgroups whose identity component is $G_{p,q}$. □

## 2. Examples of quasi-homogeneous 4-folds of $\boldsymbol{SL}(3)$

In this section, we construct various types of smooth complete quasi-homogeneous 4-folds of $\boldsymbol{SL}(3)$ by rather ad-hok methods. We use the following notations for some standard closed subgroups of $\boldsymbol{SL}(3)$:

$$B:=\left\{\begin{bmatrix}a&b&c\\0&e&f\\0&0&i\end{bmatrix}\;\middle|\;aei=1\right\},\quad B':=\left\{\begin{bmatrix}a&0&0\\d&e&0\\g&h&i\end{bmatrix}\;\middle|\;aei=1\right\},$$

$$H:=\left\{\begin{bmatrix}a&b&c\\0&e&f\\0&h&i\end{bmatrix}\;\middle|\;a(ei-fh)=1\right\},\quad H':=\left\{\begin{bmatrix}a&0&0\\d&e&f\\g&h&i\end{bmatrix}\;\middle|\;a(ei-fh)=1\right\}.$$

We note that $B$ and $B'$ are conjugate in $\boldsymbol{SL}(3)$, whereas $H$ and $H'$ are not. Now, for the construction of examples, we need to know the explicit description of $\boldsymbol{SL}(3)/B$.

Let $\boldsymbol{SL}(3)$ act on $\boldsymbol{P}^2$ in the standard way. Namely, for $A=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \in \boldsymbol{SL}(3)$ and $P=[x\colon y\colon z]\in\boldsymbol{P}^2$, $A\circ P:=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} ax+by+cz \\ dx+ey+fz \\ gx+hy+iz \end{bmatrix}$. We also consider the dual projective plane $(\boldsymbol{P}^2)^*$ with the induced $\boldsymbol{SL}(3)$-action. Namely, for $Q=[u\colon v\colon w]\in(\boldsymbol{P}^2)^*$, $A\circ Q={}^t\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1}\begin{bmatrix} u \\ v \\ w \end{bmatrix}$. We define an $\boldsymbol{SL}(3)$-action on $\boldsymbol{P}^2\times(\boldsymbol{P}^2)^*$ by $A\circ(P, Q)=(A\circ P, A\circ Q)$ for $(P, Q)\in\boldsymbol{P}^2\times(\boldsymbol{P}^2)^*$, and we set $W:=\{xu+yv+zw=0\}\subset\boldsymbol{P}^2\times(\boldsymbol{P}^2)^*$. $W$ is a flag manifold $\{(x, l)\in\boldsymbol{P}^2\times(\boldsymbol{P}^2)^* \mid x\in L\}$, where $L\subset\boldsymbol{P}^2$ is a line corresponding to $l$. The following lemma is standard and well-known. However, we give a proof since the calculation in it is frequently referred to later in this note.

**Lemma 2.** (1) *$W$ is $\boldsymbol{SL}(3)$-stable and isomorphic to $\boldsymbol{SL}(3)/B$.*

(2) *Let $p_1\colon W\to\boldsymbol{P}^2$ (resp. $p_2\colon W\to(\boldsymbol{P}^2)^*$) be the projection to the first (resp. second) factor. Then $p_1\colon W\to\boldsymbol{P}^2$ (resp. $p_2\colon W\to(\boldsymbol{P}^2)^*$) is isomorphic to the projectivized tangent bundle $\boldsymbol{P}(T_{\boldsymbol{P}^2})\to\boldsymbol{P}^2$ (resp. $\boldsymbol{P}(T_{(\boldsymbol{P}^2)^*})\to(\boldsymbol{P}^2)^*$).*

(3) *Let $\mathcal{O}_{\boldsymbol{P}}(1)$ (resp. $\mathcal{O}_{\boldsymbol{P}^*}(1)$) be the tautological line bundle of $\boldsymbol{P}(T_{\boldsymbol{P}^2})$ (resp. $\boldsymbol{P}(T_{(\boldsymbol{P}^2)^*})$). Then $\mathcal{O}_{\boldsymbol{P}}(1)\simeq\mathcal{O}_W(-2, 1)$ and $\mathcal{O}_{\boldsymbol{P}^*}(1)\simeq\mathcal{O}_W(1, -2)$, where $\mathcal{O}_W(a,b)=p_1^*(\mathcal{O}_{\boldsymbol{P}^2}(a))\otimes p_2^*(\mathcal{O}_{(\boldsymbol{P}^2)^*}(b))$.*

Proof. (1) It is clear that $W$ is $\boldsymbol{SL}(3)$-stable. Take a point $R:=([1\colon 0\colon 0], [0\colon 0\colon 1])\in W$. Then the isotropy group $\boldsymbol{SL}(3)_R$ at $R$ is $B$. In fact, it is clear that $\boldsymbol{SL}(3)_R\subset H$. Take $A=\begin{bmatrix} a & b & c \\ 0 & e & f \\ 0 & h & i \end{bmatrix}\in H$. Since ${}^t(A)^{-1}=\begin{bmatrix} 1/a & 0 & 0 \\ * & ai & -ah \\ * & -af & ae \end{bmatrix}$, $A$ fixes $R$ if and only if $h=0$, namely $A\in B$. Hence $W$ contains a 3-dimensional orbit $O(R)$ isomorphic to $\boldsymbol{SL}(3)/B$ which is complete. It follows that $W=O(R)\simeq\boldsymbol{SL}(3)/B$.

(2) We show that $p_1\colon W\to\boldsymbol{P}^2$ is isomorphic to $\boldsymbol{P}(T_{\boldsymbol{P}^2})\to\boldsymbol{P}^2$. Let $(k^3)^*$ be an affine 3-space endowed with the dual $\boldsymbol{SL}(3)$-action. We set $W':=\{xu'+yv'+zw'=0\}\subset\boldsymbol{P}^2\times(k^3)^*$, $([x\colon y\colon z], [u', v', w'])\in\boldsymbol{P}^2\times(k^3)^*$. Then $p_1'\colon W'\to\boldsymbol{P}^2$ ($p_1'$ is the projection to the first factor) is an $\boldsymbol{SL}(3)$-vector bundle of rank 2 whose projectivization is $p_1\colon W\to\boldsymbol{P}^2$. We note that $\boldsymbol{SL}(3)$-vector bundles over the homogeneous space $\boldsymbol{P}^2\simeq\boldsymbol{SL}(3)/H$ are determined by the slice representations of $H$ on the fiber over $P=[1\colon 0\colon 0]\in\boldsymbol{P}^2$ (Kraft [3; 6. 3.]). Now, take $A=$

$\begin{bmatrix} a & b & c \\ 0 & e & f \\ 0 & h & i \end{bmatrix} \in H$. Then $A$ acts on the fiber $W'_P$ over $P$ by $\begin{bmatrix} v' \\ w' \end{bmatrix} \mapsto \begin{bmatrix} ai & -ah \\ -af & ae \end{bmatrix} \begin{bmatrix} v' \\ w' \end{bmatrix}$.
On the other hand, let $\eta = y/x$, $\zeta = z/x$ be the inhomogeneous coordinates around $P$. Since $A^*\eta = (e\eta + f\zeta)(a + b\eta + c\zeta)^{-1}$, $A^*\zeta = (h\eta + i\zeta)(a + b\eta + c\zeta)^{-1}$, we get $A^*d\eta = (e/a)d\eta + (f/a)d\zeta$, $A^*d\zeta = (h/a)d\eta + (i/a)d\zeta$. It follows that $A_*\colon T_{P^2,P} \to T_{P^2,P}$ is represented by $\begin{bmatrix} e/a & f/a \\ h/a & i/a \end{bmatrix}$ with respect to the basis $\{\partial/\partial\eta, \partial/\partial\zeta\}$. Let $\mathcal{O}_{P^2}(-1) \subset P^2 \times k^3$ be the universal subbundle. Since $H$ acts on the line $\mathcal{O}_{P^2}(-1)_P$ by multiplication by $a$, we find that $W' \simeq T_{P^2} \otimes \mathcal{O}_{P^2}(-2)$. Hence $p_1\colon W = P(W') \to P^2$ is isomorphic to $P(T_{P^2}) \to P^2$. We can verify that $p_2\colon W \to (P^2)^*$ is isomorphic to $P(T_{(P^2)^*}) \to (P^2)^*$ similarly.

(3) We take a point $S = [1\colon 0] \in P(T_{P^2})_P$ whose isotropy group is $B$: $SL(3)_S = B$. Let $\mathcal{O}_P(-1) \subset \pi_1^*(T_{P^2})$ be the universal subbundle over $P(T_{P^2}) \simeq W$, where $\pi_1\colon P(T_{P^2}) \to P^2$ is the projection. Then $\mathcal{O}_P(-1)_S = k \cdot [1, 0] \subset T_{P^2,P} \simeq k^2$. Since for $A = \begin{bmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & i \end{bmatrix} \in B$, $A_*\colon T_{P^2,P} \to T_{P^2,P}$ is represented by $\begin{bmatrix} e/a & f/a \\ 0 & i/a \end{bmatrix}$, $A$ acts on the line $\mathcal{O}_P(-1)_S$ by multiplication by $e/a$. On the other hand, take a point $R = (P, Q) = ([1\colon 0\colon 0], [0\colon 0\colon 1]) \in W$ at which the isotropy group is $B$. Since $A$ acts on the line $\mathcal{O}_{P^2}(-1)_P$ (resp. $\mathcal{O}_{(P^2)^*}(-1)_Q$) by multiplication by $a$ (resp. $ae$), $A$ acts on the line $\mathcal{O}_W(p, q)_R \simeq \mathcal{O}_{P^2}(-1)_P^{\otimes(-p)} \otimes \mathcal{O}_{(P^2)^*}(-1)_Q^{\otimes(-q)}$ by multiplication by $a^{-(p+q)}e^{-q}$. Therefore we get $\mathcal{O}_P(1) \cong \mathcal{O}_W(-2, 1)$. Similar calculations show that $\mathcal{O}_{P^*}(1) \simeq \mathcal{O}_W(1, -2)$. □

Now, we construct 9 types of examples of smooth complete (actually projective) quasi-homogeneous 4-folds of $SL(3)$. The examples (a), (b), (c), (d) deal with quasi-homogeneous 4-folds whose open orbits are of the form $SL(3)/G_{p,q}$.

(a) Let $W = SL(3)/B$ be as in Lemma 2. The $SL(3)$-line bundles on $W$ are in one-to-one correspondence with the characters of $B$. Let $\varphi_{p,q}\colon B \to k^\times$ be the character of $B$ defined by $\begin{bmatrix} a & * & * \\ 0 & e & * \\ 0 & 0 & i \end{bmatrix} \mapsto a^p e^q$, and $L_{p,q}$ be the $SL(3)$-line bundle corresponding to $\varphi_{p,q}$. We note $L_{p,q} \simeq \mathcal{O}_W(-p+q, -q)$ in view of the proof of Lemma 2. Consider the $SL(3)$-action on the total space of $L_{p,q}$. If we take a non-zero vector $v$ of the fiber of $L_{p,q}$ over $I_3B \in W = SL(3)/B$ ($I_3$ is the identity matrix of degree 3), then the isotropy group at $v$ is equal to $G_{p,q}$. Hence $L_{p,q}$ contains a 4-dimensional orbit isomorphic to $SL(3)/G_{p,q}$. We projectivize $L_{p,q}$ equivariantly to a $P^1$-bundle by adding the infinite section. More precisely, let $\mathcal{O}_W$ be the trivial bundle of rank 1 over $W$, where $SL(3)$ acts on the fiber trivially, and we set $X_{p,q}:=\boldsymbol{P}(L_{p,q}\oplus\mathcal{O}_W)$ endowed with the induced $\boldsymbol{SL}(3)$-action. The orbit decomposition of $X_{p,q}$ is given by $X_{p,q}=X^4_{p,q}\cup U_0\cup U_\infty$, where $X^4_{p,q}$ is the open dense orbit isomorphic to $\boldsymbol{SL}(3)/G_{p,q}$, $U_0$ is the 0-section of $L_{p,q}$ isomorphic to $\boldsymbol{SL}(3)/B$, and $U_\infty$ is the infinite section of $X_{p,q}$ isomorphic to $\boldsymbol{SL}(3)/B$.

**Lemma 3.** *Let $X_{p,q}$ be as above, and let the notation be the same as in Lemma 2.*

(1) *$X_{p,q}$ can be blown-down to a smooth algebraic space along $U_0\simeq W$ in the $p_1$-direction (resp. $p_2$-direction) if and only if $q=1$ (resp. $p-q=1$).*

(2) *$X_{p,q}$ can be blown-down to a smooth algebraic space along $U_\infty\simeq W$ in the $p_1$-direction (resp. $p_2$-direction) if and only if $q=-1$ (resp. $q-p=1$).*

Proof. (1) Let $l_1$ (resp. $l_2$)$\subset W$ be a fiber of $p_1$ (resp. $p_2$), and $N(U_0/X_{p,q})$ be the normal bundle of $U_0$ in $X_{p,q}$. Then we have

$$(N(U_0/X_{p,q}), l_1)=(L_{p,q}, l_1)=(\mathcal{O}_W(-p+q, -q), l_1)=-q\,,$$

and similarly, $(N(U_0/X_{p,q}), l_2)=-p+q$. Hence (1) holds from the criterion for smooth blow-downs.

(2) Since $N(U_\infty/X_{p,q})\simeq L^{-1}_{p,q}$, (2) follows from (1). $\square$

(b) Let $\boldsymbol{SL}(3)$ act on $\boldsymbol{P}^2$ in the standard way. Take a point $P=[1:0:0]\in\boldsymbol{P}^2$ at which the isotropy group is $H$. Let $\rho_\alpha\colon H\to\boldsymbol{GL}(2)$ be a 2-dimensional representation of $H$ defined by $\begin{bmatrix} a & b & c\\ 0 & e & f\\ 0 & h & i\end{bmatrix}\mapsto a^\alpha\begin{bmatrix} e & f\\ h & i\end{bmatrix}$, and $E_\alpha$ be the $\boldsymbol{SL}(3)$-vector bundle of rank 2 corresponding to $\rho_\alpha(E_\alpha\simeq T_{\boldsymbol{P}^2}\otimes\mathcal{O}_{\boldsymbol{P}^2}(-\alpha-1))$. If we take a point $Q=[1, 0]\in E_{\alpha,P}=k^2$, then $\boldsymbol{SL}(3)_Q=\{A\in H\,|\,a^\alpha e=1,\ a^\alpha h=0\}=G_{\alpha,1}$. We projectivize $E_\alpha$ to a $\boldsymbol{P}^2$-bundle by adding infinite lines. More precisely, let $\mathcal{O}_{\boldsymbol{P}^2}$ be the trivial bundle of rank 1, where $\boldsymbol{SL}(3)$ acts on the fiber trivially, and we set $Y_\alpha:=\boldsymbol{P}(E_\alpha\oplus\mathcal{O}_{\boldsymbol{P}^2})$. Since $H$ acts on the infinite line by $\begin{bmatrix} v\\ w\end{bmatrix}\mapsto\begin{bmatrix} e & f\\ h & i\end{bmatrix}\begin{bmatrix} v\\ w\end{bmatrix}$, the isotropy group at $[1:0]$ on the infinite line is $B$. Hence we have a following orbit decomposition of $Y_\alpha$: $Y_\alpha=Y^4_\alpha\cup Y^3_\alpha\cup Y^2_\alpha$, where $Y^4_\alpha$ is a 4-dimensional orbit isomorphic to $\boldsymbol{SL}(3)/G_{\alpha,1}$, $Y^3_\alpha$ is a 3-dimensional orbit consisting of infinite lines isomorphic to $W=\boldsymbol{SL}(3)/B$, and $Y^2_\alpha$ is the 0-section of $E_\alpha$ isomorphic to $\boldsymbol{SL}(3)/H$.

**Lemma 4.** *$Y_\alpha$ cannot be blown-down to a smooth algebraic space along $Y^3_\alpha\simeq W$ in the $p_1$-direction, and can be blown-down in the $p_2$-direction if and only if $\alpha=0$.*

Proof. An easy calculation shows that $A=\begin{bmatrix} a & b & c\\ 0 & e & f\\ 0 & 0 & i\end{bmatrix}\in B$ acts on $N(Y^3_\alpha/Y_\alpha)_P$

($P:=I_3B\in \boldsymbol{SL}(3)/B\simeq Y_\alpha^3$) by multiplication by $ia^{1-\alpha}$. Hence we have $N(Y_\alpha^3/Y_\alpha)\simeq\mathcal{O}_W(\alpha-1,1)$ (see the proof of Lemma 2). Now, $(N(Y_\alpha^3/Y_\alpha), l_1)=(\mathcal{O}_W(\alpha-1,1), l_1)=1$, and $(N(Y_\alpha^3/Y_\alpha), l_2)=\alpha-1$. Therefore our assertion is verified by the criterion for smooth blow-downs. $\square$

(c) We consider the standard $\boldsymbol{SL}(3)$-action on the dual projective plane $(\boldsymbol{P}^2)^*$. The isotropy group at $P=[1:0:0]\in(\boldsymbol{P}^2)^*$ is $H'$. Take the 2-dimensional representation $\lambda_\alpha\colon H'\to\boldsymbol{GL}(2)$ given by $\begin{bmatrix} a&0&0\\ d&e&f\\ g&h&i\end{bmatrix}\mapsto a^\alpha\begin{bmatrix} e&f\\ h&i\end{bmatrix}$, and let $F_\alpha\to(\boldsymbol{P}^2)^*$ be the $\boldsymbol{SL}(3)$-bundle of rank 2 corresponding to $\lambda_\alpha$. If we take a point $R=[0,1]\in E_{\alpha,P}=k^2$, then $\boldsymbol{SL}(3)_R=\{A\in H'\mid a^\alpha i=1, f=0\}=\{\begin{bmatrix} a&0&0\\ d&e&0\\ g&h&i\end{bmatrix}\mid a^\alpha i=1\}=C^{-1}G_{-\alpha+1,-\alpha}C$, where $C=\begin{bmatrix} 0&0&1\\ 0&1&0\\ 1&0&0\end{bmatrix}$. Hence the isotropy group $\boldsymbol{SL}(3)_{C\circ R}$ at $C\circ R$ is equal to $G_{-\alpha+1,-\alpha}$. We projectivize $F_\alpha$ to a $\boldsymbol{P}^2$-bundle $Z_\alpha:=\boldsymbol{P}(F_\alpha\oplus\mathcal{O}_{(\boldsymbol{P}^2)^*})$. The orbit decomposition of $Z_\alpha$ is given by $Z_\alpha=Z_\alpha^4\cup Z_\alpha^3\cup Z_\alpha^2$, where $Z_\alpha^4$ is an open dense orbit isomorphic to $\boldsymbol{SL}(3)/G_{-\alpha+1,-\alpha}$, $Z_\alpha^3$ is a 3-dimensionl orbit consisting of the infinite lines isomorphic to $\boldsymbol{SL}(3)/B$, and $Z_\alpha^2$ is the 0-section of $F_\alpha$ isomorphic to $\boldsymbol{SL}(3)/H'$.

**Lemma 5.** *$Z_\alpha$ cannot be blown-down to a smooth algebraic space along $Z_\alpha^3\simeq W$ in the $p_2$-direction, and can be blown-down in the $p_1$-direction if and only if $\alpha=1$.*

Proof. We have $N(Z_\alpha^3/Z_\alpha)\simeq\mathcal{O}_W(1,\alpha-2)$. The rest of the proof is similar to Lemma 4. $\square$

(d) Let $[x_0:x_1:x_2:y_0:y_1:y_2]$ be the homogeneous coordinates of $\boldsymbol{P}^5$, and define an $\boldsymbol{SL}(3)$-action on $\boldsymbol{P}^5$ by $A\circ[x_0:x_1:x_2:y_0:y_1:y_2]=[x_0':x_1':x_2':y_0':y_1':y_2']$ for $A\in\boldsymbol{SL}(3)$, where ${}^t[x_0':x_1':x_2']=A\cdot{}^t[x_0:x_1:x_2]$ and ${}^t[y_0':y_1':y_2']=({}^tA)^{-1}\cdot{}^t[y_0:y_1:y_2]$. We set $Q:=\{x_0y_0+x_1y_1+x_2y_2=0\}\subset\boldsymbol{P}^5$. $Q$ is an $\boldsymbol{SL}(3)$-stable nonsingular quadric 4-fold. If we take a point $P:=[1:0:0:0:0:1]\in Q$, then $\boldsymbol{SL}(3)_P=G_{0,1}$. In fact, it is clear that $H\supset\boldsymbol{SL}(3)_P$. Take $A=\begin{bmatrix} a&b&c\\ 0&e&f\\ 0&h&i\end{bmatrix}\in H$. Since $({}^tA)^{-1}=\begin{bmatrix} *&*&0\\ *&*&-ah\\ *&*&ae\end{bmatrix}$, $A\circ P=[a:0:0:0:-ah:ae]$. Hence $\boldsymbol{SL}(3)_P=\{A\in H\mid h=0, e=1\}=G_{0,1}$. Set $Q^2:=\{y_0=y_1=y_2=0\}\simeq\boldsymbol{P}^2$ and $Q^{2\prime}:=\{x_0=x_1=x_2=0\}\simeq(\boldsymbol{P}^2)^*$. Then $Q^2$ (resp. $Q^{2\prime}$) is a closed orbit isomorphic to $\boldsymbol{SL}(3)/H$ (resp.

$\boldsymbol{SL}(3)/H')$. The orbit decomposition of $Q$ is given by $Q=Q^4\cup Q^2\cup Q^{2\prime}$, where $Q^4=Q-(Q^2\cup Q^{2\prime})$ is a 4-dimensional orbit isomorphic to $\boldsymbol{SL}(3)/G_{0,1}$. In fact, take any point $R=[p:q:r:s:t:u]\in Q^4$. If, for instance, $p\neq 0$, then $A\circ P=R$ for $A:=\begin{bmatrix} p & 0 & * \\ q & u/p & * \\ r & -t/p & * \end{bmatrix}\in \boldsymbol{SL}(3)$. Thus we find that $Q^4$ is an orbit.

**Lemma 6.** $N(Q^2/Q)\simeq T_{\boldsymbol{P}^2}\otimes\mathcal{O}_{\boldsymbol{P}^2}(-1)$, $N(Q^{2\prime}/Q)\simeq T_{(\boldsymbol{P}^2)^*}\otimes\mathcal{O}_{(\boldsymbol{P}^2)^*}(-1)$.

Proof. We consider the following exact sequence of normal bundles:

$$(*)\quad 0\to N(Q^2/Q)\to N(Q^2/\boldsymbol{P}^5)\to N(Q/\boldsymbol{P}^5)\,|_{Q^2}\to 0\,.$$

Since $N(Q^2/\boldsymbol{P}^5)\simeq\mathcal{O}_{\boldsymbol{P}^2}(1)^{\oplus 3}$ and $N(Q/\boldsymbol{P}^5)\,|_{Q^2}\simeq\mathcal{O}_{\boldsymbol{P}^2}(2)$, we have $N(Q^2/Q)\cong\Omega_{\boldsymbol{P}^2}\otimes\mathcal{O}_{\boldsymbol{P}^2}(2)\simeq T_{\boldsymbol{P}^2}\otimes\mathcal{O}_{\boldsymbol{P}^2}(-1)$ by comparing $(*)$ with the dual of the standard Euler sequence. □

The relation of quasi-homogeneous 4-folds in examples (a)~(d) is given in the following proposition. We denote by $B_Z(X)$ the blowing-up of a variety $X$ along a subvariety $Z$.

**Proposition 7.** *$B_{Y_p^2}(Y_p)\simeq X_{p,1}$, $B_{Z_q^2}(Z_q)\simeq X_{q-1,q}$ $(q\geq 1)$, $B_{Z_{-q}^2}(Z_{-q})\simeq X_{q+1,q}$ $(q\geq 0)$, and $B_{Q^2}(Q)\simeq Y_0$, $B_{Q^{2\prime}}(Q)\simeq Z_1$.*

Proof. We show $B_{Y_p^2}(Y_p)\simeq X_{p,1}$. In fact, the exceptional divisor $C\subset B_{Y_p^2}(Y)$ is isomorphic to $W\simeq\boldsymbol{P}(T_{\boldsymbol{P}^2})$ since $N(Y_p^2/Y_p)\simeq E_p\simeq T_{\boldsymbol{P}^2}\otimes\mathcal{O}_{\boldsymbol{P}^2}(-p-1)$. Let $F\colon B_{Y_p^2}(Y_p)\cdots\to X_{p,1}$ be a birational map induced by identifying the open dense orbits$\simeq\boldsymbol{SL}(3)/G_{p,1}$. Let $I$ (resp. $J$) be the indeterminacy locus of $F$ (resp. $F^{-1}$). Then, since $I$ and $J$ are $\boldsymbol{SL}(3)$-stable closed subsets of codimension equal to or larger than 2, we find that $I$ and $J$ are empty, and $F$ is an isomorphism. The other isomorphisms are proved similarly. □

(e) $G_1$-case. We consider the standard $\boldsymbol{SL}(3)$-action on the dual projective plane $(\boldsymbol{P}^2)^*$ and set $M_1:=(\boldsymbol{P}^2)^*\times(\boldsymbol{P}^2)^*$ endowed with the diagonal $\boldsymbol{SL}(3)$-action. If we take a point $P:=([1:0:0],[0:1:0])\in M_1$, then clearly $H'\supset \boldsymbol{SL}(3)_P$. Take $A:=\begin{bmatrix} a & 0 & 0 \\ d & e & f \\ g & h & i \end{bmatrix}\in H'$. Then, since ${}^t(A)^{-1}=\begin{bmatrix} * & fg-di & * \\ * & ai & * \\ * & -af & * \end{bmatrix}$, $A\in \boldsymbol{SL}(3)_P$ if and only if $f=d=0$. Hence $\boldsymbol{SL}(3)_P$ consists of the matrices of the form $\begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ * & * & * \end{bmatrix}$. It follows that $D^{-1}G_1D=\boldsymbol{SL}(3)_P$, where $D=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, and we get $\boldsymbol{SL}(3)_{D\circ P}=G_1$. The orbit decomposition of $M_1$ is given by $M_1=\Delta\cup(M_1-\Delta)$, where $\Delta$ is the diagonal isomorphic to $\boldsymbol{SL}(3)/H'$ and $M_1-\Delta$ is a 4-dimensional orbit isomorphic to $\boldsymbol{SL}(3)/G_1$.

Let $\pi: \bar{M}_1 \to M_1$ be the blowing-up of $M_1$ along $\Delta$. Since $\Delta$ is a closed orbit, we can define a regular $\boldsymbol{SL}(3)$-action on $\bar{M}_1$ such that $\pi$ is $\boldsymbol{SL}(3)$-equivariant. Since $N(\Delta/M_1) \simeq T_\Delta \simeq T_{(\boldsymbol{P}^2)^*}$, the exceptional divisor $E \subset \bar{M}_1$ is isomorphic to $\boldsymbol{P}(T_{(\boldsymbol{P}^2)^*}) \simeq W$, and the orbit decomposition of $\bar{M}_1$ is given by $\bar{M}_1 = \bar{M}_1^4 \cup E$, where $\bar{M}_1^4 = \bar{M}_1 - E$ is a 4-dimensional orbit isomorphic to $\boldsymbol{SL}(3)/G_1$. We note that $\bar{M}_1$ cannot be blown-down to a smooth algebraic space along $E \simeq W$ in the $p_1$-direction since $(N(E/\bar{M}_1), l_1) = (\mathcal{O}_W(-1, 2), l_1) = 2$ (notations are the same as in Lemma 2).

(f) $N(G_1)$-case. We consider the standard $\boldsymbol{SL}(3)$-action on $\boldsymbol{P}^2$. Let $S^2(T_{\boldsymbol{P}^2})$ be the symmetric tensor bundle of degree 2 of $T_{\boldsymbol{P}^2}$, and we set $N_1 := \boldsymbol{P}(S^2(T_{\boldsymbol{P}^2}))$ endowed with the induced $\boldsymbol{SL}(3)$-action. Take a point $P := [1:0:0] \in \boldsymbol{P}^2$ at which the isotropy group is $H$. Take $A = \begin{bmatrix} a & b & c \\ 0 & e & f \\ 0 & h & i \end{bmatrix} \in H$ and let $[y, z]$ be the inhomogeneous affine coordinates around the origin $P$. We recall that the $H$-action on $T_{\boldsymbol{P}^2, P}$ is represented by $a^{-1}\begin{bmatrix} e & f \\ h & i \end{bmatrix}$ with respect to the basis $\{\partial/\partial y, \partial/\partial z\}$ (cf. Lemma 2). Hence the $H$-action on $S^2(T_{\boldsymbol{P}^2})_P$ is represented by $a^{-2}\begin{bmatrix} e^2 & ef & f^2 \\ 2eh & ie+fh & 2fi \\ h^2 & ih & i^2 \end{bmatrix}$ with respect to the basis $\{(\partial/\partial y)^{\otimes 2}, (\partial/\partial y) \otimes (\partial/\partial z), (\partial/\partial z)^{\otimes 2}\}$. Thus the isotropy group at $[0:1:0] \in \boldsymbol{P}_P := \boldsymbol{P}(S^2(T_{\boldsymbol{P}^2}))_P$ is given by $\{A \in H \mid ef = ih = 0\} = \{A \in H \mid e = i = 0 \text{ or } f = h = 0\} = N(G_1)$. The orbit decomposition of $\boldsymbol{P}_P$ with respect to the $H$-action is given by $\boldsymbol{P}_P = C \cup (\boldsymbol{P}_P - C)$, where $C$ is a conic defined by $\{\eta^2 - 4\xi\zeta = 0\}$ and $[\xi:\eta:\zeta]$ are the homogeneous coordinates of $\boldsymbol{P}_P$. $C$ is the orbit through $[1:0:0] \in \boldsymbol{P}_P$ and hence isomorphic to $H/B$. Therefore the orbit decomposition of $N_1$ with respect to the $\boldsymbol{SL}(3)$-action is given by $N_1 = N_1^4 \cup F$, where $N_1^4$ is a 4-dimensional orbit isomorphic to $\boldsymbol{SL}(3)/N(G_1)$ and $F$ is a 3-dimensional orbit isomorphic to $\boldsymbol{SL}(3)/B \simeq W$.

**Proposition 8.** *Let $\bar{M}_1$ and $N_1$ be as in (e), (f).*

(1) *There exists an $\boldsymbol{SL}(3)$-equivariant finite morphism $\varphi: \bar{M}_1 \to N_1$ of degree 2. The ramification locus of $\varphi$ is $E \subset \bar{M}_1$ and the branch locus is $F \subset N_1$.*

(2) *Let $l_1$ (resp. $l_2$) be a fiber of $p_1: F = W \to \boldsymbol{P}^2$ (resp. $p_2: F \to (\boldsymbol{P}^2)^*$). Then $(F, l_1) = 4$ and $(F, l_2) = -2$. In particular, $N_1$ cannot be blown-down to a smooth algebraic space along $F$ in neither directions.*

Proof. (1) From the inclusion $G_1 \subset N(G_1)$, an $\boldsymbol{SL}(3)$-equivariant étale morphism $\nu: \bar{M}_1^4 \simeq \boldsymbol{SL}(3)/G_1 \to N_1^4 \simeq \boldsymbol{SL}(3)/N(G_1)$ of degree 2 is induced. We note that $\nu$ is the unique $\boldsymbol{SL}(3)$-equivariant morphism from $\bar{M}_1^4$ to $N_1^4$ since $\{a \in \boldsymbol{SL}(3) \mid aG_1a^{-1} \subset N(G_1)\} = N(G_1)$. Let $\varphi: \bar{M}_1 \cdots\!\!\to N_1$ be a rational map induced by $\nu$ with the indeterminacy locus $I$. Since $I$ is an $\boldsymbol{SL}(3)$-stable closed subset of codimension $\geq 2$, $I$ is empty and $\varphi$ is a morphism. Since $\varphi$ is $\boldsymbol{SL}(3)$-equivariant, $\varphi(E)=F$. We note that $\varphi|_E: E \rightarrow F$ is an isomorphsm. In fact, since $N_{\boldsymbol{SL}(3)}(B)=B$, identity is the unique $\boldsymbol{SL}(3)$-equivariant morphism from $W=\boldsymbol{SL}(3)/B$ to $W$. The assertion (1) is thus proved.

(2) We note $N(E/\bar{M}_1) \simeq \mathcal{O}_{\boldsymbol{P}(T_{(\boldsymbol{P}^2)^*})}(-1) \simeq \mathcal{O}_W(-1, 2)$. Hence $(E, l_1)=((N(E/\bar{M}_1), l_1)=(\mathcal{O}_W(-1, 2), l_1)=2$, and $(E, l_2)=-1$ similarly. Now, we have $(F, l_1)=(\varphi^*(F), l_1)=(2E, l_1)=4$, and $(F, l_2)=-2$ similarly. The assertion (2) is proved. □

REMARK. We have $[F] \simeq \mathcal{O}_{\boldsymbol{P}}(2) \otimes \pi^*(\mathcal{O}_{\boldsymbol{P}^2}(6))$, where $[F]$ is the line bundle associated to the divisor $F$, $\mathcal{O}_{\boldsymbol{P}}(1)$ is the tautological line bundle of $\boldsymbol{P}(S^2(T_{\boldsymbol{P}^2}))$, and $\pi: \boldsymbol{P}(S^2(T_{\boldsymbol{P}^2})) \rightarrow \boldsymbol{P}^2$ is the projection. Indeed, if we take a point $R=[1:0:0] \in \boldsymbol{P}_P$, then $B=\boldsymbol{SL}(3)_R$ acts on the line $\mathcal{O}_{\boldsymbol{P}}(-1)_R \subset \pi^*(S^2(T_{\boldsymbol{P}^2}))_R$ by multiplication by $e^2/a^2$. Hence we find that $\mathcal{O}_{\boldsymbol{P}}(1))|_F \simeq \mathcal{O}_W(-4, 2)$. Now, if we set $[F] \simeq \mathcal{O}_{\boldsymbol{P}}(2) \otimes \pi^*(\mathcal{O}_{\boldsymbol{P}^2}(\alpha))$ $(\alpha \in \boldsymbol{Z})$, then $-2=(F, l_2)=2(\mathcal{O}_{\boldsymbol{P}}(1), l_2)+(\pi^*(\mathcal{O}_{\boldsymbol{P}^2}(\alpha)), l_2)=2(\mathcal{O}_W(-4, 2), l_2)+(\mathcal{O}_{\boldsymbol{P}^2}(\alpha), \text{line})=-8+\alpha$. Hence $\alpha=6$.

(g) $G_2$-case. Consider the standard $\boldsymbol{SL}(3)$-action on $\boldsymbol{P}^2$ and let $\boldsymbol{SL}(3)$ act on $M_2:=\boldsymbol{P}^2 \times \boldsymbol{P}^2$ diagonally. If we take a point $S:=([1:0:0], [0:1:0]) \in M_2$, then it is clear that $\boldsymbol{SL}(3)_S=\left\{\begin{bmatrix} * & 0 & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}\right\}=G_2$. The orbit decomposition of $M_2$ is given by $M_2=(M_2-\Delta) \cup \Delta$, where $M_2-\Delta$ is a 4-dimensional orbit isomorphic to $\boldsymbol{SL}(3)/G_2$ and $\Delta$ is the diagonal isomorphic to $\boldsymbol{SL}(3)/H$.

Next, we denote by $\bar{M}_2$ the blowing-up of $M_2$ along the diagonal $\Delta$. The orbit decomposition of $\bar{M}_2$ is given by $\bar{M}_2=\bar{M}_2^4 \cup E'$, where $E'$ is the exceptional divisor isomorphic to $\boldsymbol{SL}(3)/B$, and $\bar{M}_2^4=\bar{M}_2-E'$ is a 4-dimensional orbit isomorphic to $\boldsymbol{SL}(3)/G_2$. We note that $\bar{M}_2$ cannot be blown-down to a smooth algebraic space along $E' \simeq W$ in the $p_2$-direction. Details are similar to (e).

(h) $N(G_2)$-case. We consider the dual projective plane $(\boldsymbol{P}^2)^*$. Let $S^2(T_{(\boldsymbol{P}^2)^*})$ be the symmetric tensor bundle of degree 2 of $T_{(\boldsymbol{P}^2)^*}$, and we set $N_2:=\boldsymbol{P}(S^2(T_{(\boldsymbol{P}^2)^*}))$. Take a point $P:=[1:0:0] \in (\boldsymbol{P}^2)^*$ at which the isotropy group is $H'$, and take $A=\begin{bmatrix} a & 0 & 0 \\ d & e & f \\ g & h & i \end{bmatrix} \in H'$. Let $[y, z]$ be the inhomogeneous affine coordinates around the origin $P$. Since $({}^tA)^{-1}=\begin{bmatrix} 1/a & fg-di & eg-dh \\ 0 & ai & -ah \\ 0 & -af & ae \end{bmatrix}$, an easy calculation shows that the $H'$-action on $T_{(\boldsymbol{P}^2)^*, P}$ is represented by $a^2\begin{bmatrix} i & -h \\ -f & e \end{bmatrix}$ with respect to the basis $\{\partial/\partial y, \partial/\partial z\}$. Hence the $H'$-action on $S^2(T_{(\boldsymbol{P}^2)^*})_P$ is represented by $a^4\begin{bmatrix} i^2 & -ih & h^2 \\ -2if & ie+fh & -2he \\ f^2 & -fe & e^2 \end{bmatrix}$ with respect to the basis $\{(\partial/\partial y)^{\otimes 2}, (\partial/\partial y)\otimes(\partial/\partial z), (\partial/\partial z)^{\otimes 2}\}$. Thus the isotropy group at $[0:1:0]\in\boldsymbol{P}(S^2(T_{(\boldsymbol{P}^2)^*})_P)$ is given by $\{A\in H' \mid ih=fe=0\}=\{A\in H' \mid i=e=0 \text{ or } f=h=0\}=N(G_2)$. The orbit decomposition of $N_2$ is given by $N_2=N_2^4\cup F'$, where $N_2^4$ is a 4-dimensional orbit isomorphic to $\boldsymbol{SL}(3)/N(G_2)$ and $F'$ is a 3-dimensional closed orbit isomorphic to $\boldsymbol{SL}(3)/B$ such that $[F']\simeq\mathcal{O}_{\boldsymbol{P}^*}(2)\otimes\pi^*(\mathcal{O}_{(\boldsymbol{P}^2)^*}(6))$, where $\mathcal{O}_{\boldsymbol{P}^*}(1)$ is the tautological line bundle of $\boldsymbol{P}(S^2(T_{(\boldsymbol{P}^2)^*}))$ and $\pi\colon \boldsymbol{P}(S^2(T_{(\boldsymbol{P}^2)^*}))\to(\boldsymbol{P}^2)^*$ is the projection. Details are similar to (f).

**Proposition 9.** *Let $\bar{M}_2$ and $N_2$ be as in (g), (h).*

(1) *There exists an $\boldsymbol{SL}(3)$-equivariant finite morphism $\psi\colon \bar{M}_2\to N_2$ of degree 2. The ramification locus of $\psi$ is $E'\subset\bar{M}_2$ and the branch locus is $F'\subset N_2$.*

(2) *Let $l_1$ (resp. $l_2$) be a fiber of $p_1\colon F'=W\to\boldsymbol{P}^2$ (resp. $p_2\colon F'\to(\boldsymbol{P}^2)^*$). Then $(F', l_1)=-2$ and $(F', l_2)=4$. In particular, $N_2$ cannot be blown-down to a smooth algebraic space along $F'$ in neither directions.*

The proof of this proposition is similar to that of Proposition 8.

(i) $G_0$-case. Consider the standard $\boldsymbol{SL}(3)$-actions on $\boldsymbol{P}^2$ and $(\boldsymbol{P}^2)^*$ and set $X_0:=\boldsymbol{P}^2\times(\boldsymbol{P}^2)^*$. Define an $\boldsymbol{SL}(3)$-action on $X_0$ by $A\circ(P, Q)=(A\circ P, A\circ Q)$ for $(P, Q)\in X_0$, $A\in\boldsymbol{SL}(3)$. Take a point $P:=([0:0:1], [0:0:1])\in X_0$. Then an easy calculation shows that $\boldsymbol{SL}(3)_P=G_0$. The orbit decomposition of $X_0$ is given by $X_0=X_0^4\cup X_0^3$, where $X_0^4$ is a 4-dimensional orbit isomorphic to $\boldsymbol{SL}(3)/G_0$, and $X_0^3$ is a closed orbit isomorphic to $\boldsymbol{SL}(3)/B$, which is defined by $x_0y_0+x_1y_1+x_2y_2=0$, $([x_0:x_1:x_2], [y_0:y_1:y_2])\in\boldsymbol{P}^2\times(\boldsymbol{P}^2)^*$.

## 3. Classification of quasi-homogeneous 4-folds of $\boldsymbol{SL}(3)$

In this section, we classify smooth complete quasi-homogeneous 4-folds of $\boldsymbol{SL}(3)$ up to isomorphisms. First, we need a lemma:

**Lemma 10.** *Let $V$ be a smooth complete quasi-homogeneous 4-fold of $\boldsymbol{SL}(3)$. Then $V$ has no fixed points, no 1-dimensional orbits. The possible 2-dimensional orbits are isomorphic to $\boldsymbol{P}^2$ or $(\boldsymbol{P}^2)^*$ with the standard actions.*

Proof. Assume that $x\in V$ is a fixed point. We consider the induced linear action $\rho$ of $\boldsymbol{SL}(3)$ on $T_{V,x}$. Since $V$ is smooth, $\dim T_{V,x}=4$ and $\rho$ is represented as one of the following three types:

$$A\mapsto\begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} ({}^tA)^{-1} & 0 \\ 0 & 1 \end{bmatrix} \text{ or } I_4 \text{ (identity matrix) for } A\in\boldsymbol{SL}(3).$$

Now, by Luna's étale slice theorem [4], there exists an $\boldsymbol{SL}(3)$-stable affine subvariety $S$ containing $x$ such that there is an étale $\boldsymbol{SL}(3)$-equivariant morphism $\nu\colon S \to T_{V,x}$. But then, $S$ has a 4-dimensional orbit, whereas $T_{V,x}$ has no 4-dimensional orbits in any case. Thus, we got a contradiction and $V$ has no fixed points. Since $\boldsymbol{SL}(3)$ has no closed subgroups of codimension 1, and any closed subgroup of codimension 2 is conjugate to $H$ or $H'$ (Mabuchi [6; Theorem 2.2.1]), $V$ has no orbits of dimension 1 and any 2-dimensional orbit is isomorphic to $\boldsymbol{P}^2$ or $(\boldsymbol{P}^2)^*$. □

Now, we state the main theorem of this note. For a closed subgroup $G \subset \boldsymbol{SL}(3)$ of codimension 4, we denote by $\mathcal{C}(G)$ the set of all isomorphism classes of smooth complete quasi-homogeneous 4-folds of $\boldsymbol{SL}(3)$ whose open dense orbit is of the form $\boldsymbol{SL}(3)/G$.

**Theorem 11.** *Let $X$ be a smooth complete quasi-homogeneous 4-fold of $\boldsymbol{SL}(3)$. Then $X$ is classified completely as follows:*

(1) *Assume $X \in \mathcal{C}(G_{p,q})$. Then $X \simeq X_{p,q}$ if $|p-q| \neq 1$, $q \neq 1$; $X \simeq X_{p,1}, Y_p$ if $q=1$; $X \simeq X_{q-1,q}, Z_q$ if $q-p=1$ $(q \geq 1)$; $X \simeq X_{q+1,q}, Z_{-q}$ if $p-q=1$ $(q \geq 0)$; $X \simeq X_{0,1}, Y_0, Z_1, Q$ if $p=0$, $q=1$.*

(2) *If $X \in \mathcal{C}(G_1)$, then $X \simeq (\boldsymbol{P}^2)^* \times (\boldsymbol{P}^2)^*$, $B_\Delta((\boldsymbol{P}^2)^* \times (\boldsymbol{P}^2)^*)$.*

(3) *If $X \in \mathcal{C}(N(G_1))$, then $X \simeq \boldsymbol{P}(S^2(T_{\boldsymbol{P}^2}))$.*

(4) *If $X \in \mathcal{C}(G_2)$, then $X \simeq \boldsymbol{P}^2 \times \boldsymbol{P}^2$, $B_\Delta(\boldsymbol{P}^2 \times \boldsymbol{P}^2)$.*

(5) *If $X \in \mathcal{C}(N(G_2))$, then $X \simeq \boldsymbol{P}(S^2(T_{(\boldsymbol{P}^2)^*}))$.*

(6) *If $X \in \mathcal{C}(G_0)$, then $X \simeq \boldsymbol{P}^2 \times (\boldsymbol{P}^2)^*$.*

Proof. We verify the assertion (1). Let $X$ be a smooth complete quasi-homogeneous 4-fold of $\boldsymbol{SL}(3)$ which belongs to $\mathcal{C}(G_{p,q})$. Let $\nu\colon X \dashrightarrow X_{p,q}$ be a birational map induced by identifying the open dense orbits isomorphic to $\boldsymbol{SL}(3)/G_{p,q}$. By Hironaka [1], we resolve the indeterminacy locus $I$ of $\nu$ by successive blowing-ups along smooth centers. Since $I$ is an $\boldsymbol{SL}(3)$-stable closed subset of codimension $\geq 2$, each center is isomorphic to $\boldsymbol{P}^2$ or $(\boldsymbol{P}^2)^*$ by Lemma 10. Let $\sigma\colon \tilde{X} \to X$ be the composition of these blowing-ups and $\mu = \nu \circ \sigma\colon \tilde{X} \to X_{p,q}$ be the resolution of $\nu$. Since the indeterminacy locus $J$ of $\mu^{-1}$ is $\boldsymbol{SL}(3)$-stable and has codimension greater than or equal to 2, $J$ is empty and $\mu$ is an isomorphism. Therefore, $X$ is isomorphic to $X_{p,q}$ or its smooth blow-downs. (1) is thus proved by Lemmas 3, 4, 5 and Proposition 7. Assertions (2)~(6) can be proved similarly. □

Remark. We note that in the $\boldsymbol{SL}(2)$-case, some interesting minimal rational 3-folds are constructed as smooth projective quasi-homogeneous 3-folds of $\boldsymbol{SL}(2)$ (Mukai-Umemura [7]). Here, a rational $n$-fold $X$ is called minimal if the identity component $\mathrm{Aut}^\circ(X)$ of the automorphism group of $X$ is maximal in the Cremona group $\mathrm{Bir}(\boldsymbol{P}^n)$ of $n$ variables. Therefore, to determine whether our quasi-homogeneous 4-folds of $\boldsymbol{SL}(3)$ are minimal rational 4-folds or not will be an interesting problem, which we plan to discuss elsewhere.

As an easy corollary to our theorem, the Picard groups of 4-dimensional homogeneous spaces of $\boldsymbol{SL}(3)$ are determined from the orbit decomposition of these quasi-homogeneous 4-folds.

**Corollary.** $\mathrm{Pic}(\boldsymbol{SL}(3)/G_{p,q}) \simeq \boldsymbol{Z} \oplus \boldsymbol{Z}/(g.c.d.(p,q))$, $\mathrm{Pic}(\boldsymbol{SL}(3)/G_i) \simeq \boldsymbol{Z}^2$ $(i=1,2)$, $\mathrm{Pic}(\boldsymbol{SL}(3)/N(G_i)) \simeq \boldsymbol{Z} \oplus \boldsymbol{Z}/(2)$ $(i=1,2)$ and $\mathrm{Pic}(\boldsymbol{SL}(3)/G_0) \simeq \boldsymbol{Z}$.

## References


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