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HALF NEARFIELD PLANES

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1. Introduction

Let Π be an affine translation plane of order p^n. Let Π admit an affine homology group of order k with center P and axis $0Q$ where PQ is the line at infinity, P, Q are infinite points and 0 is the zero vector (or any affine point). We shall say that Π admits *symmetric homology groups* provided there is also an affine homology group of order k with center Q and axis $0P$.

A nearfield plane is an affine translation plane of order p^n that admits symmetric homology groups of order p^n-1. Actually, if a translation plane admits one affine homology group of order p^n-1 then it admits symmetric homology groups of order p^n-1. Of course, there are many examples of translation planes that admit an affine homology group of order k that do not admit symmetric homology groups. For example, the j-planes of order q^2 (see [9] or [12]) admit homology groups of orders $q+1$ and $q-1$ but do not always admit symmetric homology groups of either order.

Let Σ denote a translation plane of even order 2^r that admits an affine elation group of order $2^r/2$. Then Jha, Johnson, and Wilke [7] have shown that there is also an elation group of order 2^r and, in this case, the plane is a semifield plane. Is a similar result valid for translation planes of odd order k that admit an affine homology group of order $(k-1)/2$? If yes, then there would be an affine homology group of order $k-1$ which would imply that the plane is a nearfield plane.

The theorem of Thas [18], [19], Bader, Lunardon [2] classifying the flocks of hyperbolic quadrics in PG$(3, q)$ plays a major part in the study undertaken herein.

Theorem 1.1. (Thas, Bader, Lunardon) *Let F be a flock of a hyperbolic quadric in $PG(3, q)$. Then either F is linear, a Thas flock or an irregular flock with* $q=11, 23$, *or* 59.

It is well known that corresponding to a flock of a hyperbolic quadric in PG$(3, q)$ is a translation plane with spread in PG$(3, q)$ (see the Thas-Walker construction, e.g. in [8]). Moreover, in Johnson [8], the following connection was noted:

Theorem 1.2. (Johnson [8]) *If a translation plane with spread in PG (3, q) admits an affine homology group H such that some orbit of components union the axis and coaxis of the group H forms a regulus then there is a corresponding flock of a hyperbolic quadric in* $PG(3, q)$.

From the standpoint of translation planes, we have the following corollary:

Corollary 1.3. *If a translation plane with spread in* $PG(3, q)$ *admits an affine homology group H such that some orbit of components union the axis and coaxis of the group H forms a regulus in* $PG(3, q)$ *then the plane is either Desarguesion, regular nearfield, or irregular nearfield of order* $11^2, 23^2$, *or* 59^2.

Note that the indicated translation planes correspond to the flocks classified by Thas, Bader, and Lunardon. Actually, the first constructions of the irregular flocks were given by Bader[1] and independently by Johnson [8] using the fact that the irregular nearfield planes of the indicated orders admit affine homology groups that are "regulus inducing" as above.

It is well known that a model for the Minkowski plane defined by the plane intersections with the hyperbolic quadric in PG(3, q) can be given by taking the "points" as the elements of PGL(1, q)$\times$PGL(1, q) and the "circles" defined by the equations $y=x\sigma$ for all σ in PGL(2, q). In this case, a flock is simply a sharply transitive subset of PGL(2, q). More generally, a flock of a finite Minkowski plane defined using a sharply 3-transitive set S is a sharply transitive subset of S.

In [14], Knarr showed that corresponding to a sharply transitive subset of PΓL(2, q) is a translation plane of order q^2. In this case, the spread is not necessarily in PGL(3, q) and the subset does not necessarily have to belong to a sharply 3-transitive set.

Bonisoli [3] has constructed a class of sharply transitive subsets of a sharply 3-transitive set within PΓL(2, q). The construction takes a regular subgroup E of PSL(2, q) of order $(q+1)/2$ and an element g of PΓL(2, q) such that $E \cup Eg$ is sharply transitive. Moreover, in this case, g normalizes E. In the corresponding translation plane, the group E becomes an affine homology group. Furthermore, just as in the case of the hyperbolic flock, there is an affine homology group of order $(q-1)$ which is "regulus inducing" (note however that the spread is not in PG(3, q) so the induced reguli lie in various other projective spaces). In this way, there is a coresponding affine homology group of order $(q^2\text{-}1)/2$ in a tranlation plane of order q^2 and due to the normalization of E by g, it turns out that there are symmetric homology groups of order $(q^2-1)/2$.

Thinking back to what might be called a half semifield plane (as above a translation plane of order 2^r that admits an affine elation group of order $2^r/2$), we see that a half semifield plane is a semifield plane. Define a translation plane of order p^r to be a half nearfield plane if the plane admits an affine homology group of order $(p^r-1)/2$. Is a half nearifield plane a nearfiled plane? Actually, when the planes arising from the sharply transitive sets of Bonisoli were found, it was originally thought that these planes were probably nearfield planes. However, the planes turned out to be André but not nearfield planes so half nearfield planes are not necessarily always nearifield. The class of generalized planes are not necessarily always nearfield. The class of generalized André planes or even André planes is not completely determined and the André planes of order k that admit two or even one homology group(s) of order $(k-1)/2$ were completely unknown until very recently. Actually, the planes arising from the Bonisoli sharply transitive sets provide the first non-nearfield examples of such planes.

Considering the examples above, Hiramine and Johnson [4], [5] studied translation planes of order k that admit at least two affine homology groups of order $(k-1)/2$ and also considered the class of generalized André planes of order k that admit at least one affine homology group of order $(k-1)/2$; the half nearfield generalized André planes. Define a near nearfield plane to be a translation planes of order k that admits symmetric affine homology groups of order $(k-1)/2$. Is every half nearfield plane a near nearfield plane?

Theorem 1.4. (Hiramine and Johnson [5]). *Let Π be a translation plane of order k that admits at least two affine homology groups of order $(k-1)/2$. Then one of the following occurs:*

(i) *Π is a generalized André plane,*

(ii) *the order is 23^2 and the plane is the irregular nearfield plane,*

(iii) *the order is 7^2 and the plane is the irregular nearfield plane,*

(iv) *the order is 7^2 and the plane is the exceptional Lüneburg plane admitting $SL(2, 3)$.*

Not all half nearfield planes are near nearfield planes although the known examples which are not are all generalized André planes. In order to provide some examples, we require some further background.

Theorem 1.5. (Hiramine and Johnson [4] (4.1) (i)). *Let Π be a generalized André plane of order k that admits an affine homology group of order $(k-1)/2$. Then there is a Dickson pair $\{q, n\}$ (all prime divisors of n also divide $(q-1)$ and if n is even then $q\equiv 1 \bmod 4$) such that $k=q^n$. Moreover, representing the translation plane as a $GF(p)$-vector space, and letting $GF(q^n)^*=\langle w\rangle$, we may represent the spread as follows:*

$\{x=0,\ y=0,\ y=x^{q^i}w^{s(q^i-1)/(q-1)}\alpha$ and $y=x^{p^tq^t}w^{q^i}w^{s(q^i-1)/(q-1)}\beta$ where α, β are in $GF(q^n)$ of orders dividing $(q^n-1)/2n$ and $i=1, 2, \cdots, n$ and $(s, n)=1$ or 2.

It turns out that, for most orders, the generalized André planes admitting such homology groups may be constructed from the Dickson nearfield planes by a natural net replacement procedure. Moreover, the generalized André planes admitting symmetric homology groups may be determined within the above class.

We shall illustrate some of these ideas by construcing a class of translation planes admitting symmetric homology groups by net repalcement in some Dickson nearfield planes and then generalize this class. This particular set of examples will occur in our main result.

Theorem 1.6. (Hiramine and Johnson (5.2) [4]) *A generalized André plane of order k odd admits symmetric homology groups of order $(k-1)/2$ if and only if there exists a Dickson pair $\{q, n\}$ such that $k=q^n$ and there exists a spread represented in the form:*

$x=0$, $y=0$, $y=x^{q^i}w^{s(q^i-1)/(q-1)}\alpha$, $y=x^{hq^i}w^{uq^i}w^{s(q^i-1)/(q-1)}\beta$ where α, β are in $GF(q^n)^*$ of order dividing $(q^n-1)/2n$, $i=1, 2, \cdots, n$ where $\langle w\rangle=GF(q^n)^*$. Further, $(s, n)=1$ or 2 for $h=p^t$, $q=p^r$ and $u=1$ or n corresponding to when s is even or odd respectively where $s(h-1)\equiv u(q-1)+2nz \bmod(q^n-1)$ for some z.

Let $\{q, 4\}$, be a Dickson pair so that $q\equiv 1 \bmod 4$. Let F be isomorphic to $GF(q^4)$ where $\langle w\rangle$ generates the multiplicative group. Let Z denote the cyclic subgroup of order $(q^4-1)/4$ in F^* and let a Dickson nearfield be denoted by $(F, +, 0)$. Then there is a generator $w^s Z$ of F^*/Z so that $F^*=w^sZ \cup w^{s(q+1)}Z \cup w^{s(q^3-1)/(s-1)}Z \cup w^{s(q^4-1)/(q-1)}Z$. By Lüneburg [16] (7.4), there are either 1 or 2 nonisomorphic nearfields depending on whether $p\equiv 1$ or $3 \bmod 4$ where $p^r=q$ for p a prime.

There is a subgroup H of the nearfield group defined as follows:

$\langle (x, y)\to(x, y^{q^{2i}}w^{s(q^{2i}-1)/(q-1)}\alpha)$ for $i=1, 2$ and for all α in $Z\rangle$. Note that H has order $(q^4-1)/2$.

Let $P_0=H(y=x)$ and $P_1=H(y=x^qw^s)$. Then, by [4] (3.4), $P^*_{1,t}=H(y=x^{p^t}w)$ is a replacement for P_1 and hence there is a corresponding and potentially new translation plane of order q^4 obtained via net replacement in a Dickson nearfield plane and which admits a homology group of order $(q^4-1)/2$. The kernel of the plane is the intersection of the fixed fields of the corresponding automorphisms defining the spread. In order to obtain a translation plane with kernel $GF(q^2)$, we need to take p^t to be q^2 or q^4. Note that are two Dickson nearfield planes of the above type for certain prime powers q so there are potentially four constructed translation planes. However, we shall see that these constructed planes are all isomorphic even if the original Dickson nearfield planes constructing them are not.

We may construct such planes more generally from a Desarguesian affine plane Σ of order h^2 coordinatized by a field F isomorphic to $GF(h^2)$ as follows. Assume that there is a corresponding translation plane Π of order h^2 which admits an affine homology group H: $\langle(x, y)\to(x, yg) \mid g\in H_0$ where H_0 is a subgroup of $\Gamma L(1, F)\rangle$. Assume that Π has kernel the subfield of F isomorphic to $GF(h)$. This forces the group H to be cyclic or have an element of the form $\sigma: (x, y)\to(x, y^h b)$ where now b is an element of F^*. Hence, the square of any such element is linear and the product of two such elements (for different b's) is linear so that there is a cyclic and linear subgroup C of order $(h^2-1)/4$. Hence, it follows that, for σ as above, b^{h+1} must have order dividing $(h^2-1)/4$. If 4 divides $h+1$ then there is a cyclic group of order $h-1$ which is regulus inducing and, as noted above, the only possible plane is the regular nearfield plane. Thus, assume that 4 divides $h-1$.

We decompose the plane Σ into its Andrè nets of degree $h+1$ which may be distinguished by the partial spreads which are orbits of the cyclic group of order $h+1$. Thus, under C there are $(h-1)/4$ Andrè nets of degree $h+1$. Let Δ denote the Desarguesian affine plane with spread $y=x^h m$, $x=0$ where $m\in F$. If we take the partial spread $y=x^h b\alpha$ for all α a 4th power in F^* and such that b^{h+1} is a square but not a 4th power in F^* then this partial spread is a set of opposite regulus nets in Σ correspoonding to the $(h-1)/4$ Andrè nets $N_\delta=\{y=xm \mid m^{h+1}=\delta$ is a square but not a 4th power in $GF(h)\}$. Let DN_δ denote the derived or opposite net of N_δ. Let P_4 denote the set of 4th power Andrè nets and DP_2 the set of opposite nets to the square but not 4th power nets. Note that if the above group H is to act on a translation plane Π, it follows that one orbit of components has the form $(y=x)H=\{y=x\alpha \mid \alpha$ is a 4th power$\} \cup \{y=x^h b\beta \mid \beta$ is a 4th power and b is a fixed element of F^* such that b^{h+1} is a square but not a 4th power in $GF(h)^*\}$. Consider $(y=xc)H$ where c^{h+1} is a nonsquare in $GF(h)$. Then this orbit is a union of $(h-1)/4$ Andrè "nonsquare" nets in Σ together with a union of $(h-1)/4$ opposite Andrè "nonsquare" nets. Depending on the type of the nonsquare chosen (for $c^{h+1}=w^{2i+1}$ or w^{4i+1}), there are two possible translation planes that admit H as a homology group. Let P_1 and P_3 denote the corresponding sets of Andrè nonsquare nets where $\log_w c\equiv 1$ or 3 mod 4 respectively.

The two possible planes are Π_1 with spread $P_4 \cup DP_2 \cup P_1 \cup DP_3$ and Π_2 with spread $P_4 \cup DP_2 \cup DP_1 \cup P_3$. Note, however, these two planes are isomorphic under the mapping $(x, y)\to(x, y^h w^3)$ which will map P_4, DP_2, DP_1, P_3 onto DP_3, P_1, P_4, DP_2 respectively. Now when h is a square, it follows as in Hiramine and Johnson [4] (4.2) that the generalized Andrè plane may be constructed from the Dickson nearfield plane of order h^2 and kernel $GF(\sqrt{h})$. The two possible Dickson nearfield planes are non-isomorphic due to the placement of generator w^s which gets attached to x^q in one plane and to x^{q^3} in another plane (when the order is q^4). Constructing a translation plane from one Dickson nearfield by either of the above net replacements and then following with the above mapping shows that the four possible translation planes obtained from the two nonisomorphic

Dickson nearfield planes are isomorphic.

Thus, we have constructed some half nearfield planes with spreads in $PG(3, q)$. When q is a square, there is a natural connection to Dickson nearfield planes which somewhat justifies the terminology.

We shall call the translation plane constructed above the *proper half nearfield plane of dimension* 2 (the word proper indicating that the plane is not a nearfield plane).

Our main result classifies the translation planes of order q^2 with spreads in $PG(3, q)$ that admit an affine homology group of order $(q^2-1)/2$.

Theorem 1.7. *Let* Π *be a translation plane of order* q^2 *with spread in* $PG(3, q)$ *thet admits an affine homology group of order* $(q^2-1)/2$. *Then one of the following possibilities occur:*

(*i*) Π *is Desarguesian,*

(ii) *the plane is regular nearfield,*

(iii) Π *is the proper half nearfield plane of dimension* 2,

(iv) *the order is* 23^2 *and the plane is the irregular nearfield plane,*

(v) *the order is* 7^2 *and the plane is the irregular nearfield plane, or*

(iv) *the order is* 7^2 *and the plane is the exceptional Lüneburg plane admitting* $SL(2, 3)$.

Note that as a corollary, we obtain:

Corollary 1.8. *Every half nearfield plane with spread in* $PG(3, q)$ *is a near nearfield plane; if a translation plane with spread in* $PG(3, q)$ *admits an affine homology group of order* $(q^2-1)/2$ *then the plane admits at least two affine homology groups of order* $(q^2-1)/2$.

We have noted above for translation planes with spreads within $PG(3, q)$, there are connections between the existence of spreads covered by reguli and the existence of certain central collineation gropus. For example, it is possible to completely determine the planes that admit affine homology groups of orders $q-1$ whose spreads also contain a regulus containing at least one of the axis and coaxis of the corresponding group.

Theorem 1.9. (Johnson [11]). *Let* Π *denote a translation plane with spread within* $PG(3, q)$ *that admits an affine homology group of order* $q-1$ *and such that the spread contains a regulus containing either the axis or coaxis of the group. Then* Π *is one of the following planes:*

(i) *Desarguesian,*

(ii) *regular nearfield,*

(iii) $q=11$, 23 or 59 *and the plane is the irregular nearfield plane of order* q^2.

Considering that the translation planes of order q^2 that admit an affine homology group are determined as above, we may consider a possible companion result involving translation planes with homology groups of order $q+1$ such that the spread contains a regulus containing at least one of the axis and the coaxis of the corresponding group. In this case, we obtain other planes:

Theorem 1.10. *Let Π denote a translation plane with spread within $PG(3, q)$ that admits an affine cyclic homology group H of order $q+1$. If the spread contains a regulus R that contains both the axis and the coaxis of the group then the plane is multiply derived from a regular nearfield plane.*

2. Background

We have recalled some of the background required for this article in the introduction. In this section, we provide the reader with a list of the remaining results required for the main result.

Theorem I. (Hiramine and Johnson [5] (3.1)). *Let Π be a translation plane of order p^m which admits an affine homology group H of order $(p^m-1)/2$ (that is, a half nearfield plane). Then one of the following situations must occur:*

(i) *There exists a prime p-primitive divisor u of p^m-1 and $H \leq \Gamma L(1, p^m)$. In particular H has a normal cyclic Sylow u-subgroup.*

(ii) *$p^m=p^2$, $p+1=2^a$ and $H=Q_{p+1} \times Z_{(p-1)/2}$ where Q_{p+1} is either cyclic or the generalized quaternion group of order $p+1=2^a$ and $Z_{(p-1)/2}$ is in the center of $GL(2, p)$.*

(iii) *$p^m=p^2$ for $p \in \{7, 23, 47\}$. Furthermore, if $p=23$ or 47 then H contains a normal subgroup $Z_{(p-1)/2}$ in the center of the group $GL(2, p)$.*

(iv) *In any of the cases, the homology group is always solvable.*

The case (ii) is handled by Hiramine and Johnson [5] (3.3) when p is not equal to 7.

Theorem II. (Hiramine and Johnson [5] (3.3)). *Under the hypothesis of Theorem I, when $p+1=2^a$ and p is not 7 and $m=2$ then the plane is either Desarguesan or the regular nearfield plane.*

Theorem III. (Hiramine and Johnson [5] (3.2)). *Under the hypothesis of Theorem I, when $p=23$ or 47 then either the homology group has a normal cyclic Sylow 3-subgroup or when $p=23$ the plane is either the regular nearfield plane, the Desarguesian plane or the irregular nearfield plane of order 23^2 or when $p=47$, the plane is either the regular nearfield plane or Desarguesian.*

Theorem IV. (Ostrom [15]). *Let Π be a translation plane of order q^n and kernel K isomorphic to $GF(q)$ where q is a prime power. Let σ be an affine homolo-*

gy whose order is a p-primitive divisor of q^n-1. *Let* Γ *denote any component orbit under* $\langle\sigma\rangle$. *Then there exists a Desarguesian affine plane* Σ *of order* q^n *coordinatized by a field extension of* K *such that* Σ *contains the axis and coaxis of* σ *and the orbit* Γ *as components.*

(Note that this result is stated over the prime field in Ostrom's work. However, since an affine homology is in the linear translation complement, the proof actually works in the more general setting. Recall that any collineation is K-semilinear where K is the kernel of the translation plane. If the collineation fixes a nontrivial K-subspace pointwise then the collineation is necessarily linear.)

Theorem V. (Jha-Johnson [6], Johnson [9], Johnson, Pomareda, Wilke [12]). *Let* Π *denote a translation plane of order* q^2. *If* Π *admits a cyclic homology group of order* $q+1$ *then every orbit of components defines a derivable partial spread. If the kernel contains a field* K *isomorphic to* $GF(q)$ *(the spread is within* $PG(3, q)$*) then the derivable partial spread is a regulus in* $PG(3, K)$.

3. The main theorem

In this section, we give the proof to the main result (1.7) on half nearfield planes stated in the introduction with the exception that we postpone the analysis of planes of orders p^2 for $p=7$, 23 47 to another section.

We assume throughout this section that Π is a translation plane with spread in $PG(3, K)$ for some field K isomorphic to $GF(q)$ and admits an affine homology group of order $(q^2-1)/2$ with $q\neq 7, 23$, or 47. Furthermore, we may assume that $q+1$ is not a power of 2 by II.

Lemma 3.1. *There is a field extension* F *of* K *which is isomorphic to* $GF(q^2)$ *such that the affine points may be identified with* $F\oplus F$,

(i) *If* Σ *is the corresponding Desarguesian affine plane coordinatized by* F, *then* Π *and* Σ *share the axis and coaxis of the homology group* H *along with an orbit of components under a normal q-primitive subgroup.*

(ii) *Representing* H *on* Π *in the form* $\langle(x, y)\rightarrow(x, yh)\mid h\in H_1$ *is a 2 by 2 matrix group over* $K\rangle$ *then* $H_1\leq\Gamma L(1, F)$,

(iii) H *contains a cyclic subgroup of order* $(q^2-1)/4$ *in* $GL(1, F)$.

Proof. By results I and IV, we can identify the affine points of Π with those of some Desarguesian plane Σ coordinatized by a field F isomorphic to $GF(q^2)$. It is also true that the field F can be chosen as a field extension of K by Passman [17] (19.8) since the homology group H is a linear K-group. This proves (i) and (ii).

To prove (iii), note that H commutes with the scalar maps acting in

$GL(4, K)$ and the scalar maps act naturally in $\Gamma L(1, F)$. Consider the map $x \to x\alpha$ for $\alpha \in K$. Since $h \in H_1$ commutes with this map and for h the map $x \to x^{p^i}b$ for $b \in F$ and $q=p^r$, it follows that $p^i=q$ or q^2 for all $h \in H_1$. Thus, it follows that $h^2 \in GL(1, F)$. Moreover, for two given elements with companion automorphisms not equal to 1, clearly their product is in $GL(1, F)$. Hence, there is an index two subgroup of order $(q^2-1)/4$ in $GL(1, F)$ which proves (iii).

Lemma 3.2. *If H contains a cyclic group of order $q-1$ isomorphic to a subgroup of $GL(1, F)$ then Π is a regular nearfield plane or Desarguesian.*

Proof. If H contains a cyclic group of order $q-1$ within $GL(1, F)$ then this group corresponds to the group of scalar maps within K. By (1.3), it follows that the cyclic group is a regulus inducing group and hence that there is a corresponding flock of a hyperbolic flock. Hence, the corresponding plane is Desarguesian, regular nearfield or irregular nearfield of order p^2 for $p=11, 23$, or 59. By Hiramine and Johnson [5], the irregular nearfield planes of orders 11^2 or 59^2 do not adimt such homology subgroups of index two. Since we are excluding the order 23^2, we have the proof to (3.2).

Now, since we have a cyclic subgroup of order $(q^2-1)/4$, if 4 divides $q+1$, we have a cyclic group of order $q-1$ of the type considered in (3.2). Thus, we have:

Lemma 3.3. *If $q \equiv -1$ mod 4 and $q \neq 7, 23$, or 47 then the theorem is proved.*

Hence, for the remainder of this section, we may assume that $q \equiv 1$ mod 4.

Lemma 3.4. *Either the plane is a regular nearfield or Desarguesian plane or we may assume that the group H has the following form: $H=\langle (x, y) \to (x, ya)$ where the order of a divides $(q^2-1)/4$, $(x, y) \to (x, y^q b)$ for some b in F such that b^{q+1} has order dividing $(q^2-1)/4\rangle$.*

Proof. By (3.1) (iii), there is a cyclic subgroup of H_1 in $GL(1, F)$ of index 2 in H_1. If H itself is cyclic, we may apply (3.2) to complete the proof. If H is not cyclic, there is an element which corresponds to a strictly semi-linear element of $\Gamma L(1, F)$ and this proves (3.4) as above.

Lemma 3.5. *Let Γ and Γ^* denote the two nontrivial H-orbits of components each of length $(q^2-1)/2$ where we choose $\Gamma=(y=x)H$ (we choose the axis of H to be $y=0$, the coaxis to be $x=0$ and a given component in Γ to have the form $y=x$). Then the partial spreads of both orbits are disjoint unions of $(q-1)/2$ reguli in $PG(3, K)$.*

Proof. Since we now have a cyclic group of order $q+1$, we may apply V.

Lemma 3.6. *Let Σ^* denote the Desarguesian plane whose components are $x=0$, $y=0$, $y=x^q m$ for $m\in F$. Let C denote the cyclic group of order $(q^2-1)/4$ which acts in Σ in the form $\langle(x,y)\to(x,ya)$ such that the order of a divides $(q^2-1)/4\rangle$. Then we may assume that $\Gamma=N\cup M^*$ where $N=(y=x)C$, and $M^*=(y=x^q b)C$. N is a a union of $(q-1)/4$ André derivable (regulus) nets in Σ and M^* is a union of $(q-1)/4$ André derivable (regulus) nets in Σ^* which are the opposite or derived nets of a set of André derivable nets in Σ.*

Proof Use result V to see that a partial spread defined by a component image under a subgroup T of C defines a regulus in $PG(3, K)$. We define the André partial spreads to be the images of components under the cyclic group T of C of order $q+1$ (see the construction in the introduction) acting in Σ. For a given André net R, the opposite regulus is defined by $y=x^q m$ where the components of R are $y=xm$ for certain vaules $m\in F$. This proves (3.6).

Lemma 3.7. *Each component L of Γ^* is either a Baer subplane of an André net of Σ (a component of Σ^*) or a Baer subplane of an André net of Σ^* (a component of an André net of Σ). Furthermore, if L is a componenet of Σ then the André net containing L belongs to* Π. *If L is a Baer subplane of Σ then the opposite André net containing L belongs to* Π.

Proof. There are $q-1$ mutually disjoint (on components) André nets of Σ and $(q-1)/2$ of these are in Γ or are the derived nets of André nets of Σ^* disjoint from the previously mentioned André nets. Thus, either a component L of Γ lies in an André net of Σ and then the T-orbit of L is an André net containing L or L is a Baer subplane of Σ. If L is a Baer subplane of Σ then L is contained in a regulus net R_L defined uniquely by the components of L. Since R_L must lie within the set of $(q-1)/2$ remaining André regulus nets of Σ, it follows that R_L must be one of these André nets for otherwise, R_L shares less than or equal to 2 components per each of the $(q-1)/2$ regulus nets and since the degree of R_L is $q+1$, this cannot occur. Then the T-orbit of L fills out the derived net to R_L which is an André net of Σ^*. This proves (3.7).

We have thus also proved:

Lemma 3.8. *The spread for* Π *is a union of the axis and coaxis of the homology group H and a set of mutually disjoint André nets of Σ or Σ^*.*

Lemma 3.9. *If* Π *is a translation plane with spread within $PG(3, K)$ for K a field isomorphic to $GF(q)$ that admits an affine homology group of order $(q^2-1)/2$ and q is not 7,23 or 47 then* Π *is either Desarguesian, the regular nearfield plane or the proper half nearfield plane of dimension 2.*

Proof. By (3.8), the spread as the form $x=0$, $y=0$, $y=xm$ or $y=x^q d$ for certain elements m, d of a field extension F of K which is isomorphic to $GF(q^2)$. Moreover, the homology group H has the form $\langle (x,y)\rightarrow(x,ya) \mid$ the order of a divides $(q^2-1)/2$, $(x,y)\rightarrow(x,y^q b)$ where b is a fixed element of $F^*\rangle$. We may now follow the analysis of the example of the plane that we called the proper half nearfield plane of dimension two described in the introduction. This proves (3.9).

4. The case $q=23$ or 47

In this section, we consider the possible translation planes of order p^2 for $p=23$ or 47 that admit an affine homology group of order $(p^2-1)/2$. By result III, the possibilities are determined unless there is a normal cyclic Sylow 3-subgroup of the homology group. Since this implies that there is an Abelian irreducible normal subgroup we may use (19.8) of Passman [17], to show that we may make the identification as in section 3 with $\Gamma L(1, F)$ where F is a field isomorphic to $GF(p^2)$. Hence, we have

Theorem 4.1. *If* Π *is a translation plane of order* p^2 *for* $p=23$ *or* 47 *that admits an affine homology group of order* $(p^2-1)/2$ *then either*

(i) $p=23$ *and the plane is Desarguesian, regular nearfield plane or irregular nearfield plane or*

(ii) $p=47$ *and the plane is Desarguesian or regular nearfield plane.*

5. The case $q=7$

In this section, we show that there are exactly four translation planes of order 7^2 that admit an affine homology group of order $(7^2-1)/2$. This section is the most combinatorial as essentially none of the classification results apply.

Theorem 5.1. *Let* Π *be a translation plane of order* 7^2 *that admits an affine homology group of order* $(7^2-1)/2$ *then* Π *is one of the following planes:*

(i) *Desarguesian,*

(ii) *regular nearfield plane,*

(iii) *irregular nearfield plane, or*

(iv) *the exceptional Lüneburg plane admitting* $SL(2, 3)$.

Proof. Let H^* denote the group induced on the coaxis. Let Z denote the center of H^* so that H^*/Z is a subgroup of $PGL(2, 7)$. Using the classification of the subgroups of $PGL(2, q)$ which contain non p-elements for $q=p^r$ and p a prime, it follows that H^*/Z is a subgroup of a dihedral group of order $2(Z\mp 1)$ or is isomorphic to A_4, S_4, or A_5 and, of course, A_5 cannot occur. Moreover, H^* is a fixed point free group and as such is a Frobenius complement. If H^* is isomorphic to S_4 then there is more than one involution in H^* and the same is true of H, a contradiction. Hence, the order H^*/Z divides 12 or 16 and either Z contains an involution or a 3-element and in either case, there is a cyclic subgroup C^* of order 6. Hence, we may diagonalize so that C^* has the form $\{\begin{bmatrix} u & 0 \\ 0 & u^j \end{bmatrix}\}$ for all $n \in GF(7)^*$ and $(j, 6)=1$ since C^* is fixed point free. Hence, $j=1$ or 5. If $j=1$ then there is a regulus inducing homology group of order 6. Hence, by result (1.3), the translation plane is Desarguesian or regular nearfield. Hence, we may assume that $j=5$. Moreover, since C^* contains an element of the center of order 2 or 3, it follows from the form of C^* that Z has order 2.

Hence, H^*/Z is isomorphic either to the dihedral group D_6 of order 12 or to A_4.

If H^*/Z is isomorphic to D_6 then there is a set of two 1-dimensional subspaces on the coaxis of H which are fixed or interchanged by H^*. Hence, there is a subgroup W of H of order 12 which fixes two 1-dimensional subspaces on the coaxis of H. By choice of a basis, we may choose coordinates so that W^* (the group induced by W on the coaxis) is diagonal and elements have the basic form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ for various elements a, b of $GF(7)$. But, since $y=x\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ then becomes a component of the translation plane, it follows that the differences of the matrices of W^* must be non-singular or zero. Since W^* has order 12, the latter cannot occur.

Hence, either the plane is Desarguesian or regular nearfield or H^*/Z is isomorphic to A_4.

So, H^* contains a normal Sylow 2-subgroup S of order 8 and since H^* is a Frobenius complement (see e.g. Lüneburg [16] (3.5), it follows that S is quaternion.

Proposition 5.2. *Let H be an affine homology group of order 24 of a translation plane of order 7^2 such that the group H^* induced on the coaxis modulo its center is isomorphic to A_4. Then H^* may be represented by the following matrix group:* $\left\langle \begin{bmatrix} 3 & 4 \\ 1 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \right\rangle$.

Proof. Let $\tau \in S$ of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for $a, b, c, d \in GF(7)$. Since $\tau^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, it follows that

$a^2+bc=-1=d^2+bc$ and $(a+d)b=0=(a+d)c$. If b or $c=0$ then $a^2=-1$ but -1 is a nonsquare in $GF(7)$. Hence, $bc \neq 0$. Thus, $a=-d$. Now conjugate τ by the subgroup $T^* = \left\langle \begin{bmatrix} u & 0 \\ 0 & u^{-1} \end{bmatrix} \middle| u \in GF(7)^* \right\rangle$ (note that $u^{-1}=u^5$ and see above remarks).

Furthermore, $-\tau=\tau^3$ so that we have the following conjugated elements in H^*: $\pm\begin{bmatrix} a & bu^2 \\ cu^{-2} & -a \end{bmatrix}$ for all elements u of $GF(7)^*$. If c is square we can take $c=1$ and if c is nonsquare then $-c$ is square and we may take $-c=1$. Thus, we may take $c=1$ in τ without loss of generality. Hence, $a^2+b=-1$.

Note that $H^*=\{\pm\begin{bmatrix} av^2 & bw^2 \\ w^{-2} & -av^2 \end{bmatrix}, \begin{bmatrix} u & 0 \\ 0 & u^{-1} \end{bmatrix}$ for $v, w, u \in GF(7)^*\}$. Since H is a homology group, the differences of the elements of H^* must be nonsingular or zero matrices.

Since $a^2+b=-1$ and the squares are 1, 2 and 4 then
$(a, b) \in \{(\pm 1, 5), (\pm 2, 2), ((\pm 4, 4)\}$.

However, $\begin{bmatrix} a & b \\ 1 & -a \end{bmatrix} - \begin{bmatrix} u & 0 \\ 0 & u^{-1} \end{bmatrix}$ is nonsingular so that $(a-u)(a+u^{-1})+b \neq 0$. Hence, we must have $a^2+b+a(u^{-1}-u)=-1+a(u^{-1}-u)$ nonzero and since $(u^{-1}-u)$ takes on the set of values $\{0, -2, 2\}$ as u varies over $GF(7)^*$ (for example, for $u=2$, $4-2=2$ and for $u=3$, $5-3=2$, etc.), we must have that a is not ± 1.

Change bases by $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ to produce the elements $\begin{bmatrix} -a & u^2 \\ bu^{-2} & a \end{bmatrix}$. We note that $b=2$ or 4 from above (as b cannot be 5 as a is not ± 1) so that we can replace a by $-a$ and retain an element of the form $\begin{bmatrix} -a & b \\ 1 & a \end{bmatrix}$ in the group. Hence, we may assume that $(a, b)=(4, 4)$ or $(2, 2)$. Note that the symbols no longer have the original meaning.

First assume that $(a, b)=(2, 2)$. Note that $\begin{bmatrix} a & b \\ 1 & -a \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$ generate A_4 modulo the center of the group $\pm I_2$. Hence, the product of these two elements must have order 3 or 6. But, when $(a, b)=(2, 2)$ a short calculation shows that the cube of $\begin{bmatrix} 2 & 2 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$ is $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$. Hence $(a, b)=(4, 4)$. Now, by changing bases back if necessary, we may assume there is an element in the group of the form $\begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix}$ and we have the proof of (5.2)

Thus, we have:

Proposition 5.3. *Under the assumptions of* (5.2), $H^*=\{\pm\begin{bmatrix} 3v^2 & 4w^2 \\ w^{-2} & 4v^{-2} \end{bmatrix}$, $\begin{bmatrix} u & 0 \\ 0 & u^{-1} \end{bmatrix}$ *for all* $v, u, w \in GF(7)^*\}$.

We now also have a partial spread of degree 24 with components $y=xM$ where M is an element of H^*. It follows that there is a component of the plane of the form $y=x\begin{bmatrix} c & d \\ 1 & 0 \end{bmatrix}$. Take the difference of $\begin{bmatrix} c & d \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} u & 0 \\ 0 & u^{-1} \end{bmatrix}$ and compute the determinant to obtain $-cu^{-1}+1-d$. If c is nonzero then $u^{-1}=(1-d)/c$ (note that d cannot be zero as the matrices are nonsingular) produces a difference which is singular but nonzero. Hence, $c=0$ and d is not 1.

So, $\begin{bmatrix} \pm 3v^2 & \pm 4w^2-d \\ \pm w^{-2}-1 & \pm 4v^{-2} \end{bmatrix}$ has nonzero determinant for all $v, w \in GF(7)^*$.

Thus, we must have when $\pm=+$, $2+(4w^2-d)(w^{-2}-1)\neq 0$ and when $\pm=-$, $-2+(w^{-2}+1)(-4w^2-d)\neq 0$. Evaluating, letting $w^2=1, 2$ or 4 in both situations shows that d cannot be 2 or 4. Also, d cannot be 0 or 1 from the above remarks. For example, when $w^2=2$ in the second situation, we obtain $2+(8-d)(4-1)\neq 0$ which implies that d is not 4.

Hence, $d=3, 5$ or 6. There are two known translation planes of this type, namely the irregular nearfield plane and the exceptional Lüneburg plane (see e.g. Lüneburg [16] p. 99). In order to obtain the nearfield plane the element $\begin{bmatrix} 0 & d \\ 1 & 0 \end{bmatrix}$ squares to an element $\begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix}$ and it follows that $d=-1$ for this to occur.

The exceptional Lüneburg plane admits symmetric homology groups of order 24. If a new coordinate structure is taken by interchanging the components $x=0$ and $y=0$ then the only difference in the spread representation is that instead of multiplying the given matrix on the right by the indicated matrix group H^*, the inverse of the matrix is multiplied on the right by H^*. See e.g. Hiramine and Johnson [5] (5.2)) to realize that one can then multiply on the left by H^* to obtain the same spread. Note that the inverse of $\begin{bmatrix} 0 & d \\ 1 & 0 \end{bmatrix}$ is $\begin{bmatrix} 0 & 1 \\ d^{-1} & 0 \end{bmatrix}$ and when the latter is multiplied on the right by $\begin{bmatrix} d & 0 \\ & d^{-1} \end{bmatrix}$, this produces the element $\begin{bmatrix} 0 & d^{-1} \\ 1 & 0 \end{bmatrix}$. In other words, the exceptional Lüneburg is obtained in either case $d=3$ or 5. This completes the proof of (5.1).

Corollary 5.4. *Every half nearfield plane with spread in $PG(3, q)$ is a near nearfield plane.*

This completes the proof to theorem (1.7) stated in the introduction.

6. Homology groups and Reguli

In this section, we classify the translation planes Π with spreads in $PG(3, q)$ that admit a cyclic homology group H of order $q+1$ and such that there is a regulus R in the spread which contains both the axis and coaxis of the group.

In the case above, note that Rh for $h\in H$ and R are reguli which share two lines so that they are either equal or share exactly these two lines. If $Rh=R$ then the order of h must divide $q-1$ so that we must have at least $(q+1)/(2, q-1)$ reguli that share two lines. In case there are $q+1$ such reguli, then the statement of the theorem of Thas, Bader-Lunardon (1.1) translated into the terminology of translation planes shows that the corresponding translation plane must be either Desarguesian, regular nearfield, or irregular nearfield of order p^2 for $p=11, 23$, or 59. However, there is a cyclic homology group of order $q-1$ in such translation planes so that for odd order the unique involution σ in H must actually leave R invariant. More generally, we have:

Theorem 6.1. *Let Π be a translation plane with spread in $PG(3, q)$ that admits a homology group G of order $q+1$ and a regulus R that contains the axis and coaxis of G.*

(i) *Is q is even then Π is Desarguesian.*

(ii) *If q is odd then there is a set of $(q+1)/2$ reguli sharing the axis and coaxis of G.*

Proof. The above argument does not use the assumption that the group is cyclic merely that there can never be an orbit of length $q+1$ when q is odd.

Of course, a more complete study would be to consider the more general case as in (6.1) where there is arbitrary homology group G. We shall state a more general result somewhat later.

It was mentioned above that when there is a set of $q+1$ reguli (regulus nets) that share two components then there is a corresponding group of order $q-1$ such that the group is "regulus inducing" in the sense that any orbit of components union the axis and coaxis of the homology group form a regulus. This is also true of partial spreads that are comprised of k reguli that share two components; there is a "homology group" of order $q-1$ of the partial spread. Thus, we note:

Propoistion 6.2. *With the assumptions of (6.1) and q odd, there is a partial spread Γ of Π consisting of $(q+1)/2$ reguli that share two components. Γ admits an affine homology gronp of order $(q^2-1)/2$.*

Now we point out that when dealing with affine homology groups, the same results hold for groups on partial spreads as on spreads.

Proposition 6.3. *Any result on the structure of homology groups acting on translation planes also holds for homology gronps which are subgroups of $GL(V)$ acting on partial spreads (vector translation nets).*

In particular, a homoloby group induces a fixed point free group on the coaxis.

Proof. Note that a translation net of degree d and order p^k admits an affine homology group acting as a linear transformation group provided there is a linear group which fixes two components and fixes one component pointwise. If a group element fixes a point P on the coaxis then since there are d parallel classes, there are d lines thru P each of which intersects the axis so that each of the parallel classes is left invariant and it follows that the corresponding group element is the identity.

The above note implies that we may utilize the results of Hiramine and Johnson listed above (see I). Actually, some of these results were obtained by applications of results of Kallaher and Ostrom [13] on groups of linear transformations.

So, we actually know somewhat of the structure of the homology group H of order $(q^2-1)/2$ acting on the translation net Γ of degree $2+(q^2-1)/2$. Since H is a subgroup of $GL(4, q)$, we may normally embedd H isomorphically into $\Gamma L(1, q^2)$ provided that q is not 7, 23 or 47, $q+1$ is not a power of 2.

We may employ most of the arguments of the introducion when we constructed the proper half nearfield plane of dimension 2 to show that there is a field K (the kernel of the original plane Π) and a field extension F of K such that the partial spread Γ has either the form $x=0$, $y=0$, $y=xm$ for all squares m in F(q is odd here) or $x=0, y=0, y=xa, y=x^q b\, a$ for all a in F of order dividing $(q^2-1)/4$ and b a fixed element of F such that b^{q+1} has order dividing $(q^2-1)/4$.

Theorem 6.4. *Let Π be a translation plane with spread in $PG(3, q)$, q odd, and not equal to 7, 23 or 47 and $q+1$ not 2^a which admits an affine homology group of order $q+1$ and such that the spread contains a regulus containing the axis and coaxis of the group.*

(i) *There is a sub partial spread Γ that admits an affine homology group H of order $(q^2-1)/2$ which is either Desarguesian or $q\equiv-1$ mod 4 and Γ has the form $x=0, y=0, y=xa, y=x^q b\, a$ for all elements a in a field F isomorphic to $GF(q^2)$ of order dividing $(q^2-1)/4$ and b is a fixed element such that b^{q+1} has order dividing $(q^2-1)/2$ and where juxtaposition denotes multiplication in F.*

(ii) *If the homology group of order $q+1$ is cyclic then the partial spread of* (i) *is Desarguesian.*

(iii) *If the partial spread of* (i) *is Desarguesian then the plane may be constructed form a regular nearfield plane by multiple derivation.*

Proof. By the above remarks, either Γ is Desarguesian and there is a cyclic homology group of order $(q^2-1)/2$ or $q+1$ does not divide $(q^2-1)/4$ so that $q\equiv-1$ mod 4 and the remaining possiblility occurs. This proves (i).

If the homology group of order $q+1$ is cyclic then since there is an induced cyclic homology group of order $q-1$, it follows that there is a cyclic homology group of order $(q^2-1)/2$ and Γ is Desarguesian. This proves (ii).

Now if Γ is Desarguesian then there is a cyclic homology group of order $(q^2-1)/2$ so that the components of Γ have the form $x=0$, $y=0$, $y=xm$ where m is a square in the field F isomorphic to $GF(q^2)$. Since $q+1$ divides the order of the group acting on the net, it follows that Γ is a union of André nets in a Desarguesian affine plane together with the components $x=0$ and $y=0$ which are intended to represent the axis and coaxis of the original homology group.

The regular nearfield may be constructed by replacement (multiple derivation) of the "square" André nets or by the replacement (multiple derivation) of the "nonsquare" André nets (that is, m^{q+1} is a square defines the components of an André square net). Note that Γ consists of the André square nets. If we think of the regular nearfield plane N as constructed from a Desarguesian plane by replacement of the nonsquare nets then Γ may be thought of a subnet of N. Since there is a cyclic homology group acting in Π, each orbit defines a regulus net. Let L be a component of $\Pi-\Gamma$ so that L is either a Baer subplane or a component of the Desarguesian plane Σ^* used to construct N. If L is a Baer subplane then since there are exactly $(q-1)/2$ remaining André nets of Σ^* not in Γ, it follows that L is a Baer subplane of an André net of Σ^*. The orbit of the cyclic homology group of order $q+1$ fills out the set of Baer subplanes incident with the zero vector of the André net in question. Hence, it follows immediately that Π may be constructed from N by the replacement of André "nonsquare" nets. This proves (iii).

Corollary 6.5. *Let* Π *denote a translation plane of order* q^2 *where* $q\equiv 1$ *mod* 4 *with spread in* $PG(3, q)$. *If* Π *admits an affine homology group of order* $q+1$ *and the spread contains a regulus which contains the axis and coaxis of the homology group then* Π *may be constructed from the regular nearfield plane by multiple derivation.*

This also proves (1.10) stated in the introduction.

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