Øksendal, B. and Zhang, T.
Osaka J. Math.
44 (2007), 207–230

THE ITÔ-VENTZELL FORMULA AND FORWARD STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY POISSON RANDOM MEASURES

BERNT ØKSENDAL and TUSHENG ZHANG

(Received March 4, 2005, revised May 15, 2006)

Abstract

In this paper we obtain existence and uniqueness of solutions of forward stochastic differential equations driven by compensated Poisson random measures. To this end, an Itô-Ventzell formula for jump processes is proved and the flow properties of solutions of stochastic differential equations driven by compensated Poisson random measures are studied.

1. Introduction

In recent years, there has been growing interests on jump processes, especially Lévy processes, partly due to the applications in mathematical finance. In [7] a Malliavin calculus was developed for Lévy processes. Among other things, the authors in [7] introduced a forward integral with respect to compensated Poisson random measures and showed that the forward integrals coincide with the Itô integrals when the integrands are non-anticipating. The purpose of this paper is to solve the following forward stochastic differential equation

$$X_t = X_0 + \int_0^t b(\omega, s, X_s)\, ds + \int_0^t \int_R \sigma(X_{s-}, z)\, \widetilde{N}(d^- s, dz) \tag{1.1}$$

with possibly anticipating coefficients and anticipating initial values, where $\widetilde{N}(d^- s, dz)$ indicates a forward integral. To this end, we adopt a same strategy as in [21] where anticipating stochastic differential equations driven by Brownian motion were studied. We first prove an Itô-Ventzell formula for jump processes and then go on to study the properties of the solution of the stochastic differential equation:

$$\phi_t(x) = x + \int_0^t \int_R \sigma(\phi_{s-}, z)\, \widetilde{N}(ds, dz). \tag{1.2}$$

Surprisingly little is known in the literature about the flow properties of $\phi_t(x)$ (see, however, [6] for the case of multidimensional Lévy processes). We obtain bounds on

2000 Mathematics Subject Classification. Primary 60H40; Secondary 60G51, 60G57, 60H07.

$\phi_t(x)$, $\phi_t'(x)$ and $(\phi_t'(x))^{-1}$ under reasonable conditions on σ, where $\phi_t'(x)$ stands for the derivative of $\phi_t(x)$ with respect to the space variable x. Finally we show that the composition of ϕ_t with a solution of a random differential equation gives rise to a solution to our equation (1.1). We also mention that a pathwise approach to forward stochastic differential equations driven by Poisson processes is considered in [13].

The rest of the paper is organized as follows. Section 2 is the preliminaries. In Section 3, we prove the Itô-Ventzell formula. The flow properties of solutions of stochastic differential equations driven by compensated Poisson random measures are studied in Section 4, where the main result is also presented.

2. Preliminaries

In this section, we recall some of the framework and preliminary results from [7], which we will use later. Let $\Omega = \mathcal{S}'(\mathbb{R})$ be the Schwartz space of tempered distributions equipped with its Borel σ-algebra $\mathcal{F} = \mathfrak{B}(\Omega)$. The space $\mathcal{S}'(\mathbb{R})$ is the dual of the Schwartz space $\mathcal{S}(\mathbb{R})$ of rapidly decreasing smooth functions on $\mathbb{R}$. We denote the action of $\omega \in \Omega = \mathcal{S}'(\mathbb{R})$ on $f \in \mathcal{S}(\mathbb{R})$ by $\langle \omega, f \rangle = \omega(f)$.

Thanks to the Bochner-Milnos-Sazonov theorem, the white noise probability measure P can be defined by the relation

$$\int_\Omega e^{i\langle \omega, f\rangle}\, dP(\omega) = e^{\int_\mathbb{R} \psi(f(x))\,dx - i\alpha \int_\mathbb{R} f(x)\,dx}, \quad f \in \mathcal{S}(\mathbb{R}),$$

where the real constant α and

$$\psi(u) = \int_\mathbb{R} (e^{iuz} - 1 - iuz1_{\{|z|<1\}})\, \nu(dz)$$

are the elements of the exponent in the characteristic functional of a pure jump Lévy process with the Lévy measure $\nu(dz)$, $z \in \mathbb{R}$, which, we recall, satisfies

$$\int_\mathbb{R} 1 \wedge z^2\, \nu(dz) < \infty. \tag{2.1}$$

Assuming that

$$M := \int_\mathbb{R} z^2\, \nu(dz) < \infty, \tag{2.2}$$

we can set $\alpha = \int_\mathbb{R} z1_{\{|z|>1\}}\, \nu(dz)$ and then we obtain that

$$E[\langle \cdot\,, f\rangle] = 0 \quad \text{and} \quad E[\langle \cdot\,, f\rangle^2] = M \int_\mathbb{R} f^2(x)\,dx, \quad f \in \mathcal{S}(\mathbb{R}).$$

Accordingly the *pure jump Lévy process with no drift*

$$\eta = \eta(\omega, t), \quad \omega \in \Omega, \ t \in \mathbb{R}_+,$$

that we do consider here and in the following, is the cadlag modification of $\langle \omega, \chi_{(0,t]} \rangle$, $\omega \in \Omega$, $t > 0$, where

$$\chi_{(0,t]}(x) = \begin{cases} 1, & 0 < x \le t \\ 0, & \text{otherwise}, \quad x \in \mathbb{R}, \end{cases} \tag{2.3}$$

with $\eta(\omega, 0) := 0$, $\omega \in \Omega$. We remark that, for all $t \in \mathbb{R}_+$, the values $\eta(t)$ belong to $L_2(P) := L_2(\Omega, \mathcal{F}, P)$.

The Lévy process η can be expressed by

$$\eta(t) = \int_0^t \int_{\mathbb{R}} z \, \widetilde{N}(ds, dz), \quad t \in \mathbb{R}_+, \tag{2.4}$$

where $\widetilde{N}(dt, dz) := N(dt, dz) - \nu(dz)\, dt$ is the *compensated Poisson random measure* associated with η.

Let $\mathcal{F}_t$, $t \in \mathbb{R}_+$, be the completed filtration generated by the Lévy process in (2.4). We fix $\mathcal{F} = \mathcal{F}_\infty$.

Let $L_2(\lambda) = L_2(\mathbb{R}_+, \mathfrak{B}(\mathbb{R}_+), \lambda)$ denote the space of the square integrable functions on $\mathbb{R}_+$ equipped with the Borel σ-algebra and the standard Lebesgue measure $\lambda(dt)$, $t \in \mathbb{R}_+$. Denote by $L_2(\nu) := L_2(\mathbb{R}, \mathfrak{B}(\mathbb{R}), \nu)$ the space of the square integrable functions on $\mathbb{R}$ equipped with the Borel σ-algebra and the Lévy measure ν. Write $L_2(P) := L_2(\Omega, \mathcal{F}, P)$ for the space of the square integrable random variables.

For the symmetric function $f \in L_2((\lambda \times \nu)^m)$ $(m = 1, 2, \dots)$, define $I_0(f) := f$ for $f \in \mathbb{R}$.

$$I_m(f) := m! \int_0^\infty \int_{\mathbb{R}} \cdots \int_0^{t_2} \int_{\mathbb{R}} f(t_1, x_1, \dots, t_m, x_m)\, \widetilde{N}(dt_1, dx_1) \cdots \widetilde{N}(dt_m, dx_m)$$
$$(m = 1, 2, \dots)$$

and set $I_0(f) := f$ for $f \in \mathbb{R}$. We have

Theorem 2.1 (Chaos expansion). *Every* $F \in L_2(P)$ *admits the* (*unique*) *representation*

$$F = \sum_{m=0}^\infty I_m(f_m) \tag{2.5}$$

via the unique sequence of symmetric functions $f_m \in L_2((\lambda \times \nu)^m)$, $m = 0, 1, \dots$.

Let $X(t,z)$, $t \in \mathbb{R}_+$, $z \in \mathbb{R}$, be a random field taking values in $L_2(P)$. Then, for all $t \in \mathbb{R}_+$ and $z \in \mathbb{R}$, Theorem 2.1 provides the chaos expansion via symmetric functions

$$X(t, z) = \sum_{m=0}^\infty I_m(f_m(t_1, z_1, \dots, t_m, z_m; t, z)).$$

Let $\hat{f}_m = \hat{f}_m(t_1, z_1, \ldots, t_{m+1}, z_{m+1})$ be the symmetrization of $f_m(t_1, z_1, \ldots, t_m, z_m; t, z)$ as a function of the $m+1$ variables $(t_1, z_1), \ldots, (t_{m+1}, z_{m+1})$ with $t_{m+1} = t$ and $z_{m+1} = z$.

DEFINITION 2.1. [11], [12] The random field $X(t,z)$, $t \in \mathbb{R}_+$, $z \in \mathbb{R}$, is *Skorohod integrable* if $\sum_{m=0}^{\infty}(m+1)!\,\|\hat{f}_m\|^2_{L^2((\lambda\times\nu)^{m+1})} < \infty$. Then its *Skorohod integral with respect to* $\widetilde{N}$, i.e.

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} X(t,z)\,\widetilde{N}(\delta t, dz),$$

is defined by

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} X(t,z)\,\widetilde{N}(\delta t, dz) := \sum_{m=0}^{\infty} I_{m+1}(\hat{f}_m). \tag{2.6}$$

The Skorohod integral is an element of $L_2(P)$ and

$$\left\|\int_{\mathbb{R}_+}\int_{\mathbb{R}} X(t,z)\,\widetilde{N}(\delta t, dz)\right\|^2_{L^2(P)} = \sum_{m=0}^{\infty}(m+1)!\,\|\hat{f}_m\|^2_{L^2((\lambda\times\nu)^{m+1})}. \tag{2.7}$$

Moreover,

$$E\int_{\mathbb{R}_+}\int_{\mathbb{R}} X(t,z)\,\widetilde{N}(\delta t, dz) = 0. \tag{2.8}$$

The Skorohod integral can be regarded as an extension of the Itô integral to *anticipating* integrands. In fact, the following result can be proved. Cf. [11], [12], [5], [7], [18] and [21].

Proposition 2.2. *Let $X(t,z)$, $t \in \mathbb{R}_+$, $z \in \mathbb{R}$, be a non-anticipating* (*adapted*) *integrand. Then the Skorohod integral and the Itô integral coincide in $L_2(P)$, i.e.*

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} X(t,z)\,\widetilde{N}(\delta t, dz) = \int_{\mathbb{R}_+}\int_{\mathbb{R}} X(t,z)\,\widetilde{N}(dt, dz).$$

DEFINITION 2.2. The space $\mathbb{D}_{1,2}$ is the set of all the elements $F \in L^2(P)$ whose chaos expansion: $F = E[F] + \sum_{m=1}^{\infty} I_m(f_m)$, satisfies

$$\|F\|^2_{\mathbb{D}_{1,2}} := \sum_{m=1}^{\infty} m\cdot m!\,\|f_m\|^2_{L^2((\lambda\times\nu)^m)} < \infty.$$

The *Malliavin derivative* $D_{t,z}$ is an operator defined on $\mathbb{D}_{1,2}$ with values in the standard L_2-space $L_2(P \times \lambda \times \nu)$ given by

$$D_{t,z}F := \sum_{m=1}^{\infty} m I_{m-1}(f_m(\cdot, t, z)), \tag{2.9}$$

where $f_m(\cdot, t, z) = f_m(t_1, z_1, \dots, t_{m-1}, z_{m-1}; t, z)$.

Note that the operator $D_{t,z}$ is proved to be closed and to coincide with a certain difference operator defined in [22].

We recall the *forward integral* with respect to the Poisson random measure $\widetilde{N}$ defined in [7].

DEFINITION 2.3. The *forward integral*

$$J(\theta) := \int_0^T \int_{\mathbb{R}} \theta(t,z)\, \widetilde{N}(d^- t, dz)$$

with respect to the Poisson random measure $\widetilde{N}$, of a caglad stochastic function $\theta(t,z)$, $t \in \mathbb{R}_+, z \in \mathbb{R}$, with

$$\theta(t,z) := \theta(t,z,\omega), \quad \omega \in \Omega,$$

is defined as

$$\int_0^T \int_{\mathbb{R}} \theta(t,z)\, \widetilde{N}(d^- t, dz) := \lim_{m\to\infty} \int_0^T \int_{\mathbb{R}} \theta(t,z) I_{U_m}\, \widetilde{N}(d^- t, dz) \tag{2.10}$$

if the limit exists in $L^2(P)$. Here U_m, $m = 1, 2, \dots$, is an increasing sequence of compact sets $U_m \subset \mathbb{R} \setminus \{0\}$ with $\nu(U_m) < \infty$ such that $\lim_{m\to\infty} U_m = \mathbb{R} \setminus \{0\}$.

The relation between the forward integral and the Skorohod integral is the following.

Lemma 2.1 ([7]). *If* $\theta(t,z) + D_{t^+,z}\theta(t,z)$ *is Skorohod integrable and* $D_{t^+,z}\theta(t,z) := \lim_{s\to t^+} D_{s,z}\theta(t,z)$ *exists in* $L^2(P \times \lambda \times \nu)$, *then the forward integral exists in* $L_2(P)$ *and*

$$\int_0^T \int_{\mathbb{R}} \theta(t,z)\, \widetilde{N}(d^- t, dz) = \int_0^T \int_{\mathbb{R}} D_{t^+,z}\theta(t,z)\, \nu(dz)\, dt + \int_0^T \int_{\mathbb{R}} (\theta(t,z) + D_{t^+,z}\theta(t,z))\, \widetilde{N}(\delta t, dz).$$

3. The Itô-Ventzell formula

Consider the following two forward processes depending on a parameter $x \in \mathbb{R}$:

$$F_t(x) = F_0(x) + \int_0^t G_s(x)\,ds + \int_0^t \int_{\mathbb{R}} H_s(z,x)\,\widetilde{N}(d^-s,dz),$$
$$Y_t(x) = Y_0(x) + \int_0^t K_s(x)\,ds + \int_0^t \int_{\mathbb{R}} J_s(z,x)\,\widetilde{N}(d^-s,dz),$$

where the integrands are such that the above integrals belong to $L^2(\Omega \times \mathbb{R}, P \times dx)$. Let $\langle\ ,\ \rangle$ denote the inner product in the space $L^2(\mathbb{R}, dx)$.

Lemma 3.1. *It holds that*

(3.11)

$$\langle F_t, Y_t\rangle = \langle Y_0, F_0\rangle + \int_0^t \langle F_s, K_s\rangle\,ds + \int_0^t \langle Y_s, G_s\rangle\,ds + \int_0^t \int_{\mathbb{R}} \langle H_s(z,\cdot), J_s(z,\cdot)\rangle\,\nu(dz)\,ds$$
$$+ \int_0^t \int_{\mathbb{R}} [\langle F_{s-}, J_s(z,\cdot)\rangle + \langle H_s(z,\cdot), Y_{s-}\rangle + \langle H_s(z,\cdot), J_s(z,\cdot)\rangle]\,\widetilde{N}(d^-s,dz).$$

Proof. Let e_i, $i \geq 1$ be an orthornormal basis of $L^2(\mathbb{R}, dx)$. For each $i \geq 1$, we have

$$\langle F_t, e_i\rangle = \langle F_0, e_i\rangle + \int_0^t \langle G_s, e_i\rangle\,ds + \int_0^t \int_{\mathbb{R}} \langle H_s(z,\cdot), e_i\rangle\,\widetilde{N}(d^-s,dz),$$
$$\langle Y_t, e_i\rangle = \langle Y_0, e_i\rangle + \int_0^t \langle K_s, e_i\rangle\,ds + \int_0^t \int_{\mathbb{R}} \langle J_s(z,\cdot), e_i\rangle\,\widetilde{N}(d^-s,dz).$$

By the Itô's formula for forward processes in [7],

$$\begin{aligned}\langle F_t, e_i\rangle\langle Y_t, e_i\rangle &= \langle F_0, e_i\rangle\langle Y_0, e_i\rangle + \int_0^t \langle F_s, e_i\rangle\langle K_s, e_i\rangle\,ds + \int_0^t \langle Y_s, e_i\rangle\langle G_s, e_i\rangle\,ds \\ &\quad + \int_0^t \int_{\mathbb{R}} [\langle F_{s-}, e_i\rangle\langle J_s(z,\cdot), e_i\rangle + \langle H_s(z,\cdot), e_i\rangle\langle Y_{s-}, e_i\rangle \\ &\qquad + \langle H_s(z,\cdot), e_i\rangle\langle J_s(z,\cdot), e_i\rangle]\,\widetilde{N}(d^-s,dz) \\ &\quad + \int_0^t \int_{\mathbb{R}} \langle H_s(z,\cdot), e_i\rangle\langle J_s(z,\cdot), e_i\rangle\,\nu(dz)\,ds.\end{aligned} \tag{3.12}$$

Taking the summation over i, we get (3.11). □

We now state and prove an Itô-Ventzell formula for forward processes. Let X_t be a forward process given by

$$X_t = X_0 + \int_0^t \alpha_s\,ds + \int_0^t \int_{\mathbb{R}} \gamma(s,z)\,\widetilde{N}(d^-s,dz). \tag{3.13}$$

Theorem 3.1. *Assume that $F_t(x)$ is C^1 w.r.t. the space variable $x \in \mathbb{R}$. Then*

(3.14)
$$
\begin{aligned}
F_t(X_t) = F_0(X_0) &+ \int_0^t F_s'(X_s)\alpha_s\,ds \\
&+ \int_0^t\int_{\mathbb{R}} [F_s(X_s+\gamma(s,z)) - F_s(X_s) - F_s'(X_s)\gamma(s,z)]\,\nu(dz)\,ds \\
&+ \int_0^t G_s(X_s)\,ds + \int_0^t\int_{\mathbb{R}} [H_s(z,X_s+\gamma(s,z)) - H_s(z,X_s)]\,\nu(dz)\,ds \\
&+ \int_0^t\int_{\mathbb{R}} [F_{s-}(X_{s-}+\gamma(s,z)) - F_{s-}(X_{s-}) + H_s(z,X_{s-}+\gamma(s,z))]\,\widetilde{N}(d^-s,dz).
\end{aligned}
$$

Here, and in the following, $F_s'(x)$ denotes the derivative of $F_s(x)$ with repect to x.

Proof. We are using the same method as in [21]. Let $\phi \in C_0^\infty(\mathbb{R},\mathbb{R}_+)$ with $\int_{\mathbb{R}} \phi(x)\,dx = 1$. Define for $\varepsilon > 0$, $\phi_\varepsilon(x) = \varepsilon^{-1}\phi(x/\varepsilon)$. It follows from Theorem 4.6 in [7] that

(3.15)
$$
\begin{aligned}
\phi_\varepsilon(X_t - x) = \phi_\varepsilon(X_0 - x) &+ \int_0^t \phi_\varepsilon'(X_s - x)\alpha_s\,ds \\
&+ \int_0^t\int_{\mathbb{R}} [\phi_\varepsilon(X_s+\gamma(s,z)-x) - \phi_\varepsilon(X_s-x) - \phi_\varepsilon'(X_s-x)\gamma(s,z)]\,\nu(dz)\,ds \\
&+ \int_0^t\int_{\mathbb{R}} [\phi_\varepsilon(X_{s-}+\gamma(s,z)-x) - \phi_\varepsilon(X_{s-}-x)]\,\widetilde{N}(d^-s,dz).
\end{aligned}
$$

Using Lemma 3.1 we get that

$$
\begin{aligned}
&\int_{\mathbb{R}} F_t(x)\phi_\varepsilon(X_t - x)\,dx \\
&= \int_{\mathbb{R}} F_0(x)\phi_\varepsilon(X_0-x)\,dx + \int_0^t\int_{\mathbb{R}} F_s(x)\alpha_s\phi_\varepsilon'(X_s-x)\,dx \\
&\quad + \int_0^t ds \int_{\mathbb{R}} F_s(x)\,dx \\
&\qquad \times \int_{\mathbb{R}} [\phi_\varepsilon(X_s+\gamma(s,z)-x) - \phi_\varepsilon(X_s-x) - \phi_\varepsilon'(X_s-x)\gamma(s,z)]\,\nu(dz) \\
&\quad + \int_0^t ds \int_{\mathbb{R}} G_s(x)\phi_\varepsilon(X_s-x)\,dx \\
&\quad + \int_0^t ds \int_{\mathbb{R}} \nu(dz) \int_{\mathbb{R}} H_s(z,x)[\phi_\varepsilon(X_s+\gamma(s,z)-x) - \phi_\varepsilon(X_s-x)]\,dx \\
&\quad + \int_0^t\int_{\mathbb{R}} \left\{\int_{\mathbb{R}} F_{s-}(x)[\phi_\varepsilon(X_{s-}+\gamma(s,z)-x) - \phi_\varepsilon(X_{s-}-x)]\,dx \right. \\
&\qquad\qquad \left. + \int_{\mathbb{R}} H_s(z,x)\phi_\varepsilon(X_{s-}+\gamma(s,z)-x)\,dx\right\}\,\widetilde{N}(d^-s,dz).
\end{aligned}
\tag{3.16}
$$

Integrating by parts,

$$
\begin{aligned}
&\int_{\mathbb{R}} F_t(x)\phi_\varepsilon(X_t - x)\,dx \\
&= \int_{\mathbb{R}} F_0(x)\phi_\varepsilon(X_0 - x)\,dx + \int_0^t \int_{\mathbb{R}} F_s'(x)\alpha_s\phi_\varepsilon(X_s - x)dx \\
&\quad + \int_0^t ds \int_{\mathbb{R}} F_s(x)\,dx \int_{\mathbb{R}} [\phi_\varepsilon(X_s + \gamma(s,z) - x) - \phi_\varepsilon(X_s - x)]\,\nu(dz) \\
&\quad - \int_0^t ds \int_{\mathbb{R}} F_s'(x)\,dx \int_{\mathbb{R}} \phi_\varepsilon(X_s - x)\gamma(s,z)]\,\nu(dz) \\
&\quad + \int_0^t ds \int_{\mathbb{R}} G_s(x)\phi_\varepsilon(X_s - x)\,dx \\
&\quad + \int_0^t ds \int_{\mathbb{R}} \nu(dz) \int_{\mathbb{R}} H_s(z,x)[\phi_\varepsilon(X_s + \gamma(s,z) - x) - \phi_\varepsilon(X_s - x)]\,dx \\
&\quad + \int_0^t \int_{\mathbb{R}} \bigg\{ \int_{\mathbb{R}} F_{s-}(x)[\phi_\varepsilon(X_{s-} + \gamma(s,z) - x) - \phi_\varepsilon(X_{s-} - x)]\,dx \\
&\qquad\qquad + \int_{\mathbb{R}} H_s(z,x)\phi_\varepsilon(X_{s-} + \gamma(s,z) - x)\,dx \bigg\}\, \widetilde{N}(d^- s, dz).
\end{aligned} \tag{3.17}
$$

Since ϕ_ε approximates to identity as $\varepsilon \to 0$, letting $\varepsilon \to 0$ we obtain that

(3.18)

$$
\begin{aligned}
F_t(X_t) &= F_0(X_0) + \int_0^t F_s'(X_s)\alpha_s\,ds \\
&\quad + \int_0^t \int_{\mathbb{R}} [F_s(X_s + \gamma(s,z)) - F_s(X_s) - F_s'(X_s)\gamma(s,z)]\,\nu(dz)\,ds \\
&\quad + \int_0^t G_s(X_s)\,ds + \int_0^t \int_{\mathbb{R}} [H_s(z, X_s + \gamma(s,z)) - H_s(z, X_s)]\,\nu(dz)\,ds \\
&\quad + \int_0^t \int_{\mathbb{R}} [F_{s-}(X_{s-} + \gamma(s,z)) - F_{s-}(X_{s-}) + H_s(z, X_{s-} + \gamma(s,z))]\, \widetilde{N}(d^- s, dz). \quad \square
\end{aligned}
$$

Next we are going to deduce an Itô-Ventzell formula for Skorohod integrals using the relation between the forward integral and the Skorohod integral. Consider

$$
\begin{aligned}
X_t &= X_0 + \int_0^t \alpha_s\,ds + \int_0^t \int_{\mathbb{R}} \gamma(s,z)\, \widetilde{N}(\delta s, dz), \\
F_t(x) &= F_0(x) + \int_0^t G_s(x)\,ds + \int_0^t \int_{\mathbb{R}} H_s(z,x)\, \widetilde{N}(\delta s, dz).
\end{aligned} \tag{3.19}
$$

The stochastic integrals here are understood as Skorohod integrals. Let $\hat{H}_s(z,x) = S_{s,z}H_s(z,x)$, $\hat{\gamma}(s,z) = S_{s,z}\gamma(s,z)$, where $S_{s,z}$ is an operator satisfying

$$
S_{s,z}G + D_{t^+,z}(S_{s,z}G) = G
$$

for any smooth random variable G. See [7] for details.

Theorem 3.2. *Assume that $F_t(x)$ is C^1 w.r.t. the space variable $x \in R$. Then*

(3.20)

$$\begin{aligned}
F_t(X_t) = F(X_0) &+ \int_0^t F_s'(X_s)\left[\alpha_s - \int_{\mathbb{R}} D_{s^+,z}\hat{\gamma}(s,z)\,\nu(dz)\right]ds + \int_0^t G_s(X_s)\,ds \\
&+ \int_0^t ds \int_{\mathbb{R}} [F_s(X_s+\hat{\gamma}(s,z)) - F_s(X_s) - F_s'(X_s)\hat{\gamma}(s,z)]\,\nu(dz) \\
&+ \int_0^t ds \int_{\mathbb{R}} [\hat{H}_s(z, X_s+\hat{\gamma}(s,z)) - \hat{H}_s(z,X_s)]\,\nu(dz) \\
&+ \int_0^t ds \int_{\mathbb{R}} D_{s^+,z}[F_{s-}(X_{s-}+\hat{\gamma}(s,z)) - F_{s-}(X_{s-}) + \hat{H}_s(z, X_s+\hat{\gamma}(s,z))]\,\nu(dz)\,ds \\
&+ \int_0^t ds \int_{\mathbb{R}} \{[F_{s-}(X_{s-}+\hat{\gamma}(s,z)) - F_{s-}(X_{s-}) + \hat{H}_s(z, X_s+\hat{\gamma}(s,z))] \\
&\qquad + D_{s^+,z}[F_{s-}(X_{s-}+\hat{\gamma}(s,z)) - F_{s-}(X_{s-}) + \hat{H}_s(z, X_s+\hat{\gamma}(s,z))]\}\,\widetilde{N}(\delta s, dz).
\end{aligned}$$

Proof. Using the relation between forward integrals and Skorohod integrals, we rewrite X_t and $F_t(x)$ as

$$X_t = X_0 + \int_0^t \left[\alpha_s - \int_{\mathbb{R}} D_{s^+,z}\hat{\gamma}(s,z)\,\nu(dz)\right]ds + \int_0^t\int_{\mathbb{R}} \hat{\gamma}(s,z)\,\widetilde{N}(d^-s, dz),$$

$$F_t(x) = F_0(x) + \int_0^t \left[G_s(x) - \int_{\mathbb{R}} D_{s^+,z}\hat{H}_s(z,x)\,\nu(dz)\right]ds + \int_0^t\int_{\mathbb{R}} \hat{H}_s(z,x)\,\widetilde{N}(d^-s, dz).$$

It follows from Theorem 3.1 that

$$\begin{aligned}
F_t(X_t) = F_0(X_0) &+ \int_0^t F_s'(X_s)\left[\alpha_s - \int_{\mathbb{R}} D_{s^+,z}\hat{\gamma}(s,z)\,\nu(dz)\right]ds \\
&+ \int_0^t ds \int_{\mathbb{R}} [F_s(X_s+\hat{\gamma}(s,z)) - F_s(X_s) - F_s'(X_s)\hat{\gamma}(s,z)]\,\nu(dz) + \int_0^t G_s(X_s)\,ds \\
&+ \int_0^t ds \int_{\mathbb{R}} [\hat{H}_s(z, X_s+\hat{\gamma}(s,z)) - \hat{H}_s(z,X_s)]\,\nu(dz) \\
&+ \int_0^t ds \int_{\mathbb{R}} [F_{s-}(X_{s-}+\hat{\gamma}(s,z)) - F_{s-}(X_{s-}) + \hat{H}_s(z, X_s+\hat{\gamma}(s,z))]\,\widetilde{N}(d^-s, dz) \\
= F(X_0) &+ \int_0^t F_s'(X_s)\left[\alpha_s - \int_{\mathbb{R}} D_{s^+,z}\hat{\gamma}(s,z)\,\nu(dz)\right]ds + \int_0^t G_s(X_s)\,ds \\
&+ \int_0^t ds \int_{\mathbb{R}} [F_s(X_s+\hat{\gamma}(s,z)) - F_s(X_s) - F_s'(X_s)\hat{\gamma}(s,z)]\,\nu(dz) \\
&+ \int_0^t ds \int_{\mathbb{R}} [\hat{H}_s(z, X_s+\hat{\gamma}(s,z)) - \hat{H}_s(z,X_s)]\,\nu(dz)
\end{aligned}$$

$$+\int_0^t ds \int_{\mathbb{R}} D_{s^+,z}[F_{s-}(X_{s-}+\hat{\gamma}(s,z)) - F_{s-}(X_{s-}) + \hat{H}_s(z, X_s+\hat{\gamma}(s,z))]\,\nu(dz)\,ds$$
$$+\int_0^t ds \int_{\mathbb{R}} \{[F_{s-}(X_{s-}+\hat{\gamma}(s,z)) - F_{s-}(X_{s-}) + \hat{H}_s(z, X_s+\hat{\gamma}(s,z))]$$
$$+ D_{s^+,z}[F_{s-}(X_{s-}+\hat{\gamma}(s,z)) - F_{s-}(X_{s-}) + \hat{H}_s(z, X_s+\hat{\gamma}(s,z))]\}\,\widetilde{N}(\delta s, dz).$$
□

EXAMPLE 3.1 (Stock price influenced by a large investor with inside information). Suppose the price $S_t = S_t(x)$ at time t of a stock is modelled by a geometric Lévy process of the form

$$dS_t(x) = S_{t-}(x)\left[\mu(t,x)\,dt + \int_{\mathbb{R}} \theta(t,z)\,\tilde{N}(dt,dz)\right], \quad S_0 > 0 \quad \text{(constant)}. \tag{3.21}$$

(See e.g. [2] for more information about the use of this type of process in financial modelling.) Here $x \in \mathbb{R}$ is a parameter and for each x and z the processes $\mu(t) = \mu(t,x,\omega)$ and $\theta(t,z) = \theta(t,z,\omega)$ are $\mathcal{F}_t$-adapted, where $\mathcal{F}_t$ is the filtration generated by the driving Lévy process

$$\eta(t) = \int_0^t \int_{\mathbb{R}} z\,\widetilde{N}(ds,dz).$$

Suppose the value of this "hidden parameter" x is influenced by a large investor with inside information, so that x can be represented by a stochastic process X_t of the form

$$x = X_t = X_0 + \int_0^t \alpha(s)\,ds + \int_0^t \int_{\mathbb{R}} \gamma(s,z)\,\widetilde{N}(d^-s,dz); \quad X_0 \in \mathbb{R} \tag{3.22}$$

where $\alpha(t)$ and $\gamma(t,z)$ are processes adapted to a larger insider filtration $\mathcal{G}_t$, satisfying $\mathcal{F}_t \subset \mathcal{G}_t$ for all $t \geq 0$. (For a justification of the use of forward integrals in the modelling of insider trading, see e.g. [7]).

Combing (3.21) and (3.22) and using Theorem 3.1 we see that the dynamics of the corresponding stock price $S_t(X_t)$ is, with $S_t'(x) = (\partial/\partial x)S_t(x)$,

(3.23)

$$\begin{aligned} d(S_t(X_t)) = {} & S_t'(X_t)\alpha(t)\,dt \\ & + \int_{\mathbb{R}} \{S_t(X_t+\gamma(t,z)) - S_t(X_t) - \gamma(t,z)S_t'(X_t)\}\,\nu(dz)\,dt \\ & + S_t(X_t)\mu(t,X_t)\,dt \\ & + \int_{\mathbb{R}} \{S_t(X_t+\gamma(t,z)) - S_t(X_t)\}\theta(t,z)\,\nu(dz)\,dt \\ & + \int_{\mathbb{R}} \{S_{t-}(X_{t-}+\gamma(t,z)) - S_{t-}(X_{t-}) + S_{t-}(X_{t-}+\gamma(t,z))\theta(t,z)\}\,\widetilde{N}(d^-t,dz). \end{aligned}$$

By the Itô formula

$$S_t(x) = S_0 \exp\left\{\int_0^t \mu(s,x)\,ds + \int_0^t\int_{\mathbb{R}} (\ln(1+\theta(s,z)) - \theta(s,z))\,\nu(dz)\,ds + \int_0^t\int_{\mathbb{R}} \ln(1+\theta(s,z))\,\widetilde{N}(ds,dz)\right\}, \tag{3.24}$$

and hence

$$S_t'(x) = S_t(x)\int_0^t \mu'(s,x)\,ds,$$

where

$$\mu'(s,x) = \frac{\partial}{\partial x}\mu(s,x).$$

Substituted into (3.23) this gives

$$\begin{aligned} dS_t(X_t) &= S_t(X_t)\left[\alpha(t) + \mu(t,X_t) + \int_0^t \mu'(s,X_t)\,ds\right]dt \\ &\quad + \int_{\mathbb{R}} \Big\{S_t(X_t+\gamma(t,z))(1+\theta(t,z)) \\ &\qquad - S_t(X_t)\left(1+\theta(t,z)+\gamma(t,z)\int_0^t \mu'(s,X_t)\,ds\right)\Big\}\,\nu(dz)\,dt \\ &\quad + \int_{\mathbb{R}} \{S_{t-}(X_{t-}+\gamma(t,z))(1+\theta(t,z)) - S_{t-}(X_{t-})\}\,\widetilde{N}(d^-t,dz). \end{aligned} \tag{3.25}$$

4. Forward SDEs driven by Poisson random measures

Let $b(\omega,s,x)\colon \Omega\times\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$, $\sigma(x,z)\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be measurable mappings (possibly anticipating). Let X_0 be a random variable. In this section, we are going to solve the following forward SDE:

$$X_t = X_0 + \int_0^t b(\omega,s,X_s)\,ds + \int_0^t\int_{\mathbb{R}} \sigma(X_{s-},z)\,\widetilde{N}(d^-s,dz). \tag{4.26}$$

Let $\phi_t(x)$, $t\ge 0$ be the stochastic flow determined by the following non-anticipating SDE:

$$\phi_t(x) = x + \int_0^t\int_{\mathbb{R}} \sigma(\phi_{s-}(x),z)\,\widetilde{N}(ds,dz). \tag{4.27}$$

Define

$$\hat{b}(\omega,s,x) = (\phi_s')^{-1}(x)b(\omega,s,\phi_s(x)).$$

Consider the differential equation:

$$\frac{dY_t}{dt} = \hat{b}(\omega, t, Y_t), \quad Y_0 = X_0. \tag{4.28}$$

Theorem 4.1. *If Y_t, $t \geq 0$ is the unique solution to equation* (4.28), *then $X_t = \phi_t(Y_t)$, $t \geq 0$ is the unique solution to equation* (4.26).

Proof. It follows from Theorem 3.1 that

$$\begin{aligned} X_t &= \phi_t(Y_t) = X_0 + \int_0^t \phi_s'(Y_s)\hat{b}(\omega, s, Y_s)\, ds + \int_0^t \int_{\mathbb{R}} \sigma(\phi_{s-}(Y_{s-}), z)\, \widetilde{N}(d^- s, dz) \\ &= X_0 + \int_0^t b(\omega, s, X_s)\, ds + \int_0^t \int_{\mathbb{R}} \sigma(X_{s-}, z)\, \widetilde{N}(d^- s, dz). \end{aligned}$$

□

Next we are going to provide appropriate conditions under which (4.28) has a unique solution. To this end, we need to study the flow generated by the solution of the following equation:

$$X_t(x) = x + \int_0^t \int_{\mathbb{R}} \sigma(X_{s-}(x), z)\, \widetilde{N}(ds, dz). \tag{4.29}$$

Let (p, D_p) denote the point process generating the Poisson random measure $N(dt, dz)$, where D_p, called the domain of the point process p, is a countable subset of $[0, \infty)$ depending on the random parameter ω.

Proposition 4.1. *Let $k \geq 1$. Assume that for $l = 1, 2, \ldots, 2k$,*

$$\int_{\mathbb{R}} |\sigma(y, z)|^l\, \nu(dz) \leq C(1 + |y|^l). \tag{4.30}$$

Let $X_t(x)$, $t \geq 0$ be the unique solution to equation (4.29). *Then, we have*

$$E\left[\sup_{0 \leq t \leq T} |X_t(x)|^{2k}\right] \leq C_{T,k}(1 + |x|^{2k}). \tag{4.31}$$

Proof. It follows from Itô's formula that

(4.32)

$$\begin{aligned} &(X_t(x))^{2k} \\ &= x^{2k} + \int_0^t \int_{\mathbb{R}} [(X_{s-}(x) + \sigma(X_{s-}(x), z))^{2k} - (X_{s-}(x))^{2k}]\widetilde{N}(ds, dz) \\ &\quad + \int_0^t \int_{\mathbb{R}} [(X_s(x) + \sigma(X_s(x), z))^{2k} - (X_s(x))^{2k} - 2k(X_s(x))^{2k-1}\sigma(X_s(x), z)]\, \nu(dz)\, ds. \end{aligned}$$

Denote by M_t the martingale part in the above equation. We have

$$
\begin{aligned}
[M]_t^{1/2} &= \left(\sum_{0\le s\le t} (\Delta M_s)^2 \right)^{1/2} \\
&= \left(\sum_{0\le s\le t, s\in D_p} [(X_{s-}(x) + \sigma(X_{s-}(x), p(s)))^{2k} - (X_{s-}(x))^{2k}]^2 \right)^{1/2} \qquad (4.33)\\
&\le \sum_{0\le s\le t, s\in D_p} |(X_{s-}(x) + \sigma(X_{s-}(x), p(s)))^{2k} - (X_{s-}(x))^{2k}|.
\end{aligned}
$$

By Burkholder's inequality,

$$
\begin{aligned}
E\left[\sup_{0\le s\le t} |M_s| \right] &\le C E\left([M]_t^{1/2}\right) \\
&\le E\left[\sum_{0\le s\le t, s\in D_p} |(X_{s-}(x) + \sigma(X_{s-}(x), p(s)))^{2k} - (X_{s-}(x))^{2k}| \right] \\
&= E\left[\int_0^t \int_{\mathbb{R}} |(X_{s-}(x) + \sigma(X_{s-}(x), z))^{2k} - (X_{s-}(x))^{2k}|\, N(ds, dz) \right] \\
&= E\left[\int_0^t \int_{\mathbb{R}} |(X_s(x) + \sigma(X_s(x), z))^{2k} - (X_s(x))^{2k}|\, ds\, \nu(dz) \right].
\end{aligned}
$$

By the Mean-Value Theorem, there exists $\theta(s, z, \omega) \in [0, 1]$ such that

$$
\begin{aligned}
&(X_s(x) + \sigma(X_s(x), z))^{2k} - (X_s(x))^{2k} \\
&= 2k(X_s(x) + \theta(s, z, \omega)\sigma(X_s(x), z))^{2k-1}\sigma(X_s(x), z).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
E\left[\sup_{0\le s\le t} |M_s| \right] &\le C_k E\left[\int_0^t ds |X_s(x)|^{2k-1} \int_{\mathbb{R}} |\sigma(X_s(x), z)|\, \nu(dz) \right] \\
&\quad + C_k E\left[\int_0^t ds \int_{\mathbb{R}} |\sigma(X_s(x), z)|^{2k}\, \nu(dz) \right] \qquad (4.34)\\
&\le C_k + C_k \int_0^t E[|X_s(x)|^{2k}]\, ds.
\end{aligned}
$$

By Taylor expansion, there exists $\eta(s, z, \omega) \in [0, 1]$ such that

(4.35)

$$
\begin{aligned}
&E\left[\int_0^t \int_{\mathbb{R}} |(X_s(x) + \sigma(X_s(x), z))^{2k} - (X_s(x))^{2k} - 2k(X_s(x))^{2k-1}\sigma(X_s(x), z)|\, \nu(dz)\, ds\right] \\
&= 2k(2k-1)E\left[\int_0^t \int_{\mathbb{R}} |(X_s(x) + \eta(s, z, \omega)\sigma(X_s(x), z))^{2k-2}|\, |\sigma(X_s(x), z)|^2\, ds\, \nu(dz)\right] \\
&\le C_k E\left[\int_0^t ds |X_s(x)|^{2k-2} \int_{\mathbb{R}} |\sigma(X_s(x), z)|^2\, \nu(dz)\right] \\
&\quad + C_k E\left[\int_0^t ds \int_{\mathbb{R}} |\sigma(X_s(x), z)|^{2k}\, \nu(dz)\right] \\
&\le C_k + C_k \int_0^t E[|X_s(x)|^{2k}]\, ds.
\end{aligned}
$$

(4.32), (4.34) and (4.35) imply that

$$E\left[\sup_{0 \le s \le t} |X_s(x)|^{2k}\right] \le C_k + C_k \int_0^t E[|X_s(x)|^{2k}]\, ds.$$

Applying Gronwall's lemma we get

$$E\left[\sup_{0 \le t \le T} |X_t(x)|^{2k}\right] \le C_{T,p}(1 + |x|^{2k}). \qquad \square$$

Proposition 4.2. *Assume that $\partial\sigma(y, z)/\partial y$ exists and*

$$\sup_y \int_{\mathbb{R}} \left|\frac{\partial\sigma(y, z)}{\partial y}\right|^l \nu(dz) < \infty, \tag{4.36}$$

for $l = 1, 2, \ldots, 2k$. Let $X_t'(x)$ denote the derivative of $X_t(x)$ w.r.t. x. Then there exists a constant $C_{T,k}$ such that

$$E\left[\sup_{0 \le t \le T} |X_t'(x)|^{2k}\right] \le C_{T,k}. \tag{4.37}$$

Proof. Differentiating both sides of the equation (4.29) we get

$$X_t'(x) = 1 + \int_0^t \int_{\mathbb{R}} \frac{\partial\sigma(X_{s-}(x), z)}{\partial y} X_{s-}'(x)\, \widetilde{N}(ds, dz). \tag{4.38}$$

Put

$$h(s, z) = \frac{\partial\sigma(X_{s-}(x), z)}{\partial y} X_{s-}'(x).$$

By Itô's formula,

(4.39)
$$\begin{aligned}(X'_t(x))^{2k} &= 1 + \int_0^t \int_{\mathbb{R}} [(X'_{s-}(x) + h(s,z))^{2k} - (X'_{s-}(x))^{2k}]\, \widetilde{N}(ds,dz) \\ &\quad + \int_0^t \int_{\mathbb{R}} [(X'_s(x) + h(s,z))^{2k} - (X'_s(x))^{2k} - 2k(X'_s(x))^{2k-1} h(s,z)]\, \nu(dz)\, ds.\end{aligned}$$

Denote the martingale part of the above equation by M. Reasoning as in the proof of Proposition 4.1 we have that

$$\begin{aligned} E\left[\sup_{0\le s\le t} |M_s|\right] &\le CE\left([M]_t^{1/2}\right) \\ &\le CE\left[\int_0^t \int_{\mathbb{R}} |(X'_{s-}(x) + h(s,z))^{2k} - (X'_{s-}(x))^{2k}|\, N(ds,dz)\right] \\ &= E\left[\int_0^t \int_{\mathbb{R}} |(X'_{s-}(x) + h(s,z))^{2k} - (X'_{s-}(x))^{2k}|\, ds\, \nu(dz)\right] \\ &\le C_k E\left[\int_0^t ds |X'_s(x)|^{2k-1} \int_{\mathbb{R}} |h(s,z)|\, \nu(dz)\right] \\ &\quad + C_k E\left[\int_0^t ds \int_{\mathbb{R}} |h(s,z)|^{2k}\, \nu(dz)\right] \\ &\le C_k E\left[\int_0^t ds |X'_s(x)|^{2k} \int_{\mathbb{R}} \left|\frac{\partial \sigma(X_{s-}(x),z)}{\partial y}\right| \nu(dz)\right] \\ &\quad + C_k E\left[\int_0^t ds |X'_s(x)|^{2k} \int_{\mathbb{R}} \left|\frac{\partial \sigma(X_{s-}(x),z)}{\partial y}\right|^{2k} \nu(dz)\right] \\ &\le \hat{C}_k + \hat{C}_k \int_0^t E[|X'_s(x)|^{2k}]\, ds, \end{aligned} \tag{4.40}$$

where

$$\hat{C}_k = C_k \left(\sup_y \int_{\mathbb{R}} \left|\frac{\partial \sigma(y,z)}{\partial y}\right| \nu(dz) + \sup_y \int_{\mathbb{R}} \left|\frac{\partial \sigma(y,z)}{\partial y}\right|^{2k} \nu(dz)\right).$$

A similar treatment applied to the second term in (4.39) yields

$$\begin{aligned} &E\left[\left|\int_0^t \int_{\mathbb{R}} [(X'_s(x) + h(s,z))^{2k} - (X'_s(x))^{2k} - 2k(X'_s(x))^{2k-1} h(s,z)]\, \nu(dz)\, ds\right|\right] \\ &\le C_k + C_k \int_0^t E[|X'_s(x)|^{2k}]\, ds. \end{aligned} \tag{4.41}$$

Combining (4.39), (4.40) and (4.41) we get

$$E\left[\sup_{0\le s\le t}|X_s'(x)|^{2k}\right]\le C_k\left(1+\int_0^t E[|X_s'(x)|^{2k}]\,ds\right).$$

An application of the Gronwall's inequality completes the proof. □

Our next step is to give estimates for $(X_t'(x))^{-1}$. Define

$$Z_t=\int_0^t\int_{\mathbb{R}}\frac{\partial\sigma(X_{s-}(x),z)}{\partial y}\,\tilde{N}(ds,dz).$$

Then we see that

$$X_t'(x)=1+\int_0^t X_{s-}'(x)\,dZ_s.$$

Define

$$W_t=:-Z_t+\int_0^t\int_{\mathbb{R}}\frac{(\partial\sigma(X_{s-}(x),z)/\partial y)^2}{1+\partial\sigma(X_{s-}(x),z)/\partial y}\,N(ds,dz).$$

Let $Y_t(x)$, $t\ge 0$ be the solution to the equation:

$$Y_t(x)=1+\int_0^t Y_{s-}(x)\,dW_s. \tag{4.42}$$

An application of Itô's formula shows that $Y_t(x)=(X_t'(x))^{-1}$.

Proposition 4.3. *Assume*

$$\sup_y\int_{\mathbb{R}}\left|\frac{(\partial\sigma(y,z)/\partial y)^2}{1+\partial\sigma(y,z)/\partial y}\right|^l\nu(dz)<\infty, \tag{4.43}$$

for $l=1,\dots,2k$. *Then there exists a constant* $C_{T,k}$ *such that*

$$E\left[\sup_{0\le t\le T}|Y_t(x)|^{2k}\right]\le C_{T,k}. \tag{4.44}$$

Proof. Note that

$$\begin{aligned}Y_t(x)=1&-\int_0^t Y_{s-}(x)\int_{\mathbb{R}}\frac{\partial\sigma(X_{s-}(x),z)}{\partial y}\,\tilde{N}(ds,dz)\\&+\int_0^t Y_{s-}(x)\int_{\mathbb{R}}\frac{(\partial\sigma(X_{s-}(x),z)/\partial y)^2}{1+\partial\sigma(X_{s-}(x),z)/\partial y}\,N(ds,dz).\end{aligned} \tag{4.45}$$

Set

$$f(s,z) = Y_{s-}(x)\frac{(\partial\sigma(X_{s-}(x),z)/\partial y)^2}{1+\partial\sigma(X_{s-}(x),z)/\partial y},$$
$$h(s,z) = -Y_{s-}(x)\frac{\partial\sigma(X_{s-}(x),z)}{\partial y}.$$

By Itô's formula,

(4.46)
$$\begin{aligned}(Y_t(x))^{2k} = 1 &+ \int_0^t\int_{\mathbb{R}}[(Y_{s-}(x)+h(s,z))^{2k} - (Y_{s-}(x))^{2k}]\,\widetilde{N}(ds,dz)\\ &+ \int_0^t\int_{\mathbb{R}}[(Y_{s-}(x)+f(s,z))^{2k} - (Y_{s-}(x))^{2k}]\,N(ds,dz)\\ &+ \int_0^t\int_{\mathbb{R}}[(Y_s(x)+h(s,z))^{2k} - (Y_s(x))^{2k} - 2k(Y_s(x))^{2k-1}h(s,z)]\,\nu(dz)\,ds.\end{aligned}$$

Denote the three terms on the right hand side of (4.46) by I_t, II_t, III_t respectively. Similar arguments as in the proof of Proposition 4.2 show that there exists a constant C_1 such that

$$E\left[\sup_{0\le s\le t}|\mathrm{I}_s|\right] \le C_1\left(1+\int_0^t E[|Y_s(x)|^{2k}]\,ds\right). \tag{4.47}$$

$$E\left[\sup_{0\le s\le t}|\mathrm{III}_s|\right] \le C_1\left(1+\int_0^t E[|Y_s(x)|^{2k}]\,ds\right). \tag{4.48}$$

By the Mean Value Theorem, we have

$$\begin{aligned}E\left[\sup_{0\le s\le t}|\mathrm{II}_s|\right] &\le E\left[\int_0^t\int_{\mathbb{R}}|(Y_{s-}(x)+f(s,z))^{2k} - (Y_{s-}(x))^{2k}|\,N(ds,dz)\right]\\ &= E\left[\int_0^t\int_{\mathbb{R}}|(Y_{s-}(x)+f(s,z))^{2k} - (Y_{s-}(x))^{2k}|\,ds\nu\,(dz)\right]\\ &\le CE\left[\int_0^t ds|Y_{s-}(x)|^{2k}\int_{\mathbb{R}}\left|\frac{(\partial\sigma(X_{s-}(x),z)/\partial y)^2}{1+\partial\sigma(X_{s-}(x),z)/\partial y}\right|\nu(dz)\right]\\ &\quad + CE\left[\int_0^t ds|Y_{s-}(x)|^{2k}\int_{\mathbb{R}}\left|\frac{(\partial\sigma(X_{s-}(x),z)/\partial y)^2}{1+\partial\sigma(X_{s-}(x),z)/\partial y}\right|^{2k}\nu(dz)\right]\\ &\le CE\left[\int_0^t ds|Y_s(x)|^{2k}\right],\end{aligned} \tag{4.49}$$

where we have used the fact that

$$\sup_y\int_{\mathbb{R}}\left|\frac{(\partial\sigma(y,z)/\partial y)^2}{1+\partial\sigma(y,z)/\partial y}\right|^l\nu(dz) < \infty,$$

for $l = 1, \ldots, 2k$. It follows from (4.46), (4.47), (4.48) and (4.49) that

$$E\left[\sup_{0\le s\le t} |Y_s(x)|^{2k}\right] \le C_k \left(1 + \int_0^t E[|Y_s(x)|^{2k}]\, ds\right).$$

The desired result follows from the Gronwall's lemma. □

Finally, we need some estimates for the derivative of $Y_t(x)$. Define

$$K(s,z) =: -Y'_{s-}(x)\frac{\partial\sigma(X_{s-}(x),z)}{\partial y} - Y_{s-}(x)X'_{s-}(x)\frac{\partial^2\sigma(X_{s-}(x),z)}{\partial y^2},$$

$$J(y,z) =: \frac{(\partial\sigma(y,z)/\partial y)^2}{1+\partial\sigma(y,z)/\partial y},$$

$$L(y,z) =: \frac{2(\partial\sigma(y,z)/\partial y)(1+\partial\sigma(y,z)/\partial y)(\partial^2\sigma(y,z)/\partial y^2)}{(1+\partial\sigma(y,z)/\partial y)^2} - \frac{(\partial^2\sigma(y,z)/\partial y^2)(\partial\sigma(y,z)/\partial y)^2}{(1+\partial\sigma(y,z)/\partial y)^2},$$

$$m(s,z) =: Y'_{s-}(x)J(X_{s-}(x),z) + Y_{s-}(x)X'_{s-}(x)L(X_{s-}(x),z).$$

Proposition 4.4. *Assume*

$$\sup_y \int_{\mathbb{R}} \left|\frac{\partial^2\sigma(y,z)}{\partial y^2}\right|^l \nu(dz) < \infty, \tag{4.50}$$

and

$$\sup_y \int_{\mathbb{R}} |L(y,z)|^l \nu(dz) < \infty, \quad \sup_y \int_{\mathbb{R}} |J(y,z)|^l \nu(dz) < \infty, \tag{4.51}$$

for $l = 1, \ldots, 2k$. Then there exists a constant C_k such that $E[\sup_{0\le s\le t} |Y'_s(x)|^{2k}] \le C_k$.

Proof. The proof is in the same nature as the proofs of previous propositions. We only sketch it. Differentiating (4.45) we see that

$$Y'_t(x) = \int_0^t\int_R K(s,z)\,\widetilde{N}(ds,dz) + \int_0^t\int_R m(s,z)\,N(ds,dz). \tag{4.52}$$

By Itô's formula,

(4.53)

$$\begin{aligned}(Y'_t(x))^{2k} &= \int_0^t\int_{\mathbb{R}} [(Y'_{s-}(x)+K(s,z))^{2k} - (Y'_{s-}(x))^{2k}]\,\widetilde{N}(ds,dz)\\ &\quad + \int_0^t\int_{\mathbb{R}} [(Y'_{s-}(x)+m(s,z))^{2k} - (Y'_{s-}(x))^{2k}]\,N(ds,dz)\\ &\quad + \int_0^t\int_{\mathbb{R}} [(Y'_s(x)+K(s,z))^{2k} - (Y'_s(x))^{2k} - 2k(Y'_s(x))^{2k-1}K(s,z)]\,\nu(dz)\,ds.\end{aligned}$$

Let us denote the three terms on the right side by I_t, II_t and III_t. Reasoning in the same way as in the proof of Proposition 4.2, we have

$$\begin{aligned}
&E\left[\sup_{0\le s\le t}|\mathrm{I}_s|\right] \\
&\le E\left[\int_0^t\int_{\mathbb{R}}|(Y_s'(x)+K(s,z))^{2k}-(Y_s'(x))^{2k}|\,ds\,\nu(dz)\right] \\
&\le CE\left[\int_0^t ds|Y_{s-}'(x)|^{2k}\int_{\mathbb{R}}\left(\left|\frac{\partial\sigma(X_{s-}(x),z)}{\partial y}\right|+\left|\frac{\partial\sigma(X_{s-}(x),z)}{\partial y}\right|^{2k}\right)\nu(dz)\right] \qquad (4.54)\\
&\quad+CE\left[\int_0^t ds|Y_{s-}'(x)|^{2k-1}|Y_s(x)X_s'(x)|\int_{\mathbb{R}}\left|\frac{\partial^2\sigma(X_{s-}(x),z)}{\partial y^2}\right|\nu(dz)\right] \\
&\quad+CE\left[\int_0^t ds|Y_s(x)X_s'(x)|^{2k}\int_{\mathbb{R}}\left|\frac{\partial^2\sigma(X_{s-}(x),z)}{\partial y^2}\right|^{2k}\nu(dz)\right].
\end{aligned}$$

Since

$$\sup_y\int_{\mathbb{R}}\left|\frac{\partial\sigma(y,z)}{\partial y}\right|^l\nu(dz)<\infty,\quad \text{for}\quad l=1,\dots,2k,$$

and

$$\sup_y\int_{\mathbb{R}}\left|\frac{\partial^2\sigma(y,z)}{\partial y^2}\right|^l\nu(dz)<\infty,\quad \text{for}\quad l=1,\dots,2k,$$

(4.54) is less than

$$\begin{aligned}
&CE\left[\int_0^t ds|(Y_{s-}'(x)|^{2k}\right]+CE\left[\int_0^t ds|(Y_{s-}'(x)|^{2k-1}|Y_s(x)X_s'(x)|\right] \qquad (4.55)\\
&\quad+CE\left[\int_0^t ds|Y_s(x)X_s'(x)|^{2k}\right].
\end{aligned}$$

Note that

$$E[|Y_{s-}'(x)|^{2k-1}|Y_s(x)X_s'(x)|]\le C_k(E[|(Y_{s-}'(x)|^{2k}]+E[|Y_s(x)X_s'(x)|^{2k}]),$$

and from Propostion 4.3,

$$E\left[\sup_{0\le s\le T}|Y_s(x)X_s'(x)|^{\alpha}\right]<\infty,\quad \text{for}\quad \alpha\le 2k.$$

It follows from (4.55) that

$$E\left[\sup_{0\le s\le t}|I_s|\right]\le C\left(1+E\left[\int_0^t|Y_{s-}'(x)|^{2k}ds\right]\right). \qquad (4.56)$$

By a similar argument, we can show that

$$E\left[\sup_{0\le s\le t}|\mathrm{III}_s|\right]\le C\left(1+E\left[\int_0^t|Y'_{s-}(x)|^{2k}ds\right]\right). \tag{4.57}$$

For the second term, we have

(4.58)

$$\begin{aligned}
E\left[\sup_{0\le s\le t}|\mathrm{II}_s|\right]&\le E\left[\int_0^t\int_{\mathbb{R}}|(Y'_{s-}(x)+m(s,z))^{2k}-(Y'_{s-}(x))^{2k}|\,ds\,\nu(dz)\right]\\
&\le C_kE\left[\int_0^t\int_{\mathbb{R}}(|Y'_{s-}(x)|^{2k-1}|m(s,z)|+|m(s,z)|^{2k})\,ds\,\nu(dz)\right]\\
&\le C_kE\left[\int_0^t\int_{\mathbb{R}}|Y'_{s-}(x)|^{2k}(|J(X_{s-}(x),z)|+|J(X_{s-}(x),z)|^{2k})\,ds\,\nu(dz)\right]\\
&\quad+C_kE\left[\int_0^t\int_{\mathbb{R}}(|Y'_{s-}(x)|^{2k-1}|Y_{s-}(x)X'_{s-}(x)|\,|L(X_{s-}(x),z)|)\,ds\,\nu(dz)\right]\\
&\quad+C_kE\left[\int_0^t\int_{\mathbb{R}}|Y_{s-}(x)X'_{s-}(x)|^{2k}|L(X_{s-}(x),z)|^{2k}\,ds\,\nu(dz)\right]\\
&\le C_kE\left[\int_0^t|Y'_{s-}(x)|^{2k}\,ds\right]+C_kE\left[\int_0^t|Y_{s-}(x)X'_{s-}(x)|^{2k}\,ds\right],\\
&\le C\left(1+E\left[\int_0^t|Y'_{s-}(x)|^{2k}\,ds\right]\right)
\end{aligned}$$

where we have used the assumptions (4.51) and the fact that

$$E\left[\sup_{0\le s\le T}|Y_{s-}(x)X'_{s-}(x)|^{2k}\right]<\infty.$$

Now (4.53), (4.56), (4.57) imply

$$E\left[\sup_{0\le s\le t}|Y'_s(x)|^{2k}\right]\le C_k\left(1+\int_0^tE[|Y'_s(x)|^{2k}]\,ds\right),$$

which yields the desired result by Gronwall's inequality. □

Let $J(y,z)$, $L(y,z)$ be defined as in Proposition 4.4.

Proposition 4.5. *Assume*

$$\sup_y\int_{\mathbb{R}}\left|\frac{\partial^j\sigma(y,z)}{\partial y^j}\right|^l\nu(dz)<\infty, \tag{4.59}$$

$$\sup_y\int_{\mathbb{R}}|L(y,z)|^l\nu(dz)<\infty,\quad \sup_y\int_{\mathbb{R}}|J(y,z)|^l\nu(dz)<\infty, \tag{4.60}$$

and

$$\sup_y \int_{\mathbb{R}} \left| \frac{\partial L(y,z)}{\partial y} \right|^l \nu(dz) < \infty, \quad \sup_y \int_{\mathbb{R}} \left| \frac{\partial J(y,z)}{\partial y} \right|^l \nu(dz) < \infty, \tag{4.61}$$

for $l = 1, \ldots, 2k$, $j = 1, 2, 3$. *Then there exists a constant* C_k *such that*

$$E\left[\sup_{0 \le s \le t} |Y_s''(x)|^{2k} \right] \le C_k.$$

The proof of this proposition is entirely similar to that of Proposition 4.4. It is omitted.

Theorem 4.2. *Assume that* $b(\omega, s, x)$ *is locally Lipschitz in* x *uniformly with respect to* (ω, s) *and*

$$|b(\omega, s, x)| \le C(1 + |x|^\delta), \tag{4.62}$$

for some constants $C > 0$ *and* $\delta < 1$. *Moreover assume that* (4.30), (4.36), (4.43), (4.59), (4.60) *and* (4.61) *hold for some* $k > (1+\delta)/(1-\delta)$. *Then the equation* (4.28) *admits a unique solution. So does the equation* (4.26).

Proof. Recall the Sobolev imbedding theorem: if $p > 1$, then

$$\sup_{x \in \mathbb{R}} |h(x)| \le c_p \|h\|_{1,p}, \tag{4.63}$$

where $\|h\|_{1,p}^p = \int_{\mathbb{R}} (|h(x)|^p + |h'(x)|^p)\, dx$. Let $\beta > 0$, $\alpha > 0$ and $p > 1$ be any parameters with $2\alpha p > 1$ and $(2\beta - 1)p > 1$. Set

$$f_s(x) = (1 + x^2)^{-\beta} X_s(x), \quad g_s(x) = (1 + x^2)^{-\alpha} Y_s(x),$$

where $Y_s(x) = (X_s'(x))^{-1}$. For any $T > 0$, using Proposition 4.2,

$$\begin{aligned} &E\left[\sup_{0 \le s \le T} \| f_s \|_{1,p}^p \right] \\ &\le C_{\beta,p} \int_{\mathbb{R}} E\left[\sup_{0 \le s \le T} |X_s(x)|^p \right] [(1+x^2)^{-\beta p} + |x|^p (1+x^2)^{-(\beta+1)p}]\, dx \\ &\quad + C_{\beta,p} \int_{\mathbb{R}} E\left[\sup_{0 \le s \le T} |X_s'(x)|^p \right] (1+x^2)^{-\beta p}\, dx \\ &\le \int_{\mathbb{R}} \{|x|^p ((1+x^2)^{-\beta p} + |x|^p (1+x^2)^{-(\beta+1)p}) + (1+x^2)^{-\beta p}\}\, dx < \infty. \end{aligned} \tag{4.64}$$

Similarly, by Proposition 4.4,

$$
\begin{aligned}
&E\left[\sup_{0\le s\le T}\|g_s\|_{1,p}^p\right]\\
&\le C_{\alpha,p}\int_{\mathbb{R}} E\left[\sup_{0\le s\le T}|Y_s(x)|^p\right][(1+x^2)^{-\alpha p}+|x|^p(1+x^2)^{-(\alpha+1)p}]\,dx \\
&\quad + C_{\alpha,p}\int_{\mathbb{R}} E\left[\sup_{0\le s\le T}|Y_s'(x)|^p\right](1+x^2)^{-\alpha p}\,dx \\
&\le \int_{\mathbb{R}}\left\{\left((1+x^2)^{-\alpha p}+|x|^p(1+x^2)^{-(\alpha+1)p}\right)+(1+x^2)^{-\alpha p}\right\}dx<\infty.
\end{aligned}
\tag{4.65}
$$

By the Sobolev imbedding theorem there exist random constants $C_{\beta,T}(\omega)$ and $C_{\alpha,T}(\omega)$ such that

$$\sup_{0\le s\le T}|X_s(x)|\le C_{\beta,T}(\omega)(1+x^2)^{\beta},$$

and

$$\sup_{0\le s\le T}|Y_s(x)|\le C_{\alpha,T}(\omega)(1+x^2)^{\alpha}.$$

The assumption (4.62) together with the above two inequalities gives

$$
\begin{aligned}
\sup_{0\le s\le T}|\hat b(\omega,s,x)| &= \sup_{0\le s\le T}\{|Y_s(x)||b(\omega,s,X_s(x))|\}\\
&\le C(\omega)(1+x^2)^{\alpha}(1+|X_s(x)|^{\delta})\\
&\le M_{\alpha,\beta,T}(\omega)(1+x^2)^{\alpha+\beta\delta}.
\end{aligned}
\tag{4.66}
$$

If $p>(1+\delta)/(1-\delta)$, it is possible to choose $\beta>0$ and $\alpha>0$ such that $2\alpha p>1$, $(2\beta-1)p>1$ and $2\alpha+2\beta\delta\le 1$. Therefore, there exists a random constant $C_T(\omega)$ such that

$$\sup_{0\le s\le T}|\hat b(\omega,s,x)|\le C_T(\omega)(1+|x|). \tag{4.67}$$

On the other hand, by the Sobolev imbedding Theorem and Proposition 4.4 we see that $(\phi_s')^{-1}(x)$ is C^1 in x and the derivative is bounded on compact sets. Combining this fact with the assumption on b, it is easily seen that for a fixed ω, $\hat b(\omega,s,x)$ is locally Lipschitz in x uniformly with respect to s on any compact sets. It follows from the general theory of ordinary differential equations that (4.28) admits a unique global solution. □

References

[1] J. Bertoin: Lévy Processes, Cambridge University Press, Cambridge, 1996.

[2] O. Barndorff-Nielsen: *Processes of normal inverse Gaussian type*, Finance and Stochastics **1** (1998), 41–68.

[3] R.F. Bass and M. Cranston: *The Malliavin calculus for pure jump processes and applications to local time*, Annals. Prob. **14** (1986), 490–532.

[4] F.E. Benth and J. Gjerde: *A remark on the equivalence between Poisson and Gaussian stochastic partial differential equations*, Pot. Anal. **8** (1998), 179–193.

[5] F.E. Benth and A.Løkka: *Anticipative calculus for Lévy processes and stochastic differential equations*, Preprint Series in Pure Mathematics, University of Oslo **6** (2002).

[6] K. Bichteler, J.B. Gravereaux and J. Jacod: Malliavin Calculus for Processes with Jumps, Gordon and Breach Science Publisher, New York, 1987.

[7] G. Di Nunno, T. Meyer-Brandis, B. Øksendal and F. Proske: *Malliavin calculus and anticipative Itô formulae for Lévy proceses*, Preprint Series in Pure Mathematics, University of Oslo **16** (2003).

[8] H. Holden, B. Øksendal, J.Ubøe and T.-S. Zhang: Stochastic Partial Differential Equations—A Modeling, White Noise Functional Approach, Birkhäuser, Boston 1996.

[9] N. Ikeda and S. Watanabe: Stochastic Differential Equations and Diffusion Processes, North-Holland/Kondansha LTD., Tokyo 1989.

[10] A.D. Kaminsky: *A white noise approach to stochastic integration for a Poisson random measure*, Theory Probab. Math. Statist. **57** (1998), 43–52.

[11] Y. Kabanov: *A generalized Itô formula for an extended stochastic integral with respect to a Poisson random measure,* Usp. Mat. Nauk **29** (1974), 167–168, (in Russian).

[12] Y. Kabanov: *On extended stochastic intervals*, Theory of Probability and its Applications **20** (1975), 710–722.

[13] J.A. León, J.L. Solé and J. Vives: *A pathwise approach to backward and forward stochastic differential equations on the Poisson space*, Stochastic Analysis and Applications **19** (2001), 821–839.

[14] P. Malliavin: Stochastic Analysis, Springer-Verlag, New York 1997.

[15] D. Nualart: The Malliavin Calculus and Related Topics, Springer, Berlin Heidelberg New York, 1995.

[16] D. Nualart and E. Pardoux: *Stochastic calculus with anticipating integrands*, Probab. Th. Rel. Fields **78** (1988), 535–581.

[17] D. Nualart and W. Schoutens: *Chaotic and predictable representations for Lévy processes*, Stochastic Process. Appl. **90** (2000), 109–122.

[18] D. Nualart and J. Vives: *A duality formula on the Poisson space and some applications*; in Seminar on Stochastic Analysis, Random Fields and Applications (Ascona, 1993), Progr. Probab. **36**, Birkhäuser, Basel, 1995, 205–213.

[19] B. Øksendal: *An introduction to Malliavin calculus with applications to economics*, Working paper, **3**/**96**, Norwegian School of Economics and Business Administration, 1996.

[20] B. Øksendal and F. Proske: *White noise for Poisson random measures,* Potential Analysis (to appear).

[21] D. Ocone and E. Pardoux: *A generalized Itô-Ventzell formula*. *Application to a class of anticipating stochastic differential equations*, Ann. Inst. Henri Poincaré, Probabilités et Statistiques **25** (1989), 39–71.

[22] J. Picard: *On the existence of smooth densities for jump processes*, Prob. Th. Rel. Fields **105** (1996) 481–511.

[23] P. Protter: Stochastic Integration and Differential Equations, Second Edition, Springer-Verlag Berlin Heidelberg New York, 2003.

[24] F. Russo and P. Valois: *Forward, backward and symmetric stochastic integration*, Prob. Th. Rel. Fields **97** (1993), 403–421.

Bernt Øksendal
Center of Mathematics for Applications (CMA)
Department of Mathematics
University of Oslo
P.O. Box 1053 Blindern, N–0316 Oslo
Norway and Norwegian School of Economics
and Business Administration
Helleveien 30, N–5045 Bergen
Norway
e-mail: oksendal@math.uio.no

Tusheng Zhang
Department of Mathematics
University of Manchester
Oxford Road, Manchester M13 9PL, England, U.K.
and
Center of Mathematics for Applications (CMA)
Department of Mathematics
University of Oslo
P.O. Box 1053 Blindern, N–0316 Oslo
Norway
e-mail: tzhang@maths.man.ac.uk