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A VARIATIONAL PROBLEM RELATED TO CONFORMAL MAPS

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Abstract

In this paper we are concerned with a variational problem for a functional related to the conformality of maps between Riemannian manifolds. We give the first variation formula, the second variation formula, a kind of the monotonicity formula and a Bochner type formula. We also consider a variational problem of minimizing the functional in each 3-homotopy class of the Sobolev space.

1. Introduction

Let (M, g), (N, h) be compact Riemannian manifolds without boundary. A smooth map f from M into N is called a *conformal map* if there exists a positive function φ on M such that $f^*h = \varphi g$, where f^*h denotes the pullback of the metric h by f, i.e.,

$$(f^*h)(X, Y) = h(df(X), df(Y)).$$

We consider a covariant symmetric tensor

$$T_f := f^*h - \frac{1}{m}\|df\|^2 g$$

where m denotes the dimension of the manifold M, and $\|df\|^2$ denotes the energy density in the harmonic map theory, i.e.,

$$\|df\|^2 = \sum_i h(df(e_i), df(e_i)).$$

(e_i denotes a local orthonormal frame on M.) Then f is conformal at x if and only if $T_f = 0$ at this point, unless $(df)_x = 0$. In this paper we are concerned with the functional

$$\Phi(f) = \int_M \|T_f\|^2 \, dv_g,$$

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where dv_g denotes the volume form of (M, g), and

$$\|T_f\|^2 = \sum_{i,j} T_f(e_i, e_j)^2.$$

The functional $\Phi(f)$ gives a quantity of the conformality of f. We give the first variation formula and the second variation formula for this functional. We also prove a kind of the monotonicity formula and a Bochner type formula. Furthermore we want to minimize the functional $\Phi(f)$ in each homotopy class of maps from M into N. Minimizers are expected to be *closest* to conformal maps, even if its homotopy class does not contain any conformal map. To this aim, we adopt the notion of 3-homotopy in the Sobolev spaces, which is given by White. We consider a variational problem of minimizing the functional $\Phi(f)$ in each 3-homotopy class of the Sobolev space.

2. The tensor T_f of the conformality and the functional $\Phi(f)$

Let (M, g), (N, h) be compact Riemannian manifolds without boundary and let f be a smooth map from M into N. In this section we give a tensor T_f of the conformality for any smooth map f. We recall here the following two notions.

DEFINITION 1. (i) A smooth map f is *weakly conformal* if there exists a *non-negative* function φ on M such that

$$f^*h = \varphi g \tag{1}$$

where f^*h denotes the pullback of the metric h by f, i.e.,

$$(f^*h)(X, Y) = h(df(X), df(Y)).$$

(ii) A smooth map f is *conformal* if there exists a *positive* function φ on M satisfying (1).

The condition (1) is equivalent to

$$f^*h = \frac{1}{m}\|df\|^2 g, \tag{2}$$

since taking the trace of the both sides of (1) (with respect to the metric g), we have $\|df\|^2 = m\varphi$, i.e., $\varphi = (1/m)\|df\|^2$. Then f is conformal if and only if it satisfies (2) with the assumption $\|df\| \neq 0$. Note that f is weakly conformal if and only if for any point $x \in M$, f is conformal at x or $df_x = 0$.

Taking the above situation into consideration, we utilize the covariant tensor

$$T_f \overset{\text{def}}{=} f^*h - \frac{1}{m}\|df\|^2 g,$$

i.e.,

$$\begin{aligned} T_f(X, Y) &\overset{\text{def}}{=} (f^*h)(X, Y) - \frac{1}{m}\|df\|^2 g(X, Y) \\ &= h(df(X), df(Y)) - \frac{1}{m}\|df\|^2 g(X, Y). \end{aligned}$$

REMARK 1. In the case of $m = 2$, the tensor T_f is equal to the stress energy tensor

$$S_f = f^*h - \frac{1}{2}\|df\|^2 g$$

in the harmonic map theory. (See Eells and Lemaire [3], p. 392.)

Lemma 1. (a) *T_f is symmetric, i.e., $T_f(X, Y) = T_f(Y, X)$.*
(b) *f is weakly conformal if and only if $T_f = 0$.*
(c) *$\|T_f\|^2 = \|f^*h\|^2 - (1/m)\|df\|^4$.*
(d) *T_f is trace-free, i.e.,*

$$\mathrm{Trace}_g\, T_f = \sum_{i,j} g(e_i, e_j) T_f(e_i, e_j) = 0,$$

where e_i denotes a local orthonormal frame on M.
(e) *The trace of T_f with respect to the pullback f^*h is equal to the norm of T_f, i.e.,*

$$\mathrm{Trace}_{f^*h}\, T_f = \sum_{i,j} (f^*h)(e_i, e_j) T_f(e_i, e_j) = \|T_f\|^2.$$

Proof. (a) follows directly from the definition of T_f.

(b): The argument mentioned above implies that f is a weakly conformal map if and only if $f^*h = (1/m)\|df\|^2 g$, which is equivalent to the condition $T_f = 0$.

(c):

$$\begin{aligned} \|T_f\|^2 &= \sum_{i,j} T_f(e_i, e_j)^2 \\ &= \sum_{i,j} \left\{ h(df(e_i), df(e_j)) - \frac{1}{m}\|df\|^2 g(e_i, e_j) \right\}^2 \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i,j} h(df(e_i), df(e_j))^2 \\
&\quad - \frac{2}{m}\|df\|^2 \sum_{i,j} h(df(e_i), df(e_j)) g(e_i, e_j) + \frac{1}{m^2}\|df\|^4 \sum_{i,j} g(e_i, e_j)^2 \\
&= \|f^*h\|^2 - \frac{2}{m}\|df\|^4 + \frac{1}{m}\|df\|^4 \\
&= \|f^*h\|^2 - \frac{1}{m}\|df\|^4.
\end{aligned}
$$

(d):

$$
\begin{aligned}
\mathrm{Trace}_g\, T_f &= \sum_{i,j} g(e_i, e_j) T_f(e_i, e_j) \\
&= \sum_{i,j} g(e_i, e_j)\left\{ h(df(e_i), df(e_j)) - \frac{1}{m}\|df\|^2 g(e_i, e_j)\right\} \\
&= \sum_{i,j} g(e_i, e_j) h(df(e_i), df(e_j)) - \frac{1}{m}\|df\|^2 \sum_{i,j} g(e_i, e_j)^2 \\
&= \|df\|^2 - \|df\|^2 \\
&= 0.
\end{aligned}
$$

(e):

$$
\begin{aligned}
\mathrm{Trace}_{f^*h}\, T_f &= \sum_{i,j} (f^*h)(e_i, e_j) T_f(e_i, e_j) \\
&= \sum_{i,j} h(df(e_i), df(e_j)) T_f(e_i, e_j) \\
&= \sum_{i,j} h(df(e_i), df(e_j))\left\{ h(df(e_i), df(e_j)) - \frac{1}{m}\|df\|^2 g(e_i, e_j)\right\} \\
&= \sum_{i,j} h(df(e_i), df(e_j))^2 - \frac{1}{m}\|df\|^2 \sum_{i,j} h(df(e_i), df(e_j)) g(e_i, e_j) \\
&= \|f^*h\|^2 - \frac{1}{m}\|df\|^4 \\
&= \|T_f\|^2 \quad \text{(by (c))}.
\end{aligned}
$$

Thus we obtain Lemma 1. □

In this paper, we are concerned with the functional

$$
\Phi(f) = \int_M \|T_f\|^2\, dv_g.
$$

This functional $\Phi(f)$ gives a quantity of the conformality of maps f. Note that if f is a conformal map, then $\Phi(f)$ vanishes.

3. First variation formula

In this section we give the first variation formula for the functional $\Phi(f)$. We define an "$f^{-1}TN$-valued" 1-form σ_f on M by

$$
\begin{aligned}
\sigma_f(X) &= \sum_j T_f(X, e_j)\, df(e_j) \\
&= \sum_j h(df(X), df(e_j))\, df(e_j) - \frac{1}{n}\|df\|^2\, df(X)
\end{aligned}
\tag{3}
$$

for any vector field X on M, where e_j denotes a local orthonormal frame on M. The 1-form σ_f plays an important role in our arguments.

Take any smooth deformation F of f, i.e., any smooth map

$$F\colon (-\varepsilon, \varepsilon) \times M \to N \quad \text{s.t.} \quad F(0, x) = f(x).$$

Let $f_t(x) = F(t, x)$, and we often say a deformation $f_t(x)$ instead of a deformation $F(t, x)$. Let

$$X = dF\left(\frac{\partial}{\partial t}\right)\bigg|_{t=0}$$

denote the variation vector fields of the deformation F. Then we have the following first variation formula.

Theorem 1 (first variation formula).

$$\frac{d\Phi(f_t)}{dt}\bigg|_{t=0} = -4\int_M h(X, \operatorname{div}_g \sigma_f)\, dv_g,$$

where dv_g denotes the volume form on M, and $\operatorname{div}_g \sigma_f$ *denotes the divergence of σ_f, i.e.,* $\operatorname{div}_g \sigma_f = \sum_{i=1}^m (\nabla_{e_i}\sigma_f)(e_i)$.

We give here the notion of stationary maps for the functional $\Phi(f)$.

DEFINITION 2. We call a smooth map f *stationary* (for the functional $\Phi(f)$) if the first variation of $\Phi(f)$ identically vanishes, i.e.,

$$\frac{d\Phi(f_t)}{dt}\bigg|_{t=0} = 0$$

for any smooth deformation f_t of f. By Theorem 1, a smooth map f is *stationary* for $\Phi(f)$ if and only if it satisfies the equation

$$\operatorname{div}_g \sigma_f = 0, \tag{4}$$

where σ_f is the covariant tensor defined by (3). It is the Euler–Lagrange equation for the functional $\Phi(f)$.

Proof of Theorem 1. We calculate $(\partial/\partial t)\| f_t^* h\|^2$ at any fixed point $x_0 \in M$. The connection ∇ is trivially extended to a connection on $(-\varepsilon, \varepsilon) \times M$. We use the same notation ∇ for this connection. The frame e_i is also trivially extended to a frame on $(-\varepsilon, \varepsilon) \times$ (the domain of the frame), and we use the same notation e_i. By a normal coordinate at x_0, we can assume $\nabla_{e_i} e_j = 0$ for any $i,\ j$ at x_0. Since $(dF)_{(t,x)}((e_i)_{(t,x)}) = (df_t)_x((e_i)_x)$, we denote them by $dF(e_i)$ simply. Note that

$$\nabla_{\partial/\partial t}(dF(e_i)) = (\nabla_{\partial/\partial t}\, dF)(e_i) = (\nabla_{e_i}\, dF)\left(\frac{\partial}{\partial t}\right) = \nabla_{e_i}\left(dF\left(\frac{\partial}{\partial t}\right)\right), \tag{5}$$

since $[\partial/\partial t, e_i] = 0$. Then we have

$$\begin{aligned}
\frac{\partial}{\partial t}\|T_{f_t}\|^2 &= \frac{\partial}{\partial t}\sum_{i,j} T_{f_t}(e_i, e_j)^2 \\
&= 2\sum_{i,j} \frac{\partial T_{f_t}(e_i, e_j)}{\partial t} T_{f_t}(e_i, e_j) \\
&= 2\sum_{i,j}\left\{\frac{\partial}{\partial t} h(df_t(e_i), df_t(e_j)) - \frac{1}{m}\frac{\partial \|df_t\|^2}{\partial t} g(e_i, e_j)\right\} T_{f_t}(e_i, e_j) \\
&= 2\sum_{i,j}\left\{\frac{\partial}{\partial t} h(df_t(e_i), df_t(e_j))\right\} T_{f_t}(e_i, e_j) - \frac{2}{n}\frac{\partial \|df_t\|^2}{\partial t}\sum_{i,j} g(e_i, e_j) T_{f_t}(e_i, e_j) \\
&= 2\sum_{i,j}\left\{\frac{\partial}{\partial t} h(df_t(e_i), df_t(e_j))\right\} T_{f_t}(e_i, e_j) \quad \text{(by Lemma 1 (d))} \\
&= 2\sum_{i,j}\left\{\frac{\partial}{\partial t} h(dF(e_i), dF(e_j))\right\} T_{f_t}(e_i, e_j) \\
&= 4\sum_{i,j} h(\nabla_{\partial/\partial t}(dF(e_i)), dF(e_j)) T_{f_t}(e_i, e_j) \quad \text{(by Lemma 1 (a))} \\
&= 4\sum_{i,j} h\left(\nabla_{e_i}\left(dF\left(\frac{\partial}{\partial t}\right)\right), df_t(e_j)\right) T_{f_t}(e_i, e_j) \quad \text{(by (5))} \\
&= 4\sum_{i} h\left(\nabla_{e_i}\left(dF\left(\frac{\partial}{\partial t}\right)\right), \sum_j T_{f_t}(e_i, e_j)\, df_t(e_j)\right) \\
&\quad (\because\ h(A, B)T_{f_t}(C, D) = h(A, T_{f_t}(C, D)B))
\end{aligned}$$

$$= 4 \sum_i h\left(\nabla_{e_i}\left(dF\left(\frac{\partial}{\partial t}\right)\right), \sigma_{f_t}(e_i)\right).$$

Thus we obtain

$$\frac{\partial}{\partial t}\|T_{f_t}\|^2 = 4 \sum_i h\left(\nabla_{e_i}\left(dF\left(\frac{\partial}{\partial t}\right)\right), \sigma_{f_t}(e_i)\right). \tag{6}$$

Integrate the both sides of (6) on M, and then we have

$$\begin{aligned}\frac{d}{dt}\int_M \|T_{f_t}\|^2\, dv_g &= \int_M \frac{\partial}{\partial t}\|T_{f_t}\|^2\, dv_g \\ &= 4\int_M \sum_i h\left(\nabla_{e_i}\left(dF\left(\frac{\partial}{\partial t}\right)\right), \sigma_{f_t}(e_i)\right) dv_g.\end{aligned}$$

Let $t = 0$ and using integration by parts, we obtain the first variation formula. □

Take a 1-parameter family φ_t $(-\varepsilon < t < \varepsilon)$ of diffeomorphisms on M. Let X be the smooth vector field on M corresponding to this 1-parameter family. We have the following first variation formula for $f_t = f \circ \varphi_t$.

Theorem 2 (first variation formula).

$$\left.\frac{d\Phi(f \circ \varphi_t)}{dt}\right|_{t=0} = -\int_M \left\{\|T_f\|^2 \operatorname{div}_g X - 4\sum_{i=1}^m h(df(\nabla_{e_i} X), \sigma_f(e_i))\right\} dv_g, \tag{7}$$

where $\{e_i\}$ denotes a local orthonormal frame on M.

Proof. Theorem 2 follows from the general form of the first variation formula (Theorem 1). Take $\tilde{X} = df(X)$ as a variation vector field X in Theorem 1 for $f_t = f \circ \varphi_t$, and then we have

$$\nabla_{e_i}\tilde{X} = (\nabla_{e_i} df)(X) + df(\nabla_{e_i} X) = (\nabla_X df)(e_i) + df(\nabla_{e_i} X). \tag{8}$$

We calculate $\sum_{i=1}^m h(\nabla_X df)(e_i), \sigma_f(e_i))$ at any fixed point $x_0 \in M$. Using a normal coordinate at x_0, we have $\nabla_{e_j} e_i = 0$ hence $\nabla_X e_i = 0$ at x_0, and then we have $(\nabla_X df)(e_i) = \nabla_X(df(e_i))$. Then we get

$$\begin{aligned}&4\sum_i h(\nabla_{e_i}\tilde{X}, \sigma_f(e_i)) \\ &= 4\sum_{i=1}^m h(\nabla_X(df(e_i)), \sigma_f(e_i)) + 4\sum_{i=1}^m h(df(\nabla_{e_i} X), \sigma_f(e_i)).\end{aligned} \tag{9}$$

We calculate $\sum_{i=1}^{m} h(\nabla_X(df(e_i)), \sigma_f(e_i))$. Let $\mathcal{L}_X$ be the Lie derivative with respect to the vector field X. We have

$$
\begin{aligned}
&4\sum_{i=1}^{m} h(\nabla_X(df(e_i)), \sigma_f(e_i)) \\
&= 4\sum_{i,j=1}^{m} h(\nabla_X(df(e_i)), df(e_j))T_f(e_i, e_j) \\
&= 2\sum_{i,j=1}^{m} \mathcal{L}_X\{h(df(e_i), df(e_j))\}T_f(e_i, e_j) \\
&= 2\sum_{i,j=1}^{m} \mathcal{L}_X\{h(df(e_i), df(e_j))\}\left\{h(df(e_i), df(e_j)) - \frac{1}{m}\|df\|^2 g(e_i, e_j)\right\} \\
&= 2\sum_{i,j=1}^{m} \mathcal{L}_X\{h(df(e_i), df(e_j))\}h(df(e_i), df(e_j)) \\
&\qquad - \frac{2}{m}\|df\|^2 \sum_{i,j=1}^{m} \mathcal{L}_X\{h(df(e_i), df(e_j))\}g(e_i, e_j) \\
&= \sum_{i,j=1}^{m} \mathcal{L}_X\{h(df(e_i), df(e_j))^2\} - \frac{2}{m}\|df\|^2\mathcal{L}_X\|df\|^2 \\
&= \mathcal{L}_X\left\{\sum_{i,j=1}^{m} h(df(e_i), df(e_j))^2\right\} - \frac{1}{m}\mathcal{L}_X\|df\|^4 \\
&= \mathcal{L}_X\|f^*h\|^2 - \frac{1}{m}\mathcal{L}_X\|df\|^4 \\
&= \mathcal{L}_X\left\{\|f^*h\|^2 - \frac{1}{m}\|df\|^4\right\} \\
&= \mathcal{L}_X\|T_f\|^2.
\end{aligned} \tag{10}
$$

Then by (9) and (10), we have

$$
4\sum_{i} h(\nabla_{e_i}\tilde{X}, \sigma_f(e_i)) = \mathcal{L}_X\|T_f\|^2 + 4\sum_{i=1}^{m} h(df(\nabla_{e_i}X), \sigma_f(e_i)) \tag{11}
$$

Therefore we get

$$
\begin{aligned}
\left.\frac{d\Phi(f\circ\varphi_t)}{dt}\right|_{t=0} = \left.\frac{d\Phi(f_t)}{dt}\right|_{t=0}
&= \int_M \mathcal{L}_X\|T_f\|^2\, dv_g + 4\int_M \sum_{i=1}^{m} h(df(\nabla_{e_i}X), \sigma_f(e_i))\, dv_g \\
&= -\int_M \|T_f\|^2\mathcal{L}_X(dv_g) + 4\int_M \sum_{i=1}^{m} h(df(\nabla_{e_i}X), \sigma_f(e_i))\, dv_g
\end{aligned}
$$

$$= -\int_M \|T_f\|^2 \operatorname{div}_g X \, dv_g + 4\int_M \sum_{i=1}^m h(df(\nabla_{e_i} X), \sigma_f(e_i)) \, dv_g.$$

Thus we obtain the conclusion of Theorem 2. □

4. Second variation formula

In this section we give the second variation formula for the functional $\Phi(f)$. Take any smooth deformation F of f with two parameters, i.e., any smooth map

$$F : (-\varepsilon, \varepsilon) \times (-\delta, \delta) \times M \to N \quad \text{s.t.} \quad F(0, 0, x) = f(x).$$

Let $f_{s,t}(x) = F(s, t, x)$, and we often say a deformation $f_{s,t}(x)$ instead of a deformation $F(s, t, x)$. Let

$$X = dF\left(\frac{\partial}{\partial s}\right)\bigg|_{s,t=0}, \quad Y = dF\left(\frac{\partial}{\partial t}\right)\bigg|_{s,t=0}$$

denote the variation vector fields of the deformation F. Then we have the following second variation formula.

Theorem 3 (second variation formula).

$$\begin{aligned}
\frac{1}{4}\frac{\partial^2 \Phi(f_{s,t})}{\partial s \partial t}\bigg|_{s,t=0} = &-\int_M h\left(\operatorname{Hess}_F\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right), \operatorname{div}_g \sigma_f\right) dv_g \\
&+ \int_M \sum_{i,j} h(\nabla_{e_i} X, \nabla_{e_j} Y) T_f(e_i, e_j) \, dv_g \\
&+ \int_M \sum_{i,j} h(\nabla_{e_i} X, df(e_j)) h(\nabla_{e_i} Y, df(e_j)) \, dv_g \\
&+ \int_M \sum_{i,j} h(\nabla_{e_i} X, df(e_j)) h(df(e_i), \nabla_{e_j} Y) \, dv_g \\
&- \frac{2}{m}\int_M \sum_i h(\nabla_{e_i} X, df(e_i)) \sum_j h(\nabla_{e_j} Y, df(e_j)) \, dv_g \\
&- \int_M \sum_{i,j} h({}^N R(df(e_i), X)Y, df(e_j)) T_f(e_i, e_j) \, dv_g,
\end{aligned}$$

where Hess_f *denotes the Hessian of* f, *i.e.*, $\operatorname{Hess}_f(Z, W) = (\nabla_Z df)(W) = (\nabla_W df)(Z)$.

REMARK 2. Note that the first term in the right hand side vanishes if f is a stationary map for the functional $\Phi(f)$.

REMARK 3. The last term of the right hand side in Theorem 3 is equal to

$$-\int_M \sum_i h({}^N R(df(e_i), X)Y, \sigma_f(e_i))\, dv_g.$$

Proof of Theorem 3. The connection ∇ is trivially extended to a connection on $(-\varepsilon, \varepsilon) \times (-\delta, \delta) \times M$. We use the same notation ∇ for this connection. The frame e_i is also trivially extended to a frame on $(-\varepsilon, \varepsilon) \times (-\delta, \delta) \times$ (the domain of the frame), and denoted by the same notation e_i. Take and fix any point $x_0 \in M$, and we calculate $(\partial^2/(\partial s\, \partial t))\| f_{s,t}^* h\|^2$ at x_0 for $s = t = 0$ (for simplicity, we abbreviate the notation "$s = t = 0$"). Using a normal coordinate at x_0, we can assume $\nabla_{e_i} e_j = 0$ for any i, j at x_0. Since

$$\left[\frac{\partial}{\partial s}, e_i\right] = \left[\frac{\partial}{\partial t}, e_i\right] = 0,$$

we see

$$\nabla_{\partial/\partial s}(dF(e_i)) = \nabla_{e_i}\left(dF\left(\frac{\partial}{\partial s}\right)\right) = \nabla_{e_i} X, \tag{12}$$

$$\nabla_{\partial/\partial t}(dF(e_i)) = \nabla_{e_i}\left(dF\left(\frac{\partial}{\partial t}\right)\right) = \nabla_{e_i} Y. \tag{13}$$

We see

$$\begin{aligned}
\frac{\partial^2}{\partial s \partial t}\|T_{f_{s,t}}\|^2 &= \frac{\partial^2}{\partial s \partial t}\sum_{i,j} T_{f_{s,t}}(e_i, e_j)^2 \\
&= 2\sum_{i,j}\left\{\frac{\partial^2 T_{f_{s,t}}(e_i, e_j)}{\partial s \partial t} T_f(e_i, e_j)\right\} + 2\sum_{i,j}\frac{\partial T_{f_{s,t}}(e_i, e_j)}{\partial s}\frac{\partial T_{f_{s,t}}(e_i, e_j)}{\partial t} \\
&\overset{\text{def}}{=:} \mathrm{I}_1 + \mathrm{I}_2.
\end{aligned}\tag{14}$$

We have

(15)

$$\begin{aligned}
\mathrm{I}_1 &= 2\sum_{i,j}\frac{\partial^2}{\partial s \partial t}\left\{h(df_{s,t}(e_i), df_{s,t}(e_j)) - \frac{1}{m}\|df_{s,t}\|^2 g(e_i, e_j)\right\} T_f(e_i, e_j) \\
&= 2\sum_{i,j}\left\{\frac{\partial^2}{\partial s \partial t} h(df_{s,t}(e_i), df_{s,t}(e_j))\right\} T_f(e_i, e_j) - \frac{2}{m}\frac{\partial^2 \|df_{s,t}\|^2}{\partial s \partial t}\sum_{i,j} g(e_i, e_j) T_f(e_i, e_j) \\
&= 2\sum_{i,j}\left\{\frac{\partial^2}{\partial s \partial t} h(df_{s,t}(e_i), df_{s,t}(e_j))\right\} T_f(e_i, e_j) \quad \text{(by Lemma 1 (d))} \\
&= 2\sum_{i,j}\left\{\frac{\partial^2}{\partial s \partial t} h(dF(e_i), dF(e_j))\right\} T_f(e_i, e_j) \\
&= 4\sum_{i,j}\{h(\nabla_{\partial/\partial s}\nabla_{\partial/\partial t}(dF(e_i)), dF(e_j))\} T_f(e_i, e_j) \\
&\quad + 4\sum_{i,j}\{h(\nabla_{\partial/\partial s}(dF(e_i)), \nabla_{\partial/\partial t}(dF(e_j)))\} T_f(e_i, e_j) \quad \text{(by Lemma 1 (a))}.
\end{aligned}$$

We get

$$
\begin{aligned}
\nabla_{\partial/\partial s}\nabla_{\partial/\partial t}(dF(e_i)) &= (\nabla_{\partial/\partial s}\nabla_{\partial/\partial t}\, dF)(e_i) = (\nabla_{\partial/\partial s}\nabla_{e_i}\, dF)\left(\frac{\partial}{\partial t}\right)\\
&= (\nabla_{e_i}\nabla_{\partial/\partial s}\, dF)\left(\frac{\partial}{\partial t}\right) - {}^N R\left(dF(e_i), dF\left(\frac{\partial}{\partial s}\right)\right) dF\left(\frac{\partial}{\partial t}\right)\\
&= \nabla_{e_i}\,\mathrm{Hess}_F\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right) - {}^N R(df(e_i), X)Y.
\end{aligned}
\tag{16}
$$

Then by (12), (13), (15) and (16), we have

$$
\begin{aligned}
\mathrm{I}_1 &= 4\sum_{i,j} h\left(\nabla_{e_i}\,\mathrm{Hess}_F\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right), df(e_j)\right) T_f(e_i, e_j)\\
&\quad - 4\sum_{i,j} h({}^N R(df(e_i), X)Y, df(e_j)) T_f(e_i, e_j)\\
&\quad + 4\sum_{i,j} h\left(\nabla_{e_i}\left(dF\left(\frac{\partial}{\partial s}\right)\right), \nabla_{e_j}\left(dF\left(\frac{\partial}{\partial t}\right)\right)\right) T_f(e_i, e_j)\\
&= 4\sum_{i} h\left(\nabla_{e_i}\,\mathrm{Hess}_F\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right), \sum_j T_f(e_i, e_j)\, df(e_j)\right)\\
&\quad - 4\sum_{i,j} h({}^N R(df(e_i), X)Y, df(e_j)) T_f(e_i, e_j)\\
&\quad + 4\sum_{i,j} h(\nabla_{e_i} X, \nabla_{e_j} Y) T_f(e_i, e_j)\\
&= 4\sum_{i} h\left(\nabla_{e_i}\,\mathrm{Hess}_F\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right), \sigma_f(e_i)\right)\\
&\quad - 4\sum_{i,j} h({}^N R(df(e_i), X)Y, df(e_j)) T_f(e_i, e_j)\\
&\quad + 4\sum_{i,j} h(\nabla_{e_i} X, \nabla_{e_j} Y) T_f(e_i, e_j)\\
&= 4\,\mathrm{div}_g\,\beta_F - 4h\left(\mathrm{Hess}_F\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right), \mathrm{div}_g\,\sigma_f\right)\\
&\quad - 4\sum_{i,j} h({}^N R(df(e_i), X)Y, df(e_j)) T_f(e_i, e_j)\\
&\quad + 4\sum_{i,j} h(\nabla_{e_i} X, \nabla_{e_j} Y) T_f(e_i, e_j),
\end{aligned}
\tag{17}
$$

where

$$
\beta_F(X) = h\left(\mathrm{Hess}_F\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right), \sigma_f(X)\right).
$$

On the other hand we have

(18)

$$
\begin{aligned}
\mathrm{I}_2 &= 2\sum_{i,j}\frac{\partial}{\partial s}\left\{h(df_{s,t}(e_i), df_{s,t}(e_j)) - \frac{1}{m}\|df_{s,t}\|^2 g(e_i, e_j)\right\}\frac{\partial T_{f_{s,t}}(e_i, e_j)}{\partial t}\\
&= 2\sum_{i,j}\left\{\frac{\partial}{\partial s}h(dF(e_i), dF(e_j))\right\}\frac{\partial T_{f_{s,t}}(e_i, e_j)}{\partial t}\\
&\quad - \frac{2}{m}\frac{\partial\|df_{s,t}\|^2}{\partial s}\sum_{i,j}g(e_i, e_j)\frac{\partial T_{f_{s,t}}(e_i, e_j)}{\partial t}\\
&= 2\sum_{i,j}\left\{\frac{\partial}{\partial s}h(dF(e_i), dF(e_j))\right\}\frac{\partial T_{f_{s,t}}(e_i, e_j)}{\partial t}\\
&\quad \left(\because \textstyle\sum_{i,j} g(e_i,e_j)\partial T_{f_{s,t}}(e_i,e_j)/\partial t = (\partial/\partial t)\left(\sum_{i,j} g(e_i,e_j)T_{f_{s,t}}(e_i,e_j)\right) = 0 \text{ by Lemma 1 (d)}\right)\\
&= 2\sum_{i,j}\left\{\frac{\partial}{\partial s}h(dF(e_i), dF(e_j))\right\}\frac{\partial}{\partial t}\left\{h(df_{s,t}(e_i), df_{s,t}(e_j)) - \frac{1}{m}\|df_{s,t}\|^2 g(e_i, e_j)\right\}\\
&= 2\sum_{i,j}\left\{\frac{\partial}{\partial s}h(dF(e_i), dF(e_j))\right\}\left\{\frac{\partial}{\partial t}h(dF(e_i), dF(e_j)) - \frac{1}{m}\frac{\partial\|df_{s,t}\|^2}{\partial t}g(e_i, e_j)\right\}\\
&= 2\sum_{i,j}\left\{\frac{\partial}{\partial s}h(dF(e_i), dF(e_j))\right\}\left\{\frac{\partial}{\partial t}h(dF(e_i), dF(e_j))\right\}\\
&\quad - \frac{2}{m}\sum_{i,j}\left\{\frac{\partial}{\partial s}h(dF(e_i), dF(e_j))\right\}g(e_i, e_j)\frac{\partial\|df_{s,t}\|^2}{\partial t}\\
&= 2\sum_{i,j}\left\{\frac{\partial}{\partial s}h(dF(e_i), dF(e_j))\right\}\left\{\frac{\partial}{\partial t}h(dF(e_i), dF(e_j))\right\}\\
&\quad - \frac{2}{m}\sum_{i}\left\{\frac{\partial}{\partial s}h(dF(e_i), dF(e_i))\right\}\sum_{j}\left\{\frac{\partial}{\partial t}h(dF(e_j), dF(e_j))\right\}\\
&\quad \left(\because\ \partial\|df_{s,t}\|^2/\partial t = (\partial/\partial t)\textstyle\sum_j h(df_{s,t}(e_j), df_{s,t}(e_j)) = \sum_j(\partial/\partial t)h(dF(e_j), dF(e_j))\right)\\
&=: \mathrm{I}_3 + \mathrm{I}_4.
\end{aligned}
$$

We have

$$
\begin{aligned}
\mathrm{I}_3 &= 2\sum_{i,j}\{h(\nabla_{\partial/\partial s}(dF(e_i)), dF(e_j)) + h(dF(e_i), \nabla_{\partial/\partial s}(dF(e_j)))\} \\
&\qquad \times \{h(\nabla_{\partial/\partial t}(dF(e_i)), dF(e_j)) + h(dF(e_i), \nabla_{\partial/\partial t}(dF(e_j)))\} \\
&= 2\sum_{i,j} h(\nabla_{\partial/\partial s}(dF(e_i)), dF(e_j)) h(\nabla_{\partial/\partial t}(dF(e_i)), dF(e_j)) \\
&\quad + 2\sum_{i,j} h(\nabla_{\partial/\partial s}(dF(e_i)), dF(e_j)) h(dF(e_i), \nabla_{\partial/\partial t}(dF(e_j))) \\
&\quad + 2\sum_{i,j} h(dF(e_i), \nabla_{\partial/\partial s}(dF(e_j))) h(\nabla_{\partial/\partial t}(dF(e_i)), dF(e_j)) \\
&\quad + 2\sum_{i,j} h(dF(e_i), \nabla_{\partial/\partial s}(dF(e_j))) h(dF(e_i), \nabla_{\partial/\partial t}(dF(e_j))) \\
&= 4\sum_{i,j} h(\nabla_{\partial/\partial s}(dF(e_i)), dF(e_j)) h(\nabla_{\partial/\partial t}(dF(e_i)), dF(e_j)) \\
&\quad + 4\sum_{i,j} h(\nabla_{\partial/\partial s}(dF(e_i)), dF(e_j)) h(dF(e_i), \nabla_{\partial/\partial t}(dF(e_j))) \\
&\quad \text{(by exchanging the indices } i \text{ and } j) \\
&= 4\sum_{i,j} h\left(\nabla_{e_i}\left(dF\left(\frac{\partial}{\partial s}\right)\right), dF(e_j)\right) h\left(\nabla_{e_i}\left(dF\left(\frac{\partial}{\partial t}\right)\right), dF(e_j)\right) \\
&\quad + 4\sum_{i,j} h\left(\nabla_{e_i}\left(dF\left(\frac{\partial}{\partial s}\right)\right), dF(e_j)\right) h\left(dF(e_i), \nabla_{e_j}\left(dF\left(\frac{\partial}{\partial t}\right)\right)\right) \\
&= 4\sum_{i,j} h(\nabla_{e_i} X, df(e_j)) h(\nabla_{e_i} Y, df(e_j)) \\
&\quad + 4\sum_{i,j} h(\nabla_{e_i} X, df(e_j)) h(df(e_i), \nabla_{e_j} Y).
\end{aligned} \tag{19}
$$

On the other hand by (12) and (13), we get

$$
\begin{aligned}
\mathrm{I}_4 &= -\frac{8}{m}\sum_i h(\nabla_{\partial/\partial s}(dF(e_i)), dF(e_i)) \sum_j h(\nabla_{\partial/\partial t}(dF(e_j)), dF(e_j)) \\
&= -\frac{8}{m}\sum_i h\left(\nabla_{e_i}\left(dF\left(\frac{\partial}{\partial s}\right)\right), dF(e_i)\right) \sum_j h\left(\nabla_{e_j}\left(dF\left(\frac{\partial}{\partial t}\right)\right), dF(e_j)\right) \\
&= -\frac{8}{m}\sum_i h(\nabla_{e_i} X, df(e_i)) \sum_j h(\nabla_{e_j} Y, df(e_j)).
\end{aligned} \tag{20}
$$

Note $(\partial^2/(\partial s\,\partial t))\Phi(f_{s,t})|_{s,t=0} = \int_M (\partial^2/(\partial s\,\partial t))\|T_{f_{s,t}}\|^2|_{s,t=0}\,dv_g$. Integrate (14) over M and use (17), (18), (19) and (20), and then we obtain the second variation formula. $\square$

5. Quasi-monotonicity formula

In this section we prove a kind of the monotonicity formula for stationary maps. We assume the following weak notion of stationary maps.

DEFINITION 3. Let f be a smooth map from M into N. We call it is *stationary* for $\Phi(f)$ *with respect to diffeomorphisms on* M if

$$\frac{d}{dt}\Phi(f\circ\varphi_t)\bigg|_{t=0} = 0$$

for any 1-parameter family φ_t of diffeomorphisms on M.

Note that the notion of stationary maps in Definition 3 is weaker than that of stationary ones in Definition 2, since $f_t(x) = f\circ\varphi_t(x)$ is a deformation in Theorem 1. Under the above weaker condition, we give the following formula.

Theorem 4 (quasi-monotonicity formula). *Let f be stationary for $\Phi(f)$ with respect to diffeomorphisms on M. Let m be the dimension of M. Then it satisfies*

$$\frac{d}{d\rho}\left\{e^{C_2\rho}\rho^{4-m}\int_{B_\rho(x_0)}\|T_f\|^2\,dv_g\right\} \geq 4e^{C_2\rho}\rho^{4-m}(\varphi'(\rho)+C_1\varphi(\rho))$$

where $B_\rho(x_0)$ denotes the open ball of a radius ρ with a center $x_0\in M$, and C_1, C_2 are constants. Here

$$\varphi(\rho) = \int_{B_\rho(x_0)} h\left(df\left(\frac{\partial}{\partial r}\right), \sigma_f\left(\frac{\partial}{\partial r}\right)\right)dv_g$$

and σ_f is defined by (3).

REMARK 4. If $\varphi(\rho)$ satisfies the condition $\varphi'(\rho)+C_1\varphi(\rho)\geq 0$, then

$$e^{C_2\rho}\rho^{4-m}\int_{B_\rho(x_0)}\|T_f\|^2\,dv_g$$

is monotone non-decreasing.

Proof of Theorem 4. We use the argument by Price [4]. (See also Xin [9], p. 43.) Let X be a smooth vector field on M, which is supported compactly in $B_r(x_0)$. Take a 1-parameter family φ_t $(-\varepsilon < t < \varepsilon)$ of diffeomorphisms on M corresponding to this vector field. By Theorem 2, we have

$$(21)\quad 0 = \left.\frac{d\Phi(f\circ\varphi_t)}{dt}\right|_{t=0} = -\int_M \left\{\|T_f\|^2 \operatorname{div}_g X - 4\sum_{i=1}^m h(df(\nabla_{e_i}X), \sigma_f(e_i))\right\} dv_g.$$

Let $r = r(x)$ denote the distance function between x_0 and x, and let $\partial/\partial r$ be the gradient vector field of the distance function r. We can take an local orthonormal frame e_i such that $e_m = \partial/\partial r$. We adopt here a smooth vector field

$$X(x) = \xi(r)r\frac{\partial}{\partial r} = \xi(r(x))r(x)\frac{\partial}{\partial r}$$

in a coordinate neighborhood U of x_0, which vanishes outside U. The function $\xi(r)$ is defined later. We see, for $1 \le i \le m-1$,

$$\nabla_{e_i}\frac{\partial}{\partial r} = \sum_{j=1}^{m-1}\mathrm{Hess}(r)(e_i, e_j)e_j,$$

where $\mathrm{Hess}(r)(X,Y) = (\nabla\, dr)(X,Y) = \nabla_X(dr(Y)) - dr(\nabla_X Y)$ denotes the Hessian of the function r. Indeed, note $dr(e_j) = g(\partial/\partial r, e_j) = 0$ $(j = 1,\dots,m-1)$ and $g(\partial/\partial r, \partial/\partial r) = 1$, and then we have

$$\begin{aligned}\nabla_{e_i}\frac{\partial}{\partial r} &= \sum_{j=1}^m g\left(\nabla_{e_i}\frac{\partial}{\partial r}, e_j\right)e_j = \sum_{j=1}^{m-1} g\left(\nabla_{e_i}\frac{\partial}{\partial r}, e_j\right)e_j + g\left(\nabla_{e_i}\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right)\frac{\partial}{\partial r}\\ &= -\sum_{j=1}^{m-1} g\left(\frac{\partial}{\partial r}, \nabla_{e_i}e_j\right)e_j = -\sum_{j=1}^{m-1} dr(\nabla_{e_i}e_j)e_j = \sum_{j=1}^{m-1}(\nabla\, dr)(e_i, e_j)e_j,\end{aligned}$$

since

$$\begin{aligned}0 &= \nabla_{e_i}\left\{g\left(\frac{\partial}{\partial r}, e_j\right)\right\} = g\left(\nabla_{e_i}\frac{\partial}{\partial r}, e_j\right) + g\left(\frac{\partial}{\partial r}, \nabla_{e_i}e_j\right),\\ 0 &= \nabla_{e_i}\left\{g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right)\right\} = 2g\left(\nabla_{e_i}\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right).\end{aligned}$$

We have

$$(22)\qquad \nabla_{\partial/\partial r}X = \nabla_{\partial/\partial r}\left(\xi(r)r\frac{\partial}{\partial r}\right) = (\xi(r)r)'\frac{\partial}{\partial r},$$

$$(23)\qquad \nabla_{e_i}X = \xi(r)r\nabla_{e_i}\frac{\partial}{\partial r} = \xi(r)r\sum_{j=1}^{m-1}\mathrm{Hess}(r)(e_i, e_j)e_j \quad (1 \le i \le m-1).$$

By the comparison theorem of Hessian, we know

$$\frac{1}{r}g(e_i, e_j)(1 - C_1 r) \leq \mathrm{Hess}(r)(e_i, e_j) \leq \frac{1}{r}g(e_i, e_j)(1 + C_1 r), \tag{24}$$

where c is a constant which depends on the upper and lower bound of the sectional curvature of M. We calculate $\mathrm{div}_g X$ and $\sum_{i=1}^{m} h(df(\nabla_{e_i} X), \sigma_f(e_i))$ in the first variation formula (21). By (22), (23) and (24), we have

$$\begin{aligned}
\mathrm{div}_g X &= \sum_{i=1}^{m-1} g(\nabla_{e_i} X, e_i) + g\left(\nabla_{\partial/\partial r} X, \frac{\partial}{\partial r}\right) \\
&= \xi(r) r \sum_{i,j=1}^{m-1} \mathrm{Hess}(r)(e_i, e_j) g(e_j, e_i) + (\xi(r) r)' \\
&\geq (m-1)\xi(r)(1 - C_1 r) + (\xi(r) r)' \\
&= \xi'(r) r + m\xi(r) - (m-1)c\xi(r) r.
\end{aligned} \tag{25}$$

We also get by (22), (23) and (24),

$$\begin{aligned}
&\sum_{i=1}^{m} h(df(\nabla_{e_i} X), \sigma_f(e_i)) \\
&= \sum_{i=1}^{m-1} h(df(\nabla_{e_i} X), \sigma_f(e_i)) + h\left(df(\nabla_{\partial/\partial r} X), \sigma_f\left(\frac{\partial}{\partial r}\right)\right) \\
&= \xi(r) r \sum_{i,j=1}^{m-1} \mathrm{Hess}(r)(e_i, e_j) h(df(e_j), \sigma_f(e_i)) + (\xi(r) r)' h\left(df\left(\frac{\partial}{\partial r}\right), \sigma_f\left(\frac{\partial}{\partial r}\right)\right) \\
&\leq \xi(r)(1 + C_1 r) \sum_{i=1}^{m-1} h(df(e_i), \sigma_f(e_i)) + (\xi'(r) r + \xi(r)) h\left(df\left(\frac{\partial}{\partial r}\right), \sigma_f\left(\frac{\partial}{\partial r}\right)\right) \\
&= \xi'(r) r h\left(df\left(\frac{\partial}{\partial r}\right), \sigma_f\left(\frac{\partial}{\partial r}\right)\right) \\
&\quad + \xi(r)\left\{\sum_{i=1}^{m-1} h(df(e_i), \sigma_f(e_i)) + h\left(df\left(\frac{\partial}{\partial r}\right), \sigma_f\left(\frac{\partial}{\partial r}\right)\right)\right\} \\
&\quad + C_1 \xi(r) r \sum_{i=1}^{m-1} h(df(e_i), \sigma_f(e_i)) \\
&= \xi'(r) r h\left(df\left(\frac{\partial}{\partial r}\right), \sigma_f\left(\frac{\partial}{\partial r}\right)\right) + \xi(r) \sum_{i=1}^{m} h(df(e_i), \sigma_f(e_i)) \\
&\quad + C_1 \xi(r) r \left\{\sum_{i=1}^{m} h(df(e_i), \sigma_f(e_i)) - h\left(df\left(\frac{\partial}{\partial r}\right), \sigma_f\left(\frac{\partial}{\partial r}\right)\right)\right\}.
\end{aligned} \tag{26}$$

We have by Lemma 1 (e)

$$
\begin{aligned}
\sum_{i=1}^{m} h(df(e_i), \sigma_f(e_i)) &= \sum_{i=1}^{m} h\left(df(e_i), \sum_{j=1}^{m} T_f(e_i, e_j)\, df(e_j) \right) \\
&= \sum_{i=1}^{m} \sum_{j=1}^{m} h(df(e_i), df(e_j)) T_f(e_i, e_j) = \|T_f\|^2.
\end{aligned} \tag{27}
$$

For simplicity we set

$$
A\left(df, \frac{\partial}{\partial r}\right) := h\left(df\left(\frac{\partial}{\partial r}\right), \sigma_f\left(\frac{\partial}{\partial r}\right)\right).
$$

Then by (26), (27), we have

$$
\begin{aligned}
&\sum_{i=1}^{m} h(df(\nabla_{e_i} X), \sigma_f(e_i)) \\
&\leq \xi'(r) r A\left(df, \frac{\partial}{\partial r}\right) + \xi(r)\|T_f\|^2 + C_1 \xi(r) r \left(\|T_f\|^2 - A\left(df, \frac{\partial}{\partial r}\right)\right).
\end{aligned} \tag{28}
$$

Therefore by (21), (25), (28), we get

$$
\begin{aligned}
0 &= \int_M \left\{ \|T_f\|^2 \operatorname{div}_g X - 4 \sum_{i=1}^{m} h(df(\nabla_{e_i} X), \sigma_f(e_i)) \right\} dv_g \\
&\geq \int_M \xi'(r) r \|T_f\|^2 dv_g + m \int_M \xi(r) \|T_f\|^2\, dv_g \\
&\quad - (m-1) C_1 \int_M \xi(r) r \|T_f\|^2\, dv_g \\
&\quad - 4 \int_M \xi'(r) r A\left(df, \frac{\partial}{\partial r}\right) dv_g - 4 \int_M \xi(r) \|T_f\|^2\, dv_g \\
&\quad - 4C_1 \int_M \xi(r) r \|T_f\|^2\, dv_g + 4C_1 \int_M \xi(r) r A\left(df, \frac{\partial}{\partial r}\right) dv_g,
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
&- \int_M \xi'(r) r \|T_f\|^2\, dv_g + (4-m) \int_M \xi(r) \|T_f\|^2\, dv_g + C_2 \int_M \xi(r) r \|T_f\|^2\, dv_g \\
&\geq -4 \int_M \xi'(r) r A\left(df, \frac{\partial}{\partial r}\right) dv_g + 4C_1 \int_M \xi(r) r A\left(df, \frac{\partial}{\partial r}\right) dv_g,
\end{aligned} \tag{29}
$$

where $C_2 = (m+3)C_1$. Take and fix a positive number ε, and let φ be a smooth function on $[0,\infty)$ such that

$$\varphi(r) = \varphi_\varepsilon(r) = \begin{cases} 1 & \text{if} \quad 0 \le r \le 1, \\ 0 & \text{if} \quad 1+\varepsilon \le r \end{cases}$$

and

$$\varphi'(r) \le 0.$$

We define

$$\xi(r) = \xi_\rho(r) \overset{\text{def}}{:=} \varphi\left(\frac{r}{\rho}\right).$$

We can verify

$$\xi'(r)r = -\rho \frac{d}{d\rho}\xi(r). \tag{30}$$

Since $\|T_f\|^2$ is independent of ρ, the above facts (29) and (30) imply

$$\begin{aligned} &\rho \frac{d}{d\rho}\int_M \xi(r)\|T_f\|^2\,dv_g + (4-m)\int_M \xi(r)\|T_f\|^2\,dv_g + C_2\int_M \xi(r)r\|T_f\|^2\,dv_g \\ &\ge 4\rho\frac{d}{d\rho}\int_M A\left(df, \frac{\partial}{\partial r}\right)\xi(r)\,dv_g + 4C_1\rho\int_M A\left(df, \frac{\partial}{\partial r}\right)\xi(r)\,dv_g. \end{aligned}$$

Let ε tend to zero, and then, since $\xi(r)$ converges to the characteristic function for the ball $B_\rho(x_0)$, we have

$$\begin{aligned} &\rho\frac{d}{d\rho}\int_{B_\rho(x_0)} \|T_f\|^2\,dv_g + (4-m)\int_{B_\rho(x_0)}\|T_f\|^2\,dv_g + C_2\rho\int_{B_\rho(x_0)}\|T_f\|^2\,dv_g \\ &\ge 4\rho\frac{d}{d\rho}\int_{B_\rho(x_0)} A\left(df, \frac{\partial}{\partial r}\right)dv_g + 4C_1\rho\int_{B_\rho(x_0)} A\left(df, \frac{\partial}{\partial r}\right)dv_g. \end{aligned}$$

Multiply $e^{C_2\rho}\rho^{3-m}$ to the both sides of this inequality, and we have

$$\begin{aligned} &\frac{d}{d\rho}\left\{e^{C_2\rho}\rho^{4-m}\int_{B_\rho(x_0)}\|T_f\|^2\,dv_g\right\} \\ &\ge 4e^{C_2\rho}\rho^{4-m}\left\{\frac{d}{d\rho}\int_M A\left(df, \frac{\partial}{\partial r}\right)dv_g + C_1\int_M A\left(df, \frac{\partial}{\partial r}\right)dv_g\right\}. \end{aligned}$$

Thus we obtain the formula. $\square$

6. Bochner type formula

In this section we prove the following formula.

Theorem 5 (Bochner type formula). *For any smooth map f from M into N, the following equality holds*:

$$
\begin{aligned}
\frac{1}{4}\,\Delta\|T_f\|^2 &= \operatorname{div}\alpha_f - h(\tau_f, \operatorname{div}\sigma_f) + \frac{1}{2}\|\nabla T_f\|^2 \\
&\quad + \sum_{i,j,k} h((\nabla_{e_k} df)(e_i), (\nabla_{e_k} df)(e_j))T_f(e_i, e_j) \\
&\quad + \sum_{i,j} h\left(df\left(\sum_k {}^M R(e_i, e_k)e_k\right), df(e_j)\right)T_f(e_i, e_j) \\
&\quad - \sum_{i,j,k} h({}^N R(df(e_i), df(e_k))\, df(e_k), df(e_j))T_f(e_i, e_j)
\end{aligned}
\tag{31}
$$

where

$$\alpha_f(X) = h(\sigma_f(X), \tau_f).$$

Here σ_f is defined by (3), *and $\tau_f = \operatorname{tr}(\nabla df) = \sum_j (\nabla_{e_j} df)(e_j)$ is the tension field of f in the harmonic map theory.* (*See Eells and Lemaire* [2], *p.* 9.)

REMARK 5. Note that the first term in the right hand side is of divergence form, and hence the integral of it over M vanishes.

REMARK 6. Note that the second term in the right hand side vanishes if f is a stationary map for the functional $\Phi(f)$.

REMARK 7. The last two terms of the right hand side in Theorem 5 are equal to

$$
\begin{aligned}
&+ \sum_i h\left(df\left(\sum_k {}^M R(e_i, e_k)e_k\right), \sigma_f(e_i)\right) \\
&- \sum_{i,k} h({}^N R(df(e_i), df(e_k))\, df(e_k), \sigma_f(e_i))
\end{aligned}
$$

respectively.

Proof of Theorem 5. We have

$$
\begin{aligned}
\Delta\|T_f\|^2 &= \Delta \sum_{i,j} T_f(e_i, e_j)^2 \\
&= 2\sum_{i,j} (\Delta T_f)(e_i, e_j)T_f(e_i, e_j) + 2\sum_{i,j}\sum_k (\nabla_{e_k} T_f)(e_i, e_j)^2 \\
&\overset{\text{def}}{=}: \mathrm{I}_1 + \mathrm{I}_2.
\end{aligned}
\tag{32}
$$

We get

$$
\begin{aligned}
\mathrm{I}_1 &= 2\sum_{i,j}(\Delta T_f)(e_i, e_j)T_f(e_i, e_j)\\
&= 2\sum_{i,j}\Biggl\{h((\Delta df)(e_i), df(e_j)) + 2\sum_k h((\nabla_{e_k} df)(e_i), (\nabla_{e_k} df)(e_j))\\
&\qquad\qquad + h(df(e_i), (\Delta df)(e_j)) - \frac{1}{m}\,\Delta\|df\|^2 g(e_i, e_j)\Biggr\}T_f(e_i, e_j)\\
&= 4\sum_{i,j} h((\Delta df)(e_i), df(e_j))T_f(e_i, e_j)\\
&\quad + 4\sum_{i,j,k} h((\nabla_{e_k} df)(e_i), (\nabla_{e_k} df)(e_j))T_f(e_i, e_j) \quad \text{(by Lemma 1 (a) and (d)).}
\end{aligned}
$$

Since by Ricci formula,

$$
\begin{aligned}
(\Delta df)(e_i) &= \sum_k(\nabla_{e_k}\nabla_{e_k} df)(e_i) = \sum_k(\nabla_{e_k}\nabla_{e_i} df)(e_k)\\
&= \sum_k(\nabla_{e_i}\nabla_{e_k} df)(e_k) + df\left(\sum_k {}^M R(e_i, e_k)e_k\right)\\
&\quad - \sum_k {}^N R(df(e_i), df(e_k))\, df(e_k)\\
&= \nabla_{e_i}\tau_f + df\left(\sum_k {}^M R(e_i, e_k)e_k\right) - \sum_k {}^N R(df(e_i), df(e_k))\, df(e_k),
\end{aligned}
$$

we have

$$
\begin{aligned}
\mathrm{I}_1 &= 4\sum_{i,j} h(\nabla_{e_i}\tau_f, df(e_j))T_f(e_i, e_j)\\
&\quad + 4\sum_{i,j} h\left(df\left(\sum_k {}^M R(e_i, e_k)e_k\right), df(e_j)\right)T_f(e_i, e_j)\\
&\quad - 4\sum_{i,j,k} h({}^N R(df(e_i), df(e_k))\, df(e_k), df(e_j))T_f(e_i, e_j)\\
&\quad + 4\sum_{i,j,k} h((\nabla_{e_k} df)(e_i), (\nabla_{e_k} df)(e_j))T_f(e_i, e_j).
\end{aligned} \tag{33}
$$

Furthermore we get

$$
\begin{aligned}
\sum_{i,j} h(\nabla_{e_i}\tau_f, df(e_j))T_f(e_i, e_j) &= \sum_i h\left(\nabla_{e_i}\tau_f, \sum_j T_f(e_i, e_j)\, df(e_j)\right)\\
&= \sum_i h(\nabla_{e_i}\tau_f, \sigma_f(e_i)) \qquad (34)\\
&= \sum_i \mathrm{div}_g\, \alpha_f - \sum_i h(\tau_f, \mathrm{div}_g\, \sigma_f).
\end{aligned}
$$

By (32), (33) and (34), we obtain Theorem 5, since $\mathrm{I}_2 = 2\|\nabla T_f\|^2$. □

7. Minimizers in homotopy classes of the Sobolev space

In this section we utilize the notion of 3-homotopy in the Sobolev spaces, which is given by White, and consider a variational problem of minimizing the functional $\Phi(f)$ in each 3-homotopy class. For any two maps f and g from M into N, these maps are k-homotopic ($k \in \mathbb{N}$) if they are homotopic to each other on k-dimensional skeletons of a triangulation on M. By Nash's isometric embedding, we may assume that N is a submanifold of a Euclidean space $\mathbb{R}^q$. Let

$$
\mathrm{L}^{1,p}(M, N) = \{ f \in \mathrm{L}^{1,p}(M, \mathbb{R}^q) \mid f(x) \in N \text{ a.e.}\},
$$

where $\mathrm{L}^{1,p}(M, \mathbb{R}^q)$ denotes the Sobolev space of $\mathbb{R}^q$-valued L^p-functions on M such that their derivatives are in L^p. Then White proved that the notion of the $[p-1]$-homotopy is compatible with the Sobolev space $\mathrm{L}^{1,p}(M, N)$, where [ ] denotes the Gauss symbol, i.e., $[r]$ is the maximum integer less than or equal to r.

Theorem S (Theorem 3.4 in White [8]. See also White [7], Schoen and Uhlenbeck [5] and Bethuel [1]).
(1) *The* $[p-1]$*-homotopy is well-defined for any map* $f \in \mathrm{L}^{1,p}(M, N)$.
(2) *If* f_j *converges weakly to* f_∞ *in* $\mathrm{L}^{1,p}(M, N)$, *then* f_j *and* f_∞ *are* $[p-1]$*-homotopic for sufficient large* j.

The functional $\Phi(f)$ is defined on $\mathrm{L}^{1,4}(M, N)$, in which the 3-homotopy is well-defined. Then for any given continuous map f_0 from M into N, we want to minimize the functional $\Phi(f)$ in the following class:

$$
\mathcal{F} = \{ f \in \mathrm{L}^{1,4}(M, N) \mid f \text{ is 3-homotopic to } f_0 \text{ and } \|f\|_{\mathrm{L}^{1,4}(M,N)} \le C_0\},
$$

where C_0 is a given positive constant. We may assume that the space $\mathcal{F}$ is not empty for sufficiently large C_0.

Theorem 6. *There exists a minimizer of the functional* $\Phi(f)$ *in* $\mathcal{F}$.

If a 3-homotopy class contains a conformal map, then the conformal map is a minimizer. Minimizers are expected to be *closest* to conformal maps, even if its 3-homotopy class does not contain any conformal map.

REMARK 8. When M is 4-dimensional and $\pi_4(N) = 0$, any *continuous* minimizer is (freely) homotopic to f_0 in the ordinary sense.

Proof of Theorem 6. Take any minimizing sequence f_j for the functional $\Phi(f)$ in the space $\mathcal{F}$, i.e., $\Phi(f_j)$ converges to the infimum in $\mathcal{F}$. We may assume that f_j converges weakly to a map f_∞ in $\mathrm{L}^{1,4}(M, N)$, since $\|f\|_{\mathrm{L}^{1,4}(M,N)} \leq C_0$. Since the weak convergence in $\mathrm{L}^{1,4}(M, N)$ preserves the 3-homotopy by Theorem S (2), f_∞ is 3-homotopic to f_j for sufficiently large j, hence to f_0. Furthermore T_{f_j} converges weakly to T_{f_∞} in L^2, since for any covariant 2-tensor S,

$$\begin{aligned}\int_M \langle T_{f_j}, S\rangle \, dv_g &= \int_M \left\langle f_j^* h - \frac{1}{m}\|df_j\|^2 g, S\right\rangle dv_g \\ &= \int_M \left\langle f_j^* h, S - \frac{1}{m}\langle g, S\rangle g\right\rangle dv_g,\end{aligned}$$

where $\langle\ ,\ \rangle$ is the pointwise pairing for covariant 2-tensors. Therefore we have

$$\Phi(f_\infty) = \|T_{f_\infty}\|_{\mathrm{L}^2} \leq \liminf_{j\to\infty} \|T_{f_j}\|_{\mathrm{L}^2} = \liminf_{j\to\infty} \Phi(f_j).$$

Then f_∞ is a minimizer in $\mathcal{F}$. □

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