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A VARIATIONAL PROBLEM RELATED TO CONFORMAL MAPS

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Abstract

In this paper we are concerned with a variational problem for a functional related to the conformality of maps between Riemannian manifolds. We give the first variation formula, the second variation formula, a kind of the monotonicity formula and a Bochner type formula. We also consider a variational problem of minimizing the functional in each 3-homotopy class of the Sobolev space.

1. Introduction

Let (M, g) , (N, h) be compact Riemannian manifolds without boundary. A smooth map f from M into N is called a *conformal map* if there exists a positive function φ on M such that $f^*h = \varphi g$, where f^*h denotes the pullback of the metric h by f , i.e.,

$$(f^*h)(X, Y) = h(df(X), df(Y)).$$

We consider a covariant symmetric tensor

$$T_f := f^*h - \frac{1}{m} \|df\|^2 g$$

where m denotes the dimension of the manifold M , and $\|df\|^2$ denotes the energy density in the harmonic map theory, i.e.,

$$\|df\|^2 = \sum_i h(df(e_i), df(e_i)).$$

(e_i denotes a local orthonormal frame on M .) Then f is conformal at x if and only if $T_f = 0$ at this point, unless $(df)_x = 0$. In this paper we are concerned with the functional

$$\Phi(f) = \int_M \|T_f\|^2 dv_g,$$

where dv_g denotes the volume form of (M, g) , and

$$\|T_f\|^2 = \sum_{i,j} T_f(e_i, e_j)^2.$$

The functional $\Phi(f)$ gives a quantity of the conformality of f . We give the first variation formula and the second variation formula for this functional. We also prove a kind of the monotonicity formula and a Bochner type formula. Furthermore we want to minimize the functional $\Phi(f)$ in each homotopy class of maps from M into N . Minimizers are expected to be *closest* to conformal maps, even if its homotopy class does not contain any conformal map. To this aim, we adopt the notion of 3-homotopy in the Sobolev spaces, which is given by White. We consider a variational problem of minimizing the functional $\Phi(f)$ in each 3-homotopy class of the Sobolev space.

2. The tensor T_f of the conformality and the functional $\Phi(f)$

Let $(M, g), (N, h)$ be compact Riemannian manifolds without boundary and let f be a smooth map from M into N . In this section we give a tensor T_f of the conformality for any smooth map f . We recall here the following two notions.

DEFINITION 1. (i) A smooth map f is *weakly conformal* if there exists a *non-negative* function φ on M such that

$$(1) \quad f^*h = \varphi g$$

where f^*h denotes the pullback of the metric h by f , i.e.,

$$(f^*h)(X, Y) = h(df(X), df(Y)).$$

(ii) A smooth map f is *conformal* if there exists a *positive* function φ on M satisfying (1).

The condition (1) is equivalent to

$$(2) \quad f^*h = \frac{1}{m} \|df\|^2 g,$$

since taking the trace of the both sides of (1) (with respect to the metric g), we have $\|df\|^2 = m\varphi$, i.e., $\varphi = (1/m)\|df\|^2$. Then f is conformal if and only if it satisfies (2) with the assumption $\|df\| \neq 0$. Note that f is weakly conformal if and only if for any point $x \in M$, f is conformal at x or $df_x = 0$.

Taking the above situation into consideration, we utilize the covariant tensor

$$T_f \stackrel{\text{def}}{=} f^*h - \frac{1}{m} \|df\|^2 g,$$

i.e.,

$$\begin{aligned} T_f(X, Y) &\stackrel{\text{def}}{=} (f^*h)(X, Y) - \frac{1}{m} \|df\|^2 g(X, Y) \\ &= h(df(X), df(Y)) - \frac{1}{m} \|df\|^2 g(X, Y). \end{aligned}$$

REMARK 1. In the case of $m = 2$, the tensor T_f is equal to the stress energy tensor

$$S_f = f^*h - \frac{1}{2} \|df\|^2 g$$

in the harmonic map theory. (See Eells and Lemaire [3], p.392.)

- Lemma 1.** (a) T_f is symmetric, i.e., $T_f(X, Y) = T_f(Y, X)$.
 (b) f is weakly conformal if and only if $T_f = 0$.
 (c) $\|T_f\|^2 = \|f^*h\|^2 - (1/m)\|df\|^4$.
 (d) T_f is trace-free, i.e.,

$$\text{Trace}_g T_f = \sum_{i,j} g(e_i, e_j) T_f(e_i, e_j) = 0,$$

where e_i denotes a local orthonormal frame on M .

- (e) The trace of T_f with respect to the pullback f^*h is equal to the norm of T_f , i.e.,

$$\text{Trace}_{f^*h} T_f = \sum_{i,j} (f^*h)(e_i, e_j) T_f(e_i, e_j) = \|T_f\|^2.$$

Proof. (a) follows directly from the definition of T_f .

(b): The argument mentioned above implies that f is a weakly conformal map if and only if $f^*h = (1/m)\|df\|^2 g$, which is equivalent to the condition $T_f = 0$.

(c):

$$\begin{aligned} \|T_f\|^2 &= \sum_{i,j} T_f(e_i, e_j)^2 \\ &= \sum_{i,j} \left\{ h(df(e_i), df(e_j)) - \frac{1}{m} \|df\|^2 g(e_i, e_j) \right\}^2 \end{aligned}$$

$$\begin{aligned}
&= \sum_{i,j} h(df(e_i), df(e_j))^2 \\
&\quad - \frac{2}{m} \|df\|^2 \sum_{i,j} h(df(e_i), df(e_j))g(e_i, e_j) + \frac{1}{m^2} \|df\|^4 \sum_{i,j} g(e_i, e_j)^2 \\
&= \|f^*h\|^2 - \frac{2}{m} \|df\|^4 + \frac{1}{m} \|df\|^4 \\
&= \|f^*h\|^2 - \frac{1}{m} \|df\|^4.
\end{aligned}$$

(d):

$$\begin{aligned}
\text{Trace}_g T_f &= \sum_{i,j} g(e_i, e_j) T_f(e_i, e_j) \\
&= \sum_{i,j} g(e_i, e_j) \left\{ h(df(e_i), df(e_j)) - \frac{1}{m} \|df\|^2 g(e_i, e_j) \right\} \\
&= \sum_{i,j} g(e_i, e_j) h(df(e_i), df(e_j)) - \frac{1}{m} \|df\|^2 \sum_{i,j} g(e_i, e_j)^2 \\
&= \|df\|^2 - \|df\|^2 \\
&= 0.
\end{aligned}$$

(e):

$$\begin{aligned}
\text{Trace}_{f^*h} T_f &= \sum_{i,j} (f^*h)(e_i, e_j) T_f(e_i, e_j) \\
&= \sum_{i,j} h(df(e_i), df(e_j)) T_f(e_i, e_j) \\
&= \sum_{i,j} h(df(e_i), df(e_j)) \left\{ h(df(e_i), df(e_j)) - \frac{1}{m} \|df\|^2 g(e_i, e_j) \right\} \\
&= \sum_{i,j} h(df(e_i), df(e_j))^2 - \frac{1}{m} \|df\|^2 \sum_{i,j} h(df(e_i), df(e_j))g(e_i, e_j) \\
&= \|f^*h\|^2 - \frac{1}{m} \|df\|^4 \\
&= \|T_f\|^2 \quad (\text{by (c)}).
\end{aligned}$$

Thus we obtain Lemma 1. □

In this paper, we are concerned with the functional

$$\Phi(f) = \int_M \|T_f\|^2 dv_g.$$

This functional $\Phi(f)$ gives a quantity of the conformality of maps f . Note that if f is a conformal map, then $\Phi(f)$ vanishes.

3. First variation formula

In this section we give the first variation formula for the functional $\Phi(f)$. We define an “ $f^{-1}TN$ -valued” 1-form σ_f on M by

$$\begin{aligned}
 \sigma_f(X) &= \sum_j T_f(X, e_j) df(e_j) \\
 (3) \qquad &= \sum_j h(df(X), df(e_j)) df(e_j) - \frac{1}{n} \|df\|^2 df(X)
 \end{aligned}$$

for any vector field X on M , where e_j denotes a local orthonormal frame on M . The 1-form σ_f plays an important role in our arguments.

Take any smooth deformation F of f , i.e., any smooth map

$$F: (-\varepsilon, \varepsilon) \times M \rightarrow N \quad \text{s.t.} \quad F(0, x) = f(x).$$

Let $f_t(x) = F(t, x)$, and we often say a deformation $f_t(x)$ instead of a deformation $F(t, x)$. Let

$$X = dF\left(\frac{\partial}{\partial t}\right)\Big|_{t=0}$$

denote the variation vector fields of the deformation F . Then we have the following first variation formula.

Theorem 1 (first variation formula).

$$\frac{d\Phi(f_t)}{dt}\Big|_{t=0} = -4 \int_M h(X, \operatorname{div}_g \sigma_f) dv_g,$$

where dv_g denotes the volume form on M , and $\operatorname{div}_g \sigma_f$ denotes the divergence of σ_f , i.e., $\operatorname{div}_g \sigma_f = \sum_{i=1}^m (\nabla_{e_i} \sigma_f)(e_i)$.

We give here the notion of stationary maps for the functional $\Phi(f)$.

DEFINITION 2. We call a smooth map f *stationary* (for the functional $\Phi(f)$) if the first variation of $\Phi(f)$ identically vanishes, i.e.,

$$\frac{d\Phi(f_t)}{dt}\Big|_{t=0} = 0$$

for any smooth deformation f_t of f . By Theorem 1, a smooth map f is *stationary* for $\Phi(f)$ if and only if it satisfies the equation

$$(4) \quad \operatorname{div}_g \sigma_f = 0,$$

where σ_f is the covariant tensor defined by (3). It is the Euler–Lagrange equation for the functional $\Phi(f)$.

Proof of Theorem 1. We calculate $(\partial/\partial t)\|f_t^*h\|^2$ at any fixed point $x_0 \in M$. The connection ∇ is trivially extended to a connection on $(-\varepsilon, \varepsilon) \times M$. We use the same notation ∇ for this connection. The frame e_i is also trivially extended to a frame on $(-\varepsilon, \varepsilon) \times$ (the domain of the frame), and we use the same notation e_i . By a normal coordinate at x_0 , we can assume $\nabla_{e_i}e_j = 0$ for any i, j at x_0 . Since $(dF)_{(t,x)}((e_i)_{(t,x)}) = (df_t)_x((e_i)_x)$, we denote them by $dF(e_i)$ simply. Note that

$$(5) \quad \nabla_{\partial/\partial t}(dF(e_i)) = (\nabla_{\partial/\partial t} dF)(e_i) = (\nabla_{e_i} dF)\left(\frac{\partial}{\partial t}\right) = \nabla_{e_i}\left(dF\left(\frac{\partial}{\partial t}\right)\right),$$

since $[\partial/\partial t, e_i] = 0$. Then we have

$$\begin{aligned} \frac{\partial}{\partial t} \|T_{f_t}\|^2 &= \frac{\partial}{\partial t} \sum_{i,j} T_{f_t}(e_i, e_j)^2 \\ &= 2 \sum_{i,j} \frac{\partial T_{f_t}(e_i, e_j)}{\partial t} T_{f_t}(e_i, e_j) \\ &= 2 \sum_{i,j} \left\{ \frac{\partial}{\partial t} h(df_t(e_i), df_t(e_j)) - \frac{1}{m} \frac{\partial \|df_t\|^2}{\partial t} g(e_i, e_j) \right\} T_{f_t}(e_i, e_j) \\ &= 2 \sum_{i,j} \left\{ \frac{\partial}{\partial t} h(df_t(e_i), df_t(e_j)) \right\} T_{f_t}(e_i, e_j) - \frac{2}{n} \frac{\partial \|df_t\|^2}{\partial t} \sum_{i,j} g(e_i, e_j) T_{f_t}(e_i, e_j) \\ &= 2 \sum_{i,j} \left\{ \frac{\partial}{\partial t} h(df_t(e_i), df_t(e_j)) \right\} T_{f_t}(e_i, e_j) \quad (\text{by Lemma 1 (d)}) \\ &= 2 \sum_{i,j} \left\{ \frac{\partial}{\partial t} h(dF(e_i), dF(e_j)) \right\} T_{f_t}(e_i, e_j) \\ &= 4 \sum_{i,j} h(\nabla_{\partial/\partial t}(dF(e_i)), dF(e_j)) T_{f_t}(e_i, e_j) \quad (\text{by Lemma 1 (a)}) \\ &= 4 \sum_{i,j} h\left(\nabla_{e_i}\left(dF\left(\frac{\partial}{\partial t}\right)\right), dF(e_j)\right) T_{f_t}(e_i, e_j) \quad (\text{by (5)}) \\ &= 4 \sum_i h\left(\nabla_{e_i}\left(dF\left(\frac{\partial}{\partial t}\right)\right), \sum_j T_{f_t}(e_i, e_j) df_t(e_j)\right) \\ &(\because h(A, B)T_{f_t}(C, D) = h(A, T_{f_t}(C, D)B)) \end{aligned}$$

$$= 4 \sum_i h\left(\nabla_{e_i}\left(dF\left(\frac{\partial}{\partial t}\right)\right), \sigma_{f_i}(e_i)\right).$$

Thus we obtain

$$(6) \quad \frac{\partial}{\partial t} \|T_{f_t}\|^2 = 4 \sum_i h\left(\nabla_{e_i}\left(dF\left(\frac{\partial}{\partial t}\right)\right), \sigma_{f_i}(e_i)\right).$$

Integrate the both sides of (6) on M , and then we have

$$\begin{aligned} \frac{d}{dt} \int_M \|T_{f_t}\|^2 dv_g &= \int_M \frac{\partial}{\partial t} \|T_{f_t}\|^2 dv_g \\ &= 4 \int_M \sum_i h\left(\nabla_{e_i}\left(dF\left(\frac{\partial}{\partial t}\right)\right), \sigma_{f_i}(e_i)\right) dv_g. \end{aligned}$$

Let $t = 0$ and using integration by parts, we obtain the first variation formula. □

Take a 1-parameter family φ_t ($-\varepsilon < t < \varepsilon$) of diffeomorphisms on M . Let X be the smooth vector field on M corresponding to this 1-parameter family. We have the following first variation formula for $f_t = f \circ \varphi_t$.

Theorem 2 (first variation formula).

$$(7) \quad \left. \frac{d\Phi(f \circ \varphi_t)}{dt} \right|_{t=0} = - \int_M \left\{ \|T_f\|^2 \operatorname{div}_g X - 4 \sum_{i=1}^m h(df(\nabla_{e_i} X), \sigma_f(e_i)) \right\} dv_g,$$

where $\{e_i\}$ denotes a local orthonormal frame on M .

Proof. Theorem 2 follows from the general form of the first variation formula (Theorem 1). Take $\tilde{X} = df(X)$ as a variation vector field X in Theorem 1 for $f_t = f \circ \varphi_t$, and then we have

$$(8) \quad \nabla_{e_i} \tilde{X} = (\nabla_{e_i} df)(X) + df(\nabla_{e_i} X) = (\nabla_X df)(e_i) + df(\nabla_{e_i} X).$$

We calculate $\sum_{i=1}^m h(\nabla_X df)(e_i), \sigma_f(e_i))$ at any fixed point $x_0 \in M$. Using a normal coordinate at x_0 , we have $\nabla_{e_j} e_i = 0$ hence $\nabla_X e_i = 0$ at x_0 , and then we have $(\nabla_X df)(e_i) = \nabla_X(df(e_i))$. Then we get

$$(9) \quad \begin{aligned} &4 \sum_i h(\nabla_{e_i} \tilde{X}, \sigma_f(e_i)) \\ &= 4 \sum_{i=1}^m h(\nabla_X(df(e_i)), \sigma_f(e_i)) + 4 \sum_{i=1}^m h(df(\nabla_{e_i} X), \sigma_f(e_i)). \end{aligned}$$

We calculate $\sum_{i=1}^m h(\nabla_X(df(e_i)), \sigma_f(e_i))$. Let \mathcal{L}_X be the Lie derivative with respect to the vector field X . We have

$$\begin{aligned}
 & 4 \sum_{i=1}^m h(\nabla_X(df(e_i)), \sigma_f(e_i)) \\
 &= 4 \sum_{i,j=1}^m h(\nabla_X(df(e_i)), df(e_j)) T_f(e_i, e_j) \\
 &= 2 \sum_{i,j=1}^m \mathcal{L}_X \{h(df(e_i), df(e_j))\} T_f(e_i, e_j) \\
 &= 2 \sum_{i,j=1}^m \mathcal{L}_X \{h(df(e_i), df(e_j))\} \left\{ h(df(e_i), df(e_j)) - \frac{1}{m} \|df\|^2 g(e_i, e_j) \right\} \\
 (10) \quad &= 2 \sum_{i,j=1}^m \mathcal{L}_X \{h(df(e_i), df(e_j))\} h(df(e_i), df(e_j)) \\
 &\quad - \frac{2}{m} \|df\|^2 \sum_{i,j=1}^m \mathcal{L}_X \{h(df(e_i), df(e_j))\} g(e_i, e_j) \\
 &= \sum_{i,j=1}^m \mathcal{L}_X \{h(df(e_i), df(e_j))^2\} - \frac{2}{m} \|df\|^2 \mathcal{L}_X \|df\|^2 \\
 &= \mathcal{L}_X \left\{ \sum_{i,j=1}^m h(df(e_i), df(e_j))^2 \right\} - \frac{1}{m} \mathcal{L}_X \|df\|^4 \\
 &= \mathcal{L}_X \|f^*h\|^2 - \frac{1}{m} \mathcal{L}_X \|df\|^4 \\
 &= \mathcal{L}_X \left\{ \|f^*h\|^2 - \frac{1}{m} \|df\|^4 \right\} \\
 &= \mathcal{L}_X \|T_f\|^2.
 \end{aligned}$$

Then by (9) and (10), we have

$$(11) \quad 4 \sum_i h(\nabla_{e_i} \tilde{X}, \sigma_f(e_i)) = \mathcal{L}_X \|T_f\|^2 + 4 \sum_{i=1}^m h(df(\nabla_{e_i} X), \sigma_f(e_i))$$

Therefore we get

$$\begin{aligned}
 \frac{d\Phi(f \circ \varphi_t)}{dt} \Big|_{t=0} &= \frac{d\Phi(f_t)}{dt} \Big|_{t=0} \\
 &= \int_M \mathcal{L}_X \|T_f\|^2 dv_g + 4 \int_M \sum_{i=1}^m h(df(\nabla_{e_i} X), \sigma_f(e_i)) dv_g \\
 &= - \int_M \|T_f\|^2 \mathcal{L}_X (dv_g) + 4 \int_M \sum_{i=1}^m h(df(\nabla_{e_i} X), \sigma_f(e_i)) dv_g
 \end{aligned}$$

$$= - \int_M \|T_f\|^2 \operatorname{div}_g X \, dv_g + 4 \int_M \sum_{i=1}^m h(df(\nabla_{e_i} X), \sigma_f(e_i)) \, dv_g.$$

Thus we obtain the conclusion of Theorem 2. □

4. Second variation formula

In this section we give the second variation formula for the functional $\Phi(f)$. Take any smooth deformation F of f with two parameters, i.e., any smooth map

$$F: (-\varepsilon, \varepsilon) \times (-\delta, \delta) \times M \rightarrow N \quad \text{s.t.} \quad F(0, 0, x) = f(x).$$

Let $f_{s,t}(x) = F(s, t, x)$, and we often say a deformation $f_{s,t}(x)$ instead of a deformation $F(s, t, x)$. Let

$$X = dF\left(\frac{\partial}{\partial s}\right)\Big|_{s,t=0}, \quad Y = dF\left(\frac{\partial}{\partial t}\right)\Big|_{s,t=0}$$

denote the variation vector fields of the deformation F . Then we have the following second variation formula.

Theorem 3 (second variation formula).

$$\begin{aligned} \frac{1}{4} \frac{\partial^2 \Phi(f_{s,t})}{\partial s \partial t} \Big|_{s,t=0} &= - \int_M h\left(\operatorname{Hess}_F\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right), \operatorname{div}_g \sigma_f\right) \, dv_g \\ &\quad + \int_M \sum_{i,j} h(\nabla_{e_i} X, \nabla_{e_j} Y) T_f(e_i, e_j) \, dv_g \\ &\quad + \int_M \sum_{i,j} h(\nabla_{e_i} X, df(e_j)) h(\nabla_{e_i} Y, df(e_j)) \, dv_g \\ &\quad + \int_M \sum_{i,j} h(\nabla_{e_i} X, df(e_j)) h(df(e_i), \nabla_{e_j} Y) \, dv_g \\ &\quad - \frac{2}{m} \int_M \sum_i h(\nabla_{e_i} X, df(e_i)) \sum_j h(\nabla_{e_j} Y, df(e_j)) \, dv_g \\ &\quad - \int_M \sum_{i,j} h({}^N R(df(e_i), X)Y, df(e_j)) T_f(e_i, e_j) \, dv_g, \end{aligned}$$

where Hess_f denotes the Hessian of f , i.e., $\operatorname{Hess}_f(Z, W) = (\nabla_Z df)(W) = (\nabla_W df)(Z)$.

REMARK 2. Note that the first term in the right hand side vanishes if f is a stationary map for the functional $\Phi(f)$.

REMARK 3. The last term of the right hand side in Theorem 3 is equal to

$$-\int_M \sum_i h^N R(df(e_i), X)Y, \sigma_f(e_i) dv_g.$$

Proof of Theorem 3. The connection ∇ is trivially extended to a connection on $(-\varepsilon, \varepsilon) \times (-\delta, \delta) \times M$. We use the same notation ∇ for this connection. The frame e_i is also trivially extended to a frame on $(-\varepsilon, \varepsilon) \times (-\delta, \delta) \times$ (the domain of the frame), and denoted by the same notation e_i . Take and fix any point $x_0 \in M$, and we calculate $(\partial^2/(\partial s \partial t))\|f_{s,t}^* h\|^2$ at x_0 for $s = t = 0$ (for simplicity, we abbreviate the notation “ $s = t = 0$ ”). Using a normal coordinate at x_0 , we can assume $\nabla_{e_i} e_j = 0$ for any i, j at x_0 . Since

$$\left[\frac{\partial}{\partial s}, e_i \right] = \left[\frac{\partial}{\partial t}, e_i \right] = 0,$$

we see

$$(12) \quad \nabla_{\partial/\partial s}(dF(e_i)) = \nabla_{e_i} \left(dF \left(\frac{\partial}{\partial s} \right) \right) = \nabla_{e_i} X,$$

$$(13) \quad \nabla_{\partial/\partial t}(dF(e_i)) = \nabla_{e_i} \left(dF \left(\frac{\partial}{\partial t} \right) \right) = \nabla_{e_i} Y.$$

We see

$$(14) \quad \begin{aligned} \frac{\partial^2}{\partial s \partial t} \|T_{f_{s,t}}\|^2 &= \frac{\partial^2}{\partial s \partial t} \sum_{i,j} T_{f_{s,t}}(e_i, e_j)^2 \\ &= 2 \sum_{i,j} \left\{ \frac{\partial^2 T_{f_{s,t}}(e_i, e_j)}{\partial s \partial t} T_f(e_i, e_j) \right\} + 2 \sum_{i,j} \frac{\partial T_{f_{s,t}}(e_i, e_j)}{\partial s} \frac{\partial T_{f_{s,t}}(e_i, e_j)}{\partial t} \\ &\stackrel{\text{def}}{=} I_1 + I_2. \end{aligned}$$

We have

$$(15) \quad \begin{aligned} I_1 &= 2 \sum_{i,j} \frac{\partial^2}{\partial s \partial t} \left\{ h(df_{s,t}(e_i), df_{s,t}(e_j)) - \frac{1}{m} \|df_{s,t}\|^2 g(e_i, e_j) \right\} T_f(e_i, e_j) \\ &= 2 \sum_{i,j} \left\{ \frac{\partial^2}{\partial s \partial t} h(df_{s,t}(e_i), df_{s,t}(e_j)) \right\} T_f(e_i, e_j) - \frac{2}{m} \frac{\partial^2 \|df_{s,t}\|^2}{\partial s \partial t} \sum_{i,j} g(e_i, e_j) T_f(e_i, e_j) \\ &= 2 \sum_{i,j} \left\{ \frac{\partial^2}{\partial s \partial t} h(df_{s,t}(e_i), df_{s,t}(e_j)) \right\} T_f(e_i, e_j) \quad (\text{by Lemma 1 (d)}) \\ &= 2 \sum_{i,j} \left\{ \frac{\partial^2}{\partial s \partial t} h(dF(e_i), dF(e_j)) \right\} T_f(e_i, e_j) \\ &= 4 \sum_{i,j} \{ h(\nabla_{\partial/\partial s} \nabla_{\partial/\partial t}(dF(e_i)), dF(e_j)) \} T_f(e_i, e_j) \\ &\quad + 4 \sum_{i,j} \{ h(\nabla_{\partial/\partial s}(dF(e_i)), \nabla_{\partial/\partial t}(dF(e_j))) \} T_f(e_i, e_j) \quad (\text{by Lemma 1 (a)}). \end{aligned}$$

We get

$$\begin{aligned}
 \nabla_{\partial/\partial s} \nabla_{\partial/\partial t} (dF(e_i)) &= (\nabla_{\partial/\partial s} \nabla_{\partial/\partial t} dF)(e_i) = (\nabla_{\partial/\partial s} \nabla_{e_i} dF) \left(\frac{\partial}{\partial t} \right) \\
 (16) \qquad \qquad \qquad &= (\nabla_{e_i} \nabla_{\partial/\partial s} dF) \left(\frac{\partial}{\partial t} \right) - {}^N R \left(dF(e_i), dF \left(\frac{\partial}{\partial s} \right) \right) dF \left(\frac{\partial}{\partial t} \right) \\
 &= \nabla_{e_i} \text{Hess}_F \left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right) - {}^N R(df(e_i), X)Y.
 \end{aligned}$$

Then by (12), (13), (15) and (16), we have

$$\begin{aligned}
 I_1 &= 4 \sum_{i,j} h \left(\nabla_{e_i} \text{Hess}_F \left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right), df(e_j) \right) T_f(e_i, e_j) \\
 &\quad - 4 \sum_{i,j} h({}^N R(df(e_i), X)Y, df(e_j)) T_f(e_i, e_j) \\
 &\quad + 4 \sum_{i,j} h \left(\nabla_{e_i} \left(dF \left(\frac{\partial}{\partial s} \right) \right), \nabla_{e_j} \left(dF \left(\frac{\partial}{\partial t} \right) \right) \right) T_f(e_i, e_j) \\
 &= 4 \sum_i h \left(\nabla_{e_i} \text{Hess}_F \left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right), \sum_j T_f(e_i, e_j) df(e_j) \right) \\
 &\quad - 4 \sum_{i,j} h({}^N R(df(e_i), X)Y, df(e_j)) T_f(e_i, e_j) \\
 &\quad + 4 \sum_{i,j} h(\nabla_{e_i} X, \nabla_{e_j} Y) T_f(e_i, e_j) \\
 (17) \qquad \qquad \qquad &= 4 \sum_i h \left(\nabla_{e_i} \text{Hess}_F \left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right), \sigma_f(e_i) \right) \\
 &\quad - 4 \sum_{i,j} h({}^N R(df(e_i), X)Y, df(e_j)) T_f(e_i, e_j) \\
 &\quad + 4 \sum_{i,j} h(\nabla_{e_i} X, \nabla_{e_j} Y) T_f(e_i, e_j) \\
 &= 4 \text{div}_g \beta_F - 4h \left(\text{Hess}_F \left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right), \text{div}_g \sigma_f \right) \\
 &\quad - 4 \sum_{i,j} h({}^N R(df(e_i), X)Y, df(e_j)) T_f(e_i, e_j) \\
 &\quad + 4 \sum_{i,j} h(\nabla_{e_i} X, \nabla_{e_j} Y) T_f(e_i, e_j),
 \end{aligned}$$

where

$$\beta_F(X) = h \left(\text{Hess}_F \left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right), \sigma_f(X) \right).$$

On the other hand we have

(18)

$$\begin{aligned}
\mathbf{I}_2 &= 2 \sum_{i,j} \frac{\partial}{\partial s} \left\{ h(df_{s,t}(e_i), df_{s,t}(e_j)) - \frac{1}{m} \|df_{s,t}\|^2 g(e_i, e_j) \right\} \frac{\partial T_{f_{s,t}}(e_i, e_j)}{\partial t} \\
&= 2 \sum_{i,j} \left\{ \frac{\partial}{\partial s} h(dF(e_i), dF(e_j)) \right\} \frac{\partial T_{f_{s,t}}(e_i, e_j)}{\partial t} \\
&\quad - \frac{2}{m} \frac{\partial \|df_{s,t}\|^2}{\partial s} \sum_{i,j} g(e_i, e_j) \frac{\partial T_{f_{s,t}}(e_i, e_j)}{\partial t} \\
&= 2 \sum_{i,j} \left\{ \frac{\partial}{\partial s} h(dF(e_i), dF(e_j)) \right\} \frac{\partial T_{f_{s,t}}(e_i, e_j)}{\partial t} \\
&\quad (\because \sum_{i,j} g(e_i, e_j) \partial T_{f_{s,t}}(e_i, e_j) / \partial t = (\partial / \partial t) (\sum_{i,j} g(e_i, e_j) T_{f_{s,t}}(e_i, e_j)) = 0 \text{ by Lemma 1 (d)}) \\
&= 2 \sum_{i,j} \left\{ \frac{\partial}{\partial s} h(dF(e_i), dF(e_j)) \right\} \frac{\partial}{\partial t} \left\{ h(df_{s,t}(e_i), df_{s,t}(e_j)) - \frac{1}{m} \|df_{s,t}\|^2 g(e_i, e_j) \right\} \\
&= 2 \sum_{i,j} \left\{ \frac{\partial}{\partial s} h(dF(e_i), dF(e_j)) \right\} \left\{ \frac{\partial}{\partial t} h(dF(e_i), dF(e_j)) - \frac{1}{m} \frac{\partial \|df_{s,t}\|^2}{\partial t} g(e_i, e_j) \right\} \\
&= 2 \sum_{i,j} \left\{ \frac{\partial}{\partial s} h(dF(e_i), dF(e_j)) \right\} \left\{ \frac{\partial}{\partial t} h(dF(e_i), dF(e_j)) \right\} \\
&\quad - \frac{2}{m} \sum_{i,j} \left\{ \frac{\partial}{\partial s} h(dF(e_i), dF(e_j)) \right\} g(e_i, e_j) \frac{\partial \|df_{s,t}\|^2}{\partial t} \\
&= 2 \sum_{i,j} \left\{ \frac{\partial}{\partial s} h(dF(e_i), dF(e_j)) \right\} \left\{ \frac{\partial}{\partial t} h(dF(e_i), dF(e_j)) \right\} \\
&\quad - \frac{2}{m} \sum_i \left\{ \frac{\partial}{\partial s} h(dF(e_i), dF(e_i)) \right\} \sum_j \left\{ \frac{\partial}{\partial t} h(dF(e_j), dF(e_j)) \right\} \\
&\quad (\because \partial \|df_{s,t}\|^2 / \partial t = (\partial / \partial t) \sum_j h(df_{s,t}(e_j), df_{s,t}(e_j)) = \sum_j (\partial / \partial t) h(dF(e_j), dF(e_j))) \\
&=: \mathbf{I}_3 + \mathbf{I}_4.
\end{aligned}$$

We have

$$\begin{aligned}
 I_3 &= 2 \sum_{i,j} \{h(\nabla_{\partial/\partial s}(dF(e_i)), dF(e_j)) + h(dF(e_i), \nabla_{\partial/\partial s}(dF(e_j)))\} \\
 &\quad \times \{h(\nabla_{\partial/\partial t}(dF(e_i)), dF(e_j)) + h(dF(e_i), \nabla_{\partial/\partial t}(dF(e_j)))\} \\
 &= 2 \sum_{i,j} h(\nabla_{\partial/\partial s}(dF(e_i)), dF(e_j))h(\nabla_{\partial/\partial t}(dF(e_i)), dF(e_j)) \\
 &\quad + 2 \sum_{i,j} h(\nabla_{\partial/\partial s}(dF(e_i)), dF(e_j))h(dF(e_i), \nabla_{\partial/\partial t}(dF(e_j))) \\
 &\quad + 2 \sum_{i,j} h(dF(e_i), \nabla_{\partial/\partial s}(dF(e_j)))h(\nabla_{\partial/\partial t}(dF(e_i)), dF(e_j)) \\
 &\quad + 2 \sum_{i,j} h(dF(e_i), \nabla_{\partial/\partial s}(dF(e_j)))h(dF(e_i), \nabla_{\partial/\partial t}(dF(e_j))) \\
 (19) \quad &= 4 \sum_{i,j} h(\nabla_{\partial/\partial s}(dF(e_i)), dF(e_j))h(\nabla_{\partial/\partial t}(dF(e_i)), dF(e_j)) \\
 &\quad + 4 \sum_{i,j} h(\nabla_{\partial/\partial s}(dF(e_i)), dF(e_j))h(dF(e_i), \nabla_{\partial/\partial t}(dF(e_j))) \\
 &\quad \text{(by exchanging the indices } i \text{ and } j) \\
 &= 4 \sum_{i,j} h\left(\nabla_{e_i}\left(dF\left(\frac{\partial}{\partial s}\right)\right), dF(e_j)\right)h\left(\nabla_{e_i}\left(dF\left(\frac{\partial}{\partial t}\right)\right), dF(e_j)\right) \\
 &\quad + 4 \sum_{i,j} h\left(\nabla_{e_i}\left(dF\left(\frac{\partial}{\partial s}\right)\right), dF(e_j)\right)h\left(dF(e_i), \nabla_{e_j}\left(dF\left(\frac{\partial}{\partial t}\right)\right)\right) \\
 &= 4 \sum_{i,j} h(\nabla_{e_i} X, df(e_j))h(\nabla_{e_i} Y, df(e_j)) \\
 &\quad + 4 \sum_{i,j} h(\nabla_{e_i} X, df(e_j))h(df(e_i), \nabla_{e_j} Y).
 \end{aligned}$$

On the other hand by (12) and (13), we get

$$\begin{aligned}
 I_4 &= -\frac{8}{m} \sum_i h(\nabla_{\partial/\partial s}(dF(e_i)), dF(e_i)) \sum_j h(\nabla_{\partial/\partial t}(dF(e_j)), dF(e_j)) \\
 (20) \quad &= -\frac{8}{m} \sum_i h\left(\nabla_{e_i}\left(dF\left(\frac{\partial}{\partial s}\right)\right), dF(e_i)\right) \sum_j h\left(\nabla_{e_j}\left(dF\left(\frac{\partial}{\partial t}\right)\right), dF(e_j)\right) \\
 &= -\frac{8}{m} \sum_i h(\nabla_{e_i} X, df(e_i)) \sum_j h(\nabla_{e_j} Y, df(e_j)).
 \end{aligned}$$

Note $(\partial^2/(\partial s \partial t))\Phi(f_{s,t})|_{s,t=0} = \int_M (\partial^2/(\partial s \partial t))\|T_{f_{s,t}}\|^2|_{s,t=0} dv_g$. Integrate (14) over M and use (17), (18), (19) and (20), and then we obtain the second variation formula. \square

5. Quasi-monotonicity formula

In this section we prove a kind of the monotonicity formula for stationary maps. We assume the following weak notion of stationary maps.

DEFINITION 3. Let f be a smooth map from M into N . We call it is *stationary* for $\Phi(f)$ with respect to diffeomorphisms on M if

$$\left. \frac{d}{dt} \Phi(f \circ \varphi_t) \right|_{t=0} = 0$$

for any 1-parameter family φ_t of diffeomorphisms on M .

Note that the notion of stationary maps in Definition 3 is weaker than that of stationary ones in Definition 2, since $f_t(x) = f \circ \varphi_t(x)$ is a deformation in Theorem 1. Under the above weaker condition, we give the following formula.

Theorem 4 (quasi-monotonicity formula). *Let f be stationary for $\Phi(f)$ with respect to diffeomorphisms on M . Let m be the dimension of M . Then it satisfies*

$$\frac{d}{d\rho} \left\{ e^{C_2\rho} \rho^{4-m} \int_{B_\rho(x_0)} \|T_f\|^2 dv_g \right\} \geq 4e^{C_2\rho} \rho^{4-m} (\varphi'(\rho) + C_1\varphi(\rho))$$

where $B_\rho(x_0)$ denotes the open ball of a radius ρ with a center $x_0 \in M$, and C_1, C_2 are constants. Here

$$\varphi(\rho) = \int_{B_\rho(x_0)} h \left(df \left(\frac{\partial}{\partial r} \right), \sigma_f \left(\frac{\partial}{\partial r} \right) \right) dv_g$$

and σ_f is defined by (3).

REMARK 4. If $\varphi(\rho)$ satisfies the condition $\varphi'(\rho) + C_1\varphi(\rho) \geq 0$, then

$$e^{C_2\rho} \rho^{4-m} \int_{B_\rho(x_0)} \|T_f\|^2 dv_g$$

is monotone non-decreasing.

Proof of Theorem 4. We use the argument by Price [4]. (See also Xin [9], p.43.) Let X be a smooth vector field on M , which is supported compactly in $B_r(x_0)$. Take

a 1-parameter family φ_t ($-\varepsilon < t < \varepsilon$) of diffeomorphisms on M corresponding to this vector field. By Theorem 2, we have

$$(21) \quad 0 = \left. \frac{d\Phi(f \circ \varphi_t)}{dt} \right|_{t=0} = - \int_M \left\{ \|T_f\|^2 \operatorname{div}_g X - 4 \sum_{i=1}^m h(df(\nabla_{e_i} X), \sigma_f(e_i)) \right\} dv_g.$$

Let $r = r(x)$ denote the distance function between x_0 and x , and let $\partial/\partial r$ be the gradient vector field of the distance function r . We can take an local orthonormal frame e_i such that $e_m = \partial/\partial r$. We adopt here a smooth vector field

$$X(x) = \xi(r)r \frac{\partial}{\partial r} = \xi(r(x))r(x) \frac{\partial}{\partial r}$$

in a coordinate neighborhood U of x_0 , which vanishes outside U . The function $\xi(r)$ is defined later. We see, for $1 \leq i \leq m - 1$,

$$\nabla_{e_i} \frac{\partial}{\partial r} = \sum_{j=1}^{m-1} \operatorname{Hess}(r)(e_i, e_j) e_j,$$

where $\operatorname{Hess}(r)(X, Y) = (\nabla dr)(X, Y) = \nabla_X(dr(Y)) - dr(\nabla_X Y)$ denotes the Hessian of the function r . Indeed, note $dr(e_j) = g(\partial/\partial r, e_j) = 0$ ($j = 1, \dots, m - 1$) and $g(\partial/\partial r, \partial/\partial r) = 1$, and then we have

$$\begin{aligned} \nabla_{e_i} \frac{\partial}{\partial r} &= \sum_{j=1}^m g\left(\nabla_{e_i} \frac{\partial}{\partial r}, e_j\right) e_j = \sum_{j=1}^{m-1} g\left(\nabla_{e_i} \frac{\partial}{\partial r}, e_j\right) e_j + g\left(\nabla_{e_i} \frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) \frac{\partial}{\partial r} \\ &= - \sum_{j=1}^{m-1} g\left(\frac{\partial}{\partial r}, \nabla_{e_i} e_j\right) e_j = - \sum_{j=1}^{m-1} dr(\nabla_{e_i} e_j) e_j = \sum_{j=1}^{m-1} (\nabla dr)(e_i, e_j) e_j, \end{aligned}$$

since

$$\begin{aligned} 0 &= \nabla_{e_i} \left\{ g\left(\frac{\partial}{\partial r}, e_j\right) \right\} = g\left(\nabla_{e_i} \frac{\partial}{\partial r}, e_j\right) + g\left(\frac{\partial}{\partial r}, \nabla_{e_i} e_j\right), \\ 0 &= \nabla_{e_i} \left\{ g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) \right\} = 2g\left(\nabla_{e_i} \frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right). \end{aligned}$$

We have

$$(22) \quad \nabla_{\partial/\partial r} X = \nabla_{\partial/\partial r} \left(\xi(r)r \frac{\partial}{\partial r} \right) = (\xi(r)r)' \frac{\partial}{\partial r},$$

$$(23) \quad \nabla_{e_i} X = \xi(r)r \nabla_{e_i} \frac{\partial}{\partial r} = \xi(r)r \sum_{j=1}^{m-1} \operatorname{Hess}(r)(e_i, e_j) e_j \quad (1 \leq i \leq m - 1).$$

By the comparison theorem of Hessian, we know

$$(24) \quad \frac{1}{r}g(e_i, e_j)(1 - C_1r) \leq \text{Hess}(r)(e_i, e_j) \leq \frac{1}{r}g(e_i, e_j)(1 + C_1r),$$

where c is a constant which depends on the upper and lower bound of the sectional curvature of M . We calculate $\text{div}_g X$ and $\sum_{i=1}^m h(df(\nabla_{e_i} X), \sigma_f(e_i))$ in the first variation formula (21). By (22), (23) and (24), we have

$$(25) \quad \begin{aligned} \text{div}_g X &= \sum_{i=1}^{m-1} g(\nabla_{e_i} X, e_i) + g\left(\nabla_{\frac{\partial}{\partial r}} X, \frac{\partial}{\partial r}\right) \\ &= \xi(r)r \sum_{i,j=1}^{m-1} \text{Hess}(r)(e_i, e_j)g(e_j, e_i) + (\xi(r)r)' \\ &\geq (m-1)\xi(r)(1 - C_1r) + (\xi(r)r)' \\ &= \xi'(r)r + m\xi(r) - (m-1)c\xi(r)r. \end{aligned}$$

We also get by (22), (23) and (24),

$$(26) \quad \begin{aligned} &\sum_{i=1}^m h(df(\nabla_{e_i} X), \sigma_f(e_i)) \\ &= \sum_{i=1}^{m-1} h(df(\nabla_{e_i} X), \sigma_f(e_i)) + h\left(df(\nabla_{\frac{\partial}{\partial r}} X), \sigma_f\left(\frac{\partial}{\partial r}\right)\right) \\ &= \xi(r)r \sum_{i,j=1}^{m-1} \text{Hess}(r)(e_i, e_j)h(df(e_j), \sigma_f(e_i)) + (\xi(r)r)'h\left(df\left(\frac{\partial}{\partial r}\right), \sigma_f\left(\frac{\partial}{\partial r}\right)\right) \\ &\leq \xi(r)(1 + C_1r) \sum_{i=1}^{m-1} h(df(e_i), \sigma_f(e_i)) + (\xi'(r)r + \xi(r))h\left(df\left(\frac{\partial}{\partial r}\right), \sigma_f\left(\frac{\partial}{\partial r}\right)\right) \\ &= \xi'(r)r h\left(df\left(\frac{\partial}{\partial r}\right), \sigma_f\left(\frac{\partial}{\partial r}\right)\right) \\ &\quad + \xi(r) \left\{ \sum_{i=1}^{m-1} h(df(e_i), \sigma_f(e_i)) + h\left(df\left(\frac{\partial}{\partial r}\right), \sigma_f\left(\frac{\partial}{\partial r}\right)\right) \right\} \\ &\quad + C_1\xi(r)r \sum_{i=1}^{m-1} h(df(e_i), \sigma_f(e_i)) \\ &= \xi'(r)r h\left(df\left(\frac{\partial}{\partial r}\right), \sigma_f\left(\frac{\partial}{\partial r}\right)\right) + \xi(r) \sum_{i=1}^m h(df(e_i), \sigma_f(e_i)) \\ &\quad + C_1\xi(r)r \left\{ \sum_{i=1}^m h(df(e_i), \sigma_f(e_i)) - h\left(df\left(\frac{\partial}{\partial r}\right), \sigma_f\left(\frac{\partial}{\partial r}\right)\right) \right\}. \end{aligned}$$

We have by Lemma 1 (e)

$$\begin{aligned}
 \sum_{i=1}^m h(df(e_i), \sigma_f(e_i)) &= \sum_{i=1}^m h\left(df(e_i), \sum_{j=1}^m T_f(e_i, e_j) df(e_j)\right) \\
 (27) \qquad \qquad \qquad &= \sum_{i=1}^m \sum_{j=1}^m h(df(e_i), df(e_j)) T_f(e_i, e_j) = \|T_f\|^2.
 \end{aligned}$$

For simplicity we set

$$A\left(df, \frac{\partial}{\partial r}\right) := h\left(df\left(\frac{\partial}{\partial r}\right), \sigma_f\left(\frac{\partial}{\partial r}\right)\right).$$

Then by (26), (27), we have

$$\begin{aligned}
 \sum_{i=1}^m h(df(\nabla_{e_i} X), \sigma_f(e_i)) \\
 (28) \qquad \qquad \qquad &\leq \xi'(r)rA\left(df, \frac{\partial}{\partial r}\right) + \xi(r)\|T_f\|^2 + C_1\xi(r)r\left(\|T_f\|^2 - A\left(df, \frac{\partial}{\partial r}\right)\right).
 \end{aligned}$$

Therefore by (21), (25), (28), we get

$$\begin{aligned}
 0 &= \int_M \left\{ \|T_f\|^2 \operatorname{div}_g X - 4 \sum_{i=1}^m h(df(\nabla_{e_i} X), \sigma_f(e_i)) \right\} dv_g \\
 &\geq \int_M \xi'(r)r\|T_f\|^2 dv_g + m \int_M \xi(r)\|T_f\|^2 dv_g \\
 &\quad - (m-1)C_1 \int_M \xi(r)r\|T_f\|^2 dv_g \\
 &\quad - 4 \int_M \xi'(r)rA\left(df, \frac{\partial}{\partial r}\right) dv_g - 4 \int_M \xi(r)\|T_f\|^2 dv_g \\
 &\quad - 4C_1 \int_M \xi(r)r\|T_f\|^2 dv_g + 4C_1 \int_M \xi(r)rA\left(df, \frac{\partial}{\partial r}\right) dv_g,
 \end{aligned}$$

i.e.,

$$\begin{aligned}
 &- \int_M \xi'(r)r\|T_f\|^2 dv_g + (4-m) \int_M \xi(r)\|T_f\|^2 dv_g + C_2 \int_M \xi(r)r\|T_f\|^2 dv_g \\
 (29) \qquad \qquad \qquad &\geq -4 \int_M \xi'(r)rA\left(df, \frac{\partial}{\partial r}\right) dv_g + 4C_1 \int_M \xi(r)rA\left(df, \frac{\partial}{\partial r}\right) dv_g,
 \end{aligned}$$

where $C_2 = (m + 3)C_1$. Take and fix a positive number ε , and let φ be a smooth function on $[0, \infty)$ such that

$$\varphi(r) = \varphi_\varepsilon(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq 1, \\ 0 & \text{if } 1 + \varepsilon \leq r \end{cases}$$

and

$$\varphi'(r) \leq 0.$$

We define

$$\xi(r) = \xi_\rho(r) \stackrel{\text{def}}{=} \varphi\left(\frac{r}{\rho}\right).$$

We can verify

$$(30) \quad \xi'(r)r = -\rho \frac{d}{d\rho} \xi(r).$$

Since $\|T_f\|^2$ is independent of ρ , the above facts (29) and (30) imply

$$\begin{aligned} & \rho \frac{d}{d\rho} \int_M \xi(r) \|T_f\|^2 dv_g + (4 - m) \int_M \xi(r) \|T_f\|^2 dv_g + C_2 \int_M \xi(r)r \|T_f\|^2 dv_g \\ & \geq 4\rho \frac{d}{d\rho} \int_M A\left(df, \frac{\partial}{\partial r}\right) \xi(r) dv_g + 4C_1\rho \int_M A\left(df, \frac{\partial}{\partial r}\right) \xi(r) dv_g. \end{aligned}$$

Let ε tend to zero, and then, since $\xi(r)$ converges to the characteristic function for the ball $B_\rho(x_0)$, we have

$$\begin{aligned} & \rho \frac{d}{d\rho} \int_{B_\rho(x_0)} \|T_f\|^2 dv_g + (4 - m) \int_{B_\rho(x_0)} \|T_f\|^2 dv_g + C_2\rho \int_{B_\rho(x_0)} \|T_f\|^2 dv_g \\ & \geq 4\rho \frac{d}{d\rho} \int_{B_\rho(x_0)} A\left(df, \frac{\partial}{\partial r}\right) dv_g + 4C_1\rho \int_{B_\rho(x_0)} A\left(df, \frac{\partial}{\partial r}\right) dv_g. \end{aligned}$$

Multiply $e^{C_2\rho} \rho^{3-m}$ to the both sides of this inequality, and we have

$$\begin{aligned} & \frac{d}{d\rho} \left\{ e^{C_2\rho} \rho^{4-m} \int_{B_\rho(x_0)} \|T_f\|^2 dv_g \right\} \\ & \geq 4e^{C_2\rho} \rho^{4-m} \left\{ \frac{d}{d\rho} \int_M A\left(df, \frac{\partial}{\partial r}\right) dv_g + C_1 \int_M A\left(df, \frac{\partial}{\partial r}\right) dv_g \right\}. \end{aligned}$$

Thus we obtain the formula. □

6. Bochner type formula

In this section we prove the following formula.

Theorem 5 (Bochner type formula). *For any smooth map f from M into N , the following equality holds:*

$$\begin{aligned}
 \frac{1}{4} \Delta \|T_f\|^2 &= \operatorname{div} \alpha_f - h(\tau_f, \operatorname{div} \sigma_f) + \frac{1}{2} \|\nabla T_f\|^2 \\
 &+ \sum_{i,j,k} h((\nabla_{e_k} df)(e_i), (\nabla_{e_k} df)(e_j)) T_f(e_i, e_j) \\
 (31) \quad &+ \sum_{i,j} h\left(df\left(\sum_k {}^M R(e_i, e_k)e_k\right), df(e_j)\right) T_f(e_i, e_j) \\
 &- \sum_{i,j,k} h({}^N R(df(e_i), df(e_k)) df(e_k), df(e_j)) T_f(e_i, e_j)
 \end{aligned}$$

where

$$\alpha_f(X) = h(\sigma_f(X), \tau_f).$$

Here σ_f is defined by (3), and $\tau_f = \operatorname{tr}(\nabla df) = \sum_j (\nabla_{e_j} df)(e_j)$ is the tension field of f in the harmonic map theory. (See Eells and Lemaire [2], p. 9.)

REMARK 5. Note that the first term in the right hand side is of divergence form, and hence the integral of it over M vanishes.

REMARK 6. Note that the second term in the right hand side vanishes if f is a stationary map for the functional $\Phi(f)$.

REMARK 7. The last two terms of the right hand side in Theorem 5 are equal to

$$\begin{aligned}
 &+ \sum_i h\left(df\left(\sum_k {}^M R(e_i, e_k)e_k\right), \sigma_f(e_i)\right) \\
 &- \sum_{i,k} h({}^N R(df(e_i), df(e_k)) df(e_k), \sigma_f(e_i))
 \end{aligned}$$

respectively.

Proof of Theorem 5. We have

$$\begin{aligned}
 \Delta \|T_f\|^2 &= \Delta \sum_{i,j} T_f(e_i, e_j)^2 \\
 (32) \quad &= 2 \sum_{i,j} (\Delta T_f)(e_i, e_j) T_f(e_i, e_j) + 2 \sum_{i,j} \sum_k (\nabla_{e_k} T_f)(e_i, e_j)^2 \\
 &\stackrel{\text{def}}{=} I_1 + I_2.
 \end{aligned}$$

We get

$$\begin{aligned}
I_1 &= 2 \sum_{i,j} (\Delta T_f)(e_i, e_j) T_f(e_i, e_j) \\
&= 2 \sum_{i,j} \left\{ h((\Delta df)(e_i), df(e_j)) + 2 \sum_k h((\nabla_{e_k} df)(e_i), (\nabla_{e_k} df)(e_j)) \right. \\
&\quad \left. + h(df(e_i), (\Delta df)(e_j)) - \frac{1}{m} \Delta \|df\|^2 g(e_i, e_j) \right\} T_f(e_i, e_j) \\
&= 4 \sum_{i,j} h((\Delta df)(e_i), df(e_j)) T_f(e_i, e_j) \\
&\quad + 4 \sum_{i,j,k} h((\nabla_{e_k} df)(e_i), (\nabla_{e_k} df)(e_j)) T_f(e_i, e_j) \quad (\text{by Lemma 1 (a) and (d)}).
\end{aligned}$$

Since by Ricci formula,

$$\begin{aligned}
(\Delta df)(e_i) &= \sum_k (\nabla_{e_k} \nabla_{e_k} df)(e_i) = \sum_k (\nabla_{e_k} \nabla_{e_i} df)(e_k) \\
&= \sum_k (\nabla_{e_i} \nabla_{e_k} df)(e_k) + df \left(\sum_k {}^M R(e_i, e_k) e_k \right) \\
&\quad - \sum_k {}^N R(df(e_i), df(e_k)) df(e_k) \\
&= \nabla_{e_i} \tau_f + df \left(\sum_k {}^M R(e_i, e_k) e_k \right) - \sum_k {}^N R(df(e_i), df(e_k)) df(e_k),
\end{aligned}$$

we have

$$\begin{aligned}
(33) \quad I_1 &= 4 \sum_{i,j} h(\nabla_{e_i} \tau_f, df(e_j)) T_f(e_i, e_j) \\
&\quad + 4 \sum_{i,j} h \left(df \left(\sum_k {}^M R(e_i, e_k) e_k \right), df(e_j) \right) T_f(e_i, e_j) \\
&\quad - 4 \sum_{i,j,k} h({}^N R(df(e_i), df(e_k)) df(e_k), df(e_j)) T_f(e_i, e_j) \\
&\quad + 4 \sum_{i,j,k} h((\nabla_{e_k} df)(e_i), (\nabla_{e_k} df)(e_j)) T_f(e_i, e_j).
\end{aligned}$$

Furthermore we get

$$\begin{aligned}
 \sum_{i,j} h(\nabla_{e_i} \tau_f, df(e_j)) T_f(e_i, e_j) &= \sum_i h\left(\nabla_{e_i} \tau_f, \sum_j T_f(e_i, e_j) df(e_j)\right) \\
 (34) \qquad \qquad \qquad &= \sum_i h(\nabla_{e_i} \tau_f, \sigma_f(e_i)) \\
 &= \sum_i \operatorname{div}_g \alpha_f - \sum_i h(\tau_f, \operatorname{div}_g \sigma_f).
 \end{aligned}$$

By (32), (33) and (34), we obtain Theorem 5, since $I_2 = 2\|\nabla T_f\|^2$. □

7. Minimizers in homotopy classes of the Sobolev space

In this section we utilize the notion of 3-homotopy in the Sobolev spaces, which is given by White, and consider a variational problem of minimizing the functional $\Phi(f)$ in each 3-homotopy class. For any two maps f and g from M into N , these maps are k -homotopic ($k \in \mathbb{N}$) if they are homotopic to each other on k -dimensional skeletons of a triangulation on M . By Nash’s isometric embedding, we may assume that N is a submanifold of a Euclidean space \mathbb{R}^q . Let

$$L^{1,p}(M, N) = \{f \in L^{1,p}(M, \mathbb{R}^q) \mid f(x) \in N \text{ a.e.}\},$$

where $L^{1,p}(M, \mathbb{R}^q)$ denotes the Sobolev space of \mathbb{R}^q -valued L^p -functions on M such that their derivatives are in L^p . Then White proved that the notion of the $[p - 1]$ -homotopy is compatible with the Sobolev space $L^{1,p}(M, N)$, where $[]$ denotes the Gauss symbol, i.e., $[r]$ is the maximum integer less than or equal to r .

Theorem S (Theorem 3.4 in White [8]. See also White [7], Schoen and Uhlenbeck [5] and Bethuel [1]).

- (1) *The $[p - 1]$ -homotopy is well-defined for any map $f \in L^{1,p}(M, N)$.*
- (2) *If f_j converges weakly to f_∞ in $L^{1,p}(M, N)$, then f_j and f_∞ are $[p - 1]$ -homotopic for sufficient large j .*

The functional $\Phi(f)$ is defined on $L^{1,4}(M, N)$, in which the 3-homotopy is well-defined. Then for any given continuous map f_0 from M into N , we want to minimize the functional $\Phi(f)$ in the following class:

$$\mathcal{F} = \{f \in L^{1,4}(M, N) \mid f \text{ is 3-homotopic to } f_0 \text{ and } \|f\|_{L^{1,4}(M,N)} \leq C_0\},$$

where C_0 is a given positive constant. We may assume that the space \mathcal{F} is not empty for sufficiently large C_0 .

Theorem 6. *There exists a minimizer of the functional $\Phi(f)$ in \mathcal{F} .*

If a 3-homotopy class contains a conformal map, then the conformal map is a minimizer. Minimizers are expected to be *closest* to conformal maps, even if its 3-homotopy class does not contain any conformal map.

REMARK 8. When M is 4-dimensional and $\pi_4(N) = 0$, any *continuous* minimizer is (freely) homotopic to f_0 in the ordinary sense.

Proof of Theorem 6. Take any minimizing sequence f_j for the functional $\Phi(f)$ in the space \mathcal{F} , i.e., $\Phi(f_j)$ converges to the infimum in \mathcal{F} . We may assume that f_j converges weakly to a map f_∞ in $L^{1,4}(M, N)$, since $\|f\|_{L^{1,4}(M,N)} \leq C_0$. Since the weak convergence in $L^{1,4}(M, N)$ preserves the 3-homotopy by Theorem S (2), f_∞ is 3-homotopic to f_j for sufficiently large j , hence to f_0 . Furthermore T_{f_j} converges weakly to T_{f_∞} in L^2 , since for any covariant 2-tensor S ,

$$\begin{aligned} \int_M \langle T_{f_j}, S \rangle dv_g &= \int_M \left\langle f_j^* h - \frac{1}{m} \|df_j\|^2 g, S \right\rangle dv_g \\ &= \int_M \left\langle f_j^* h, S - \frac{1}{m} \langle g, S \rangle g \right\rangle dv_g, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the pointwise pairing for covariant 2-tensors. Therefore we have

$$\Phi(f_\infty) = \|T_{f_\infty}\|_{L^2} \leq \liminf_{j \rightarrow \infty} \|T_{f_j}\|_{L^2} = \liminf_{j \rightarrow \infty} \Phi(f_j).$$

Then f_∞ is a minimizer in \mathcal{F} . □

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References

- [1] F. Bethuel: *The approximation problem for Sobolev maps between two manifolds*, Acta Math. **167** (1991), 153–206.
- [2] J. Eells and L. Lemaire: *A report on harmonic maps*, Bull. London Math. Soc. **10** (1978), 1–68.
- [3] J. Eells and L. Lemaire: *Another report on harmonic maps*, Bull. London Math. Soc. **20** (1988), 385–524.
- [4] P. Price: *A monotonicity formula for Yang–Mills fields*, Manuscripta Math. **43** (1983), 131–166.
- [5] R. Schoen and K. Uhlenbeck: *Approximation theorems for Sobolev mappings*, unpublished.
- [6] H. Urakawa: *Calculus of Variations and Harmonic Maps*, Translations of Mathematical Monographs **132**, Amer. Math. Soc., Providence, RI, 1993.
- [7] B. White: *Infima of energy functionals in homotopy classes of mappings*, J. Differential Geom. **23** (1986), 127–142.
- [8] B. White: *Homotopy classes in Sobolev spaces and the existence of energy minimizing maps*, Acta Math. **160** (1988), 1–17.

- [9] Y. Xin: *Geometry of Harmonic Maps*, Birkhäuser, Boston, MA, 1996.

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