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<td>Author(s)</td>
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<td>Citation</td>
<td>Osaka Journal of Mathematics. 48(3) P.719-P.741</td>
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<tr>
<td>Issue Date</td>
<td>2011-09</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/5960">https://doi.org/10.18910/5960</a></td>
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Osaka University
A VARIATIONAL PROBLEM RELATED TO CONFORMAL MAPS

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(Received June 12, 2009, revised March 8, 2010)

Abstract

In this paper we are concerned with a variational problem for a functional related to the conformality of maps between Riemannian manifolds. We give the first variation formula, the second variation formula, a kind of the monotonicity formula and a Bochner type formula. We also consider a variational problem of minimizing the functional in each 3-homotopy class of the Sobolev space.

1. Introduction

Let \((M, g), (N, h)\) be compact Riemannian manifolds without boundary. A smooth map \(f\) from \(M\) into \(N\) is called a conformal map if there exists a positive function \(\varphi\) on \(M\) such that \(f^* h = \varphi g\), where \(f^* h\) denotes the pullback of the metric \(h\) by \(f\), i.e.,

\[(f^* h)(X, Y) = h(df(X), df(Y)).\]

We consider a covariant symmetric tensor

\[T_f := f^* h - \frac{1}{m} \| df \|^2 g\]

where \(m\) denotes the dimension of the manifold \(M\), and \(\| df \|^2\) denotes the energy density in the harmonic map theory, i.e.,

\[\| df \|^2 = \sum_i h(df(e_i), df(e_i)).\]

\((e_i\) denotes a local orthonormal frame on \(M\).) Then \(f\) is conformal at \(x\) if and only if \(T_f = 0\) at this point, unless \((df)_x = 0\). In this paper we are concerned with the functional

\[\Phi(f) = \int_M \| T_f \|^2 dv_g,\]

2000 Mathematics Subject Classification. 58E99, 58E20, 53C43.
where $dv_g$ denotes the volume form of $(M, g)$, and

$$\|T_f\|^2 = \sum_{i,j} T_f(e_i, e_j)^2.$$ 

The functional $\Phi(f)$ gives a quantity of the conformality of $f$. We give the first variation formula and the second variation formula for this functional. We also prove a kind of the monotonicity formula and a Bochner type formula. Furthermore we want to minimize the functional $\Phi(f)$ in each homotopy class of maps from $M$ into $N$. Minimizers are expected to be closest to conformal maps, even if its homotopy class does not contain any conformal map. To this aim, we adopt the notion of 3-homotopy in the Sobolev spaces, which is given by White. We consider a variational problem of minimizing the functional $\Phi(f)$ in each 3-homotopy class of the Sobolev space.

2. The tensor $T_f$ of the conformality and the functional $\Phi(f)$

Let $(M, g), (N, h)$ be compact Riemannian manifolds without boundary and let $f$ be a smooth map from $M$ into $N$. In this section we give a tensor $T_f$ of the conformality for any smooth map $f$. We recall here the following two notions.

DEFINITION 1. (i) A smooth map $f$ is weakly conformal if there exists a non-negative function $\varphi$ on $M$ such that

$$f^*h = \varphi g$$

where $f^*h$ denotes the pullback of the metric $h$ by $f$, i.e.,

$$(f^*h)(X, Y) = h(df(X), df(Y)).$$

(ii) A smooth map $f$ is conformal if there exists a positive function $\varphi$ on $M$ satisfying (1).

The condition (1) is equivalent to

$$f^*h = \frac{1}{m}\|df\|^2g,$$

since taking the trace of the both sides of (1) (with respect to the metric $g$), we have $\|df\|^2 = m\varphi$, i.e., $\varphi = (1/m)\|df\|^2$. Then $f$ is conformal if and only if it satisfies (2) with the assumption $\|df\| \neq 0$. Note that $f$ is weakly conformal if and only if for any point $x \in M$, $f$ is conformal at $x$ or $df_x = 0$. 

Taking the above situation into consideration, we utilize the covariant tensor

\[ T_f \overset{\text{def}}{=} f^* h - \frac{1}{m} \|df\|^2 g, \]

i.e.,

\[ T_f(X, Y) \overset{\text{def}}{=} (f^* h)(X, Y) - \frac{1}{m} \|df\|^2 g(X, Y) \]

\[ = h(df(X), df(Y)) - \frac{1}{m} \|df\|^2 g(X, Y). \]

**Remark 1.** In the case of \( m = 2 \), the tensor \( T_f \) is equal to the stress energy tensor

\[ S_f = f^* h - \frac{1}{2} \|df\|^2 g \]

in the harmonic map theory. (See Eells and Lemaire [3], p. 392.)

**Lemma 1.** (a) \( T_f \) is symmetric, i.e., \( T_f(X, Y) = T_f(Y, X) \).
(b) \( f \) is weakly conformal if and only if \( T_f = 0 \).
(c) \( \|T_f\|^2 = \|f^* h\|^2 - (1/m) \|df\|^4 \).
(d) \( T_f \) is trace-free, i.e.,

\[ \text{Trace}_g T_f = \sum_{i,j} g(e_i, e_j) T_f(e_i, e_j) = 0, \]

where \( e_i \) denotes a local orthonormal frame on \( M \).
(e) The trace of \( T_f \) with respect to the pullback \( f^* h \) is equal to the norm of \( T_f \), i.e.,

\[ \text{Trace}_{f^* h} T_f = \sum_{i,j} (f^* h)(e_i, e_j) T_f(e_i, e_j) = \|T_f\|^2. \]

**Proof.** (a) follows directly from the definition of \( T_f \).
(b): The argument mentioned above implies that \( f \) is a weakly conformal map if and only if \( f^* h = (1/m) \|df\|^2 g \), which is equivalent to the condition \( T_f = 0 \).
(c): \[
\|T_f\|^2 = \sum_{i,j} T_f(e_i, e_j)^2
= \sum_{i,j} \left\{ h(df(e_i), df(e_j)) - \frac{1}{m} \|df\|^2 g(e_i, e_j) \right\}^2
\]
\[
\sum_{i,j} h(df(e_i), df(e_j))^2 \\
- \frac{2}{m} \|df\|^2 \sum_{i,j} h(df(e_i), df(e_j)) g(e_i, e_j) + \frac{1}{m^2} \|df\|^4 \sum_{i,j} g(e_i, e_j)^2 \\
= \|f^* h\|^2 - \frac{2}{m} \|df\|^4 + \frac{1}{m} \|df\|^4 \\
= \|f^* h\|^2 - \frac{1}{m} \|df\|^4.
\]

(d):
\[
\text{Trace}_g T_f = \sum_{i,j} g(e_i, e_j) T_f(e_i, e_j) \\
= \sum_{i,j} g(e_i, e_j) \left\{ h(df(e_i), df(e_j)) - \frac{1}{m} \|df\|^2 g(e_i, e_j) \right\} \\
= \sum_{i,j} g(e_i, e_j) h(df(e_i), df(e_j)) - \frac{1}{m} \|df\|^2 \sum_{i,j} g(e_i, e_j)^2 \\
= \|df\|^2 - \|df\|^2 \\
= 0.
\]

e):
\[
\text{Trace}_{f^* h} T_f = \sum_{i,j}(f^* h)(e_i, e_j) T_f(e_i, e_j) \\
= \sum_{i,j} h(df(e_i), df(e_j)) T_f(e_i, e_j) \\
= \sum_{i,j} h(df(e_i), df(e_j)) \left\{ h(df(e_i), df(e_j)) - \frac{1}{m} \|df\|^2 g(e_i, e_j) \right\} \\
= \sum_{i,j} h(df(e_i), df(e_j))^2 - \frac{1}{m} \|df\|^2 \sum_{i,j} h(df(e_i), df(e_j)) g(e_i, e_j) \\
= \|f^* h\|^2 - \frac{1}{m} \|df\|^4 \\
= \|T_f\|^2 \quad \text{(by (c))}.
\]

Thus we obtain Lemma 1. \qed

In this paper, we are concerned with the functional
\[
\Phi(f) = \int_M \|T_f\|^2 d\nu_g.
\]
This functional $\Phi(f)$ gives a quantity of the conformality of maps $f$. Note that if $f$ is a conformal map, then $\Phi(f)$ vanishes.

3. First variation formula

In this section we give the first variation formula for the functional $\Phi(f)$. We define an “$f^{-1}TN$-valued” 1-form $\sigma_f$ on $M$ by

$$
\sigma_f(X) = \sum_j T_f(X, e_j) df(e_j)
$$

(3)

$$
= \sum_j h(df(X), df(e_j)) df(e_j) - \frac{1}{n} \|df\|^2 df(X)
$$

for any vector field $X$ on $M$, where $e_j$ denotes a local orthonormal frame on $M$. The 1-form $\sigma_f$ plays an important role in our arguments.

Take any smooth deformation $F$ of $f$, i.e., any smooth map

$F: (-e, e) \times M \to N$ s.t. $F(0, x) = f(x)$.

Let $f_t(x) = F(t, x)$, and we often say a deformation $f_t(x)$ instead of a deformation $F(t, x)$. Let

$$
X = \left. dF \left( \frac{\partial}{\partial t} \right) \right|_{t=0}
$$

denote the variation vector fields of the deformation $F$. Then we have the following first variation formula.

**Theorem 1** (first variation formula).

$$
\left. \frac{d\Phi(f_t)}{dt} \right|_{t=0} = -4 \int_M h(X, \text{div}_g \sigma_f) dv_g,
$$

where $dv_g$ denotes the volume form on $M$, and $\text{div}_g \sigma_f$ denotes the divergence of $\sigma_f$, i.e., $\text{div}_g \sigma_f = \sum_{i=1}^m (\nabla_e \sigma_f)(e_i)$.

We give here the notion of stationary maps for the functional $\Phi(f)$.

**Definition 2.** We call a smooth map $f$ stationary (for the functional $\Phi(f)$) if the first variation of $\Phi(f)$ identically vanishes, i.e.,

$$
\left. \frac{d\Phi(f_t)}{dt} \right|_{t=0} = 0
$$
for any smooth deformation \( f_t \) of \( f \). By Theorem 1, a smooth map \( f \) is stationary for \( \Phi(f) \) if and only if it satisfies the equation

\[
(4) \quad \text{div}_x \sigma_f = 0,
\]

where \( \sigma_f \) is the covariant tensor defined by (3). It is the Euler–Lagrange equation for the functional \( \Phi(f) \).

Proof of Theorem 1. We calculate \((\partial/\partial t)\|f_t^*h\|^2\) at any fixed point \( x_0 \in M \). The connection \( \nabla \) is trivially extended to a connection on \((-\varepsilon, \varepsilon) \times M\). We use the same notation \( \nabla \) for this connection. The frame \( e_i \) is also trivially extended to a frame on \((-\varepsilon, \varepsilon) \times \) (the domain of the frame), and we use the same notation \( e_i \). By a normal coordinate at \( x_0 \), we can assume \( \nabla_{e_i} e_j = 0 \) for any \( i, j \) at \( x_0 \).

Since \((dF)_{t,x}(e_i)(t,x) = (df_t)_x((e_i)_x)\), we denote them by \( dF(e_i) \) simply. Note that

\[
(5) \quad \nabla_{\partial/\partial t}(dF(e_i)) = (\nabla_{\partial/\partial t} dF)(e_i) = (\nabla_{e_i} dF)\left(\frac{\partial}{\partial t}\right) = \nabla_{e_i}\left(dF\left(\frac{\partial}{\partial t}\right)\right).
\]

Since \([\partial/\partial t, e_i] = 0\). Then we have

\[
\frac{\partial}{\partial t}\left\|T_{f_t}\right\|^2 = \frac{\partial}{\partial t}\sum_{i,j} T_{f_t}(e_i, e_j)^2
\]

\[
= 2\sum_{i,j} \frac{\partial T_{f_t}(e_i, e_j)}{\partial t} T_{f_t}(e_i, e_j)
\]

\[
= 2\sum_{i,j} \left\{ \frac{\partial}{\partial t} h(df_t(e_i), df_t(e_j)) - \frac{1}{m} \frac{\partial \|df_t\|^2}{\partial t} g(e_i, e_j) \right\} T_{f_t}(e_i, e_j)
\]

\[
= 2\sum_{i,j} \left\{ \frac{\partial}{\partial t} h(df_t(e_i), df_t(e_j)) \right\} T_{f_t}(e_i, e_j) - \frac{2}{n} \frac{\partial \|df_t\|^2}{\partial t} \sum_{i,j} g(e_i, e_j) T_{f_t}(e_i, e_j)
\]

\[
= 2\sum_{i,j} \left\{ \frac{\partial}{\partial t} h(df_t(e_i), df_t(e_j)) \right\} T_{f_t}(e_i, e_j) \quad \text{(by Lemma 1 (d))}
\]

\[
= 2\sum_{i,j} \left\{ \frac{\partial}{\partial t} h(dF(e_i), dF(e_j)) \right\} T_{f_t}(e_i, e_j)
\]

\[
= 4\sum_{i,j} h(\nabla_{\partial/\partial t}(dF(e_i)), dF(e_j)) T_{f_t}(e_i, e_j) \quad \text{(by Lemma 1 (a))}
\]

\[
= 4\sum_{i,j} h\left(\nabla_{e_i}\left(dF\left(\frac{\partial}{\partial t}\right)\right), df_t(e_j)\right) T_{f_t}(e_i, e_j) \quad \text{(by (5))}
\]

\[
= 4\sum_{i,j} h\left(\nabla_{e_i}\left(dF\left(\frac{\partial}{\partial t}\right)\right), \sum_j T_{f_t}(e_i, e_j) df_t(e_j)\right)
\]

\((\ast) \quad h(A, B) T_{f_t}(C, D) = h(A, T_{f_t}(C, D) B))\)
\[ = 4 \sum_i h\left( \nabla e_i \left( dF\left( \frac{\partial}{\partial t} \right) \right), \sigma_f(e_i) \right). \]

Thus we obtain

\[ \frac{\partial}{\partial t} \|T_f\|^2 = 4 \sum_i h\left( \nabla e_i \left( dF\left( \frac{\partial}{\partial t} \right) \right), \sigma_f(e_i) \right). \]  

Integrate the both sides of (6) on \( M \), and then we have

\[
\frac{d}{dt} \int_M \|T_f\|^2 \, dv_g = \int_M \frac{\partial}{\partial t} \|T_f\|^2 \, dv_g \\
= 4 \int_M \sum_i h\left( \nabla e_i \left( dF\left( \frac{\partial}{\partial t} \right) \right), \sigma_f(e_i) \right) \, dv_g.
\]

Let \( t = 0 \) and using integration by parts, we obtain the first variation formula. \( \square \)

Take a 1-parameter family \( \varphi_t (\varepsilon < t < \varepsilon) \) of diffeomorphisms on \( M \). Let \( X \) be the smooth vector field on \( M \) corresponding to this 1-parameter family. We have the following first variation formula for \( f_t = f \circ \varphi_t \).

**Theorem 2** (first variation formula).

\[ \frac{d}{dt} \Phi(f \circ \varphi_t) \bigg|_{t=0} = - \int_M \left( \|T_f\|^2 \, \text{div}_g X - 4 \sum_{i=1}^m h(df(\nabla e_i, X), \sigma_f(e_i)) \right) \, dv_g, \]

where \( \{e_i\} \) denotes a local orthonormal frame on \( M \).

Proof. Theorem 2 follows from the general form of the first variation formula (Theorem 1). Take \( \tilde{X} = df(X) \) as a variation vector field \( X \) in Theorem 1 for \( f_t = f \circ \varphi_t \), and then we have

\[ \nabla e_i \tilde{X} = (\nabla e_i df)(X) + df(\nabla e_i, X) = (\nabla_X df)(e_i) + df(\nabla e_i, X). \]

We calculate \( \sum_{i=1}^m h(\nabla_X df)(e_i), \sigma_f(e_i)) \) at any fixed point \( x_0 \in M \). Using a normal coordinate at \( x_0 \), we have \( \nabla e_i e_i = 0 \) hence \( \nabla_X e_i = 0 \) at \( x_0 \), and then we have \( \nabla_X df)(e_i) = \nabla_X(df(e_i)). \) Then we get

\[ 4 \sum_i h(\nabla e_i \tilde{X}, \sigma_f(e_i)) \]

\[ = 4 \sum_{i=1}^m h(\nabla_X(df(e_i)), \sigma_f(e_i)) + 4 \sum_{i=1}^m h(df(\nabla e_i, X), \sigma_f(e_i)). \]
We calculate \( \sum_{j=1}^{m} h(\nabla_X(df(e_i)), \sigma_f(e_i)) \). Let \( \mathcal{L}_X \) be the Lie derivative with respect to the vector field \( X \). We have

\[
4 \sum_{j=1}^{m} h(\nabla_X(df(e_i)), \sigma_f(e_i)) \\
= 4 \sum_{i,j=1}^{m} h(\nabla_X(df(e_i)), df(e_j)) T_f(e_i, e_j) \\
= 2 \sum_{i,j=1}^{m} \mathcal{L}_X[h(df(e_i), df(e_j))] T_f(e_i, e_j) \\
= 2 \sum_{i,j=1}^{m} \mathcal{L}_X[h(df(e_i), df(e_j))] \left\{ h(df(e_i), df(e_j)) - \frac{1}{m} \| df \|^2 g(e_i, e_j) \right\} \\
= 2 \sum_{i,j=1}^{m} \mathcal{L}_X[h(df(e_i), df(e_j))] \cdot h(df(e_i), df(e_j)) - \frac{2}{m} \| df \|^2 \mathcal{L}_X \| df \|^2 \\
= \sum_{i,j=1}^{m} \mathcal{L}_X\left\{ \sum_{i,j=1}^{m} h(df(e_i), df(e_j))^2 \right\} - \frac{2}{m} \mathcal{L}_X \| df \|^4 \\
= \mathcal{L}_X \| f^* h \|^2 - \frac{1}{m} \mathcal{L}_X \| df \|^4 \\
= \mathcal{L}_X \left\{ \| f^* h \|^2 - \frac{1}{m} \| df \|^4 \right\} \\
= \mathcal{L}_X \| T_f \|^2.
\]

Then by (9) and (10), we have

\[
4 \sum_{i} h(\nabla e_i, \bar{X}, \sigma_f(e_i)) = \mathcal{L}_X \| T_f \|^2 + 4 \sum_{i=1}^{m} h(df(\nabla e_i, X), \sigma_f(e_i))
\]

Therefore we get

\[
\frac{d\Phi(f \circ \varphi_t)}{dt} \bigg|_{t=0} = \left. \frac{d\Phi(f_t)}{dt} \right|_{t=0} \\
= \int_M \mathcal{L}_X \| T_f \|^2 \, dv_g + 4 \int_M \sum_{i=1}^{m} h(df(\nabla e_i, X), \sigma_f(e_i)) \, dv_g \\
= - \int_M \| T_f \|^2 \mathcal{L}_X(dv_g) + 4 \int_M \sum_{i=1}^{m} h(df(\nabla e_i, X), \sigma_f(e_i)) \, dv_g
\]
\[ -\int_M \| T_f \|^2 \, \text{div}_g X \, dv_g + 4 \int_M \sum_{i=1}^m h(df(\nabla_{e_i}X), \sigma_f(e_i)) \, dv_g. \]

Thus we obtain the conclusion of Theorem 2.

4. Second variation formula

In this section we give the second variation formula for the functional \( \Phi(f) \). Take any smooth deformation \( F \) of \( f \) with two parameters, i.e., any smooth map \( F: (-\varepsilon, \varepsilon) \times (-\delta, \delta) \times M \to N \) s.t. \( F(0,0,x) = f(x) \).

Let \( f_{s,t}(x) = F(s,t,x) \), and we often say a deformation \( f_{s,t}(x) \) instead of a deformation \( F(s,t,x) \). Let

\[ X = dF\left( \frac{\partial}{\partial s} \right) \bigg|_{s,t=0}, \quad Y = dF\left( \frac{\partial}{\partial t} \right) \bigg|_{s,t=0} \]

denote the variation vector fields of the deformation \( F \). Then we have the following second variation formula.

**Theorem 3** (second variation formula).

\[
\frac{1}{4} \frac{\partial^2 \Phi(f_{s,t})}{\partial s \partial t} \bigg|_{s,t=0} = -\int_M h \left( \text{Hess}_F \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right), \text{div}_g \sigma_f \right) \, dv_g \\
+ \int_M \sum_{i,j} h(\nabla_{e_i}X, \nabla_{e_j}Y)T_f(e_i, e_j) \, dv_g \\
+ \int_M \sum_{i,j} h(\nabla_{e_i}X, df(e_j))h(\nabla_{e_j}Y, df(e_j)) \, dv_g \\
+ \int_M \sum_{i,j} h(\nabla_{e_i}X, df(e_j))h(df(e_i), \nabla_{e_j}Y) \, dv_g \\
- \frac{2}{m} \int_M \sum_{i,j} h(\nabla_{e_i}X, df(e_i)) \sum_j h(\nabla_{e_j}Y, df(e_j)) \, dv_g \\
- \int_M \sum_{i,j} h(R(df(e_i), X)Y, df(e_j))T_f(e_i, e_j) \, dv_g,
\]

where \( \text{Hess}_f \) denotes the Hessian of \( f \), i.e., \( \text{Hess}_f(Z,W) = (\nabla_Z df)(W) = (\nabla_W df)(Z) \).

**Remark 2.** Note that the first term in the right hand side vanishes if \( f \) is a stationary map for the functional \( \Phi(f) \).
Remark 3. The last term of the right hand side in Theorem 3 is equal to
\[- \int_M \sum_i h(NR(df(e_i), X), \sigma_j(e_i)) \, dv_x.\]

Proof of Theorem 3. The connection \(\nabla\) is trivially extended to a connection on \((-\varepsilon, \varepsilon) \times (-\delta, \delta) \times M\). We use the same notation \(\nabla\) for this connection. The frame \(e_i\) is also trivially extended to a frame on \((-\varepsilon, \varepsilon) \times (-\delta, \delta) \times (\text{the domain of the frame})\), and denoted by the same notation \(e_i\). Take and fix any point \(x_0 \in M\), and we calculate \((\partial^2/\partial s \partial t) \| f_s^t h \|^2\) at \(x_0\) for \(s = t = 0\) (for simplicity, we abbreviate the notation “\(s = t = 0\)”). Using a normal coordinate at \(x_0\), we can assume \(\nabla_{e_i} e_j = 0\) for any \(i, j\) at \(x_0\). Since
\[
\left[ \frac{\partial}{\partial s}, e_i \right] = \left[ \frac{\partial}{\partial t}, e_i \right] = 0,
\]
we see
\[
\nabla_{\partial^2} (d F(e_i)) = \nabla_{e_i} \left( d F \left( \frac{\partial}{\partial s} \right) \right) = \nabla_{e_i} X,
\]
\[
\nabla_{\partial^2} (d F(e_i)) = \nabla_{e_i} \left( d F \left( \frac{\partial}{\partial t} \right) \right) = \nabla_{e_i} Y.
\]

We see
\[
\frac{\partial^2}{\partial s \partial t} \| T_{f_s, t} \|^2 = \frac{\partial^2}{\partial s \partial t} \sum_{i,j} T_{f_s, t}(e_i, e_j)^2
\]
\[
= 2 \sum_{i,j} \left\{ \frac{\partial^2}{\partial s \partial t} T_{f_s, t}(e_i, e_j) T_f(e_i, e_j) \right\} + 2 \sum_{i,j} \frac{\partial T_{f_s, t}(e_i, e_j)}{\partial s} \frac{\partial T_{f_s, t}(e_i, e_j)}{\partial t}
\]
\[
\overset{\text{def}}{=} I_1 + I_2.
\]

We have
\[
I_1 = 2 \sum_{i,j} \frac{\partial^2}{\partial s \partial t} \left\{ h(d f_{s,t}(e_i), d f_{s,t}(e_j)) - \frac{1}{m} \| df_{s,t} \|^2 g(e_i, e_j) \right\} T_f(e_i, e_j)
\]
\[
= 2 \sum_{i,j} \left\{ \frac{\partial^2}{\partial s \partial t} h(d f_{s,t}(e_i), d f_{s,t}(e_j)) \right\} T_f(e_i, e_j) - \frac{2}{m} \| df_{s,t} \|^2 \sum_{i,j} g(e_i, e_j) T_f(e_i, e_j)
\]
\[
= 2 \sum_{i,j} \left\{ \frac{\partial^2}{\partial s \partial t} h(d f_{s,t}(e_i), d f_{s,t}(e_j)) \right\} T_f(e_i, e_j) \quad \text{(by Lemma 1 (d))}
\]
\[
= 2 \sum_{i,j} \left\{ \frac{\partial^2}{\partial s \partial t} h(d F(e_i), d F(e_j)) \right\} T_f(e_i, e_j)
\]
\[
= 4 \sum_{i,j} \left\{ h(\nabla_{\partial s} \nabla_{\partial t} (d F(e_i)), d F(e_j)) \right\} T_f(e_i, e_j)
\]
\[
+ 4 \sum_{i,j} \left\{ h(\nabla_{\partial s} (d F(e_i)), \nabla_{\partial t} (d F(e_j))) \right\} T_f(e_i, e_j) \quad \text{(by Lemma 1 (a))}.
\]
We get
\[ \nabla_{\partial/\partial s} \nabla_{\partial/\partial t} (dF(e_i)) = (\nabla_{\partial/\partial s} \nabla_{\partial/\partial t} dF)(e_i) = (\nabla_{\partial/\partial s} \nabla_{\partial/\partial t} dF) \left( \frac{\partial}{\partial t} \right) \]
\[ = (\nabla_{\partial/\partial t} dF) \left( \frac{\partial}{\partial t} \right) - \mathcal{H} F \left( dF(e_i), dF \left( \frac{\partial}{\partial s} \right) \right) dF \left( \frac{\partial}{\partial t} \right) \]
\[ = \nabla_{\partial/\partial t} \mathcal{H} F \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right) - \mathcal{H} F (dF(e_i), X) Y. \]

Then by (12), (13), (15) and (16), we have
\[ I_1 = 4 \sum_{i,j} h \left( \nabla_{\partial/\partial t} \mathcal{H} F \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right), dF(e_j) \right) T_f(e_i, e_j) \]
\[ - 4 \sum_{i,j} h(\mathcal{H} F (dF(e_i), X) Y, dF(e_j)) T_f(e_i, e_j) \]
\[ + 4 \sum_{i,j} h \left( \nabla_{\partial/\partial t} \left( dF \left( \frac{\partial}{\partial s} \right) \right), \nabla_{\partial/\partial t} \left( dF \left( \frac{\partial}{\partial t} \right) \right) \right) T_f(e_i, e_j) \]
\[ = 4 \sum_i h \left( \nabla_{\partial/\partial t} \mathcal{H} F \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right), \sum_j T_f(e_i, e_j) dF(e_j) \right) \]
\[ - 4 \sum_{i,j} h(\mathcal{H} F (dF(e_i), X) Y, dF(e_j)) T_f(e_i, e_j) \]
\[ + 4 \sum_{i,j} h(\nabla_{\partial/\partial t} X, \nabla_{\partial/\partial t} Y) T_f(e_i, e_j) \]
\[ = 4 \sum_i h \left( \nabla_{\partial/\partial t} \mathcal{H} F \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right), \sigma_f(e_i) \right) \]
\[ - 4 \sum_{i,j} h(\mathcal{H} F (dF(e_i), X) Y, dF(e_j)) T_f(e_i, e_j) \]
\[ + 4 \sum_{i,j} h(\nabla_{\partial/\partial t} X, \nabla_{\partial/\partial t} Y) T_f(e_i, e_j) \]
\[ = 4 \text{div}_g \beta_F - 4h \left( \mathcal{H} F \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right), \text{div}_g \sigma_f \right) \]
\[ - 4 \sum_{i,j} h(\mathcal{H} F (dF(e_i), X) Y, dF(e_j)) T_f(e_i, e_j) \]
\[ + 4 \sum_{i,j} h(\nabla_{\partial/\partial t} X, \nabla_{\partial/\partial t} Y) T_f(e_i, e_j), \]

where
\[ \beta_F(X) = h \left( \mathcal{H} F \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right), \sigma_f(X) \right). \]
On the other hand we have

\[(18)\]

\[I_2 = 2 \sum_{i,j} \frac{\partial}{\partial s} \left\{ h(d_{f,i}(e_i), d_{f,j}(e_j)) - \frac{1}{m} \|d_{f,i}\|^2 g(e_i, e_j) \right\} \frac{\partial T_{f_i}(e_i, e_j)}{\partial t} \]

\[= 2 \sum_{i,j} \left\{ \frac{\partial}{\partial s} h(dF(e_i), dF(e_j)) \right\} \frac{\partial T_{f_i}(e_i, e_j)}{\partial t} \]

\[\quad - \frac{2}{m} \frac{\partial \|d_{f,i}\|^2}{\partial s} \sum_{i,j} g(e_i, e_j) \frac{\partial T_{f_i}(e_i, e_j)}{\partial t} \]

\[= 2 \sum_{i,j} \left\{ \frac{\partial}{\partial s} h(dF(e_i), dF(e_j)) \right\} \frac{\partial T_{f_i}(e_i, e_j)}{\partial t} \]

\[\quad (\because \sum_{i,j} g(e_i, e_j) \frac{\partial T_{f_i}(e_i, e_j)}{\partial t} = (\partial / \partial t) \left( \sum_{i,j} g(e_i, e_j) T_{f_i}(e_i, e_j) \right) = 0 \text{ by Lemma 1 (d)}) \]

\[= 2 \sum_{i,j} \left\{ \frac{\partial}{\partial s} h(dF(e_i), dF(e_j)) \right\} \frac{\partial}{\partial t} \left\{ h(d_{f,i}(e_i), d_{f,j}(e_j)) - \frac{1}{m} \|d_{f,i}\|^2 g(e_i, e_j) \right\} \]

\[= 2 \sum_{i,j} \left\{ \frac{\partial}{\partial s} h(dF(e_i), dF(e_j)) \right\} \left\{ \frac{\partial}{\partial t} h(dF(e_i), dF(e_j)) - \frac{1}{m} \frac{\partial \|d_{f,i}\|^2}{\partial t} g(e_i, e_j) \right\} \]

\[= 2 \sum_{i,j} \left\{ \frac{\partial}{\partial s} h(dF(e_i), dF(e_j)) \right\} \left\{ \frac{\partial}{\partial t} h(dF(e_i), dF(e_j)) \right\} \]

\[- \frac{2}{m} \sum_{i,j} \left\{ \frac{\partial}{\partial s} h(dF(e_i), dF(e_j)) \right\} g(e_i, e_j) \frac{\partial \|d_{f,i}\|^2}{\partial t} \]

\[= 2 \sum_{i,j} \left\{ \frac{\partial}{\partial s} h(dF(e_i), dF(e_j)) \right\} \left\{ \frac{\partial}{\partial t} h(dF(e_i), dF(e_j)) \right\} \]

\[- \frac{2}{m} \sum_{i} \left\{ \frac{\partial}{\partial s} h(dF(e_i), dF(e_i)) \right\} \sum_{j} \left\{ \frac{\partial}{\partial t} h(dF(e_j), dF(e_j)) \right\} \]

\[\quad (\because \frac{\partial \|d_{f,i}\|^2}{\partial t} = (\partial / \partial t) \sum_{j} h(d_{f,i}(e_j), d_{f,i}(e_j)) = \sum_{j} (\partial / \partial t) h(dF(e_j), dF(e_j)) \]

\[= I_3 + I_4.\]
We have

\[
I_3 = 2 \sum_{i,j} \left\{ h(\nabla_{\partial s}(dF(e_i)), dF(e_j)) + h(dF(e_i), \nabla_{\partial s}(dF(e_j))) \right\}
\times \left\{ h(\nabla_{\partial t}(dF(e_i)), dF(e_j)) + h(dF(e_i), \nabla_{\partial t}(dF(e_j))) \right\}
\]

\[
= 2 \sum_{i,j} h(\nabla_{\partial s}(dF(e_i)), dF(e_j)) h(\nabla_{\partial s}(dF(e_i)), dF(e_j))
+ 2 \sum_{i,j} h(\nabla_{\partial t}(dF(e_i)), dF(e_j)) h(\nabla_{\partial t}(dF(e_i)), dF(e_j))
\]

\[
+ 2 \sum_{i,j} h(dF(e_i), \nabla_{\partial s}(dF(e_j))) h(dF(e_i), \nabla_{\partial s}(dF(e_j)))
+ 2 \sum_{i,j} h(dF(e_i), \nabla_{\partial t}(dF(e_j))) h(dF(e_i), \nabla_{\partial t}(dF(e_j)))
\]

\[
(19)
\]

\[
= 4 \sum_{i,j} h(\nabla_{\partial s}(dF(e_i)), dF(e_j)) h(\nabla_{\partial s}(dF(e_i)), dF(e_j))
+ 4 \sum_{i,j} h(\nabla_{\partial t}(dF(e_i)), dF(e_j)) h(\nabla_{\partial t}(dF(e_i)), dF(e_j))
\]

(by exchanging the indices \(i\) and \(j\))

\[
= 4 \sum_{i,j} h\left( \nabla_{e_i} \left( dF \left( \frac{\partial}{\partial s} \right) \right), dF(e_j) \right) h\left( \nabla_{e_i} \left( dF \left( \frac{\partial}{\partial t} \right) \right), dF(e_j) \right)
\]

\[
+ 4 \sum_{i,j} h\left( \nabla_{e_i} \left( dF \left( \frac{\partial}{\partial s} \right) \right), dF(e_j) \right) h\left( dF(e_i), \nabla_{e_j} \left( dF \left( \frac{\partial}{\partial t} \right) \right) \right)
\]

\[
= 4 \sum_{i,j} h(\nabla_{e_i} X, df(e_j)) h(\nabla_{e_j} Y, df(e_j))
\]

\[
+ 4 \sum_{i,j} h(\nabla_{e_i} X, df(e_j)) h(df(e_i), \nabla_{e_j} Y).
\]

On the other hand by (12) and (13), we get

\[
I_4 = -\frac{8}{m} \sum_i h(\nabla_{\partial s}(dF(e_i)), dF(e_j)) \sum_j h(\nabla_{\partial t}(dF(e_j)), dF(e_j))
\]

\[
(20)
\]

\[
= -\frac{8}{m} \sum_i h\left( \nabla_{e_i} \left( dF \left( \frac{\partial}{\partial s} \right) \right), dF(e_i) \right) \sum_j h\left( \nabla_{e_j} \left( dF \left( \frac{\partial}{\partial t} \right) \right), dF(e_j) \right)
\]

\[
= -\frac{8}{m} \sum_i h(\nabla_{e_i} X, df(e_i)) \sum_j h(\nabla_{e_j} Y, df(e_j)).
\]
Note \((\partial^2/(\partial s \partial t)) \Phi(f_{s,t})(x, t=0) = \int_M (\partial^2/(\partial s \partial t)) \|T_{f_{s,t}}\|^2_{k, t=0} dv_g\). Integrate (14) over \(M\) and use (17), (18), (19) and (20), and then we obtain the second variation formula.

5. Quasi-monotonicity formula

In this section we prove a kind of the monotonicity formula for stationary maps. We assume the following weak notion of stationary maps.

**Definition 3.** Let \(f\) be a smooth map from \(M\) into \(N\). We call it is stationary for \(\Phi(f)\) with respect to diffeomorphisms on \(M\) if

\[
\frac{d}{dt} \Phi(f \circ \varphi_t) \bigg|_{t=0} = 0
\]

for any 1-parameter family \(\varphi_t\) of diffeomorphisms on \(M\).

Note that the notion of stationary maps in Definition 3 is weaker than that of stationary ones in Definition 2, since \(f_t(x) = f \circ \varphi_t(x)\) is a deformation in Theorem 1. Under the above weaker condition, we give the following formula.

**Theorem 4** (quasi-monotonicity formula). Let \(f\) be stationary for \(\Phi(f)\) with respect to diffeomorphisms on \(M\). Let \(m\) be the dimension of \(M\). Then it satisfies

\[
\frac{d}{d\rho} \left\{ e^{C_2 \rho} \rho^{4-m} \int_{B_{\rho}(x_0)} \|T_f\|^2 dv_g \right\} \geq 4e^{C_2 \rho} \rho^{4-m}(\varphi' (\rho) + C_1 \varphi(\rho))
\]

where \(B_{\rho}(x_0)\) denotes the open ball of a radius \(\rho\) with a center \(x_0 \in M\), and \(C_1, C_2\) are constants. Here

\[
\varphi(\rho) = \int_{B_{\rho}(x_0)} h \left( df \left( \frac{\partial}{\partial r} \right), \sigma_f \left( \frac{\partial}{\partial r} \right) \right) dv_g
\]

and \(\sigma_f\) is defined by (3).

**Remark 4.** If \(\varphi(\rho)\) satisfies the condition \(\varphi' (\rho) + C_1 \varphi(\rho) \geq 0\), then

\[
e^{C_2 \rho} \rho^{4-m} \int_{B_{\rho}(x_0)} \|T_f\|^2 dv_g
\]

is monotone non-decreasing.

**Proof of Theorem 4.** We use the argument by Price [4]. (See also Xin [9], p.43.) Let \(X\) be a smooth vector field on \(M\), which is supported compactly in \(B_{\rho}(x_0)\). Take
a 1-parameter family \( \varphi_t \) \((-\varepsilon < t < \varepsilon)\) of diffeomorphisms on \( M \) corresponding to this vector field. By Theorem 2, we have

\[
(21) \quad 0 = \frac{d \Phi(f \circ \varphi_t)}{d t} \Big|_{t=0} = - \int_M \left\{ \| T_f \|^2 \text{div}_g X - 4 \sum_{i=1}^m h(\partial_t (\nabla e_i^X), \sigma_f(e_i)) \right\} dv_g.
\]

Let \( r = r(x) \) denote the distance function between \( x_0 \) and \( x \), and let \( \partial / \partial r \) be the gradient vector field of the distance function \( r \). We can take an local orthonormal frame \( e_i \) such that \( e_m = \partial / \partial r \). We adopt here a smooth vector field \( X \) in a coordinate neighborhood \( U \) of \( x_0 \), which vanishes outside \( U \). The function \( \xi(r) \) is defined later. We see, for \( 1 \leq i \leq m - 1 \),

\[
\nabla_{e_i} \frac{\partial}{\partial r} = \sum_{j=1}^{m-1} \text{Hess}(r)(e_i, e_j)e_j,
\]

where \( \text{Hess}(r)(X, Y) = (\nabla dr)(X, Y) = \nabla_X(dr(Y)) - dr(\nabla_X Y) \) denotes the Hessian of the function \( r \). Indeed, note \( dr(e_j) = g(\partial / \partial r, e_j) = 0 \) \((j = 1, \ldots, m-1)\) and \( g(\partial / \partial r, \partial / \partial r) = 1 \), and then we have

\[
\nabla_{e_i} \frac{\partial}{\partial r} = \sum_{j=1}^m g \left( \nabla_{e_i} \frac{\partial}{\partial r}, e_j \right) e_j = \sum_{j=1}^{m-1} g \left( \nabla_{e_i} \frac{\partial}{\partial r}, e_j \right) e_j + g \left( \nabla_{e_i} \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right) \frac{\partial}{\partial r} = - \sum_{j=1}^{m-1} g \left( \frac{\partial}{\partial r}, \nabla_{e_i} e_j \right) e_j = - \sum_{j=1}^{m-1} dr(\nabla_{e_i} e_j)e_j = \sum_{j=1}^{m-1} (\nabla dr)(e_i, e_j)e_j,
\]

since

\[
0 = \nabla_{e_i} \left\{ g \left( \frac{\partial}{\partial r}, e_j \right) \right\} = g \left( \nabla_{e_i} \frac{\partial}{\partial r}, e_j \right) + g \left( \nabla_{e_i} \frac{\partial}{\partial r}, \nabla_{e_i} e_j \right),
\]

\[
0 = \nabla_{e_i} \left\{ g \left( \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right) \right\} = 2 g \left( \nabla_{e_i} \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right).
\]

We have

\[
(22) \quad \nabla_{\partial / \partial r} X = \nabla_{\partial / \partial r} \left( \xi(r) \frac{\partial}{\partial r} \right) = (\xi(r)) \frac{\partial}{\partial r},
\]

\[
(23) \quad \nabla_{e_i} X = \xi(r) \frac{\partial}{\partial r} \sum_{j=1}^{m-1} \text{Hess}(r)(e_i, e_j)e_j \quad (1 \leq i \leq m - 1).
\]
By the comparison theorem of Hessian, we know

\begin{equation}
\frac{1}{r} g(e_i, e_j)(1 - C_1 r) \leq \text{Hess}(r)(e_i, e_j) \leq \frac{1}{r} g(e_i, e_j)(1 + C_1 r),
\end{equation}

where \( c \) is a constant which depends on the upper and lower bound of the sectional curvature of \( M \). We calculate \( \text{div}_g X \) and \( \sum_{i=1}^m h(df(\nabla e_i X), \sigma_f(e_i)) \) in the first variation formula \( (21) \). By \( (22), (23) \) and \( (24) \), we have

\[
\text{div}_g X = \sum_{i=1}^{m-1} g(\nabla e_i X, e_i) + g \left( \nabla \frac{\partial}{\partial r} X, \frac{\partial}{\partial r} \right)
\]

\begin{equation}
= \xi(r) r \sum_{i,j=1} h(df(\nabla e_i X), \sigma_f(e_i)) + (\xi(r)r) \sum_{i,j=1} h(df(\nabla e_i X), \sigma_f(e_i))
\end{equation}

\[
\geq (m - 1)\xi(r)(1 - C_1 r) + (\xi(r)r) \sum_{i,j=1} h(df(\nabla e_i X), \sigma_f(e_i))
\]

\[
= \xi'(r) r + m\xi(r) - (m - 1)c\xi(r)r.
\]

We also get by \( (22), (23) \) and \( (24) \),

\[
\sum_{i=1}^m h(df(\nabla e_i X), \sigma_f(e_i))
\]

\[
= \sum_{i=1}^{m-1} h(df(\nabla e_i X), \sigma_f(e_i)) + h\left( df(\nabla \frac{\partial}{\partial r} X), \sigma_f \left( \frac{\partial}{\partial r} \right) \right)
\]

\[
= \xi(r) r \sum_{i,j=1} h(df(\nabla e_i X), \sigma_f(e_i)) + (\xi(r)r) \sum_{i,j=1} h(df(\nabla e_i X), \sigma_f(e_i))
\]

\[
\leq \xi(r)(1 + C_1 r) \sum_{i=1}^{m-1} h(df(e_i), \sigma_f(e_i)) + (\xi(r)r + \xi(r)) h\left( df(\nabla \frac{\partial}{\partial r} X), \sigma_f \left( \frac{\partial}{\partial r} \right) \right)
\]

\begin{equation}
= \xi'(r) r h\left( df \left( \frac{\partial}{\partial r} \right), \sigma_f \left( \frac{\partial}{\partial r} \right) \right)
\end{equation}

\[
+ \xi(r) \left\{ \sum_{i=1}^{m-1} h(df(e_i), \sigma_f(e_i)) + h\left( df \left( \frac{\partial}{\partial r} \right), \sigma_f \left( \frac{\partial}{\partial r} \right) \right) \right\}
\]

\[
+ C_1 \xi(r) r \sum_{i=1}^{m-1} h(df(e_i), \sigma_f(e_i))
\]

\[
= \xi'(r) r h\left( df \left( \frac{\partial}{\partial r} \right), \sigma_f \left( \frac{\partial}{\partial r} \right) \right) + \xi(r) \sum_{i=1}^m h(df(e_i), \sigma_f(e_i))
\]

\[
+ C_1 \xi(r) r \left\{ \sum_{i=1}^m h(df(e_i), \sigma_f(e_i)) - h\left( df \left( \frac{\partial}{\partial r} \right), \sigma_f \left( \frac{\partial}{\partial r} \right) \right) \right\}.
\]
We have by Lemma 1 (e)

\[
\sum_{i=1}^{m} h(df(e_i), \sigma_f(e_i)) = \sum_{i=1}^{m} h \left( df(e_i), \sum_{j=1}^{m} T_f(e_i, e_j) df(e_j) \right) \\
= \sum_{i=1}^{m} \sum_{j=1}^{m} h(df(e_i), df(e_j)) T_f(e_i, e_j) = \|T_f\|^2.
\]

(27)

For simplicity we set

\[
A \left( df, \frac{\partial}{\partial r} \right) := h \left( df \left( \frac{\partial}{\partial r} \right), \sigma_f \left( \frac{\partial}{\partial r} \right) \right).
\]

Then by (26), (27), we have

\[
\sum_{i=1}^{m} h(df(\nabla_{e_i} X), \sigma_f(e_i)) \\
\leq \xi'(r) r A \left( df, \frac{\partial}{\partial r} \right) + \xi(r) \|T_f\|^2 + C_1 \xi(r) r \left( \|T_f\|^2 - A \left( df, \frac{\partial}{\partial r} \right) \right).
\]

(28)

Therefore by (21), (25), (28), we get

\[
0 = \int_M \left\{ \|T_f\|^2 \text{div}_g X - 4 \sum_{i=1}^{m} h(df(\nabla_{e_i} X), \sigma_f(e_i)) \right\} dv_g \\
\geq \int_M \xi'(r) r \|T_f\|^2 dv_g + m \int_M \xi(r) \|T_f\|^2 dv_g \\
\geq (m - 1) C_1 \int_M \xi(r) r \|T_f\|^2 dv_g \\
- 4 \int_M \xi'(r) r A \left( df, \frac{\partial}{\partial r} \right) dv_g - 4 \int_M \xi(r) \|T_f\|^2 dv_g \\
- 4 C_1 \int_M \xi(r) r \|T_f\|^2 dv_g + 4 C_1 \int_M \xi(r) r A \left( df, \frac{\partial}{\partial r} \right) dv_g,
\]

i.e.,

\[
- \int_M \xi'(r) r \|T_f\|^2 dv_g + (4 - m) \int_M \xi(r) \|T_f\|^2 dv_g + C_2 \int_M \xi(r) r \|T_f\|^2 dv_g \\
\geq -4 \int_M \xi'(r) r A \left( df, \frac{\partial}{\partial r} \right) dv_g + 4 C_1 \int_M \xi(r) r A \left( df, \frac{\partial}{\partial r} \right) dv_g,
\]

(29)
where \( C_2 = (m + 3)C_1 \). Take and fix a positive number \( \varepsilon \), and let \( \varphi \) be a smooth function on \([0, \infty)\) such that
\[
\varphi(r) = \varphi_\varepsilon(r) = \begin{cases} 
1 & \text{if} \ 0 \leq r \leq 1, \\
0 & \text{if} \ 1 + \varepsilon \leq r
\end{cases}
\]
and
\[
\varphi'(r) \leq 0.
\]
We define
\[
\xi(r) = \xi_\rho(r) := \varphi\left(\frac{r}{\rho}\right).
\]
We can verify
\[
(30) \quad \xi'(r) = -\rho \frac{d}{d\rho} \xi(r).
\]
Since \( \|T_f\|^2 \) is independent of \( \rho \), the above facts (29) and (30) imply
\[
\rho \frac{d}{d\rho} \int_M \xi(r)\|T_f\|^2 d\nu_g + (4 - m) \int_M \xi(r)\|T_f\|^2 d\nu_g + C_2 \int_M \xi(r)r\|T_f\|^2 d\nu_g \\
\geq 4\rho \frac{d}{d\rho} \int_M A\left(df, \frac{\partial}{\partial r}\right) \xi(r) d\nu_g + 4C_1 \rho \int_M A\left(df, \frac{\partial}{\partial r}\right) \xi(r) d\nu_g.
\]
Let \( \varepsilon \) tend to zero, and then, since \( \xi(r) \) converges to the characteristic function for the ball \( B_\rho(x_0) \), we have
\[
\rho \frac{d}{d\rho} \int_{B_\rho(x_0)} \|T_f\|^2 d\nu_g + (4 - m) \int_{B_\rho(x_0)} \|T_f\|^2 d\nu_g + C_2\rho \int_{B_\rho(x_0)} \|T_f\|^2 d\nu_g \\
\geq 4\rho \frac{d}{d\rho} \int_{B_\rho(x_0)} A\left(df, \frac{\partial}{\partial r}\right) d\nu_g + 4C_1 \rho \int_{B_\rho(x_0)} A\left(df, \frac{\partial}{\partial r}\right) d\nu_g.
\]
Multiply \( e^{C_2\rho} \rho^{4-m} \) to the both sides of this inequality, and we have
\[
\frac{d}{d\rho} \left\{ e^{C_2\rho} \rho^{4-m} \int_{B_\rho(x_0)} \|T_f\|^2 d\nu_g \right\} \\
\geq 4e^{C_2\rho} \rho^{4-m} \left\{ \frac{d}{d\rho} \int_M A\left(df, \frac{\partial}{\partial r}\right) d\nu_g + C_1 \int_M A\left(df, \frac{\partial}{\partial r}\right) d\nu_g \right\}.
\]
Thus we obtain the formula. \(\square\)

6. Bochner type formula

In this section we prove the following formula.
Theorem 5 (Bochner type formula). For any smooth map $f$ from $M$ into $N$, the following equality holds:

$$
\frac{1}{4} \Delta \|T_f\|^2 = \text{div} \alpha_f - h(\tau_f, \text{div} \sigma_f) + \frac{1}{2} \|\nabla T_f\|^2
$$

$$
+ \sum_{i,j,k} h(\nabla e_i df)(e_i), (\nabla e_k df)(e_j)) T_f(e_i, e_j)
$$

$$
+ \sum_{i,j} h\left(df\left(\sum_k M_R(e_i, e_k)e_k\right), df(e_j)\right) T_f(e_i, e_j)
$$

$$
- \sum_{i,j,k} h(N^R(df(e_i), df(e_k)) df(e_j), df(e_j)) T_f(e_i, e_j)
$$

where

$$
\alpha_f(X) = h(\sigma_f(X), \tau_f).
$$

Here $\sigma_f$ is defined by (3), and $\tau_f = \text{tr}(\nabla df) = \sum_j(\nabla e_j df)(e_j)$ is the tension field of $f$ in the harmonic map theory. (See Eells and Lemaire [2], p. 9.)

**Remark 5.** Note that the first term in the right hand side is of divergence form, and hence the integral of it over $M$ vanishes.

**Remark 6.** Note that the second term in the right hand side vanishes if $f$ is a stationary map for the functional $\Phi(f)$.

**Remark 7.** The last two terms of the right hand side in Theorem 5 are equal to

$$
+ \sum_{i,j} h\left(df\left(\sum_k M_R(e_i, e_k)e_k\right), \sigma_f(e_i)\right)
$$

$$
- \sum_{i,j,k} h(N^R(df(e_i), df(e_k)) df(e_j), \sigma_f(e_j))
$$

respectively.

**Proof of Theorem 5.** We have

$$
\Delta \|T_f\|^2 = \Delta \sum T_f(e_i, e_j)^2
$$

$$
= 2 \sum (\Delta T_f)(e_i, e_j) T_f(e_i, e_j) + 2 \sum \sum_k (\nabla_{e_k} T_f)(e_i, e_j)^2
def \Delta = I_1 + I_2.
$$
We get

\[ I_1 = 2 \sum_{i,j} (\Delta T_f)(e_i, e_j) T_f(e_i, e_j) \]

\[ = 2 \sum_{i,j} \left\{ h((\Delta df)(e_i), df(e_j)) + 2 \sum_k h((\nabla_{e_i} df)(e_i), (\nabla_{e_k} df)(e_j)) \right. \]

\[ + \left. h(df(e_i), (\Delta df)(e_j)) - \frac{1}{m} \Delta df\|^2 g(e_i, e_j) \right\} T_f(e_i, e_j) \]

\[ = 4 \sum_{i,j} h((\Delta df)(e_i), df(e_j)) T_f(e_i, e_j) \]

\[ + 4 \sum_{i,j,k} h((\nabla_{e_i} df)(e_i), (\nabla_{e_k} df)(e_j)) T_f(e_i, e_j) \quad \text{(by Lemma 1 (a) and (d))}. \]

Since by Ricci formula,

\[ (\Delta df)(e_i) = \sum_k (\nabla_{e_k} \nabla_{e_i} df)(e_i) = \sum_k (\nabla_{e_k} \nabla_{e_i} df)(e_k) \]

\[ = \sum_k (\nabla_{e_i} \nabla_{e_k} df)(e_k) + df \left( \sum_k M R(e_i, e_k) e_k \right) \]

\[ - \sum_k N R(df(e_i), df(e_k)) df(e_k) \]

\[ = \nabla_{e_i} \tau_f + df \left( \sum_k M R(e_i, e_k) e_k \right) - \sum_k N R(df(e_i), df(e_k)) df(e_k). \]

we have

\[ I_1 = 4 \sum_{i,j} h(\nabla_{e_i} \tau_f, df(e_j)) T_f(e_i, e_j) \]

\[ + 4 \sum_{i,j} h\left( df \left( \sum_k M R(e_i, e_k) e_k \right), df(e_j) \right) T_f(e_i, e_j) \]

\[ - 4 \sum_{i,j,k} h(N R(df(e_i), df(e_k)) df(e_k), df(e_j)) T_f(e_i, e_j) \]

\[ + 4 \sum_{i,j,k} h((\nabla_{e_i} df)(e_i), (\nabla_{e_k} df)(e_j)) T_f(e_i, e_j). \]
Furthermore we get
\[
\sum_{i,j} h(\nabla_i \tau_f, d f(e_j)) T_f(e_i, e_j) = \sum_i h \left( \nabla_i \tau_f, \sum_j T_f(e_i, e_j) d f(e_j) \right) = \sum_i h(\nabla_i \tau_f, \sigma_f(e_i)) = \sum_i \text{div}_g \sigma_f - \sum_i h(\tau_f, \text{div}_g \sigma_f).
\]

By (32), (33) and (34), we obtain Theorem 5, since \( I_2 = 2 \| \nabla T_f \|^2 \).

7. Minimizers in homotopy classes of the Sobolev space

In this section we utilize the notion of 3-homotopy in the Sobolev spaces, which is given by White, and consider a variational problem of minimizing the functional \( \Phi(f) \) in each 3-homotopy class. For any two maps \( f \) and \( g \) from \( M \) into \( N \), these maps are \( k \)-homotopic (\( k \in \mathbb{N} \)) if they are homotopic to each other on \( k \)-dimensional skeletons of a triangulation on \( M \). By Nash’s isometric embedding, we may assume that \( N \) is a submanifold of a Euclidean space \( \mathbb{R}^q \). Let
\[
L^{1,p}(M, N) = \{ f \in L^{1,p}(M, \mathbb{R}^q) \mid f(x) \in N \text{ a.e.} \},
\]
where \( L^{1,p}(M, \mathbb{R}^q) \) denotes the Sobolev space of \( \mathbb{R}^q \)-valued \( L^p \)-functions on \( M \) such that their derivatives are in \( L^p \). Then White proved that the notion of the \([p - 1]\)-homotopy is compatible with the Sobolev space \( L^{1,p}(M, N) \), where \([\ ]\) denotes the Gauss symbol, i.e., \([r]\) is the maximum integer less than or equal to \( r \).

**Theorem 5** (Theorem 3.4 in White [8]. See also White [7], Schoen and Uhlenbeck [5] and Bethuel [1]).

1. The \([p - 1]\)-homotopy is well-defined for any map \( f \in L^{1,p}(M, N) \).
2. If \( f_j \) converges weakly to \( f_\infty \) in \( L^{1,p}(M, N) \), then \( f_j \) and \( f_\infty \) are \([p - 1]\)-homotopic for sufficient large \( j \).

The functional \( \Phi(f) \) is defined on \( L^{1,4}(M, N) \), in which the 3-homotopy is well-defined. Then for any given continuous map \( f_0 \) from \( M \) into \( N \), we want to minimize the functional \( \Phi(f) \) in the following class:
\[
\mathcal{F} = \{ f \in L^{1,4}(M, N) \mid f \text{ is 3-homotopic to } f_0 \text{ and } \| f \|_{L^{1,4}(M,N)} \leq C_0 \},
\]
where \( C_0 \) is a given positive constant. We may assume that the space \( \mathcal{F} \) is not empty for sufficiently large \( C_0 \).

**Theorem 6.** There exists a minimizer of the functional \( \Phi(f) \) in \( \mathcal{F} \).
If a 3-homotopy class contains a conformal map, then the conformal map is a minimizer. Minimizers are expected to be closest to conformal maps, even if its 3-homotopy class does not contain any conformal map.

**Remark 8.** When $M$ is 4-dimensional and $\pi_4(N) = 0$, any continuous minimizer is (freely) homotopic to $f_0$ in the ordinary sense.

Proof of Theorem 6. Take any minimizing sequence $f_j$ for the functional $\Phi(f)$ in the space $\mathcal{F}$, i.e., $\Phi(f_j)$ converges to the infimum in $\mathcal{F}$. We may assume that $f_j$ converges weakly to a map $f_\infty$ in $L^{1,4}(M,N)$, since $\|f\|_{L^{1,4}(M,N)} \leq C_0$. Since the weak convergence in $L^{1,4}(M,N)$ preserves the 3-homotopy by Theorem S (2), $f_\infty$ is 3-homotopic to $f_j$ for sufficiently large $j$, hence to $f_0$. Furthermore $T_{f_j}$ converges weakly to $T_{f_\infty}$ in $L^2$, since for any covariant 2-tensor $S$,

$$
\int_M \langle T_{f_j}, S \rangle \, dv_g = \int_M \left( f_j^* h - \frac{1}{m} \| df_j \|^2 g, S \right) \, dv_g
$$

$$
= \int_M \left( f_j^* h, S - \frac{1}{m} \langle g, S \rangle g \right) \, dv_g,
$$

where $\langle \cdot, \cdot \rangle$ is the pointwise pairing for covariant 2-tensors. Therefore we have

$$
\Phi(f_\infty) = \| T_{f_\infty} \|_{L^2} \leq \liminf_{j \to \infty} \| T_{f_j} \|_{L^2} = \liminf_{j \to \infty} \Phi(f_j).
$$

Then $f_\infty$ is a minimizer in $\mathcal{F}$.

**Acknowledgement.** The author would like to thank Professor Takakuwa and the referee for valuable comments.

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**References**


