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ON COMPACT COMPLEX PARALLELISABLE SOLVMANIFOLDS

Dedicated to the memory of Taira Honda

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1. Introduction

This paper deals with compact complex solvmanifolds. Our main purpose is to generalize the theory on the divisor group of a complex torus to these manifolds. By a solvmanifold we mean a homogeneous space of solvable Lie group. Let G be a simply connected complex solvable Lie group and Γ be a lattice of G, that is, a discrete subgroup of G such that G/Γ is compact. The de Rham cohomology group and the Dolbeault cohomology group of a compact complex manifold G/Γ play an important role in studying the divisor group of a complex manifold G/Γ. The de Rham cohomology group of a compact solvmanifold G/Γ has been discussed by Matsushima [7], Nomizu [10] and Mostow [8].

Let M be a compact connected complex manifold and $H^{p,q}_{\bar{\partial}}(M)$ denote the Dolbeault cohomology group of M of type (p, q). Let $\mathfrak{g}$ be a complex Lie algebra and I be the canonical complex structure of $\mathfrak{g}$. Then $\mathfrak{g}^C=\mathfrak{g}^+\oplus\mathfrak{g}^-$, where $\mathfrak{g}^\pm=\{X\in\mathfrak{g}^C \mid IX=\pm\sqrt{-1}X\}$. In section 2, we prove:

Theorem 1. *Let G be a simply connected complex nilpotent Lie group and Γ be a lattice of G. Then there is a canonical isomorphism*

$$H^{p,q}_{\bar{\partial}}(G/\Gamma)\cong H^q(\mathfrak{g}^-)\otimes\Lambda^p(\mathfrak{g}^+)^*$$

where $H^q(\mathfrak{g}^-)$ denotes the Lie algebra cohomology group of $\mathfrak{g}^-$ and $(\mathfrak{g}^+)^*$ denotes the dual vector space of $\mathfrak{g}^+$.

Let G be a simply connected complex solvable Lie group and Γ be a lattice of G which has the following property:

(M) *$Ad(G)$ and $Ad(\Gamma)$ have the same Zariski closure in the group $Aut(\mathfrak{g}^C)$.*

This condition has been used by Mostow in his study of lattices of solvable

1) This work was presented to the Graduate School of the University of Notre Dame in partial fulfillment of the requirement for the Ph. D. degree.

Lie group [8]. Denote by $[G, G]$ the commutator group of G and let $\pi: G\to G/[G, G]$ be the projection. Then $\Gamma\cap[G, G]$ is a lattice of $[G, G]$, so that $\pi(\Gamma)$ is a lattice of $G/[G, G]$ and $(G/\Gamma, \pi, (G/[G, G])/\pi(\Gamma), [G, G]/([G, G]\cap\Gamma))$ is a homlomorphic fiber bundle. Let T denote the complex torus $(G/[G, G])/\pi(\Gamma)$. In section 3, we study Chern classes of holomorphic line bundles over these compact complex solvmanifolds.

Let M and N be complex manifolds and $\phi: M\to N$ be a surjective holomorphic map. For a divisor $\tilde{D}$ on N let $\phi^*(\tilde{D})$ denote the divisor on M defined by $\phi_x^{-1}(\tilde{D}_{\phi(x)})$ for all $x\in M$. We call the divisor $\phi^*(\tilde{D})$ on M the pull back of the divisor $\tilde{D}$ on N [15]. In section 4, we prove:

Theorem 2. *Let G be a simply connected complex solvable Lie group and Γ be a lattice of G. Assume that Γ satisfies the condition (M) and that $H_{\bar\partial}^{0,1}(G/\Gamma)\cong H^1(\mathfrak{g}^-)$ canonically. Then, under the notation introduced above, for each positive divisor D on G/Γ, there exists a positive divisor $\tilde{D}$ on the complex torus T such that the divisor D is the pull back of the divisor $\tilde{D}$ on T by the projection $\pi: G/\Gamma\to T$, i.e., $D=\pi^*\tilde{D}$.*

Note that our assumption in Theorem 2 is always satisfied if G is a simply connected complex nilpotent Lie group and Γ is a lattice of G.

If M is a compact connected complex manifold, $K(M)$ will denote the field of all meromorphic functions on M.

Corollary. *Under the condition of Theorem 2, there is a canonical isomorphism*

$$\pi^*: K(T)\cong K(G/\Gamma)\,.$$

In particular, the transcendence degree of $K(G/\Gamma)$ over $\mathbf{C}$ is not larger than the complex dimension of the complex torus T.

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2. Dolbeault cohomology groups of compact complex nilmanifolds

Let M be a complex manifold and $H_{\bar\partial}^{p,q}(M)$ denote the Dolbeault cohomology of M of type (p, q). Let G be a simply connected complex Lie group and Γ be a uniform lattice of G. Let $\mathfrak{g}$ denote the Lie algebra of all right invariant vector fields on G, I denote the complex structure of $\mathfrak{g}$ and $\mathfrak{g}^+$ (resp. $\mathfrak{g}^-$) denote the vector space of the $\sqrt{-1}$ (resp. $-\sqrt{-1}$) eigenvectors of I in the complexification $\mathfrak{g}^C$ of $\mathfrak{g}$. We identify $\mathfrak{g}^+$ to the Lie algebra of all right invariant holomorphic vector fields on G and the dual space $(\mathfrak{g}^+)^*$ to the space of all right invariant holomorphic 1-forms on G. Moreover we may identify an element of

$\mathfrak{g}^+$ (resp. $(\mathfrak{g}^+)^*$) to a holomorphic vector field (resp. a holomorphic 1-form) on G/Γ. Let $\Lambda^p T^*(G/\Gamma)$ be the p-th exterior product bundle of the holomorphic cotangent bundle $T^*(G/\Gamma)$ of G/Γ. Since G/Γ is a compact complex parallelisable manifold, the holomorphic vector bundle $\Lambda^p T^*(G/\Gamma)$ on G/Γ is the trivial vector bundle $G/\Gamma \times \Lambda^p(\mathfrak{g}^+)^*$. Thus we have an isomorphism

$$H^{p,q}_{d''}(G/\Gamma) \cong H^{0,q}_{d''}(G/\Gamma) \otimes \Lambda^p(\mathfrak{g}^+)^* . \tag{2.1}$$

Theorem 1. *Let G be a simply connected complex nilpotent Lie group and Γ be a lattice of G. Then we have a canonical isomorphism*

$$H^{p,q}_{d''}(G/\Gamma) \cong H^q(\mathfrak{g}^-) \otimes \Lambda^p(\mathfrak{g}^+)^*$$

where $H^q(\mathfrak{g}^-)$ denoted the q-th Lie algebra cohomology of with the trivial representation ρ_0: $\mathfrak{g}^- \to \boldsymbol{C}$.

We need some preparations to prove Theorem 1. Consider the descending central series $\{C^k(G)\}$ of G, where $C^k(G)=[G, C^{k-1}(G)]$ and $C^0(G)=G$. Since G is nilpotent, there is an integer $m \in \boldsymbol{N}$ such that $C^m(G) \neq (e)$ and $C^{m+1}(G)=(e)$. Let A denote the group $C^m(G)$. Then A is contained in the center $Z(G)$ of G. Since G is a simply connected nilpotent Lie group and A is connected, A is a simply connected closed Lie subgroup. Let Γ be a lattice of G. Then $A \cap \Gamma$ is a lattice of A ([11] p. 31 Corollary 1) and $A\Gamma$ is closed in G ([11] p. 23 Theorem 1.13). Let π: $G \to G/A$ be the canonical map. Then $\pi(\Gamma)$ is a lattice of G/A. Since $A/(A \cap \Gamma) \cong A\Gamma/\Gamma$ is a complex torus, we have a holomorphic principal fiber bundle $(G/\Gamma, (G/A)/\pi(\Gamma), \pi, A/(A \cap \Gamma))$.

Let $C^\infty(G, \boldsymbol{C})$ be the vector space of all complex valued C^∞-functions on G. Define the subspaces $\underline{C}$ and $\underline{C}'$ of $C^\infty(G, \boldsymbol{C})$ by

$$\underline{C} = \{f \in C^\infty(G, \boldsymbol{C}) \mid f(g\gamma) = f(g) \qquad \text{for all } \gamma \in \Gamma\}$$

and

$$\underline{C}' = \{f \in \underline{C} \mid f(ga) = f(g) \qquad \text{for all } a \in A\} .$$

For a right invariant vector field $X \in \mathfrak{g}$ and $f \in C^\infty(G, \boldsymbol{C})$, put

$$(Xf)(g) = \frac{d}{dt} f(a(t)g) \big|_{t=0}$$

where $a(t)$ is the one parameter subgroup corresponding to X. Then $C^\infty(G, \boldsymbol{C})$ is a $\mathfrak{g}$-module, and hence $\underline{C}$ and $\underline{C}'$ are $\mathfrak{g}^C$-submodules of $C^\infty(G, \boldsymbol{C})$.

Let $\mathfrak{a}$ be the Lie subalgebra of $\mathfrak{g}$ corresponding to the complex Lie subgroup A of G. Then $\mathfrak{a}^C$ has the decomposition $\mathfrak{a}^C = \mathfrak{a}^+ \oplus \mathfrak{a}^-$ with respect to the complex structure I, and $\underline{C}$ and $\underline{C}'$ are $\mathfrak{a}^-$-modules. Let $\{A^q(\mathfrak{a}^-, \underline{C}), d\}$ (resp. $\{A^q(\mathfrak{a}^-, \underline{C}'), d\}$ denote the cochain complex of $\mathfrak{a}^-$-module $\underline{C}$ (resp. $\underline{C}'$) and $H^*(\mathfrak{a}^-, \underline{C})$ (resp. $H^*(\mathfrak{a}^-, \underline{C}')$) denote the Lie algebra cohomology of $\mathfrak{a}^-$-module

$\underline{C}$ (resp. $\underline{C}'$). Since $\mathfrak{a}^-$ is an ideal of $\mathfrak{g}^-$, $A^q(\mathfrak{a}^-, \underline{C})$ (resp. $A^q(\mathfrak{a}^-, \underline{C}')$ is $\mathfrak{g}^-$-module by

$$(L_{\bar{X}}\omega(\bar{X}_1, \cdots, \bar{X}_q) = \bar{X}(\omega(\bar{X}_1, \cdots, \bar{X}_q)) - \sum_{j=1}^{q} \omega(\bar{X}_1, \cdots, [\bar{X}, \bar{X}_j], \cdots, \bar{X}_q)$$

where $\bar{X}\in\mathfrak{g}^-$, $\omega\in A^q(\mathfrak{a}^-, \underline{C})$ (resp. $\omega\in A^q(\mathfrak{a}^-, \underline{C}')$) and $\bar{X}_1, \cdots, \bar{X}_q\in\mathfrak{a}^-$. Moreover $L_{\bar{X}}\circ d = d\circ L_{\bar{X}}$ for all $\bar{X}\in\mathfrak{g}^-$. Thus $H^*(\mathfrak{a}^-, \underline{C})$ and $H^*(\mathfrak{a}^-, \underline{C}')$ are $\mathfrak{g}^-$-modules.

Proposition 2.1. *The inclusion map ι_0: $\underline{C}' \to \underline{C}$ induces an isomorphism ι_0^* of $\mathfrak{g}^-$-modules*

$$\iota_0^*: H^q(\mathfrak{a}^-, \underline{C}') \to H^q(\mathfrak{a}^-, \underline{C}).$$

This follows from Kodaira and Spencer [6] §2, but we shall give an elementary proof (cf. [11] VII §4).

Let $\{X_1, \cdots, X_l\}$ be a basis of $\mathfrak{a}^+$ and $\{\omega_1, \cdots, \omega_l\}$ be the dual basis. We reagrd ω_j $(j=1, \cdots, l)$ as the holomorphic invariant 1-forms on the complex torus $A/(A\cap\Gamma)$. Define an invariant hermitian metric h on $A/(A\cap\Gamma)$ by $h=\sum_{j=1}^{l} \omega_j\cdot\bar{\omega}_j$. Let Ω be the associated form of type (1, 1). Then

$$\Omega = \sqrt{-1}\sum_{j=1}^{l} \omega_j\wedge\bar{\omega}_j,$$

and $\frac{1}{l!}\Omega^l$ defines a Haar measure da on $A/A\cap\Gamma$. We may assume that $\int_{A/A\cap\Gamma}\frac{1}{l!}\Omega^l=1$ by changing the choice of a basis of $\mathfrak{a}^+$ if necessary. For $f\in\underline{C}$ and $x\in G$, let $f_x(a)=f(xa)$ for $a\in A$. Then we can define a $\mathfrak{g}^C$-module homomorphism $\boldsymbol{H}$: $\underline{C}\to\underline{C}'$ by

$$\boldsymbol{H}(f)(x) = \int_{A/A\cap\Gamma} f_x(a)\frac{\Omega^l}{l!} = \int_{A/A\cap\Gamma} f(xa)da.$$

Let $Y_j=\frac{1}{2}(X_j+\bar{X}_j)$ and $Y_{j+l}=\frac{\sqrt{-1}}{2}(X_j-\bar{X}_j)$ for $j=1, \cdots, l$. Then $\{Y_1, \cdots, Y_{2l}\}$ is a basis of $\mathfrak{a}$. Let $\{\theta_1, \cdots, \theta_{2l}\}$ be its dual basis. Let $A^r(\mathfrak{a}, \underline{C})$ denote the vector space of all $\underline{C}$-valued r-forms on $A/A\cap\Gamma$. Note that each element $\omega\in A^r(\mathfrak{a}, \underline{C})$ can be written uniquely as

$$\omega = \sum_{k_1<\cdots<k_r} f_{k_1\cdots k_r}\theta_{k_1}\wedge\cdots\wedge\theta_{k_r} \qquad \text{where } f_{k_1\cdots k_r}\in\underline{C}.$$

For simplicity, let $\theta_K=\theta_{k_1}\wedge\cdots\wedge\theta_{k_r}$ and $f_K=f_{k_1\cdots k_r}$ for $K=(k_1, \cdots, k_r)$ $(1\leqq k_1<\cdots<k_r\leqq 2l)$. Then $\omega=\sum_K f_K\theta_K$.

Let $A^{p,q}(\mathfrak{a}, \underline{C})$ denote the vector space of all $\underline{C}$-valued forms of type (p, q) on $A/A\cap\Gamma$. Each element $\omega\in A^{p,q}(\mathfrak{a}, \underline{C})$ can be written uniquely as

$$\omega = \sum_{I,J} f_{I\bar{J}} \omega_I \wedge \overline{\omega}_J$$

where $I=(i_1, \cdots, i_p)$ $(1 \leqq i_1 < \cdots < i_p \leqq l)$, $J=(j_1, \cdots, j_q)$ $(1 \leqq j_1 < \cdots < j_q \leqq l)$, $f_{I\bar{J}} \in \underline{C}$, $\omega_I = \omega_{i_1} \wedge \cdots \wedge \omega_{i_p}$ and $\overline{\omega}_J = \overline{\omega}_{j_1} \wedge \cdots \wedge \overline{\omega}_{j_q}$.

Define operators d: $A^r(\mathfrak{a}, \underline{C}) \to A^{r+1}(\mathfrak{a}, \underline{C})$ by

$$d\omega = \sum_K \left(\sum_{j=1}^{2l} Y_j f_K\right) \theta_j \wedge \theta_K$$

for $\omega = \sum_K f_K \theta_K \in A^r(\mathfrak{a}, \underline{C})$, d': $A^{p,q}(\mathfrak{a}, \underline{C}) \to A^{p+1,q}(\mathfrak{a}, \underline{C})$ by

$$d'\omega = \sum_{I,J} \left(\sum_{k=1}^{l} X_k f_{I\bar{J}}\right) \omega_k \wedge \omega_I \wedge \overline{\omega}_J$$

for $\omega = \sum_{I,J} f_{I\bar{J}} \omega_I \wedge \overline{\omega}_J \in A^{p,q}(\mathfrak{a}, \underline{C})$ and d'': $A^{p,q}(\mathfrak{a}, \underline{C}) \to A^{p,q+1}(\mathfrak{a}, \underline{C})$ by

$$d''\omega = \sum_{I,J} \left(\sum_{k=1}^{l} \overline{X}_k f_{I\bar{J}}\right) \overline{\omega}_k \wedge \omega_I \wedge \overline{\omega}_J$$

for $\omega = \sum_{I,J} f_{I\bar{J}} \omega_I \wedge \overline{\omega}_J \in A^{p,q}(\mathfrak{a}, C)$. Then $d \circ d = d' \circ d' = d'' \circ d'' = 0$.

Define $\langle \omega, \eta \rangle \in \underline{C}'$ for ω, $\eta \in A^{p,q}(\mathfrak{a}, \underline{C})$ by

$$\langle \omega, \eta \rangle(x) = \sum_{I,J} \int_{A/A \cap \Gamma} f_{I\bar{J}}(xa) \overline{g}_{I\bar{J}}(xa) da = \int_{A/A \cap \Gamma} \omega \wedge \overline{*\eta},$$

where $\omega = \sum_{I,J} f_{I\bar{J}} \omega_I \wedge \overline{\omega}_J$, $\eta = \sum_{I,J} g_{I\bar{J}} \omega_I \wedge \overline{\omega}_J$ and $*$ is the operation defined by the natural orientation of $A/A \cap \Gamma$ and the metric h on $A/A \cap \Gamma$.

Let $\tilde{f} \in C^\infty(G/A\Gamma, C)$ denote the function corresponding to $f \in \underline{C}'$. Define a hermitian inner product (,) on $A^{p,q}(\mathfrak{a}, \underline{C})$ by

$$(\omega, \eta) = \int_{G/A\Gamma} \widetilde{\langle \omega, \eta \rangle}(x) dx$$

where dx denotes an invariant measure on $G/A\Gamma$.

Define $(\omega, \eta) = 0$ if $\omega \in A^{p,q}(\mathfrak{a}, \underline{C})$, $\eta \in A^{p',q'}(\mathfrak{a}, \underline{C})$ for $(p, q) \neq (p', q')$. Since $A^r(\mathfrak{a}, \underline{C}) = \sum_{p+q=r} A^{p,q}(\mathfrak{a}, \underline{C})$, we have thus an hermitian inner product (,) on $A^r(\mathfrak{a}, \underline{C})$.

Now define the adjoint operators δ, δ', δ'' of d, d', d'' by $\delta = -*d*$, $\delta' = -*d''*$, $\delta'' = -*d'*$ respectively. We then have

$$\begin{aligned}
(d\omega, \eta) &= (\omega, \delta\eta) \quad \text{for} \quad \omega \in A^r(\mathfrak{a}, \underline{C}) \quad \text{and} \quad \eta \in A^{r+1}(\mathfrak{a}, \underline{C}), \\
(d'\omega, \eta) &= (\omega, \delta'\eta) \quad \text{for} \quad \omega \in A^{p,q}(\mathfrak{a}, \underline{C}) \quad \text{and} \quad \eta \in A^{p+1,q}(\mathfrak{a}, \underline{C}), \\
(d''\omega, \eta) &= (\omega, \delta''\eta) \quad \text{for} \quad \omega \in A^{p,q}(\mathfrak{a}, \underline{C}) \quad \text{and} \quad \eta \in A^{p,q+1}(\mathfrak{a}, \underline{C}).
\end{aligned}$$

with respect to the hermidian inner product (,).

Define Laplacians Δ, $\square'$, $\square''$ by

$$\Delta = d\delta + \delta d, \quad \square' = d'\delta' + \delta' d', \quad \square'' = d\delta'' + \delta'' d''.$$

Then, by a direct computation we get

$$\Delta\omega = -\sum_K \left(\sum_{j=1}^{2l} Y_j^2 f_K\right)\theta_K$$

for $\omega = \sum_K f_K \theta_K$, and

$$\Box'\omega = \Box''\omega = -\sum_{I,J}\left(\sum_{j=1}^{l} X_j \bar{X}_j\right) f_{I\bar{J}}\omega_I \wedge \bar{\omega}_J$$

for $\omega = \sum_{I,J} f_{I\bar{J}}\omega_I \wedge \bar{\omega}_J$.

Since $X_j\bar{X}_j f = (Y_j^2 + Y_{j+l}^2) f$ for each $f \in \underline{C}$, we see $\Delta = \Box' = \Box''$.

Since A is abelian and simply connected, we may identify A (resp. the lattice $A \cap \Gamma$ of A) with Euclidean space $(\boldsymbol{R}^n, \langle\ ,\ \rangle)$ (resp. a lattice D in $\boldsymbol{R}^n$). For a fixed $x \in G$ and $f \in \underline{C}$, f_x can be regarded as a function on the torus $\boldsymbol{R}^n/D$. Consider the Fourier expansion of f_x,

$$f_x(a) = f(xa) = \sum_{\alpha \in D'} C_\alpha(x) \exp 2\pi\sqrt{-1}\langle\alpha, a\rangle$$

where $D' = \{\alpha \in \boldsymbol{R}^n \mid \langle\alpha, d\rangle \in \boldsymbol{Z}$ for any $d \in D\}$ and $C_\alpha(x) = \int_{A/A\cap\Gamma} f(xa) \exp -2\pi\sqrt{-1}\langle\alpha, a\rangle da$ for $\alpha \in D'$. Note that $\boldsymbol{H}(f)(x) = C_0(x) = \int_{A/A\cap\Gamma} f(xa)\, da$.

For $Y \in \mathfrak{a}$, $f \in \underline{C}$ and $x \in G$, we have

$$(Yf)(xa) = \frac{d}{dt} f(a(t)xa)\,|_{t=0}$$

where $a(t)$ is the one parameter subgroup corresponding to Y. Since A is contained in the center of G,

$$\begin{aligned}
(Yf)(xa) &= \frac{d}{dt}\Big|_{t=0} f(xa(t)a) \\
&= \frac{d}{dt}\Big|_{t=0}\Big\{\sum_{\alpha\in D'} C_\alpha(x) \exp 2\pi\sqrt{-1}\langle\alpha, a(t)a\rangle\Big\} \\
&= \frac{d}{dt}\Big|_{t=0}\Big\{\sum_{\alpha\in D'} C_\alpha(x) \exp 2\pi\sqrt{-1}(\langle\alpha, a\rangle + \langle\alpha, a(t)\rangle)\Big\} \\
&= 2\pi\sqrt{-1}\sum_{\alpha\in D'} C_\alpha(x)\langle\alpha, Y\rangle \exp 2\pi\sqrt{-1}\langle\alpha, a\rangle\,.
\end{aligned}$$

Since $\langle Y_j, Y_k\rangle = \frac{1}{4}\delta_{jk}$ for $j, k = 1, \cdots, 2l$, it follows that $4(\Delta f)(xa) = -4\sum_{j=1}^{2l}(Y_j^2 f)(xa) = (2\pi)^2 \sum_{\alpha\in D'} C_\alpha(x)\|\alpha\|^2 \exp 2\pi\sqrt{-1}\langle\alpha, a\rangle$ where $\|\alpha\|^2 = \langle\alpha, \alpha\rangle$.

Define an operator $\boldsymbol{G}\colon \underline{C} \to \underline{C}$ by

$$\boldsymbol{G}(f)(xa) = \frac{1}{(2\pi^2)}\sum_{\alpha\in D'-(0)} \frac{C_\alpha(x)}{\|\alpha\|^2} \exp 2\pi\sqrt{-1}\langle\alpha, a\rangle$$

for $x \in G$ and $f \in \underline{C}$. We can show that $\boldsymbol{G}(f)(xa) = \boldsymbol{G}(f)(yb)$ if $xa = yb$ where $a, b \in A$ ([11] p. 118). Thus $\boldsymbol{G}(f) \in C^\infty(G, \boldsymbol{C})$. We also have $\boldsymbol{G}(f)(x\gamma) = \boldsymbol{G}(f)(x)$ for any $\gamma \in \Gamma$. Hence, $\boldsymbol{G}(f) \in \underline{C}$. It is obvious that

$$4\Delta \boldsymbol{G}(f) = 4\boldsymbol{G}\Delta(f) = f \qquad \text{if} \quad \boldsymbol{H}(f) = 0\,,$$

and $\boldsymbol{G}\circ\boldsymbol{H}(f)=\boldsymbol{H}\circ\boldsymbol{G}(f)=0$ for any $f\in\underline{C}$. Therefore

$$f = \boldsymbol{H}(f)+4\Delta\boldsymbol{G}(f) = \boldsymbol{H}(f)+4\boldsymbol{G}\Delta(f) \qquad \text{for any} \quad f\in\underline{C}\,.$$

Define $\boldsymbol{H}$: $A^{p,q}(\mathfrak{a}, \underline{C})\to A^{p,q}(\mathfrak{a}, \underline{C}')$ and $\boldsymbol{G}$: $A^{p,q}(\mathfrak{a}, \underline{C})\to A^{p,q}(\mathfrak{a}, \underline{C})$ by

$$\boldsymbol{H}(\omega) = \sum_{I,J} \boldsymbol{H}(f_{I\bar{J}})\omega_I\wedge\bar{\omega}_J \qquad \text{for} \quad \omega = \sum_{I,J} f_{I\bar{J}}\omega_I\wedge\bar{\omega}_J$$

and

$$\boldsymbol{G}(\omega) = \sum_{I,J} \boldsymbol{G}(f_{I\bar{J}})\omega_I\wedge\bar{\omega}_J \qquad \text{for} \quad \omega = \sum_{I,J} f_{I\bar{J}}\omega_I\wedge\bar{\omega}_J\,.$$

Then we have

$$\omega = \boldsymbol{H}(\omega)+4\boldsymbol{G}\Delta(\omega) = \boldsymbol{H}(\omega)+4\Delta\boldsymbol{G}(\omega)$$

and

$$\omega = \boldsymbol{H}(\omega)+4\boldsymbol{G}\Box''(\omega) = \boldsymbol{H}(\omega)+4\Box''\boldsymbol{G}(\omega)\,.$$

Obviously $d''\circ\boldsymbol{H}=d'\circ\boldsymbol{H}=0$. Since $\int_{A/A\cap\Gamma}(\bar{X}_jf)(xa)da=\int_{A/A\cap\Gamma}(X_jf)(xa)da$ $=0$ for $j=1, \cdots, l$ and $f\in\underline{C}$, $\boldsymbol{H}\circ d''=\boldsymbol{H}\circ d'=0$. By the definition of $\boldsymbol{H}$, it is obvious that $*\circ\boldsymbol{H}=\boldsymbol{H}\circ *$, so that $\delta''\circ\boldsymbol{H}=\boldsymbol{H}\circ\delta''=0$.

Let $A^*(\mathfrak{a}, \underline{C})=\sum_{p,q} A^{p,q}(\mathfrak{a}, \underline{C})$.

Lemma 4.2. *Let F: $A^*(\mathfrak{a}, \underline{C})\to A^*(\mathfrak{a}, \underline{C})$ be an additive operator which commutes with $\Box''$. Then F commutes with $\boldsymbol{H}$ and $\boldsymbol{G}$. In particular, $\boldsymbol{G}$ commutes with d'' and δ''.*

Proof. See [15] Chapter IV lemma 3.

Proof of Proposition 2.1. Note that the cochain complex $\{A^{0,q}(\mathfrak{a}, \underline{C}), d''\}$ is exactely the cochain complex of $\mathfrak{a}^-$-module $\underline{C}$. The inclusion map ι_0: $\underline{C}'\to\underline{C}$ induces a cochain map ι_0^*: $A^*(\mathfrak{a}^-, \underline{C}')\to A^*(\mathfrak{a}^-, \underline{C})$. In particular, the following diagram commutes

$$\begin{array}{ccc} A^{0,q}(\mathfrak{a}, \underline{C}') & \xrightarrow{\iota_0^*} & A^{0,q}(\mathfrak{a}, \underline{C}) \\ \big\downarrow d'' & & \big\downarrow d'' \\ A^{0,q+1}(\mathfrak{a}, \underline{C}') & \xrightarrow{\iota_0^*} & A^{0,q+1}(\mathfrak{a}, \underline{C})\,. \end{array}$$

Since $d''(\omega)=0$ for any $\omega\in A^{0,q}(\mathfrak{a}, \underline{C}')$, $H^q(\mathfrak{a}^-, \underline{C}')=A^{0,q}(\mathfrak{a}, \underline{C}')$.

Let ι_0^*: $H^q(\mathfrak{a}^-, \underline{C}')\to H^q(\mathfrak{a}^-, \underline{C})$ denote the map induced from the cochain map ι_0^*: $A^*(\mathfrak{a}^-, \underline{C}')\to A^*(\mathfrak{a}^-, \underline{C})$. Since $\boldsymbol{H}\circ d''=d''\circ\boldsymbol{H}$, $\boldsymbol{H}$: $A^{0,q}(\mathfrak{a}, \underline{C})\to A^{0,q}(\mathfrak{a}, \underline{C}')$ induces a linear map $\boldsymbol{H}$: $H^q(\mathfrak{a}^-, \underline{C})\to H^q(\mathfrak{a}^-, \underline{C}')$.

We claim that $\iota_0^*\circ\boldsymbol{H}=id$ and $\boldsymbol{H}\circ\iota_0^*=id$. By definition $\boldsymbol{H}\circ\iota_0^*[\omega]=[\omega]$ for $[\omega]\in H^q(\mathfrak{a}^-, \underline{C}')$. Since $\omega=\boldsymbol{H}(\omega)+4\boldsymbol{G}\Box''(\omega)=\boldsymbol{H}(\omega)+4\boldsymbol{G}d''\delta''\omega=\boldsymbol{H}(\omega)+4d''\boldsymbol{G}\delta''\omega$ for any $\omega\in A^{0,q}(\mathfrak{a}, \underline{C})$ such that $d''\omega=0$, $\iota_0^*\boldsymbol{H}[\omega]=[\omega]$ for any $[\omega]\in H^q(\mathfrak{a}^-, \underline{C})$. It is now obvious that ι_0^* is a $\mathfrak{g}^-$-module homomorphism. q.e.d.

Proof of Theorem 1. Let $A^{0,q}(G/\Gamma, \boldsymbol{C})$ be the space of all $\boldsymbol{C}$-valued C^∞-differential forms on G/Γ of type $(0, q)$. Take a basis $\{X_1, \cdots, X_n\}$ of $\mathfrak{g}^+$ and let $\{\omega_1, \cdots, \omega_n\}$ be the dual basis of $(\mathfrak{g}^+)^*$. We regard an element $\omega \in (\mathfrak{g}^+)^*$ as a holomorphic 1-form on G/Γ. Then any element $\omega \in A^{0,q}(G/\Gamma, \boldsymbol{C})$ can be written as $\omega = \sum f_{\bar{J}} \bar{\omega}_J$ where $\bar{\omega}_J = \bar{\omega}_{j_1} \wedge \cdots \wedge \bar{\omega}_{j_q}$,

$$J = (j_1, \cdots, j_q)\,(1 \leqq j_1 < \cdots < j_q \leqq n) \quad \text{and} \quad f_{\bar{J}} \in \underline{\boldsymbol{C}}\,.$$

The operator $d''\colon A^{0,q}(G/\Gamma, \boldsymbol{C}) \to A^{0,q+1}(G/\Gamma, \boldsymbol{C})$ can be written as

$$d''\omega = \sum_J \Big(\sum_{k=1}^{n} \bar{X}_k f_{\bar{J}}\Big) \bar{\omega}_k \wedge \bar{\omega}_J + f_{\bar{J}} \overline{d\omega_J}$$

for $\omega = \sum_J f_{\bar{J}} \bar{\omega}_J$.

Therefore the Dolbeault cohomology group $H^{0,q}_{d''}(G/\Gamma)$ can be regarded as the Lie algebra cohomology $H^q(\mathfrak{g}^-, \underline{C})$ of $\mathfrak{g}^-$-module $\underline{C}$.

$$\text{(2.2)} \qquad H^{0,q}_{d''}(G/\Gamma) \cong H^q(\mathfrak{g}^-, \underline{C})\,.$$

Regarding $\boldsymbol{C}$ as constant functions on G, we have the inclusion map $\iota\colon \boldsymbol{C} \to \underline{\boldsymbol{C}}$ of $\mathfrak{g}^-$-modules. Now by (2.1), Theorem 1 is equivalent to assert that ι induces an isomorphism on the cohomology groups

$$\iota^*\colon H^q(\mathfrak{g}^-) \to H^q(\mathfrak{g}^-, \underline{C})\,.$$

We prove th the isomorphism $\iota^*\colon H^q(\mathfrak{g}^-) \to H^q(\mathfrak{g}^-, \underline{C})$ by the induction on the dimension of G/Γ. If G is abelian, G/Γ is a complex torus and our claim is well-known. As before, let A be the normal subgroup of G contained in the center of G and $\mathfrak{a}$ be the ideal in $\mathfrak{g}$ corresponding to A. Consider the Hochschild and Serre spectral sequences for $\mathfrak{g}^-$-modules $\boldsymbol{C}$ and $\underline{C}$, and a homomorphism of these spectral sequences induced by the inclusion map $\iota\colon \boldsymbol{C} \to \underline{\boldsymbol{C}}$ [2];

$$E_2(\iota)\colon H^t(\mathfrak{g}^-/\mathfrak{a}^-, H^s(\mathfrak{a}^-, \boldsymbol{C})) \to H^t(\mathfrak{g}^-/\mathfrak{a}^-, H^s/(\mathfrak{a}^-, \underline{C}))$$

for $t, s = 0, 1, 2, \cdots$.

Consider also the $\mathfrak{g}^-$-module $\underline{C}'$. Then we have a commutative diagram of $\mathfrak{g}^-$-modules

$$\begin{tikzcd} \boldsymbol{C} \arrow[rr, "\iota"] \arrow[dr, "j"'] & & \underline{\boldsymbol{C}} \\ & \underline{C}' \arrow[ur, "\iota_0"'] & \end{tikzcd}.$$

This commutative diagram induces the corresponding commutative diagram of spectral sequences

$$\begin{array}{ccc} H^t(\mathfrak{g}^-/\mathfrak{a}^-, H^s(\mathfrak{a}^-, \boldsymbol{C})) & \xrightarrow{E_2(\iota)} & H^t(\mathfrak{g}^-/\mathfrak{a}^-, H^s(\mathfrak{a}^-, \underline{\boldsymbol{C}})) \\ & E_2(j)\searrow \quad \nearrow E_2(\iota_0) & \\ & H^t(\mathfrak{g}^-/\mathfrak{a}^-, H^s(\mathfrak{a}^-, \underline{\boldsymbol{C}}')) \,. & \end{array}$$

By proposition 2.1, we have an isomorphism of $\mathfrak{g}^-$-modules $\iota_0^*\colon H^s(\mathfrak{a}^-, \underline{\boldsymbol{C}}')\to H^s(\mathfrak{a}^-, \underline{\boldsymbol{C}})$. Hence,

$$E_2(\iota_0)\colon H^t(\mathfrak{g}^-/\mathfrak{a}^-, H^s(\mathfrak{a}^-, \underline{\boldsymbol{C}}')) \to H^t(\mathfrak{g}^-/\mathfrak{a}^-, H^s(\mathfrak{a}^-, \underline{\boldsymbol{C}}))$$

is an isomorphism.

We shall show that $E_2(j)$ is an isomorphism. Since $\mathfrak{a}^-$ is contained in the center of $\mathfrak{g}^-$, $\mathfrak{g}^-$ acts trivially on $H^s(\mathfrak{a}^-, \boldsymbol{C})=A^s(\mathfrak{a}^-, \boldsymbol{C})$. Hence,

$$H^t(\mathfrak{g}^-/\mathfrak{a}^-, H^s(\mathfrak{a}^-, \boldsymbol{C})) = H^t(\mathfrak{g}^-/\mathfrak{a}^-, \boldsymbol{C})\otimes H^s(\mathfrak{a}^-, \boldsymbol{C})\,.$$

Since $\mathfrak{a}^-$ acts trivially on $\underline{\boldsymbol{C}}'$, $H^s(\mathfrak{a}^-, \underline{\boldsymbol{C}}')=A^s(\mathfrak{a}^-, \underline{\boldsymbol{C}}')$. Consider the action of $\mathfrak{g}^-$ on $H^s(\mathfrak{a}, \underline{\boldsymbol{C}}')$. For an s-cochain $\omega=\sum_J f_J\overline{\omega}_J\in A^s(\mathfrak{a}^-, \underline{\boldsymbol{C}}')$ and $\overline{X}\in\mathfrak{g}^-$, $L_{\overline{X}}\omega=\sum_J (\overline{X}f_{\overline{J}})\overline{\omega}_J$, since $\mathfrak{a}^-$ is contained in the center of $\mathfrak{g}^-$. Hence, $H^s(\mathfrak{a}^-, \underline{\boldsymbol{C}}')$ and $\underline{\boldsymbol{C}}'\otimes H^s(\mathfrak{a}, \boldsymbol{C})$ are isomorphic as $\mathfrak{g}^-$-modules. Hence, we have

$$\begin{aligned} &H^t(\mathfrak{g}^-/\mathfrak{a}^-, H^s(\mathfrak{a}^-, \underline{\boldsymbol{C}}')) \cong H^t(\mathfrak{g}^-/\mathfrak{a}^-, \underline{\boldsymbol{C}}'\otimes H^s(\mathfrak{a}^-, \boldsymbol{C})) \\ &\cong H^t(\mathfrak{g}^-/\mathfrak{a}^-, \underline{\boldsymbol{C}}')\otimes H^s(\mathfrak{a}^-, \boldsymbol{C})\,. \end{aligned}$$

We now regard $\underline{\boldsymbol{C}}'$ as the vector space of all $\boldsymbol{C}$-valued C^∞-functions on $(G/A)/\pi(\Gamma)$. It is easy to see that this identification is compatible with $\mathfrak{g}^-/\mathfrak{a}^-$-module structure. Thus we have

$$H^t(\mathfrak{g}^-/\mathfrak{a}^-, C^\infty((G/A)/\pi(\Gamma), \boldsymbol{C})) = H^t(\mathfrak{g}^-/\mathfrak{a}^-, \underline{\boldsymbol{C}}')\,.$$

By the assumption of the induction, we get

$$H^t(\mathfrak{g}^-/\mathfrak{a}^-, C^\infty((G/A)/\pi(\Gamma), \boldsymbol{C})) = H^t(\mathfrak{g}^-/\mathfrak{a}^-, \boldsymbol{C})\,.$$

Hence, we have an isomorphism

$$E_2(j)\colon H^t(\mathfrak{g}^-/\mathfrak{a}^-, H^s(\mathfrak{a}^-, \boldsymbol{C})) \to H^t(\mathfrak{g}^-/\mathfrak{a}^-, H^s(\mathfrak{a}^-, \underline{\boldsymbol{C}}'))\,.$$

Thus $E_2(\iota)\colon H^t(\mathfrak{g}^-/\mathfrak{a}^-, H^s(\mathfrak{a}^-, \boldsymbol{C}))\to H^t({}^-\mathfrak{g}/\mathfrak{a}^-, H^s(\mathfrak{a}^-, \underline{\boldsymbol{C}}))$ is an isomorphism. By a theroem on spectral sequence ([13] Chapter 9, §1 Theorem 3), this implies an existence of an isomorphism

$$\iota^*\colon H^q(\mathfrak{g}^-, \boldsymbol{C}) \cong H^q(\mathfrak{g}^-, \underline{\boldsymbol{C}})\,.$$

Combining this (2.1) and (2.2), we get

$$H^{p,q}_{\bar{\partial}}(G/\Gamma) \cong H^q(\mathfrak{g}^-)\otimes\Lambda^p(\mathfrak{g}^+)^*. \qquad \text{q.e.d.}$$

Corollary 1 (Kodaira [9]). *Let r be the dimension of the vector space of all closed holomorphic* 1*-forms on a compact complex parallelisable nilmanifold* G/Γ. *Then* $\dim H^{0,1}_{d''}(G/\Gamma)=r$.

Proof. Let ω be a closed holomorphic 1-form on G/Γ. Then $\omega=\sum_{j=1}^{n}f_j\phi_j$ where $(\phi_1, \cdots, \phi_n)$ is a basis of $(\mathfrak{g}^+)^*$ and f_j $(j=1, \cdots, n)$ are holomorphic functions on G/Γ. Since G/Γ is compact, f_j are constant. Hence, $\omega\in(\mathfrak{g}^+)^*$. Moreover $d\omega=0$ if and only if $\omega([\mathfrak{g}^+, \mathfrak{g}^+])=(0)$. Thus $r=\dim(\mathfrak{g}^+/[\mathfrak{g}^+, \mathfrak{g}^+])$. Since $\dim H^1(\mathfrak{g}^-)=\dim(\mathfrak{g}^-/[\mathfrak{g}^-, \mathfrak{g}^-])=\dim(\mathfrak{g}^+/[\mathfrak{g}^+, \mathfrak{g}^+])$, we have $r=\dim H^{0,1}_{d''}(G/\Gamma)$ by Theorem 1. q.e.d.

Let M be a compact connected complex manifold. Let b_r (resp. $h^{p,q}$) denote $\dim_{\boldsymbol{R}}H^r(M, \boldsymbol{R})$ (resp. $\dim_{\boldsymbol{C}}H^{p,q}_{d''}(M)$).

Corollary 2. *If M is a compact complex parallelisable nilmanifold G/Γ,*

$$b_{2k+1} = 2(h^{0,2k+1}+h^{0,2k}h^{0,1}+\cdots+h^{0,k+1}h^{0,k})$$
$$b_{2k} = 2(h^{0,2k}+h^{0,2k-1}h^{0,1}+\cdots+h^{0,k+1}h^{0,k-1})+(h^{0,k})^2$$

for $2k+1,\ 2k\leqq n=\dim_{\boldsymbol{C}}G$.

Proof. By a theorem of Nomizu [10] (See [11] Corollary 7.28.), $H^r(G/\Gamma, \boldsymbol{R})\simeq H^r(\mathfrak{g}, \boldsymbol{R})$. Thus $H^r(G/\Gamma, \boldsymbol{C})\simeq H^r(\mathfrak{g}, \boldsymbol{C})\simeq H^r(\mathfrak{g}^{\boldsymbol{C}})$. Since $\mathfrak{g}^{\boldsymbol{C}}=\mathfrak{g}^+\oplus\mathfrak{g}^-$ and $[\mathfrak{g}^+, \mathfrak{g}^-]=(0)$, $H^r(\mathfrak{g}^{\boldsymbol{C}})\simeq\sum_{p+q=r}H^p(\mathfrak{g}^+)\otimes H^q(\mathfrak{g}^-)$. Since $\dim H^q(\mathfrak{g}^+)=\dim H^q(\mathfrak{g}^-)=h^{0,q}$ and $\dim H^r(\mathfrak{g}^{\boldsymbol{C}})=b_r$, $b_r=\sum_{p+q=r}h^{0,p}h^{0,q}$. q.e.d.

Example. Let G be a nilpotent Lie group defined by

$$G=\left\{\begin{pmatrix}1 & z_1 & z_3\\ 0 & 1 & z_2\\ 0 & 0 & 1\end{pmatrix}\middle|\, z_1, z_2, z_3\in\boldsymbol{C}\right\}$$

Let Γ be a lattice in G, for example,

$$\Gamma=\left\{\begin{pmatrix}1 & a_1 & a_3\\ 0 & 1 & a_2\\ 0 & 0 & 1\end{pmatrix}\middle|\, a_{,1}\ a_2,\ a_3\in\boldsymbol{Z}+\sqrt{-1}\boldsymbol{Z}\right\}.$$

We can take a basis $\{X_1, X_2, X_3\}$ of $\mathfrak{g}^+$ such that

$$[X_1, X_2]=X_3\,,\quad [X_2, X_3]=[X_1, X_3]=0\,.$$

Then the dual basis $\{\omega_1, \omega_2, \omega_3\}$ satisfies that

$$d\omega_3=-\omega_1\wedge\omega_2\,,\quad d\omega_1=d\omega_2=0\,.$$

Now it follows easily from Theorem 1 that $h^{0,1}=h^{0,2}=2$. Note that $h^{1,0}=3$. By corollary 2, we get

$$b_0=b_6=1\,,\quad b_1=b_5=4\,,\quad b_2=b_4=8\quad\text{and}\quad b_3=10\,.$$

3. Chern classes of holomorphic line bundles over a compact complex parallelisable solvmanifold

Let G be a simply connected complex solvable Lie group and Γ be a lattice of G. We assume the following condition:

(M) *$Ad(G)$ and $Ad(\Gamma)$ have the same Zariski closure in $Aut(\mathfrak{g}^C)$.*

Lemma 3.1. *If G is non-abelian, we have $\Gamma\cap[G,G]\neq\{e\}$.*

Proof. Suppose that $\Gamma\cap[G,G]=\{e\}$. Since $[\Gamma,\Gamma]\subset\Gamma\cap[G,G]$, Γ is abelian, so is $Ad(\Gamma)$. Since $Ad(\Gamma)$ and $Ad(G)$ have the same Zariski closure, $\overline{Ad([G,G])}^z=\overline{[Ad(G),Ad(G)]}^z=[\overline{Ad(G)}^z,\overline{Ad(G)}^z]=[\overline{Ad(\Gamma)}^z=\overline{Ad(\Gamma)}^z]=Ad(\overline{[\Gamma,\Gamma]}^z)=\{e\}$, where $\overline{X}^z$ denotes the Zariski closure of X in $Aut(\mathfrak{g}^C)$. Hence, $[G,G]$ is contained in the center Z of G. Thus G is nilpotent. Since Γ is abelian, G is abelian [11]. This is a contradiction. q.e.d.

Proposition 3.2. $\Gamma_1=\Gamma\cap[G,G]$ *is a lattice of* $[G,G]$.

Proof. At first note the following:
If $\mathfrak{m}$ is an ideal of $\mathfrak{g}$ and ρ_1 (resp. ρ_2) is the representation on $\mathfrak{m}^C$ (resp. $\mathfrak{g}^C/\mathfrak{m}^C$) induced by the adjoint representation $Ad\colon G\to Aut(\mathfrak{g}^C)$, $\rho_1(G)$ and $\rho_1(\Gamma)$ (resp. $\rho_2(G)$ and $\rho_2(\Gamma)$) have the same Zariski closure in $Aut(\mathfrak{m}^C)$ (resp. $Aut(\mathfrak{g}/\mathfrak{m}^C)$).

Now $[G,G]$ is a simply connected nilpotent closed Lie subgroup of G and Γ_1 is a discrete subgroup of $[G,G]$. Let H be the connected closed subgroup of $[G,G]$ such that H/Γ_1 is compact ([11] Proposition 2.5.). We claim that H is a normal subgroup of G. Let $\exp\colon[\mathfrak{g},\mathfrak{g}]\to[G,G]$ be the exponential map. Then $\exp^{-1}(\Gamma_1)=\mathfrak{l}$ is a lattice in the Lie algebra $\mathfrak{h}$ of H and $\mathfrak{l}\otimes\boldsymbol{R}=\mathfrak{h}$ ([11] Theorem 2.12). Since $\Gamma_1=\Gamma\cap[G,G]$ is a normal subgroup of Γ, $\exp Ad(\gamma)L=\gamma(\exp L)\gamma^{-1}\in\Gamma_1$ for any $L\in\mathfrak{l}=\exp^{-1}(\Gamma_1)$ and $\gamma\in\Gamma$. Hence, $Ad(\gamma)\mathfrak{l}\subset\mathfrak{l}$ and $Ad(\gamma)\mathfrak{h}=\mathfrak{h}$ for any $\gamma\in\Gamma$. Since $Ad(G)$ and $Ad(\Gamma)$ have the same Zariski closure in $Aut(\mathfrak{g}^C)$, $Ad(G)\mathfrak{h}=\mathfrak{h}$. Hence, $\mathfrak{h}$ is an ideal in $\mathfrak{g}$. Thus H is a normal subgroup of G.

Since $H\subset[G,G]$ and $\Gamma_1\subset H$, $H\cap\Gamma=H\cap[G,G]\cap\Gamma=H\cap\Gamma_1=\Gamma_1$. Thus $H/H\cap\Gamma$ is compact and $H\cdot\Gamma$ is closed in G ([11] Theorem 1.13). Hence, $H\cdot\Gamma/H$ is a lattice of G/H. We claim that $\Gamma H/H\cap[G/H,G/H]=\{e\}$. Let $a\in\Gamma H/H\cap[G/H,G/H]$. Since $[G/H,G/H]=[G,G]H/H=[G,G]/H$, $a=\gamma H=g_1H$ for some $\gamma\in\Gamma$ and $g_1\in[G,G]$, that is, $\gamma=g_1h$ for some $h\in H$. Since $H\subset[G,G]$, $\gamma\in[G,G]\cap\Gamma=\Gamma_1\subset H$. Hence, $a=\gamma H=H$.

Since $Ad(G/H)$ and $Ad(\Gamma H/H)$ have the same Zariski closure in $Aut(\mathfrak{g}^C/\mathfrak{h}^C)$, G/H is abelian by Lemma 3.1. Hence $H\supset[G, G]$. Thus $H=[G, G]$ and is Γ_1 a lattice of $[G, G]$. q.e.d.

Since $\Gamma\cap[G, G]$ is a lattice of $[G, G]$, $[G, G]\Gamma$ is closed in G ([11] Theorem 1.13.) and $\pi(\Gamma)=\Gamma[G, G]/[G, G]$ is a lattice of $G/[G, G]$. Note that $G/[G, G]\Gamma =(G/[G, G])/\pi(\Gamma)$ is a complex torus. Thus we have a holomorphic fiber bundle $(G/\Gamma, \pi, (G/[G, G])/\pi(\Gamma), [G, G]/[G, G]\cap\Gamma)$. Let T denote the complex torus $G/[G, G]\Gamma$.

Now we denote by $A^{1,1}(G/\Gamma, \boldsymbol{R})$ the vector space of all real differential forms of type (1, 1) on G/Γ. Let $H^{1,1}(G/\Gamma, \boldsymbol{R})$ be the vector space

$$\frac{\{\omega\in A^{1,1}(G/\Gamma, \boldsymbol{R})\,|\,d\omega=0\}}{\{\omega\in A^{1,1}(G/\Gamma, \boldsymbol{R})\,|\,\omega=d\theta,\ \theta \text{ is a real 1-form}\}}\,.$$

We shall characterize $H^{1,1}(G/\Gamma, \boldsymbol{R})$ in terms of the Lie algebra $\mathfrak{g}$ of G.

Proposition 3.3. *Suppose that a lattice Γ of G satisfies the condition (M). Then, for any real closed form α of type $(1, 1)$ on G/Γ, there is a unique real right invariant closed form $\beta\in\Lambda^2(\mathfrak{g}^*)$ of type $(1, 1)$ on G such that $\alpha=\beta+d\eta$ on G/Γ where η is a real 1-form on G/Γ.*

Proof. According to a theorem of Mostow ([8], [11]), for a given real closed 2-form α, there is a real right invariant closed 2-form $\beta\in\Lambda^2\mathfrak{g}^*$ such that

$$\alpha=\beta+d\gamma \tag{3.1}$$

where γ is a real 1-form on G/Γ. Let $\beta=\beta^{0,2}+\beta^{1,1}+\beta^{2,0}$ where $\beta^{p,q}$ is the component of β of type (p, q). Since β is a real form, $\beta^{2,0}=\bar{\beta}^{0,2}$ and $\beta^{1,1}$ is a real form. Let $\gamma=\gamma^{1,0}+\gamma^{0,1}$, $\gamma^{1,0}=\bar{\gamma}^{0,1}$. Taking a basis $\{X_1, \cdots, X_n\}$ of $\mathfrak{g}^+$, let $\{\omega_1, \cdots, \omega_n\}$ be its dual basis of $(\mathfrak{g}^+)^*$. We identify ω_j $(j=1, \cdots, n)$ as holomorphic 1-forms on G/Γ. We then have

$$\gamma^{0,1}=\sum_{j=1}^{n}f_j\bar{\omega}_j$$

where $f_j\in C^\infty(G/\Gamma, \boldsymbol{C})$ for $j=1, \cdots, n$ and

$$\beta^{0,2}=\sum_{j<k}a_{jk}\bar{\omega}_j\wedge\bar{\omega}_k$$

where $a_{jk}\in\boldsymbol{C}$. Since α is of type (1,1), we get

$$\beta^{0,2}+d''\gamma^{0,1}=0 \quad\text{and}\quad \alpha=\beta^{1,1}+d''\gamma^{1,0}+\overline{d''\gamma^{1,0}} \tag{3.2}$$

by comparing the type of forms of both hands. We now have

$$d''\gamma^{0,1} = d''(\sum_{j=1}^{n} f_j\bar{\omega}_j) = \sum_{j=1}^{n}(d''f_j\wedge\bar{\omega}_j + f_j d\bar{\omega}_j)$$
$$= \sum_{j,k=1}^{n}\bar{X}_k f_j\bar{\omega}_k\wedge\bar{\omega}_j - \sum_{j=1}^{n}\sum_{k<l} f_j\bar{C}_{kl}^{j}\bar{\omega}_k\wedge\bar{\omega}_l$$

where C_{kl}^{j} are the structure constant of Lie algebra $\mathfrak{g}^+$ with respect to the basis $\{X_1, \cdots, X_n\}$. By (3.2), we get the equalities

$$a_{kl} = \bar{X}_k f_l - \bar{X}_l f_k - \sum_{j=1}^{n} f_j\bar{C}_{kl}^{j} \qquad \text{for } 1\leqq k<l\leqq n\,. \tag{3.3}$$

Integrating (3.3) on G/Γ, we have

$$\int_{G/\Gamma} a_{kl}\,dg = \int_{G/\Gamma}(\bar{X}_k f_l)dg - \int_{G/\Gamma}(\bar{X}_l f_k)\,dg - \sum_{j=1}^{n}\int_{G/\Gamma} f_j\bar{C}_{kl}^{j}dg \tag{3.4}$$

where dg is an invariant measure on G/Γ. Since G is unimodular, $\int_{G/\Gamma}(\bar{X}_k f_l)dg = \int_{G/\Gamma}(\bar{X}_l f_k)dg = 0$, and we get

$$a_{kl}\int_{G/\Gamma} dg = -\sum_{j=1}^{n}\bar{C}_{kl}^{j}\int_{G/\Gamma} f_j dg\,. \tag{3.5}$$

Let $b_j\in \boldsymbol{C}$ denote $\int_{G/\Gamma} f_j dg/\int_{G/\Gamma} dg$. Then (3.5) can be written as

$$a_{kl} = -\sum_{j=1}^{n} b_j\bar{C}_{kl}^{j}\,. \tag{3.6}$$

$$\beta^{0,2} = \sum_{k<l} a_{kl}\bar{\omega}_k\wedge\bar{\omega}_l = -\sum_{k<l}\sum_{j=1}^{n} b_j\bar{C}_{kl}^{j}\bar{\omega}_k\wedge\bar{\omega}_l$$
$$= \sum_{j=1}^{n} b_j(-\sum_{k<l}\bar{C}_{lk}^{j}\bar{\omega}_k\wedge\bar{\omega}_l) = \sum_{j=1}^{n} b_j(d\bar{\omega}_j) = d(\sum_{j=1}^{n} b_j\bar{\omega}_j)\,.$$

Put $\eta = \sum_{j=1}^{n} b_j\bar{\omega}_j$. We then see that η is of type (0, 1), $\beta^{0,2} = d\eta$ and $\beta^{2,0} = d\bar{\eta}$. By (3.1), we get

$$\alpha = \beta^{1,1} + d(\eta+\bar{\eta}) + d\gamma = \beta^{1,1} + d\theta$$

where $\theta = \eta+\bar{\eta}+\gamma$ is a real 1-form on G/Γ.

It remains to show the uniqueness of $\beta^{1,1}$. It is sufficient to see that if $\beta^{1,1} = d\theta$, θ is a real 1-form, then $\beta^{1,1} = 0$. Put $\beta^{1,1} = \sum_{j,k=1}^{n} a_{jk}\omega_j\wedge\bar{\omega}_k$ and $\theta = \theta^{0,1} + \bar{\theta}^{0,1}$ where $\theta^{0,1} = \sum_{j=1}^{n} g_j\bar{\omega}_j$, $g_j \in C^\infty(G/\Gamma, \boldsymbol{C})$ $(j=1, \cdots, n)$. Since $d'\theta^{0,1} = \sum_{k,j=1}^{n} X_k g_j\omega_k\wedge\bar{\omega}_j$ and $d''\overline{\theta^{0,1}} = \overline{d'\theta^{0,1}} = \sum_{k,j=1}^{n}\bar{X}_k\bar{g}_j\bar{\omega}_j\wedge\omega_j$, we get

$$a_{jk} = X_j g_k - \bar{X}_k\bar{g}_j\,. \tag{3.7}$$

Integrating (3.7) on G/Γ, we have

$$a_{jk}\int_{G/\Gamma}dg=\int_{G/\Gamma}(X_j g_k)dg-\int_{G/\Gamma}(\bar{X}_k\bar{g}_j)dg=0\,.$$

Hence, $a_{jk}=0$ for $j,\ k=1, \cdots, n$ and $\beta^{1,1}=0$. q.e.d.

We now determine real closed right invariant forms of type (1, 1) on G/Γ. Take a basis $\{X_1, \cdots, X_n\}$ of $\mathfrak{g}^+$ such that $\{X_{r+1}, \cdots, X_n\}$ is a basis of $[\mathfrak{g}^+, \mathfrak{g}^+]$. Let $\{\omega_1, \cdots, \omega_n\}$ be its dual basis of $(\mathfrak{g}^+)^*$.

Proposition 3.4. *Let α be a right invariant real 2-form of type* (1, 1) *on G. Then $d\alpha=0$ if and only if $\alpha=\frac{1}{2\sqrt{-1}}\sum_{j,k=1}^{r}h_{jk}\omega_j\wedge\bar{\omega}_k$ where $H=(h_{jk})\in M(r, \boldsymbol{C})$ is a hermitian matrix, and $r=\dim\mathfrak{g}^+/[\mathfrak{g}^+, \mathfrak{g}^+]$.*

Proof. Since α is a right invariant form on G, α defines a bilinear form on $\mathfrak{g}^+\times\mathfrak{g}^-$. Now $d\alpha=0$ if and only if

$$\alpha([X, Y], \bar{Z})=0 \quad\text{and}\quad \alpha([\bar{X}, \bar{Y}], Z)=0 \qquad \text{for } X, Y, Z\in\mathfrak{g}^+\,,$$

since $(d\alpha)(X, Y, Z)=-\alpha([X, Y], Z)+\alpha([X, Z], Y)-\alpha([Y, Z], X)$ for X, Y, $Z\in\mathfrak{g}^C$ and since $[\mathfrak{g}^+, \mathfrak{g}^-]=(0)$. In particular, for a real form α of type (1, 1), we get

$$\text{(3.8)}\qquad d\alpha=0 \text{ if and only if } \iota([X, Y])\alpha=0 \qquad \text{for } X, Y\in\mathfrak{g}^+\,.$$

Note that $d\omega_j=0$ for $j=1, \cdots, r$. Therefore, if $\alpha=\frac{1}{2\sqrt{-1}}\sum_{j,k=1}^{r}h_{jk}\omega_j\wedge\bar{\omega}_k$ then $d\alpha=0$. Conversely, put $\alpha=\frac{1}{2\sqrt{-1}}\sum_{j,k=1}^{n}h_{jk}\omega_j\wedge\bar{\omega}_k$. If α is closed, then $\iota(X_j)\alpha=0$ for $j=r+1, \cdots, n$ by (3.8).

Since $(\iota(X_j)\alpha)(\bar{X}_k)=\alpha(X_j, \bar{X}_k)=\frac{1}{2\sqrt{-1}}h_{jk}$ and $H=(h_{jk})$ is a hermitian matrix, we have $h_{jk}=0$ for $j=r+1, \cdots, n$; $k=1, \cdots, n$ and $j=1, \cdots, n$; $k=r+1, \cdots, n$, so that $\alpha=\frac{1}{2\sqrt{-1}}\sum_{j,k=1}^{r}h_{jk}\omega_j\wedge\bar{\omega}_k$. q.e.d.

Consider a holomorphic line bundle L on G/Γ. Let $C(L)$ denote the Chern class of L. Then we have $C(L)\in H^{1,1}(G/\Gamma, \boldsymbol{R})$ ([15], Chapter V, n°4.).

Proposition 3.5. *Let G be a simply connected complex solvable Lie group and Γ be a lattice of G satisfying the condition* (*M*) *and such that $H^{0,1}_{\bar{\partial}}(G/\Gamma)\cong H^1(\mathfrak{g}^-)$* (*canonically*). *Let L be a holomorphic line bundle on G/Γ. Then there is a unique real invariant form $\alpha\in\Lambda^2\mathfrak{g}^*$ of type* (1, 1) *in $C(L)$, and this is a curvature form of a connection η of type* (1, 0).

Proof. It is easy to see that there is a real closed 2-form β of type (1, 1) in

$C(L)$ which is a curvature form of a connection ω of type (1,0) ([15], Chapter V, n°4).

According to Proposition 3.3, we have $\beta=\alpha+d\gamma$ where is γ a real 1-form on G/Γ. Decompose $\gamma=\gamma^{1,0}+\gamma^{0,1}$ where $\gamma^{1,0}$ (resp. $\gamma^{0,1}$) is the component of type (1, 0) (resp. (0, 1)) of γ. Then we have $d''\gamma^{0,1}=0$, since β and α are of type (1, 1). By the assumption (2), there is a right invariant 1-form θ of type (0, 1) such that $\gamma^{0,1}-\theta=d''f$ where $f\in C^\infty(G/\Gamma, \boldsymbol{C})$.

We can write $\theta=\sum_{j=1}^{r} a_j\bar{\omega}_j$, $a_j\in\boldsymbol{C}$ ($j=1, \cdots, r$), where $\{\omega_1, \cdots, \omega_n\}$ is the same as before, since $H^1(\mathfrak{g}^-)=(\mathfrak{g}^-/[\mathfrak{g}^-, \mathfrak{g}^-])^*$. We then have $d\theta=\sum_{j=1}^{r} a_j d\bar{\omega}_j=0$, so that $\beta=\alpha+d'\gamma^{0,1}+d''\gamma^{1,0}=\alpha+d'\gamma^{0,1}+\overline{d'\gamma^{0,1}}=\alpha+d'(\theta+d''f)+\overline{d'(\theta+d''f)}$ $=\alpha+d'd''(f-\bar{f})$. Put $\psi=d'(\bar{f}-f)$. We then have $\beta=\alpha+dd'(\bar{f}-f)=\alpha+d\psi$. Since β is a curvature from ω of a connection of type (1, 0) by definition and ψ is of type (1, 0), α is a curvature form of a connection $\eta=\omega-\psi$ of type (1, 0).
q.e.d.

From now on we always assume that G and Γ satisfies the assumptions of Proposition 3.5.

Consider a holomorphic line bundle L on G/Γ. We fix a (sufficiently fine) simple covering $\{U_i\}$ on G/Γ and choose a connected component U_{i0} of $p^{-1}(U_i)$ for each i, p: $G\to G/\Gamma$ being the canonical map; let $U_{i\gamma}$ denote the image of U_{i0} under the right translation $R_\gamma(g)=g\gamma$ for $\gamma\in\Gamma$. Then $p^{-1}(U_i)=\bigcup_{\gamma\in\Gamma} U_{i\gamma}$ is a disjoint union and p maps each $U_{i\gamma}$ biholomorphically to U_i.

We may consider a holomorphic line bundle L on G/Γ is given by a system of transition functions $\{g_{jk}\}$ relative to the covering $\{U_i\}$ of G/Γ. Let $C(L)$ be the Chern class of L and α be the unique real right invariant form of type (1, 1) in $C(L)$. By Proposition 3.5, α is a curvature form of a connection η of type (1, 0), so that there is an element $\eta_j\in A^{1,0}(U_j)$ for each j satisfying $\eta_k-\eta_j=\frac{\sqrt{-1}}{2\pi}d\log g_{jk}$ on $U_j\cap U_k\neq\phi$ and $\alpha=d\eta_j$ on U_j.

Proposition 3.6. *Identify* $\mathfrak{g}^+$ *to the complex Lie algebra* $(\mathfrak{g}, I)$. *Then we can take a basis* $\{X_1, \cdots, X_n\}$ *of* $\mathfrak{g}^+$ *such that a map* ψ: $\mathfrak{g}^+\to G$ *defined by*

$$\psi(\sum_{i=1}^{n} z_iX_i) = (\exp z_1X_1) \cdots (\exp z_nX_n)$$

is biholomorphic. In particular, G is biholomorphic to $\boldsymbol{C}^n$. *Moreover G has a system of coordinates* $(z_1, \cdots, z_n)$ *such that, for* $j=1, \cdots, r$, $z_j(gg')=z_j(g)+z_j(g')$ *for any* $g, g'\in G$, *where* $r=\dim \mathfrak{g}^+/[\mathfrak{g}^+, \mathfrak{g}^+]$.

Proof. We prove this proposition by induction on the dimension n of $\mathfrak{g}^+$. Assume that it has been proved for all dimensions $<n$. Since $\mathfrak{g}^+$ is solva-

ble, it has an abelian ideal $\mathfrak{a}^+$ of dimension >0. Let A be the connected complex abelian subgroup of G whose Lie algebra is $\mathfrak{a}^+$; A is simply connected and G/A is a simply connected complex solvable Lie group of complex dimension $<n$. Applying our proposition to G/A, we get a basis $\{X_1^*, \cdots, X_m^*\}$ of ${}^+\mathfrak{g}/\mathfrak{a}^+$ such that a map $\psi^*\colon \mathfrak{g}^+/\mathfrak{a}^+ \to G/A$ defined by

$$\psi^*(\sum_{i=1}^{m} z_i X_i^*) = (\exp z_1 X_1^*) \cdots (\exp z_m X_m^*)$$

is biholomorphic. Take elements $X_1, \cdots, X_m \in \mathfrak{g}^+$ such that $\pi_*(X_i)=X_i^*$ where $\pi_*\colon \mathfrak{g}^+ \to \mathfrak{g}^+/\mathfrak{a}^+$ is a projection. Choose also a basis $\{X_{m+1}, \cdots, X_n\}$ of $\mathfrak{a}^+$. Then every element of A can be written uniquely in the form $(\exp z_{m+1}X_{m+1}) \cdots (\exp z_n X_n)$. Let g be any element of G and $g^*=\pi(g)$ where $\pi\colon G \to G/A$ is a projection. Then we can write uniquely g^* in the form $(\exp z_1 X_1^*)\cdots(\exp z_m X_m^*)$. Hence, we have $g=(\exp z_1 X_1)\cdots(\exp z_m X_m)a$ $(a\in A)$ and a can be written in the form $(\exp z_{m+1}X_{m+1})\cdots(\exp z_n X_n)$, which proves that g is in the form $(\exp z_1 X_1)(\exp z_2 X_2)\cdots(\exp z_n X_n)$. Moreover $z_1, \cdots, z_m$ are uniquely determined by $\pi(g)$ (and a fortiori by g); hence a is determined by g and $z_{m+1}, \cdots, z_n$ are uniquely determined by g. Since exp is holomorphic, z_j $(j=1, \cdots, n)$ are holomorphic functions on G and $\psi\colon \mathfrak{g}^+ \to G$ is biholomorphic.

Since we can choose a basis $\{X_1, \cdots, X_n\}$ of $\mathfrak{g}^+$ in such a way that $\{X_{r+1}, \cdots, X_n\}$ is a basis of $[\mathfrak{g}^+, \mathfrak{g}^+]$ and $\psi\colon \mathfrak{g}^+ \to G$ is biholomorphic, the last assertion follows from the Campbell-Hausdorff formula ([4] p. 170). q.e.d.

We may asrsme that $\omega_j=dz_j$ for $j=1, \cdots, r$ by changing a basis of $\mathfrak{g}^+$ if necessary. Then by Proposition 3.4, we get

$$\alpha = \frac{1}{2\sqrt{-1}} \sum_{j,k=1}^{r} h_{jk}\omega_j \wedge \overline{\omega}_k = \frac{1}{2\sqrt{-1}} \sum_{j,k=1}^{r} h_{jk} dz_j \wedge d\overline{z_k}$$

where (h_{jk}) is a hermitian matrix.

4. Divisors on a compact complex parallelisable solvmanifold

Let M and N be complex manifolds and $\Phi\colon M \to N$ be a surjective holomorphic map. For a divisor $\tilde{D}$ on N, $\Phi^*(\tilde{D})$ denotes the divisor on M defined by $\Phi_x^{-1}(\tilde{D}_{\Phi(x)})$ for all $x\in M$ ([15] Appendice n°7). We call this divisor $\Phi^*(\tilde{D})$ on M the pull back of the divisor $\tilde{D}$ on N. In this section we prove the following theorem.

Theorem 2. *Let G be a simply connected complex solvable Lie group. Let Γ be a lattice of G. Assume that Γ satisfies the condition (M) and that $H_{\bar{\partial}}^{0,1}(G/\Gamma) \cong H^1(\mathfrak{g}^-)$ (canonically). Then, for each positive divisor D on G/Γ, there exists a positive divisor $\tilde{D}$ on the complex torus T such that the divisor D is the pull back of the divisor $\tilde{D}$ on T by the projection $\pi\colon G/\Gamma \to T$, i.e., $D=\pi^*\tilde{D}$.*

If G is nilpotent, the condition (M) is always satisfied ([11] Theorem 2.1). Moreover, by Theorem 1 in the section 2, $H^{0,1}_{d''}(G/\Gamma)\cong H^1(\mathfrak{g}^-)$. Thus we get:

Corollary. *Let G be a simply connected complex nilpotent Lie group and Γ be a lattice of G. Then the conclusion of Theorem 2 holds.*

Let D denote a positive divisor on G/Γ. Take a representative $\{(U_i, f_i)\}$ of D, where $f_i\colon U_i\to \boldsymbol{C}$ is a holomorphic function. Let $L=\{D\}$ denote the holomorphic line bundle corresponding to the divisor D. ([15] Chapter V, n° 6). Let $\{g_{jk}\}$ denote the system of transition functions of $L=\{D\}$ with respect to $\{(U_i, f_i)\}$. We then have $f_j=g_{jk}f_k$ on $U_j\cap U_k\neq\phi$ by definition.

Let M be a complex manifold, $\tilde{M}$ be the universal covering of M and $p\colon \tilde{M}\to M$ be the covering map. Let Π denote the fundamental group $\pi_1(M)$ of M.

A map $j\colon \Pi\times\tilde{M}\to \boldsymbol{C}^*$ is said to be *an automorphic factor* if

(1) the function $z\to j(\sigma, z)$ is holomorphic for any $\sigma\in\Pi$, and

(2) $j(\sigma\tau, z)=j(\sigma, \tau(z))\cdot j(\tau, z)$ for any σ, $\tau\in\Pi$ and any $z\in\tilde{M}$.

Let f be a holomorphic function on $\tilde{M}$ which is not identically zero. f is said to be *automorphic of type j* if

$$f(\sigma(z)) = j(\sigma, z)f(z) \qquad \text{for } z\in\tilde{M} \text{ and } \sigma\in\Pi\,.$$

Proposition 4.1. *Let D be a positive divisor of G/Γ. Then D is the divisor of a holomorphic automorphic function θ on G, for which the automorphic factor $j(\gamma, g)\colon \Gamma\times G\to \boldsymbol{C}^*$ is given by*

$$j(\gamma, g) = \exp 2\pi\sqrt{-1}\Big(\frac{1}{2\sqrt{-1}}\sum_{k,l=1}^{r} h_{kl}z_k(g)\bar{z}_l(\gamma)+C(\gamma)\Big)\,,$$

where $H=(h_{jk})$ is a hermitian matrix determined by the form α in the Chern class $C(L)=C(\{D\})$:

$$\alpha=\frac{1}{2\sqrt{-1}}\sum_{k,l=1}^{r} h_{kl}\,dz_k\wedge d\bar{z}_l\,,$$

and $C(\gamma)\in\boldsymbol{C}$ is a constant depending only on $\gamma\in\Gamma$.

Proof. Let us define $\varphi_{i\gamma}(g)$ for $g\in U_{i\gamma}$ by

$$\varphi_{i\gamma}(g) = \eta_i(p(g))+\frac{1}{2\sqrt{-1}}\sum_{k,l=1}^{r} h_{kl}\,\bar{z}_l(g\gamma^{-1})dz_k\,,$$

where η_i is the component of the connection introduced before. Then $\varphi_{i\gamma}$ is an element of $A^{1,0}(U_{i\gamma}))$ satisfying $d\varphi_{i\gamma}=0$. Since $U_{i\gamma}$ is simply connected, there is a holomorphic function $\psi_{i\gamma}$ satisfying $d\psi_{i\gamma}=\varphi_{i\gamma}$. Define $\theta_{i\gamma}(g)$ for $g\in U_{i\gamma}$ by

$$\theta_{i\gamma}(g) = f_i(p(g)) \exp 2\pi\sqrt{-1}(\psi_{i\gamma}(g))\,.$$

We then have

$$\theta_{i\gamma}(g)=\theta_{j\delta}(g)\exp 2\pi\sqrt{-1}\Big(\frac{1}{2\sqrt{-1}}\sum_{k,l=1}^{r}h_{kl}z_k(g)\bar{z}_l(\gamma\delta^{-1})+C_{i\gamma,j\delta}\Big)$$

on $U_{i\gamma}\cap U_{j\delta}$, where $C_{i\gamma,j\delta}\in\boldsymbol{C}$ is a constant. Applying Proposition 3.6, we get

$$\frac{\sqrt{-1}}{2}d\log g_{ij}(p(g))+\varphi_{i\gamma}(g)-\varphi_{j\delta}(g)=\frac{1}{2\sqrt{-1}}\sum_{k,l=1}^{r}h_{kl}(\bar{z}_l(\delta)-\bar{z}_l(\gamma))dz_k\,.$$

Put $a_{i\gamma,j\delta}=\exp 2\pi\sqrt{-1}\,C_{i\gamma,j\delta}$. $\{a_{i\gamma,j\delta}\}$ satisfies relations

$$a_{i\gamma,j\delta}\cdot a_{j\delta,k\nu}=a_{i\gamma,k\nu}\quad\text{on}\quad U_{i\gamma}\cap U_{j\delta}\cap U_{k\nu}\neq\phi\,, \tag{4.1}$$

since

$$\begin{aligned}a_{i\gamma,j\delta}&=\exp 2\pi\sqrt{-1}\,C_{i\gamma,j\delta}\\&=g_{ij}^{-1}(p(g))\exp 2\pi\sqrt{-1}\Big\{(\psi_{i\gamma}-\psi_{j\delta})+\frac{1}{2\sqrt{-1}}\sum_{k,l=1}^{r}h_{kl}(\bar{z}_l(\gamma)-z_l(\delta))\Big\}\,.\end{aligned}$$

By the principal of monodromy ([15], Chapter V, n°1), there is a system of constant functions $\{b_{i\gamma}\}$ such that

$$a_{i\gamma,j\delta}=b_{i\gamma}^{-1}\cdot b_{j\delta}\,,$$

since G is simply connected and $\{U_i\}$ is an open covering of G. We define a holomorphic function θ on G by

$$\theta(g)=\theta_{i\gamma}(g)\exp 2\pi\sqrt{-1}\Big(\frac{1}{2\sqrt{-1}}\sum_{k,l=1}^{r}h_{kl}z_k(g)\bar{z}_l(\gamma)+b_{i\gamma}\Big)$$

on $g\in U_{i\gamma}$.

We can see easily that θ is well defined and θ is different from zero.

Note that

$$\theta_{i\gamma}(g\gamma)=\theta_{i0}(g)\exp 2\pi\sqrt{-1}\,d_{i\gamma}$$

for $g\in U_{i0}$, where $d_{i\gamma}$ is a constant. In fact, we have

$$d(R_\gamma^*\psi_{i\gamma})-d\psi_{0i}=R_\gamma^*\varphi_{i\gamma}-\varphi_{i0}=0\qquad\text{on}\ U_{i0}\,,$$

and

$$\psi_{i\gamma}(g\gamma)-\psi_{i0}(g)=d_{i\gamma}\qquad\text{on}\ U_{i0}\,.$$

We now show that

$$\theta(g\gamma)=\theta(g)\cdot\exp 2\pi\sqrt{-1}\Big(\frac{1}{2\sqrt{-1}}\sum_{j,k=1}^{r}h_{jk}z_j(g)\bar{z}_k(\gamma)+C(\gamma)\Big)$$

for $g\in G$ and $\gamma\in\Gamma$, where $C(\gamma)$ is a constant. For $g\in U_{i0}$, we have

$$\begin{aligned}\theta(g\gamma) &= \theta_{i\gamma}(g\gamma)\cdot\exp 2\pi\sqrt{-1}\Big(\frac{1}{2\sqrt{-1}}\sum_{k,l=1}^{r} h_{kl}z_k(g\gamma)\bar{z}_l(\gamma)+b_{i\gamma}\Big)\\ &= \theta_{i_0}(g)\cdot\exp 2\pi\sqrt{-1}\Big\{d_{i\gamma}+\frac{1}{2\sqrt{-1}}\sum_{k,l=1}^{r} h_{kl}z_k(g\gamma)\bar{z}_l(\gamma)+b_{i\gamma}\Big\}\,.\end{aligned}$$

Since $\theta(g)=\theta_{i_0}(g)\exp 2\pi\sqrt{-1}\,b_{i_0}$ on U_{i_0}, and since $z_k(g\gamma)=z_k(g)+z_k(\gamma)$ by Proposition 3.6,

$$\theta(g\gamma) = \theta(g)\exp 2\pi\sqrt{-1}\Big\{\frac{1}{2\sqrt{-1}}\sum_{k,l=1}^{r} h_{kl}\,z_k(g)\bar{z}_l(\gamma)+C_i(\gamma)\Big\}$$

for $g\in U_{i_0}$, where $C_i(\gamma)$ is a constant. Since $\theta(g\gamma)$ and

$$\theta(g)\exp 2\pi\sqrt{-1}\Big\{\frac{1}{2\sqrt{-1}}\sum_{k,l=1}^{r} h_{kl}\,z_k(g)\bar{z}_l(\gamma)+C_i(\gamma)\Big\}$$

are holomorphic functions on G, we have

$$\theta(g\gamma) = \theta(g)\exp 2\pi\sqrt{-1}\Big(\frac{1}{2\sqrt{-1}}\sum_{k,l=1}^{r} h_{kl}z_k(g)\bar{z}_l(\gamma)+C(\gamma)\Big)$$

for $g\in G$ and $\gamma\in\Gamma$. By the definition of θ, we have $p^*D=\mathrm{div}(\theta)$. q.e.d.

From now on, let $\boldsymbol{e}$ denote $\exp 2\pi\sqrt{-1}$ and $H(g_1, g_2)=\sum_{k,l=1}^{r} h_{kl}z_k(g_1)\bar{z}_l(g_2)$. Then $j(\gamma, g)=\boldsymbol{e}\Big(\frac{1}{2\sqrt{-1}}H(g, \gamma)+C(\gamma)\Big)$ for $g\in G$ and $\gamma\in\Gamma$.

Since $j(\gamma_1\gamma_2, g)=j(\gamma_1, g)\,j(\gamma_2, g\gamma_1)$, we get

$$C(\gamma_1\gamma_2)\equiv C(\gamma_1)+C(\gamma_2)+\frac{1}{2\sqrt{-1}}H(\gamma_1, \gamma_2) \qquad (\mathrm{mod}\ 1)\,.$$

In particular, $C(e)\in\boldsymbol{Z}$ and

$$C(\gamma^{-1})\equiv -C(\gamma)+\frac{1}{2\sqrt{-1}}H(\gamma, \gamma) \qquad \text{for } \gamma\in\Gamma\,.$$

Lemma 4.2. *$C(\gamma)\in\boldsymbol{R}$ for $\gamma\in[\Gamma, \Gamma]$.*

Proof. Since $[\Gamma, \Gamma]\subset[G, G]$, $H(g, \gamma)=0$ for $\gamma\in[\Gamma, \Gamma]$ and $g\in G$. It is enough to show that $C(\gamma)\in\boldsymbol{R}$ for $\gamma=\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}$, $\gamma_1, \gamma_2\in\Gamma$. In this case,

$$\begin{aligned}C(\gamma)&\equiv C(\gamma_1\gamma_2)+C(\gamma_1^{-1}\gamma_2^{-1})+\frac{1}{2\sqrt{-1}}H(\gamma_1\gamma_2, \gamma_1^{-1}\gamma_2^{-1})\\ &\equiv C(\gamma_1)+C(\gamma_2)+\frac{1}{2\sqrt{-1}}H(\gamma_1, \gamma_2)+C(\gamma_1^{-1})+C(\gamma_2^{-1})+\frac{1}{2\sqrt{-1}}H(\gamma_1, \gamma_2)\\ &\quad+\frac{1}{2\sqrt{-1}}\{-H(\gamma_1, \gamma_1)-H(\gamma_2, \gamma_2)-H(\gamma_2, \gamma_1)-H(\gamma_1, \gamma_2)\}\end{aligned}$$

$$\equiv \frac{1}{2\sqrt{-1}}(H(\gamma_1, \gamma_2)-H(\gamma_2, \gamma_1)) = \frac{1}{2\sqrt{-1}}(H(\gamma_1, \gamma_2)-\overline{H(\gamma_1, \gamma_2)})\in \boldsymbol{R}.$$

q.e.d.

Proposition 4.3. $[\Gamma, \Gamma]$ *is a lattice of* $[G, G]$ *and* $[\Gamma, \Gamma]$ *is a subgroup of finite index of* $\Gamma \cap [G, G]$.

Proof. It follows from Porposition 3.2 that $\Gamma[G, G]/[G, G]$ is a lattice of $G/[G, G]$. Since $G/[G, G]$ is a vector group of dimension $2r=\dim_{\boldsymbol{R}} \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$, $\Gamma[G, G]/[G,G]=\Gamma/\Gamma \cap [G, G]$ is a free abelian group of rank $2r$. On the other hand, since G is simply connected, $\pi_1(G/\Gamma)=\Gamma$ and is Γ finitely generated. It follow that $H_1(G/\Gamma, \boldsymbol{Z})\cong\Gamma/[\Gamma, \Gamma]$. Since $\dim H^1(G/\Gamma, \boldsymbol{R}) = \dim H^1(\mathfrak{g}, \boldsymbol{R}) = \dim_{\boldsymbol{R}} \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] = 2r$ by a theorem of Mostow (cf. [8], [11] Corollary 7.29.), $\Gamma/[\Gamma, \Gamma]$ is then direct sum of a free abelian group of rank $2r$ and a finite group. The group $(\Gamma \cap [G, G])/[\Gamma, \Gamma]$ is finite, because $\Gamma/\Gamma \cap [G, G] \approx (\Gamma/[\Gamma, \Gamma])/(\Gamma \cap [G, G]/[\Gamma, \Gamma])$ is a free abelian group of rank $2r$. Since $[G, G]/\Gamma \cap [G, G]$ is compact by Proposition 3.2, $[G, G]/[\Gamma, \Gamma]$ is compact q.e.d.

Proposition 4.4. $C(\gamma)\in \boldsymbol{Z}$ *for* $\gamma \in [\Gamma, \Gamma]$.

Proof. Let θ be a holomorphic automorphic function on G of type $j(\gamma, g)$. We then have

$$\theta(g\gamma_1) = \theta(g)\boldsymbol{e}(C(\gamma_1)) \qquad \text{for } g\in G \text{ and } \gamma_1\in[\Gamma, \Gamma].$$

Since θ is not identically zero, there is a point $g_0\in G$ such that $\theta(g_0)\neq 0$.

Define a holomorphic function F: $[G, G]\to \boldsymbol{C}$ by $F(g_1)=\theta(g_0 g_1)$. Then F is different from zero and satisfies $F(g_1\gamma_1)=\theta(g_0 g_1 \gamma_1)=\theta(g_0 g_1)\boldsymbol{e}(C(\gamma_1))=F(g_1)\boldsymbol{e}(C(\gamma_1))$ for $g_1\in[G, G]$ and $\gamma_1\in[\Gamma, \Gamma]$ and $F(e)\neq 0$.

Let f: $[G, G]\to \boldsymbol{R}$ denote C^∞-function $|F(g_1)|$. Then $f(g_1\gamma_1)=f(g_1)$ for $\gamma_1\in[\Gamma, \Gamma]$ since $C(\gamma_1)\in \boldsymbol{R}$ by Lemma 4.2.

We also denote by f the function on $[G, G]/[\Gamma, \Gamma]$ induced by f: $[G, G]\to \boldsymbol{R}$. Since $[\Gamma, \Gamma]$ is a lattice of $[G, G]$, $[G, G]/[\Gamma, \Gamma]$ is a compact complex manifold. Hence, f: $[G, G]/[\Gamma, \Gamma]\to \boldsymbol{R}$ is bounded:

$$|F(g_1)| = f(g_1) = f(p(g_1))\leq c$$

for some constant $c>0$.
Since $[G, G]$ is biholomorphic onto $\boldsymbol{C}^m$, a holomorphic bounded function F: $[G, G]\to \boldsymbol{C}$ is constant. Since $F(\gamma_1)=F(e)\boldsymbol{e}(C(\gamma_1))$, $C(\gamma_1)\in \boldsymbol{Z}$. q.e.d.

Let $A(g_1, g_2)=\frac{1}{2\sqrt{-1}}(H(g_1, g_2)-\overline{H(g_1, g_2)})$. We then get

$$\begin{aligned} A(\gamma_1, \gamma_2) &= \frac{1}{2\sqrt{-1}}(H(\gamma_1, \gamma_2)-\overline{H(\gamma_1, \gamma_2)}) \\ &\equiv C(\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1})\equiv 0 \qquad (\text{mod } 1). \end{aligned}$$

We put $d(\gamma)=C(\gamma)-\frac{1}{4\sqrt{-1}}H(\gamma, \gamma)$ for $\gamma\in\Gamma$. We have then

$$d(\gamma\delta)\equiv\frac{1}{2}A(\gamma, \delta)+d(\gamma)+d(\delta) \pmod 1 \quad \text{for } \gamma, \delta\in\Gamma .$$

Let $\rho(\gamma)$ be the imaginary part of $d(\gamma)$. We see that $\rho(\gamma\delta)=\rho(\gamma)+\rho(\delta)$ for $\gamma, \delta\in\Gamma$, that is, $\rho\colon \Gamma\to\boldsymbol{R}$ is a homomorphism. It is clear that Ker $\rho\supset[\Gamma, \Gamma]$. Moreover we have Ker $\rho\supset\Gamma\cap[G, G]$, since $[\Gamma, \Gamma]$ is a subgroup of finite index of $\Gamma\cap[G, G]$. Hence ρ induces a homomorphism $\rho\colon \Gamma/\Gamma\cap[G, G]\to\boldsymbol{R}$.

Since $\pi(\Gamma)=\Gamma\cdot[G, G]/[G, G]\approx\Gamma/\Gamma\cap[G, G]$ and $\pi(\Gamma)$ is a lattice of $G/[G, G]$, $\tilde{\rho}$ can be extended to a homomorphism from $G/[G, G]$ to $\boldsymbol{R}$, so that $\rho\colon \Gamma\to\boldsymbol{R}$ can be extended to a homomorphism $\rho\colon G\to\boldsymbol{R}$.

Consider now the biholomorphic map $\Phi\colon G\to\boldsymbol{C}^n$ given by $\Phi(g)=(z_1(g), \cdots, z_n(g))$. Let $z_j(g)=x_j(g)+\sqrt{-1}\,y_j(g)$ for $j=1, \cdots, r$. Note that $\Phi\colon G\to\boldsymbol{C}^n$ induces a map from $G/[G, G]$ onto $\boldsymbol{C}^r$ given by $\pi(g)\to(z_1(g), \cdots, z_r(g))$. We can write $\rho\colon G\to\boldsymbol{R}$ as

$$\rho(g)=\sum_{j=1}^{r}a_jx_j(g)+\sum_{j=1}^{r}b_jy_j(g)$$

for $g\in G$, where $a_j, b_j\in\boldsymbol{R}, j=1, \cdots, r$.

Define $l\colon G\to\boldsymbol{C}$ by

$$l(g)==\sqrt{-1}\cdot\sum_{j=1}^{r}a_jz_j(g)+\sum_{j=1}^{r}b_jz_j(g) .$$

We have Im $l(g)=\rho(g)$ and $d(\gamma)-l(\gamma)\in\boldsymbol{R}$ for $\gamma\in\Gamma$.
Note that $l\colon G\to\boldsymbol{C}$ is a holomorphic homomorphism.

Since we can regard $A(g_1, g_2)$ as an alternating form on a vector group $G/[G, G]$ such that $A(g_1, g_2)$ takes integers on the lattice $\pi(\Gamma)$, there is a $\boldsymbol{R}$-bilinear form B which is $\boldsymbol{Z}$-valued on the lattice $\pi(\Gamma)$ and $A(g_1, g_2)=B(g_1, g_2)-B(g_2, g_1)$ ([15] Chapter VI, n°2).

Define $\chi\colon \Gamma\to\{z\in\boldsymbol{C}\mid |z|=1\}$ by

$$\chi(\gamma)=\boldsymbol{e}\Big(d(\gamma)-l(\gamma)-\frac{1}{2}B(\gamma, \gamma)\Big) .$$

χ is a character of Γ, since $A(\gamma_1, \gamma_2)\in\boldsymbol{Z}$ for $\gamma_1, \gamma_2\in\Gamma$. Put

$$\psi(\gamma)=\chi(\gamma)\boldsymbol{e}\Big(\frac{1}{2}B(\gamma, \gamma)\Big) \qquad \text{for } \gamma\in\Gamma .$$

We get

$$j(\gamma, g)=\boldsymbol{e}\Big(\frac{1}{2\sqrt{-1}}H(g, \gamma)+\frac{1}{4\sqrt{-1}}H(\gamma, \gamma)+l(\gamma)\Big)\cdot\psi(\gamma)$$

for $\gamma\in\Gamma$ and $g\in G$.

Since $l(g)\colon G\to \boldsymbol{C}$ is a holomorphic map which satisfies $l(g\gamma)=l(g)+l(\gamma)$ for $g\in G$ and $\gamma\in\Gamma$, $j(\gamma, g)$ is equivalent to the automorphic factor

$$\boldsymbol{e}\Big(\frac{1}{2\sqrt{-1}}H(g,\gamma)+\frac{1}{4\sqrt{-1}}H(\gamma,\gamma)\Big)\psi(\gamma)\,.$$

We need the following proposition to show that $\psi\,|\,\Gamma\cap[G, G]=id$.

Proposition 4.5. *Let θ be a holomorphic automorphic function on G of type*

$$j(\gamma, g)=\boldsymbol{e}\Big(\frac{1}{2\sqrt{-1}}H(g,\gamma)+\frac{1}{4\sqrt{-1}}H(\gamma,\gamma)\Big)\cdot\psi(\gamma)\,.$$

Then the hermitian form $H=(h_{jk})$ is non-negative. Moreover $\theta(g\cdot g_0)=\theta(g)$ for $g\in G$, if $g_0\in G$ satisfies $H(g_0, g_0)=0$.

Proof. Let $f\colon G\to\boldsymbol{R}$ denote the function defined by

$$f(g)=|\theta(g)|^2\boldsymbol{e}\Big(\frac{-1}{2\sqrt{-1}}H(g, g)\Big)=|\theta(g)|^2\exp(-\pi H(g, g))\,.$$

We have $f(g\gamma)=f(g)$ for $\gamma\in\Gamma$, so that f induces a function $F\colon G/\Gamma\to\boldsymbol{R}$. Since G/Γ is compact, there is a constant $c>0$ such that $0\leqq F(p(g))\leqq c$ for $g\in G$. Therefore we get

$$f(g)=|\theta(g)|^2\exp(-\pi H(g, g))\leqq c\qquad\text{for } g\in G\,.$$

Thus we have

$$|\theta(g)|^2\leqq c\exp\pi H(g, g)\qquad\text{for } g\in G\,.$$

Suppose that $H(g_1, g_1)<0$ for some $g_1\in G$. Define $g(\tau)\in G$ $(\tau\in\boldsymbol{C})$ by

$$g(\tau)=\Phi^{-1}(\tau z_1(g_1)+z_1(g),\cdots,\tau z_n(g_1)+z_n(g))\,.$$

Then we have $g(0)=g$ and

$$|\theta(g(\tau))|^2\leqq c\exp\pi H(g(\tau), g(\tau))\,.$$

Put $\rho=H(g(\tau), g(\tau))$.

$$\begin{aligned}\rho&=\sum_{j,k=1}^{r}h_{jk}(\tau z_j(g_1)+z_j(g))\cdot\overline{(\tau z_k(g_1)+z_k(g))}\\&=|\tau|^2\cdot\sum_{j,k=1}^{r}h_{jk}z_j(g_1)\bar{z}_k(g_1)+2\operatorname{Re}(\tau H(g_1, g))+H(g, g)\\&=|\tau|^2H(g_1, g_1)+2\operatorname{Re}(\tau H(g_1, g))+H(g, g)\,.\end{aligned}$$

For any $\varepsilon>0$, there is $R>0$ such that $\pi\rho\leqq\log\varepsilon$ for every τ satisfying $|\tau|\geqq R$.

Fix $g_1, g\in G$, and we have

$$|\theta(g(\tau))|^2 \leqq c\varepsilon \qquad \text{for } |\tau| \geqq R\,.$$

Therefore $\theta(g(\tau))$ is a bounded holomorphic function on $\boldsymbol{C}$. Hence $\theta(g(\tau))$ is constant with respect to $\tau \in \boldsymbol{C}$. Tending $\varepsilon \to 0$, we get $|\theta(g(\tau))|^2 = 0$. In particular,

$$|\theta(g)|^2 = |\theta(g(0))|^2 = 0\,.$$

Hence $\theta \equiv 0$ on G, since g can be any element of G. This is a contradiction. Therefore $H = (h_{jk})$ is a non-negative hermitian form.

Take an element $g_0 \in G$ satisfying $H(g_0, g_0) = 0$. Then we have $H(g, g_0) = 0$ for any $g \in G$ since $H(g, g) \geqq 0$ for any $g \in G$. Put

$$g_0(\tau) = \Phi^{-1}(\tau z_1(g_0), \cdots, \tau z_n(g_0)) \in G$$

for $\tau \in \boldsymbol{C}$. Then we have

$$\begin{aligned} &|\theta(g \cdot g_0(\tau))|^2 \leqq c \cdot \exp \tau H(g \cdot g_0(\tau), g \cdot g_0(\tau)) \\ &= c \cdot \exp \pi(H(g, g) + 2 \operatorname{Re} \tau H(g, g_0) + |\tau|^2 H(g_0, g_0)) \\ &= c \cdot \exp \pi H(g, g)\,. \end{aligned}$$

This shows that $\theta(g \cdot g_0((\tau)$ is a bounded holomorphic function with respect to $\tau \in \boldsymbol{C}$. Hence $\theta(g \cdot g_0(\tau))$ is constant with respect to $\tau \in \boldsymbol{C}$. In particular, $\theta(g) = \theta(g \cdot g_0(0)) = \theta(g \cdot g_0(1)) = \theta(g \cdot g_0)$. q.e.d.

Take an element $g_1 \in G$ satisfying $\theta(g_1) \neq 0$. Since $H(g_0, g_0) = 0$ for $g_0 \in [G, G]$, $\theta(gg_0) = \theta(g)$ for $g \in G$. In particular, $\theta(g \cdot \gamma) = \theta(g)$ for $\gamma \in \Gamma \cap [G, G]$. Put $g = g_1 \gamma^{-1}$. Then $0 \neq \theta(g_1) = \theta(g \cdot \gamma) = \theta(g)$. Since

$$\theta(g \cdot \gamma) = \theta(g) \cdot \boldsymbol{e}\Big(\frac{1}{2\sqrt{-1}} H(g, \gamma) + \frac{1}{4\sqrt{-1}} H(\gamma, \gamma)\Big)\psi(\gamma) = \theta(g)\psi(\gamma)\,,$$

$\psi(\gamma) = 1$, for $\gamma \in \Gamma \cap [G, G]$. Note that $B(g, g) = 0$ for $g \in [G, G]$. Hence, $\chi\colon \Gamma \to \{z \in \boldsymbol{C} \mid |z| = 1\}$ satisfies that

$$\chi \mid \Gamma \cap [G, G] \equiv 1\,.$$

Since $\pi(\Gamma) \cong \Gamma/\Gamma \cap [G, G]$, χ induces a character

$$\tilde{\chi}\colon \pi(\Gamma) \to \{z \in \boldsymbol{C} \mid |z| = 1\}\,.$$

Let $\Theta\colon G/[G, G] \to \boldsymbol{C}$ denote the holomorphic function on $G/[G, G]$ induced by $\theta\colon G \to \boldsymbol{C}$ and $\tilde{j}\colon \pi(\Gamma) \times G/[G, G] \to \boldsymbol{C}^*$ the automorphic factor induced by $j\colon \Gamma \times G \to \boldsymbol{C}^*$.

Denote $\tilde{D}$ the divisor on $(G/[G, G])/\pi(\Gamma)$ denfied by the holomorphic automorphic function Θ on $G/[G, G]$. We then get $D = \pi^* \tilde{D}$. Therefore we have proved Theorem 2.

Let D be a divisor on G/Γ. Then there exist positive divisors D^+, D^- on G/Γsuch that D^+ and D^- are relatively prime and $D=D^+-D^-$ ([15], Appendix n°6). By Theorem 2, there are holomorphic theta functions Θ_1, Θ_2 on the complex torus T such that $D^+=\pi^*(\operatorname{div}\Theta_1)$ and $D^-=\pi^*(\operatorname{div}\Theta_2)$.
Since $\pi\colon G/\Gamma\to T$ is onto holomorphic,

$$D = D^+-D^- = \pi^*(\operatorname{div}\Theta_1)-\pi^*(\operatorname{div}\Theta_2)$$
$$= \pi^* \operatorname{div}\left(\frac{\Theta_1}{\Theta_2}\right).$$

Note that $\frac{\Theta_1}{\Theta_2}$ is a memorphic theta function on the complex torus T.

It is easy to see that if the divisor $D=0$ the corresponding automorphic function θ is trivial.

Take a meromorphic function ψ on G/Γ. Let $D=\operatorname{div}(\psi)$. Since $D=\pi^*\operatorname{div}\left(\frac{\Theta_1}{\Theta_2}\right)$, we get that $\psi=\frac{\Theta_1\circ\pi}{\Theta_2\circ\pi}$. Since $\psi(g\gamma)=\psi(\gamma)$ for $g\in G$ are $\gamma\in\Gamma$, $\frac{\Theta_1\circ\pi(g\gamma)}{\Theta_2\circ\pi(g\gamma)}=\frac{\Theta_1\circ\pi(g)}{\Theta_2\circ\pi(g)}$, hence $\frac{\Theta_1}{\Theta_2}$ is a meromorphic function on T. Thus we get that if ψ is a meromorphic function on G/Γ, there is a meromorphic function $\tilde{\psi}$ on the torus T such that $\psi=\pi^*\tilde{\psi}$.

Let $K(G/\Gamma)$ (resp. $K(T)$) denote the field of all meromorphic functions on G/Γ (resp. on T).

We now get the following corollary of Theorem 2.

Corollary *Under the assumptions of Theorem 2, there is a canonical isomorphism $\pi^*\colon K(T)\to K(G/\Gamma)$. In particular, the transcendence degree of $K(G/\Gamma)$ over $\boldsymbol{C}$ is not more than the complex dimention of complex torus T.*

5. Remarks and examples of compact complex parallelisable nilmanifolds

Proposition 5.1. *Let M be a compact complex parallelisable manifold of complex dimension 2. Then M is a complex torus.*

Proof. By a theorem of Wang [14], $M=G/\Gamma$ where G is a simply connected complex Lie group of dimension 2 and Γ is a lattice of G. Let $\{X_1, X_2\}$ be a basis of $\mathfrak{g}^+$ and $\{\omega_1, \omega_2\}$ be the dual basis of $(\mathfrak{g}^+)^*$. We may consider ω_1, ω_2 as holomorphic 1-forms on G/Γ. Since G/Γ is 2 dimensional, ω_1, ω_2 are d-closed; $d\omega_1=d\omega_2=0$. Thus $[X_1, X_2]=0$. Hence, G is abelian and G/Γ is a complex torus. q.e.d.

Now we shall give some examples of compact complex parallelisable nilmanifolds.

(1) Let G be a simply connected complex nilpotent Lie group defined by

$$G=\left\{\left(\begin{array}{ccccc} 1 & z_{12} & \cdots & \cdots & z_{1n} \\ & 1 & \ddots & & z_{2n} \\ & & \ddots & \ddots & \vdots \\ & 0 & & 1 & z_{n-1n} \\ & & & & 1 \end{array}\right) \middle| \, z_{ij}\in \boldsymbol{C},\ i<j \right\}$$

and Γ be a lattice of G defined by

$$\Gamma=\left\{\left(\begin{array}{ccccc} 1 & a_{12} & \cdots & \cdots & a_{1n} \\ & 1 & \ddots & & a_{2n} \\ & & \ddots & \ddots & \vdots \\ & 0 & & 1 & a_{n-1n} \\ & & & & 1 \end{array}\right) \middle| \, a_{ij}\in \boldsymbol{Z}+\sqrt{-1}\boldsymbol{Z},\ i<j \right\}.$$

Then G/Γ is a compact complex parallelisable nilmanifold. In this case, we see that the transcendence degree of $K(G/\Gamma)$ over $\boldsymbol{C}$ is $n-1$.

(2) Let G be a simply connected complex nilpotent Lie group defined by

$$G=\left\{\left(\begin{array}{cccccc} 1 & z_1 & z_2 & \cdots & z_{n-1} & w \\ & 1 & 0 & \cdots & 0 & y_{n-1} \\ & & \ddots & & \vdots & \vdots \\ & & 1 & \ddots & \vdots & \vdots \\ & & & \ddots & \vdots & \vdots \\ & & & 1 & 0 & y_2 \\ & 0 & & & 1 & y_1 \\ & & & & & 1 \end{array}\right) \middle| \begin{array}{l} z_j,\, y_j,\, w\in \boldsymbol{C} \\ j=1, 2, \cdots, n-1 \end{array} \right\}$$

for $n\geqq 2$, and Γ be a lattice of G defined by

$$\Gamma=\left\{\left(\begin{array}{cccccc} 1 & a_1 & a_2 & \cdots & a_{n-1} & c \\ & 1 & 0 & \cdots & 0 & b_{n-1} \\ & & \ddots & & \vdots & \vdots \\ & & 1 & \ddots & \vdots & \vdots \\ & & & \ddots & \vdots & \vdots \\ & & & 1 & 0 & b_2 \\ & 0 & & & 1 & b_1 \\ & & & & & 1 \end{array}\right) \middle| \begin{array}{l} a_j,\, b_j,\, c\in \boldsymbol{Z}+\sqrt{-1}\boldsymbol{Z} \\ j=1, 2, \cdots, n-1 \end{array} \right\}.$$

Then G/Γ is a compact complex parallelisable nilmanifold. In this case, we see that the transcendence degree of $K(G/\Gamma)$ over $\boldsymbol{C}$ is $2(n-1)$.

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