ON COMPACT COMPLEX PARALLELISABLE
SOLVMANIFOLDS

Dedicated to the memory of Taira Honda

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1. Introduction

This paper deals with compact complex solvmanifolds. Our main purpose is to generalize the theory on the divisor group of a complex torus to these manifolds. By a solvmanifold we mean a homogeneous space of solvable Lie group. Let G be a simply connected complex solvable Lie group and Γ be a lattice of G, that is, a discrete subgroup of G such that G/Γ is compact. The de Rham cohomology group and the Dolbeault cohomology group of a compact complex manifold G/Γ play an important role in studying the divisor group of a complex manifold G/Γ. The de Rham cohomology group of a compact solvmanifold G/Γ has been discussed by Matsushima [7], Nomizu [10] and Mostow [8].

Let M be a compact connected complex manifold and $H^{p,q}_{\bar{\partial}}(M)$ denote the Dolbeault cohomology group of M of type (p, q). Let $\mathfrak{g}$ be a complex Lie algebra and I be the canonical complex structure of $\mathfrak{g}$. Then $\mathfrak{g}^C = \mathfrak{g}^+ \oplus \mathfrak{g}^-$, where $\mathfrak{g}^\pm = \{X \in \mathfrak{g}^C \,|\, IX = \pm\sqrt{-1}\,X\}$. In section 2, we prove:

Theorem 1. *Let G be a simply connected complex nilpotent Lie group and Γ be a lattice of G. Then there is a canonical isomorphism*

$$H^{p,q}_{\bar{\partial}}(G/\Gamma) \cong H^q(\mathfrak{g}^-) \otimes \Lambda^p(\mathfrak{g}^+)^*$$

where $H^q(\mathfrak{g}^-)$ denotes the Lie algebra cohomology group of $\mathfrak{g}^-$ and $(\mathfrak{g}^+)^*$ denotes the dual vector space of $\mathfrak{g}^+$.

Let G be a simply connected complex solvable Lie group and Γ be a lattice of G which has the following property:

(M) *$Ad(G)$ and $Ad(\Gamma)$ have the same Zariski closure in the group $Aut(\mathfrak{g}^C)$.*

This condition has been used by Mostow in his study of lattices of solvable

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Lie group [8]. Denote by $[G, G]$ the commutator group of G and let $\pi\colon G \to G/[G, G]$ be the projection. Then $\Gamma \cap [G, G]$ is a lattice of $[G, G]$, so that $\pi(\Gamma)$ is a lattice of $G/[G, G]$ and $(G/\Gamma, \pi, (G/[G, G])/\pi(\Gamma), [G, G]/([G, G] \cap \Gamma))$ is a homlomorphic fiber bundle. Let T denote the complex torus $(G/[G, G])/\pi(\Gamma)$. In section 3, we study Chern classes of holomorphic line bundles over these compact complex solvmanifolds.

Let M and N be complex manifolds and $\phi\colon M \to N$ be a surjective holomorphic map. For a divisor $\tilde{D}$ on N let $\phi^*(\tilde{D})$ denote the divisor on M defined by $\phi_x^{-1}(\tilde{D}_{\phi(x)})$ for all $x \in M$. We call the divisor $\phi^*(\tilde{D})$ on M the pull back of the divisor $\tilde{D}$ on N [15]. In section 4, we prove:

Theorem 2. *Let G be a simply connected complex solvable Lie group and Γ be a lattice of G. Assume that Γ satisfies the condition (M) and that $H^{0,1}_{\bar{\partial}}(G/\Gamma) \cong H^1(\mathfrak{g}^-)$ canonically. Then, under the notation introduced above, for each positive divisor D on G/Γ, there exists a positive divisor $\tilde{D}$ on the complex torus T such that the divisor D is the pull back of the divisor $\tilde{D}$ on T by the projection $\pi\colon G/\Gamma \to T$, i.e., $D = \pi^*\tilde{D}$.*

Note that our assumption in Theorem 2 is always satisfied if G is a simply connected complex nilpotent Lie group and Γ is a lattice of G.

If M is a compact connected complex manifold, $K(M)$ will denote the field of all meromorphic functions on M.

Corollary. *Under the condition of Theorem 2, there is a canonical isomorphism*

$$\pi^*\colon K(T) \cong K(G/\Gamma).$$

In particular, the transcendence degree of $K(G/\Gamma)$ over C is not larger than the complex dimension of the complex torus T.

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2. Dolbeault cohomology groups of compact complex nilmanifolds

Let M be a complex manifold and $H^{p,q}_{\bar{\partial}}(M)$ denote the Dolbeault cohomology of M of type (p, q). Let G be a simply connected complex Lie group and Γ be a uniform lattice of G. Let $\mathfrak{g}$ denote the Lie algebra of all right invariant vector fields on G, I denote the complex structure of $\mathfrak{g}$ and $\mathfrak{g}^+$ (resp. $\mathfrak{g}^-$) denote the vector space of the $\sqrt{-1}$ (resp. $-\sqrt{-1}$) eigenvectors of I in the complexification $\mathfrak{g}^C$ of $\mathfrak{g}$. We identify $\mathfrak{g}^+$ to the Lie algebra of all right invariant holomorphic vector fields on G and the dual space $(\mathfrak{g}^+)^*$ to the space of all right invariant holomorphic 1-forms on G. Moreover we may identify an element of

$\mathfrak{g}^+$ (resp. $(\mathfrak{g}^+)^*$) to a holomorphic vector field (resp. a holomorphic 1-form) on G/Γ. Let $\Lambda^p T^*(G/\Gamma)$ be the p-th exterior product bundle of the holomorphic cotangent bundle $T^*(G/\Gamma)$ of G/Γ. Since G/Γ is a compact complex parallelisable manifold, the holomorphic vector bundle $\Lambda^p T^*(G/\Gamma)$ on G/Γ is the trivial vector bundle $G/\Gamma \times \Lambda^p(\mathfrak{g}^+)^*$. Thus we have an isomorphism

$$(2.1) \qquad\qquad H^{p,q}_{\bar{\partial}}(G/\Gamma) \cong H^{0,q}_{\bar{\partial}}(G/\Gamma) \otimes \Lambda^p(\mathfrak{g}^+)^* .$$

Theorem 1. *Let G be a simply connected complex nilpotent Lie group and Γ be a lattice of G. Then we have a canonical isomorphism*

$$H^{p,q}_{\bar{\partial}}(G/\Gamma) \cong H^q(\mathfrak{g}^-) \otimes \Lambda^p(\mathfrak{g}^+)^*$$

where $H^q(\mathfrak{g}^-)$ denoted the q-th Lie algebra cohomology of with the trivial representation $\rho_0: \mathfrak{g}^- \to \mathbf{C}$.

We need some preparations to prove Theorem 1. Consider the descending central series $\{C^k(G)\}$ of G, where $C^k(G) = [G, C^{k-1}(G)]$ and $C^0(G) = G$. Since G is nilpotent, there is an integer $m \in N$ such that $C^m(G) \neq (e)$ and $C^{m+1}(G) = (e)$. Let A denote the group $C^m(G)$. Then A is contained in the center $Z(G)$ of G. Since G is a simply connected nilpotent Lie group and A is connected, A is a simply connected closed Lie subgroup. Let Γ be a lattice of G. Then $A \cap \Gamma$ is a lattice of A ([11] p. 31 Corollary 1) and $A\Gamma$ is closed in G ([11] p. 23 Theorem 1.13). Let $\pi: G \to G/A$ be the canonical map. Then $\pi(\Gamma)$ is a lattice of G/A. Since $A/(A \cap \Gamma) \cong A\Gamma/\Gamma$ is a complex torus, we have a holomorphic principal fiber bundle $(G/\Gamma, (G/A)/\pi(\Gamma), \pi, A/(A \cap \Gamma))$.

Let $C^\infty(G, \mathbf{C})$ be the vector space of all complex valued C^∞-functions on G. Define the subspaces $\underline{C}$ and $\underline{C}'$ of $C^\infty(G, \mathbf{C})$ by

$$\underline{C} = \{f \in C^\infty(G, \mathbf{C}) \mid f(g\gamma) = f(g) \quad \text{for all } \gamma \in \Gamma\}$$

and

$$\underline{C}' = \{f \in \underline{C} \mid f(ga) = f(g) \quad \text{for all } a \in A\} .$$

For a right invariant vector field $X \in \mathfrak{g}$ and $f \in C^\infty(G, \mathbf{C})$, put

$$(Xf)(g) = \frac{d}{dt} f(a(t)g)|_{t=0}$$

where $a(t)$ is the one parameter subgroup corresponding to X. Then $C^\infty(G, \mathbf{C})$ is a $\mathfrak{g}$-module, and hence $\underline{C}$ and $\underline{C}'$ are $\mathfrak{g}^c$-submodules of $C^\infty(G, \mathbf{C})$.

Let $\mathfrak{a}$ be the Lie subalgebra of $\mathfrak{g}$ corresponding to the complex Lie subgroup A of G. Then $\mathfrak{a}^c$ has the decomposition $\mathfrak{a}^c = \mathfrak{a}^+ \oplus \mathfrak{a}^-$ with respect to the complex structure I, and $\underline{C}$ and $\underline{C}'$ are $\mathfrak{a}^-$-modules. Let $\{A^q(\mathfrak{a}^-, \underline{C}), d\}$ (resp. $\{A^q(\mathfrak{a}^-, \underline{C}'), d\}$ denote the cochain complex of $\mathfrak{a}^-$-module $\underline{C}$ (resp. $\underline{C}'$) and $H^*(\mathfrak{a}^-, \underline{C})$ (resp. $H^*(\mathfrak{a}^-, \underline{C}')$) denote the Lie algebra cohomology of $\mathfrak{a}^-$-module

$\underline{C}$ (resp. $\underline{C}'$).   Since $\mathfrak{a}^-$ is an ideal of $\mathfrak{g}^-$, $A^q(\mathfrak{a}^-, \underline{C})$ (resp. $A^q(\mathfrak{a}^-, \underline{C}')$ is $\mathfrak{g}^-$-module by

$$(L_{\bar{X}}\omega(\bar{X}_1, \cdots, \bar{X}_q) = \bar{X}(\omega(\bar{X}_1, \cdots, \bar{X}_q)) - \sum_{j=1}^{q} \omega(\bar{X}_1, \cdots, [\bar{X}, \bar{X}_j], \cdots, \bar{X}_q)$$

where $\bar{X} \in \mathfrak{g}^-$, $\omega \in A^q(\mathfrak{a}^-, \underline{C})$ (resp. $\omega \in A^q(\mathfrak{a}^-, \underline{C}')$) and $\bar{X}_1, \cdots, \bar{X}_q \in \mathfrak{a}^-$.   Moreover $L_{\bar{X}} \circ d = d \circ L_{\bar{X}}$ for all $\bar{X} \in \mathfrak{g}^-$.   Thus $H^*(\mathfrak{a}^-, \underline{C})$ and $H^*(\mathfrak{a}^-, \underline{C}')$ are $\mathfrak{g}^-$-modules.

Proposition 2.1. *The inclusion map* $\iota_0 \colon \underline{C}' \to \underline{C}$ *induces an isomorphism* ι_0^* *of* $\mathfrak{g}^-$-*modules*

$$\iota_0^* \colon H^q(\mathfrak{a}^-, \underline{C}') \to H^q(\mathfrak{a}^-, \underline{C}).$$

This follows from Kodaira and Spencer [6] §2, but we shall give an elementary proof (cf. [11] VII §4).

Let $\{X_1, \cdots, X_l\}$ be a basis of $\mathfrak{a}^+$ and $\{\omega_1, \cdots, \omega_l\}$ be the dual basis. We reagrd ω_j $(j=1, \cdots, l)$ as the holomorphic invariant 1-forms on the complex torus $A/(A \cap \Gamma)$.   Define an invariant hermitian metric h on $A/(A \cap \Gamma)$ by $h = \sum_{j=1}^{l} \omega_j \cdot \bar{\omega}_j$.   Let Ω be the associated form of type $(1, 1)$.   Then

$$\Omega = \sqrt{-1} \sum_{j=1}^{l} \omega_j \wedge \bar{\omega}_j,$$

and $\frac{1}{l!}\Omega^l$ defines a Haar measure da on $A/A \cap \Gamma$. We may assume that $\int_{A/A \cap \Gamma} \frac{1}{l!}\Omega^l = 1$ by changing the choice of a basis of $\mathfrak{a}^+$ if necessary.   For $f \in \underline{C}$ and $x \in G$, let $f_x(a) = f(xa)$ for $a \in A$. Then we can define a $\mathfrak{g}^C$-module homomorphism $H \colon \underline{C} \to \underline{C}'$ by

$$H(f)(x) = \int_{A/A \cap \Gamma} f_x(a) \frac{\Omega^l}{l!} = \int_{A/A \cap \Gamma} f(xa) da.$$

Let $Y_j = \frac{1}{2}(X_j + \bar{X}_j)$ and $Y_{j+l} = \frac{\sqrt{-1}}{2}(X_j - \bar{X}_j)$ for $j = 1, \cdots, l$.   Then $\{Y_1, \cdots, Y_{2l}\}$ is a basis of $\mathfrak{a}$. Let $\{\theta_1, \cdots, \theta_{2l}\}$ be its dual basis. Let $A^r(\mathfrak{a}, \underline{C})$ denote the vector space of all $\underline{C}$-valued r-forms on $A/A \cap \Gamma$. Note that each element $\omega \in A^r(\mathfrak{a}, \underline{C})$ can be written uniquely as

$$\omega = \sum_{k_1 < \cdots < k_r} f_{k_1 \cdots k_r} \theta_{k_1} \wedge \cdots \wedge \theta_{k_r}, \qquad \text{where } f_{k_1 \cdots k_r} \in \underline{C}.$$

For simplicity, let $\theta_K = \theta_{k_1} \wedge \cdots \wedge \theta_{k_r}$ and $f_K = f_{k_1 \cdots k_r}$ for $K = (k_1, \cdots, k_r)$ $(1 \le k_1 < \cdots < k_r \le 2l)$.   Then $\omega = \sum_K f_K \theta_K$.

Let $A^{p,q}(\mathfrak{a}, \underline{C})$ denote the vector space of all $\underline{C}$-valued forms of type (p, q) on $A/A \cap \Gamma$.   Each element $\omega \in A^{p,q}(\mathfrak{a}, \underline{C})$ can be written uniquely as

$$\omega = \sum_{I,J} f_{I\bar{J}}\omega_I \wedge \overline{\omega}_J$$

where $I = (i_1, \cdots, i_p)$ $(1 \leqq i_1 < \cdots < i_p \leqq l)$, $J = (j_1, \cdots, j_q)$ $(1 \leqq j_1 < \cdots < j_q \leqq l)$, $f_{I\bar{J}} \in \underline{C}$, $\omega_I = \omega_{i_1} \wedge \cdots \wedge \omega_{i_p}$ and $\overline{\omega}_J = \overline{\omega}_{j_1} \wedge \cdots \wedge \overline{\omega}_{j_q}$.
Define operators $d: A^r(\mathfrak{a}, \underline{C}) \to A^{r+1}(\mathfrak{a}, \underline{C})$ by

$$d\omega = \sum_K (\sum_{j=1}^{2l} Y_j f_K)\theta_j \wedge \theta_K$$

for $\omega = \sum_K f_K \theta_K \in A^r(\mathfrak{a}, \underline{C})$, $d': A^{p,q}(\mathfrak{a}, \underline{C}) \to A^{p+1,q}(\mathfrak{a}, \underline{C})$ by

$$d'\omega = \sum_{I,J}^l (\sum_{k=1}^l X_k f_{I\bar{J}})\omega_k \wedge \omega_I \wedge \overline{\omega}_J$$

for $\omega = \sum_{I,J} f_{I\bar{J}}\omega_I \wedge \overline{\omega}_J \in A^{p,q}(\mathfrak{a}, \underline{C})$ and $d'': A^{p,q}(\mathfrak{a}, \underline{C}) \to A^{p,q+1}(\mathfrak{a}, \underline{C})$ by

$$d''\omega = \sum_{I,J}^l (\sum_{k=1}^l \overline{X}_k f_{I\bar{J}})\overline{\omega}_k \wedge \omega_I \wedge \overline{\omega}_J$$

for $\omega = \sum_{I,J} f_{I\bar{J}}\omega_I \wedge \overline{\omega}_J \in A^{p,q}(\mathfrak{a}, \mathbf{C})$. Then $d \circ d = d' \circ d' = d'' \circ d'' = 0$.
Define $\langle \omega, \eta \rangle \in \underline{C}'$ for $\omega, \eta \in A^{p,q}(\mathfrak{a}, \underline{C})$ by

$$\langle \omega, \eta \rangle(x) = \sum_{I,J} \int_{A/A \cap \Gamma} f_{I\bar{J}}(xa)\overline{g}_{I\bar{J}}(xa)da = \int_{A/A \cap \Gamma} \omega \wedge \overline{*\eta},$$

where $\omega = \sum_{I,J} f_{I\bar{J}}\omega_I \wedge \overline{\omega}_J$, $\eta = \sum_{I,J} g_{I\bar{J}}\omega_I \wedge \overline{\omega}_J$ and $*$ is the operation defined by the natural orientation of $A/A \cap \Gamma$ and the metric h on $A/A \cap \Gamma$.

Let $\tilde{f} \in C^\infty(G/A\Gamma, \mathbf{C})$ denote the function corresponding to $f \in \underline{C}'$. Define a hermitian inner product $(,)$ on $A^{p,q}(\mathfrak{a}, \underline{C})$ by

$$(\omega, \eta) = \int_{G/A\Gamma} \widetilde{\langle \omega, \eta \rangle}(x)dx$$

where dx denotes an invariant measure on $G/A\Gamma$.

Define $(\omega, \eta) = 0$ if $\omega \in A^{p,q}(\mathfrak{a}, \underline{C})$, $\eta \in A^{p',q'}(\mathfrak{a}, \underline{C})$ for $(p, q) \neq (p', q')$. Since $A^r(\mathfrak{a}, \underline{C}) = \sum_{p+q=r} A^{p,q}(\mathfrak{a}, \underline{C})$, we have thus an hermitian inner product $(,)$ on $A^r(\mathfrak{a}, \underline{C})$.

Now define the adjoint operators $\delta, \delta', \delta'''$ of d, d', d'' by $\delta = -*d*$, $\delta' = -*d''*$, $\delta'' = -*d'*$ respectively. We then have

$$(d\omega, \eta) = (\omega, \delta\eta) \quad \text{for} \quad \omega \in A^r(\mathfrak{a}, \underline{C}) \quad \text{and} \quad \eta \in A^{r+1}(\mathfrak{a}, \underline{C}),$$
$$(d'\omega, \eta) = (\omega, \delta'\eta) \quad \text{for} \quad \omega \in A^{p,q}(\mathfrak{a}, \underline{C}) \quad \text{and} \quad \eta \in A^{p+1,q}(\mathfrak{a}, \underline{C}),$$
$$(d''\omega, \eta) = (\omega, \delta''\eta) \quad \text{for} \quad \omega \in A^{p,q}(\mathfrak{a}, \underline{C}) \quad \text{and} \quad \eta \in A^{p,q+1}(\mathfrak{a}, \underline{C}).$$

with respect to the hermidian inner product $(,)$.
Define Laplacians $\Delta, \square', \square''$ by

$$\Delta = d\delta + \delta d, \quad \square' = d'\delta' + \delta'd', \quad \square'' = d''\delta'' + \delta''d''.$$

Then, by a direct computation we get

$$\Delta\omega = -\sum_K (\sum_{j=1}^{2l} Y_j^2 f_K)\theta_K$$

for $\omega = \sum_K f_K \theta_K$, and

$$\square'\omega = \square''\omega = -\sum_{I,J}(\sum_{j=1}^{l} X_j \bar{X}_j)f_{I\bar{J}}\omega_I \wedge \bar{\omega}_J$$

for $\omega = \sum_{I,J} f_{I\bar{J}}\omega_I \wedge \bar{\omega}_J$.

Since $X_j\bar{X}_j f = (Y_j^2 + Y_{j+l}^2)f$ for each $f \in \underline{C}$, we see $\Delta = \square' = \square''$.

Since A is abelian and simply connected, we may identify A (resp. the lattice $A \cap \Gamma$ of A) with Euclidean space $(\mathbf{R}^n, \langle\,,\,\rangle)$ (resp. a lattice D in $\mathbf{R}^n$). For a fixed $x \in G$ and $f \in \underline{C}$, f_x can be regarded as a function on the torus $\mathbf{R}^n/D$. Consider the Fourier expansion of f_x,

$$f_x(a) = f(xa) = \sum_{\alpha \in D'} C_\alpha(x) \exp 2\pi\sqrt{-1}\langle \alpha, a\rangle$$

where $D' = \{\alpha \in \mathbf{R}^n | \langle\alpha, d\rangle \in \mathbf{Z}$ for any $d \in D\}$ and $C_\alpha(x) = \int_{A/A\cap\Gamma} f(xa) \exp -2\pi\sqrt{-1}\langle\alpha, a\rangle da$ for $\alpha \in D'$. Note that $\mathbf{H}(f)(x) = C_0(x) = \int_{A/A\cap\Gamma} f(xa)da$.

For $Y \in \mathfrak{a}$, $f \in \underline{C}$ and $x \in G$, we have

$$(Yf)(xa) = \frac{d}{dt}f(a(t)xa)|_{t=0}$$

where $a(t)$ is the one parameter subgroup corresponding to Y. Since A is contained in the center of G,

$$(Yf)(xa) = \frac{d}{dt}|_{t=0}f(xa(t)a)$$

$$= \frac{d}{dt}|_{t=0}\{\sum_{\alpha \in D'} C_\alpha(x) \exp 2\pi\sqrt{-1}\langle\alpha, a(t)a\rangle\}$$

$$= \frac{d}{dt}|_{t=0}\{\sum_{\alpha \in D'} C_\alpha(x) \exp 2\pi\sqrt{-1}(\langle\alpha, a\rangle + \langle\alpha, a(t)\rangle)\}$$

$$= 2\pi\sqrt{-1}\sum_{\alpha \in D'} C_\alpha(x)\langle\alpha, Y\rangle \exp 2\pi\sqrt{-1}\langle\alpha, a\rangle .$$

Since $\langle Y_j, Y_k\rangle = \frac{1}{4}\delta_{jk}$ for $j, k = 1, \cdots, 2l$, it follows that $4(\Delta f)(xa) = -4\sum_{j=1}^{2l}(Y_j^2 f)(xa) = (2\pi)^2 \sum_{\alpha \in D'} C_\alpha(x)\|\alpha\|^2 \exp 2\pi\sqrt{-1}\langle\alpha, a\rangle$ where $\|\alpha\|^2 = \langle\alpha, \alpha\rangle$.

Define an operator $\mathbf{G}: \underline{C} \to \underline{C}$ by

$$\mathbf{G}(f)(xa) = \frac{1}{(2\pi^2)} \sum_{\alpha \in D'-(0)} \frac{C_\alpha(x)}{\|\alpha\|^2} \exp 2\pi\sqrt{-1}\langle\alpha, a\rangle$$

for $x \in G$ and $f \in \underline{C}$. We can show that $\mathbf{G}(f)(xa) = \mathbf{G}(f)(yb)$ if $xa = yb$ where $a, b \in A$ ([11] p. 118). Thus $\mathbf{G}(f) \in C^\infty(G, \mathbf{C})$. We also have $\mathbf{G}(f)(x\gamma) = \mathbf{G}(f)(x)$ for any $\gamma \in \Gamma$. Hence, $\mathbf{G}(f) \in \underline{C}$. It is obvious that

$$4\Delta G(f) = 4G\Delta(f) = f \qquad \text{if} \quad H(f) = 0 \,,$$

and $G \circ H(f) = H \circ G(f) = 0$ for any $f \in \underline{C}$. Therefore

$$f = H(f) + 4\Delta G(f) = H(f) + 4G\Delta(f) \qquad \text{for any} \quad f \in \underline{C} \,.$$

Define $H: A^{p,q}(\mathfrak{a}, \underline{C}) \to A^{p,q}(\mathfrak{a}, \underline{C}')$ and $G: A^{p,q}(\mathfrak{a}, \underline{C}) \to A^{p,q}(\mathfrak{a}, \underline{C})$ by

$$H(\omega) = \sum_{I,J} H(f_{I\bar{J}}) \omega_I \wedge \bar{\omega}_J \qquad \text{for} \quad \omega = \sum_{I,J} f_{I\bar{J}} \omega_I \wedge \bar{\omega}_J$$

and

$$G(\omega) = \sum_{I,J} G(f_{I\bar{J}}) \omega_I \wedge \bar{\omega}_J \qquad \text{for} \quad \omega = \sum_{I,J} f_{I\bar{J}} \omega_I \wedge \bar{\omega}_J \,.$$

Then we have

$$\omega = H(\omega) + 4G\Delta(\omega) = H(\omega) + 4\Delta G(\omega)$$

and

$$\omega = H(\omega) + 4G\square''(\omega) = H(\omega) + 4\square'' G(\omega) \,.$$

Obviously $d'' \circ H = d' \circ H = 0$. Since $\int_{A/A \cap \Gamma} (\bar{X}_j f)(xa) da = \int_{A/A \cap \Gamma} (X_j f)(xa) da$ $= 0$ for $j = 1, \cdots, l$ and $f \in \underline{C}$, $H \circ d'' = H \circ d' = 0$. By the definition of H, it is obvious that $* \circ H = H \circ *$, so that $\delta'' \circ H = H \circ \delta'' = 0$.

Let $A^*(\mathfrak{a}, \underline{C}) = \sum_{p,q} A^{p,q}(\mathfrak{a}, \underline{C})$.

Lemma 4.2. *Let* $F: A^*(\mathfrak{a}, \underline{C}) \to A^*(\mathfrak{a}, \underline{C})$ *be an additive operator which commutes with* $\square''$. *Then* F *commutes with* H *and* G. *In particular,* G *commutes with* d'' *and* δ''.

Proof.   See [15] Chapter IV lemma 3.

Proof of Proposition 2.1.   Note that the cochain complex $\{A^{0,q}(\mathfrak{a}, \underline{C}), d''\}$ is exactly the cochain complex of $\mathfrak{a}^-$-module $\underline{C}$. The inclusion map $\iota_0: \underline{C}' \to \underline{C}$ induces a cochain map $\iota_0^*: A^*(\mathfrak{a}^-, \underline{C}') \to A^*(\mathfrak{a}^-, \underline{C})$. In particular, the following diagram commutes

$$
\begin{array}{ccc}
A^{0,q}(\mathfrak{a}, \underline{C}') & \xrightarrow{\iota_0^*} & A^{0,q}(\mathfrak{a}, \underline{C}) \\
\downarrow{\scriptstyle d''} & & \downarrow{\scriptstyle d''} \\
A^{0,q+1}(\mathfrak{a}, \underline{C}') & \xrightarrow{\iota_0^*} & A^{0,q+1}(\mathfrak{a}, \underline{C}) \,.
\end{array}
$$

Since $d''(\omega) = 0$ for any $\omega \in A^{0,q}(\mathfrak{a}, \underline{C}')$, $H^q(\mathfrak{a}^-, \underline{C}') = A^{0,q}(\mathfrak{a}, \underline{C}')$.

Let $\iota_0^*: H^q(\mathfrak{a}^-, \underline{C}') \to H^q(\mathfrak{a}^-, \underline{C})$ denote the map induced from the cochain map $\iota_0^*: A^*(\mathfrak{a}^-, \underline{C}') \to A^*(\mathfrak{a}^-, \underline{C})$. Since $H \circ d'' = d'' \circ H$, $H: A^{0,q}(\mathfrak{a}, \underline{C}) \to A^{0,q}(\mathfrak{a}, \underline{C}')$ induces a linear map $H: H^q(\mathfrak{a}^-, \underline{C}) \to H^q(\mathfrak{a}^-, \underline{C}')$.

We claim that $\iota_0^* \circ H = id$ and $H \circ \iota_0^* = id$. By definition $H \circ \iota_0^*[\omega] = [\omega]$ for $[\omega] \in H^q(\mathfrak{a}^-, \underline{C}')$. Since $\omega = H(\omega) + 4G\square''(\omega) = H(\omega) + 4Gd''\delta''\omega = H(\omega) + 4d''G\delta''\omega$ for any $\omega \in A^{0,q}(\mathfrak{a}, \underline{C})$ such that $d''\omega = 0$, $\iota_0^* H[\omega] = [\omega]$ for any $[\omega] \in H^q(\mathfrak{a}^-, \underline{C})$. It is now obvious that ι_0^* is a $\mathfrak{g}^-$-module homomorphism.   q.e.d.

Proof of Theorem 1. Let $A^{0,q}(G/\Gamma, C)$ be the space of all C-valued C^∞-differential forms on G/Γ of type $(0, q)$. Take a basis $\{X_1, \cdots, X_n\}$ of $\mathfrak{g}^+$ and let $\{\omega_1, \cdots, \omega_n\}$ be the dual basis of $(\mathfrak{g}^+)^*$. We regard an element $\omega \in (\mathfrak{g}^+)^*$ as a holomorphic 1-form on G/Γ. Then any element $\omega \in A^{0,q}(G/\Gamma, C)$ can be written as $\omega = \sum f_{\bar{J}} \bar{\omega}_J$ where $\bar{\omega}_J = \bar{\omega}_{j_1} \wedge \cdots \wedge \bar{\omega}_{j_q}$,

$$J = (j_1, \cdots, j_q) \, (1 \leq j_1 < \cdots < j_q \leq n) \quad \text{and} \quad f_{\bar{J}} \in \underline{C}.$$

The operator $d'': A^{0,q}(G/\Gamma, C) \to A^{0,q+1}(G/\Gamma, C)$ can be written as

$$d''\omega = \sum_J (\sum_{k=1}^n \bar{X}_k f_{\bar{J}}) \bar{\omega}_k \wedge \bar{\omega}_J + f_{\bar{J}} \overline{d\omega}_J$$

for $\omega = \sum_J f_{\bar{J}} \bar{\omega}_J$.

Therefore the Dolbeault cohomology group $H^{0,q}_{d''}(G/\Gamma)$ can be regarded as the Lie algebra cohomology $H^q(\mathfrak{g}^-, \underline{C})$ of $\mathfrak{g}^-$-module $\underline{C}$.

(2.2) $$H^{0,q}_{d''}(G/\Gamma) \cong H^q(\mathfrak{g}^-, \underline{C}).$$

Regarding C as constant functions on G, we have the inclusion map $\iota: C \to \underline{C}$ of $\mathfrak{g}^-$-modules. Now by (2.1), Theorem 1 is equivalent to assert that ι induces an isomorphism on the cohomology groups

$$\iota^*: H^q(\mathfrak{g}^-) \to H^q(\mathfrak{g}^-, \underline{C}).$$

We prove th the isomorphism $\iota^*: H^q(\mathfrak{g}^-) \to H^q(\mathfrak{g}^-, \underline{C})$ by the induction on the dimension of G/Γ. If G is abelian, G/Γ is a complex torus and our claim is well-known. As before, let A be the normal subgroup of G contained in the center of G and $\mathfrak{a}$ be the ideal in $\mathfrak{g}$ corresponding to A. Consider the Hochschild and Serre spectral sequences for $\mathfrak{g}^-$-modules C and $\underline{C}$, and a homomorphism of these spectral sequences induced by the inclusion map $\iota: C \to \underline{C}$ [2];

$$E_2(\iota): H^t(\mathfrak{g}^-/\mathfrak{a}^-, H^s(\mathfrak{a}^-, C)) \to H^t(\mathfrak{g}^-/\mathfrak{a}^-, H^s/(\mathfrak{a}^-, \underline{C}))$$

for $t, s = 0, 1, 2, \cdots$.

Consider also the $\mathfrak{g}^-$-module $\underline{C}'$. Then we have a commutative diagram of $\mathfrak{g}^-$-modules

$$\begin{array}{ccc} C & \overset{\iota}{\longrightarrow} & \underline{C} \\ {\scriptstyle j} \searrow & & \nearrow {\scriptstyle \iota_0} \\ & \underline{C}'. & \end{array}$$

This commutative diagram induces the corresponding commutative diagram of spectral sequences

$$H^t(\mathfrak{g}^-/\mathfrak{a}^-, H^s(\mathfrak{a}^-, C)) \xrightarrow{E_2(\iota)} H^t(\mathfrak{g}^-/\mathfrak{a}^-, H^s(\mathfrak{a}^-, \underline{C}))$$

$$E_2(j)\searrow \qquad \nearrow E_2(\iota_0)$$

$$H^t(\mathfrak{g}^-/\mathfrak{a}^-, H^s(\mathfrak{a}^-, \underline{C}'))\,.$$

By proposition 2.1, we have an isomorphism of $\mathfrak{g}^-$-modules $\iota_0^*\colon H^s(\mathfrak{a}^-, \underline{C}') \to H^s(\mathfrak{a}^-, \underline{C})$. Hence,

$$E_2(\iota_0)\colon H^t(\mathfrak{g}^-/\mathfrak{a}^-, H^s(\mathfrak{a}^-, \underline{C}')) \to H^t(\mathfrak{g}^-/\mathfrak{a}^-, H^s(\mathfrak{a}^-, \underline{C}))$$

is an isomorphism.

We shall show that $E_2(j)$ is an isomorphism. Since $\mathfrak{a}^-$ is contained in the center of $\mathfrak{g}^-$, $\mathfrak{g}^-$ acts trivially on $H^s(\mathfrak{a}^-, C) = A^s(\mathfrak{a}^-, C)$. Hence,

$$H^t(\mathfrak{g}^-/\mathfrak{a}^-, H^s(\mathfrak{a}^-, C)) = H^t(\mathfrak{g}^-/\mathfrak{a}^-, C) \otimes H^s(\mathfrak{a}^-, C)\,.$$

Since $\mathfrak{a}^-$ acts trivially on $\underline{C}'$, $H^s(\mathfrak{a}^-, \underline{C}') = A^s(\mathfrak{a}^-, \underline{C}')$. Consider the action of $\mathfrak{g}^-$ on $H^s(\mathfrak{a}, \underline{C}')$. For an s-cochain $\omega = \sum_J f_J \bar\omega_J \in A^s(\mathfrak{a}^-, \underline{C}')$ and $\bar{X} \in \mathfrak{g}^-$, $L_{\bar{X}}\omega = \sum_J (\bar{X}f_{\bar{J}})\bar\omega_J$, since $\mathfrak{a}^-$ is contained in the center of $\mathfrak{g}^-$. Hence, $H^s(\mathfrak{a}^-, \underline{C}')$ and $\underline{C}' \otimes H^s(\mathfrak{a}, C)$ are isomorphic as $\mathfrak{g}^-$-modules. Hence, we have

$$H^t(\mathfrak{g}^-/\mathfrak{a}^-, H^s(\mathfrak{a}^-, \underline{C}')) \cong H^t(\mathfrak{g}^-/\mathfrak{a}^-, \underline{C}' \otimes H^s(\mathfrak{a}^-, C))$$
$$\cong H^t(\mathfrak{g}^-/\mathfrak{a}^-, \underline{C}') \otimes H^s(\mathfrak{a}^-, C)\,.$$

We now regard $\underline{C}'$ as the vector space of all C-valued C^∞-functions on $(G/A)/\pi(\Gamma)$. It is easy to see that this identification is compatible with $\mathfrak{g}^-/\mathfrak{a}^-$-module structure. Thus we have

$$H^t(\mathfrak{g}^-/\mathfrak{a}^-, C^\infty((G/A)/\pi(\Gamma), C)) = H^t(\mathfrak{g}^-/\mathfrak{a}^-, \underline{C}')\,.$$

By the assumption of the induction, we get

$$H^t(\mathfrak{g}^-/\mathfrak{a}^-, C^\infty((G/A)/\pi(\Gamma), C)) = H^t(\mathfrak{g}^-/\mathfrak{a}^-, C)\,.$$

Hence, we have an isomorphism

$$E_2(j)\colon H^t(\mathfrak{g}^-/\mathfrak{a}^-, H^s(\mathfrak{a}^-, C)) \to H^t(\mathfrak{g}^-/\mathfrak{a}^-, H^s(\mathfrak{a}^-, \underline{C}'))\,.$$

Thus $E_2(\iota)\colon H^t(\mathfrak{g}^-/\mathfrak{a}^-, H^s(\mathfrak{a}^-, C)) \to H^t(^-\mathfrak{g}/\mathfrak{a}^-, H^s(\mathfrak{a}^-, \underline{C}))$ is an isomorphism. By a theroem on spectral sequence ([13] Chapter 9, §1 Theorem 3), this implies an existence of an isomorphism

$$\iota^*\colon H^q(\mathfrak{g}^-, C) \cong H^q(\mathfrak{g}^-, \underline{C})\,.$$

Combining this (2.1) and (2.2), we get

$$H^{p,q}_{\bar\partial}(G/\Gamma) \cong H^q(\mathfrak{g}^-) \otimes \Lambda^p(\mathfrak{g}^+)^*. \qquad\qquad \text{q.e.d.}$$

Corollary 1 (Kodaira [9]). *Let r be the dimension of the vector space of all closed holomorphic 1-forms on a compact complex parallelisable nilmanifold G/Γ. Then dim $H_{\bar{\partial}}^{0,1}(G/\Gamma)=r$.*

Proof. Let ω be a closed holomorphic 1-form on G/Γ. Then $\omega=\sum_{j=1}^{n}f_j\phi_j$ where $(\phi_1, \cdots, \phi_n)$ is a basis of $(\mathfrak{g}^+)^*$ and f_j $(j=1, \cdots, n)$ are holomorphic functions on G/Γ. Since G/Γ is compact, f_j are constant. Hence, $\omega\in(\mathfrak{g}^+)^*$. Moreover $d\omega=0$ if and only if $\omega([\mathfrak{g}^+, \mathfrak{g}^+])=(0)$. Thus $r=\dim(\mathfrak{g}^+/[\mathfrak{g}^+, \mathfrak{g}^+])$. Since $\dim H^1(\mathfrak{g}^-)=\dim(\mathfrak{g}^-/[\mathfrak{g}^-, \mathfrak{g}^-])=\dim(\mathfrak{g}^+/[\mathfrak{g}^+, \mathfrak{g}^+])$, we have $r=\dim H_{\bar{\partial}}^{0,1}(G/\Gamma)$ by Theorem 1.   q.e.d.

Let M be a compact connected complex manifold. Let b_r (resp. $h^{p,q}$) denote $\dim_R H^r(M, R)$ (resp. $\dim_C H_{\bar{\partial}}^{p,q}(M)$).

Corollary 2. *If M is a compact complex parallelisable nilmanifold G/Γ,*

$$b_{2k+1} = 2(h^{0,2k+1}+h^{0,2k}h^{0,1}+\cdots+h^{0,k+1}h^{0,k})$$
$$b_{2k} = 2(h^{0,2k}+h^{0,2k-1}h^{0,1}+\cdots+h^{0,k+1}h^{0,k-1})+(h^{0,k})^2$$

for $2k+1, 2k\leq n=\dim_C G$.

Proof. By a theorem of Nomizu [10] (See [11] Corollary 7.28.), $H^r(G/\Gamma, R)\simeq H^r(\mathfrak{g}, R)$. Thus $H^r(G/\Gamma, C)\simeq H^r(\mathfrak{g}, C)\simeq H^r(\mathfrak{g}^C)$. Since $\mathfrak{g}^C=\mathfrak{g}^+\oplus\mathfrak{g}^-$ and $[\mathfrak{g}^+, \mathfrak{g}^-]=(0)$, $H^r(\mathfrak{g}^C)\simeq\sum_{p+q=r}H^p(\mathfrak{g}^+)\otimes H^q(\mathfrak{g}^-)$. Since $\dim H^q(\mathfrak{g}^+)=\dim H^q(\mathfrak{g}^-)=h^{0,q}$ and $\dim H^r(\mathfrak{g}^C)=b_r$, $b_r=\sum_{p+q=r}h^{0,p}h^{0,q}$.   q.e.d.

EXAMPLE. Let G be a nilpotent Lie group defined by

$$G = \left\{\begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix}\middle| z_1, z_2, z_3 \in C\right\}$$

Let Γ be a lattice in G, for example,

$$\Gamma = \left\{\begin{pmatrix} 1 & a_1 & a_3 \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \end{pmatrix}\middle| a_1, a_2, a_3 \in Z+\sqrt{-1}Z\right\}.$$

We can take a basis $\{X_1, X_2, X_3\}$ of $\mathfrak{g}^+$ such that

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = [X_1, X_3] = 0.$$

Then the dual basis $\{\omega_1, \omega_2, \omega_3\}$ satisfies that

$$d\omega_3 = -\omega_1\wedge\omega_2, \quad d\omega_1 = d\omega_2 = 0.$$

Now it follows easily from Theorem 1 that $h^{0,1}=h^{0,2}=2$. Note that $h^{1,0}=3$. By corollary 2, we get

$$b_0 = b_6 = 1 \ , \quad b_1 = b_5 = 4 \ , \quad b_2 = b_4 = 8 \quad \text{and} \quad b_3 = 10 \ .$$

3. Chern classes of holomorphic line bundles over a compact complex parallelisable solvmanifold

Let G be a simply connected complex solvable Lie group and Γ be a lattice of G. We assume the following condition:

(M) $Ad(G)$ and $Ad(\Gamma)$ have the same Zariski closure in $Aut(\mathfrak{g}^C)$.

Lemma 3.1. *If G is non-abelian, we have $\Gamma \cap [G, G] \neq \{e\}$.*

Proof. Suppose that $\Gamma \cap [G, G] = \{e\}$. Since $[\Gamma, \Gamma] \subset \Gamma \cap [G, G]$, Γ is abelian, so is $Ad(\Gamma)$. Since $Ad(\Gamma)$ and $Ad(G)$ have the same Zariski closure, $\overline{Ad([G, G])}^z = \overline{[Ad(G), Ad(G)]}^z = [\overline{Ad(G)}^z, \overline{Ad(G)}^z] = [\overline{Ad(\Gamma)}^z = \overline{Ad(\Gamma)}^z] = Ad(\overline{[\Gamma, \Gamma]}^z) = \{e\}$, where $\bar{X}^z$ denotes the Zariski closure of X in $Aut(\mathfrak{g}^C)$. Hence, $[G, G]$ is contained in the center Z of G. Thus G is nilpotent. Since Γ is abelian, G is abelian [11]. This is a contradiction.   q.e.d.

Proposition 3.2. *$\Gamma_1 = \Gamma \cap [G, G]$ is a lattice of $[G, G]$.*

Proof. At first note the following:
If $\mathfrak{m}$ is an ideal of $\mathfrak{g}$ and ρ_1 (resp. ρ_2) is the representation on $\mathfrak{m}^C$ (resp. $\mathfrak{g}^C/\mathfrak{m}^C$) induced by the adjoint representation $Ad: G \to Aut(\mathfrak{g}^C)$, $\rho_1(G)$ and $\rho_1(\Gamma)$ (resp. $\rho_2(G)$ and $\rho_2(\Gamma)$) have the same Zariski closure in $Aut(\mathfrak{m}^C)$ (resp. $Aut(\mathfrak{g}/\mathfrak{m}^C)$).

Now $[G, G]$ is a simply connected nilpotent closed Lie subgroup of G and Γ_1 is a discrete subgroup of $[G, G]$. Let H be the connected closed subgroup of $[G, G]$ such that H/Γ_1 is compact ([11] Proposition 2.5.). We claim that H is a normal subgroup of G. Let $\exp: [\mathfrak{g}, \mathfrak{g}] \to [G, G]$ be the exponential map. Then $\exp^{-1}(\Gamma_1) = \mathfrak{l}$ is a lattice in the Lie algebra $\mathfrak{h}$ of H and $\mathfrak{l} \otimes \mathbf{R} = \mathfrak{h}$ ([11] Theorem 2.12). Since $\Gamma_1 = \Gamma \cap [G, G]$ is a normal subgroup of Γ, $\exp Ad(\gamma)L = \gamma(\exp L)\gamma^{-1} \in \Gamma_1$ for any $L \in \mathfrak{l} = \exp^{-1}(\Gamma_1)$ and $\gamma \in \Gamma$. Hence, $Ad(\gamma)\mathfrak{l} \subset \mathfrak{l}$ and $Ad(\gamma)\mathfrak{h} = \mathfrak{h}$ for any $\gamma \in \Gamma$. Since $Ad(G)$ and $Ad(\Gamma)$ have the same Zariski closure in $Aut(\mathfrak{g}^C)$, $Ad(G)\mathfrak{h} = \mathfrak{h}$. Hence, $\mathfrak{h}$ is an ideal in $\mathfrak{g}$. Thus H is a normal subgroup of G.

Since $H \subset [G, G]$ and $\Gamma_1 \subset H$, $H \cap \Gamma = H \cap [G, G] \cap \Gamma = H \cap \Gamma_1 = \Gamma_1$. Thus $H/H \cap \Gamma$ is compact and $H \cdot \Gamma$ is closed in G ([11] Theorem 1.13). Hence, $H \cdot \Gamma/H$ is a lattice of G/H. We claim that $\Gamma H/H \cap [G/H, G/H] = \{e\}$. Let $a \in \Gamma H/H \cap [G/H, G/H]$. Since $[G/H, G/H] = [G, G]H/H = [G, G]/H$, $a = \gamma H = g_1 H$ for some $\gamma \in \Gamma$ and $g_1 \in [G, G]$, that is, $\gamma = g_1 h$ for some $h \in H$. Since $H \subset [G, G]$, $\gamma \in [G, G] \cap \Gamma = \Gamma_1 \subset H$. Hence, $a = \gamma H = H$.

Since $Ad(G/H)$ and $Ad(\Gamma H/H)$ have the same Zariski closure in $Aut(\mathfrak{g}^c/\mathfrak{h}^c)$, G/H is abelian by Lemma 3.1. Hence $H \supset [G, G]$. Thus $H = [G, G]$ and is Γ_1 a lattice of $[G, G]$.   q.e.d.

Since $\Gamma \cap [G, G]$ is a lattice of $[G, G]$, $[G, G]\Gamma$ is closed in G ([11] Theorem 1.13.) and $\pi(\Gamma) = \Gamma[G, G]/[G, G]$ is a lattice of $G/[G, G]$. Note that $G/[G, G]\Gamma = (G/[G, G])/\pi(\Gamma)$ is a complex torus. Thus we have a holomorphic fiber bundle $(G/\Gamma, \pi, (G/[G, G])/\pi(\Gamma), [G, G]/[G, G]\cap\Gamma)$. Let T denote the complex torus $G/[G, G]\Gamma$.

Now we denote by $A^{1,1}(G/\Gamma, \mathbf{R})$ the vector space of all real differential forms of type $(1, 1)$ on G/Γ. Let $H^{1,1}(G/\Gamma, \mathbf{R})$ be the vector space

$$\frac{\{\omega \in A^{1,1}(G/\Gamma, \mathbf{R}) \,|\, d\omega = 0\}}{\{\omega \in A^{1,1}(G/\Gamma, \mathbf{R}) \,|\, \omega = d\theta,\ \theta \text{ is a real 1-form}\}}.$$

We shall characterize $H^{1,1}(G/\Gamma, \mathbf{R})$ in terms of the Lie algebra $\mathfrak{g}$ of G.

Proposition 3.3. *Suppose that a lattice Γ of G satisfies the condition (M). Then, for any real closed form α of type $(1, 1)$ on G/Γ, there is a unique real right invariant closed form $\beta \in \Lambda^2(\mathfrak{g}^*)$ of type $(1, 1)$ on G such that $\alpha = \beta + d\eta$ on G/Γ where η is a real 1-form on G/Γ.*

Proof. According to a theorem of Mostow ([8], [11]), for a given real closed 2-form α, there is a real right invariant closed 2-form $\beta \in \Lambda^2\mathfrak{g}^*$ such that

$$(3.1) \qquad\qquad\qquad \alpha = \beta + d\gamma$$

where γ is a real 1-form on G/Γ. Let $\beta = \beta^{0,2} + \beta^{1,1} + \beta^{2,0}$ where $\beta^{p,q}$ is the component of β of type (p, q). Since β is a real form, $\beta^{2,0} = \bar{\beta}^{0,2}$ and $\beta^{1,1}$ is a real form. Let $\gamma = \gamma^{1,0} + \gamma^{0,1}$, $\gamma^{1,0} = \bar{\gamma}^{0,1}$. Taking a basis $\{X_1, \cdots, X_n\}$ of $\mathfrak{g}^+$, let $\{\omega_1, \cdots, \omega_n\}$ be its dual basis of $(\mathfrak{g}^+)^*$. We identify ω_j $(j = 1, \cdots, n)$ as holomorphic 1-forms on G/Γ. We then have

$$\gamma^{0,1} = \sum_{j=1}^{n} f_j \bar{\omega}_j$$

where $f_j \in C^{\infty}(G/\Gamma, \mathbf{C})$ for $j = 1, \cdots, n$ and

$$\beta^{0,2} = \sum_{j<k} a_{jk} \bar{\omega}_j \wedge \bar{\omega}_k$$

where $a_{jk} \in \mathbf{C}$. Since α is of type $(1,1)$, we get

$$(3.2) \qquad \beta^{0,2} + d''\gamma^{0,1} = 0 \quad \text{and} \quad \alpha = \beta^{1,1} + d''\gamma^{1,0} + \overline{d''\gamma^{1,0}}$$

by comparing the type of forms of both hands. We now have

$$d''\gamma^{0,1} = d'' \left(\sum_{j=1}^{n} f_j \overline{\omega}_j\right) = \sum_{j=1}^{n} (d''f_j \wedge \overline{\omega}_j + f_j d\overline{\omega}_j)$$

$$= \sum_{j,k=1}^{n} \overline{X}_k f_j \overline{\omega}_k \wedge \overline{\omega}_j - \sum_{j=1}^{n} \sum_{k<l} f_j \overline{C}_{kl}^{j} \overline{\omega}_k \wedge \overline{\omega}_l$$

where C_{kl}^{j} are the structure constant of Lie algebra $\mathfrak{g}^+$ with respect to the basis $\{X_1, \cdots, X_n\}$. By (3.2), we get the equalities

$$(3.3) \qquad a_{kl} = \overline{X}_k f_l - \overline{X}_l f_k - \sum_{j=1}^{n} f_j \overline{C}_{kl}^{j} \qquad \text{for } 1 \leq k < l \leq n.$$

Integrating (3.3) on G/Γ, we have

$$(3.4) \qquad \int_{G/\Gamma} a_{kl} \, dg = \int_{G/\Gamma} (\overline{X}_k f_l) dg - \int_{G/\Gamma} (\overline{X}_l f_k) dg - \sum_{j=1}^{n} \int_{G/\Gamma} f_j \overline{C}_{kl}^{j} dg$$

where dg is an invariant measure on G/Γ. Since G is unimodular, $\int_{G/\Gamma} (\overline{X}_k f_l) dg$
$= \int_{G/\Gamma} (\overline{X}_l f_k) dg = 0$, and we get

$$(3.5) \qquad a_{kl} \int_{G/\Gamma} dg = -\sum_{j=1}^{n} \overline{C}_{kl}^{j} \int_{G/\Gamma} f_j \, dg.$$

Let $b_j \in C$ denote $\int_{G/\Gamma} f_j \, dg / \int_{G/\Gamma} dg$. Then (3.5) can be written as

$$(3.6) \qquad a_{kl} = -\sum_{j=1}^{n} b_j \overline{C}_{kl}^{j}.$$

$$\beta^{0,2} = \sum_{k<l} a_{kl} \overline{\omega}_k \wedge \overline{\omega}_l = -\sum_{k<l} \sum_{j=1}^{n} b_j \overline{C}_{kl}^{j} \overline{\omega}_k \wedge \overline{\omega}_l$$

$$= \sum_{j=1}^{n} b_j \left(-\sum_{k<l} \overline{C}_{lk}^{j} \overline{\omega}_k \wedge \overline{\omega}_l\right) = \sum_{j=1}^{n} b_j (d\overline{\omega}_j) = d\left(\sum_{j=1}^{n} b_j \overline{\omega}_j\right).$$

Put $\eta = \sum_{j=1}^{n} b_j \overline{\omega}_j$. We then see that η is of type $(0, 1)$, $\beta^{0,2} = d\eta$ and $\beta^{2,0} = d\overline{\eta}$.
By (3.1), we get

$$\alpha = \beta^{1,1} + d(\eta + \overline{\eta}) + d\gamma = \beta^{1,1} + d\theta$$

where $\theta = \eta + \overline{\eta} + \gamma$ is a real 1-form on G/Γ.

It remains to show the uniqueness of $\beta^{1,1}$. It is sufficient to see that if $\beta^{1,1} = d\theta$, θ is a real 1-form, then $\beta^{1,1} = 0$. Put $\beta^{1,1} = \sum_{j,k=1}^{n} a_{jk} \omega_j \wedge \overline{\omega}_k$ and $\theta = \theta^{0,1} + \overline{\theta}^{0,1}$ where $\theta^{0,1} = \sum_{j=1}^{n} g_j \overline{\omega}_j$, $g_j \in C^{\infty}(G/\Gamma, C)$ $(j = 1, \cdots, n)$. Since $d'\theta^{0,1} = \sum_{k,j=1}^{n} X_k g_j \omega_k \wedge \overline{\omega}_j$ and $d''\overline{\theta}^{0,1} = \overline{d'\theta^{0,1}} = \sum_{k,j=1}^{n} \overline{X}_k \overline{g}_j \overline{\omega}_j \wedge \omega_j$, we get

$$(3.7) \qquad a_{jk} = X_j g_k - \overline{X}_k \overline{g}_j.$$

Integrating (3.7) on G/Γ, we have

$$a_{jk} \int_{G/\Gamma} dg = \int_{G/\Gamma} (X_j g_k) dg - \int_{G/\Gamma} (\bar{X}_k \bar{g}_j) dg = 0 \,.$$

Hence, $a_{jk}=0$ for $j, k=1, \cdots, n$ and $\beta^{1,1}=0$.   q.e.d.

We now determine real closed right invariant forms of type $(1, 1)$ on G/Γ. Take a basis $\{X_1, \cdots, X_n\}$ of $\mathfrak{g}^+$ such that $\{X_{r+1}, \cdots, X_n\}$ is a basis of $[\mathfrak{g}^+, \mathfrak{g}^+]$. Let $\{\omega_1, \cdots, \omega_n\}$ be its dual basis of $(\mathfrak{g}^+)^*$.

Proposition 3.4. *Let α be a right invariant real 2-form of type $(1, 1)$ on G. Then $d\alpha=0$ if and only if $\alpha = \dfrac{1}{2\sqrt{-1}} \sum\limits_{j,k=1}^{r} h_{jk} \omega_j \wedge \bar{\omega}_k$ where $H=(h_{jk}) \in M(r, \boldsymbol{C})$ is a hermitian matrix, and $r=\dim \mathfrak{g}^+/[\mathfrak{g}^+, \mathfrak{g}^+]$.*

Proof. Since α is a right invariant form on G, α defines a bilinear form on $\mathfrak{g}^+ \times \mathfrak{g}^-$. Now $d\alpha=0$ if and only if

$$\alpha([X, Y], \bar{Z}) = 0 \quad \text{and} \quad \alpha([\bar{X}, \bar{Y}], Z) = 0 \qquad \text{for } X, Y, Z \in \mathfrak{g}^+ \,,$$

since $(d\alpha)(X, Y, Z) = -\alpha([X, Y], Z) + \alpha([X, Z], Y) - \alpha([Y, Z], X)$ for $X, Y, Z \in \mathfrak{g}^c$ and since $[\mathfrak{g}^+, \mathfrak{g}^-] = (0)$. In particular, for a real form α of type $(1, 1)$, we get

(3.8) $d\alpha = 0$ if and only if $\iota([X, Y])\alpha = 0 \qquad$ for $X, Y \in \mathfrak{g}^+$.

Note that $d\omega_j=0$ for $j=1, \cdots, r$. Therefore, if $\alpha = \dfrac{1}{2\sqrt{-1}} \sum\limits_{j,k=1}^{r} h_{jk} \omega_j \wedge \bar{\omega}_k$ then $d\alpha=0$. Conversely, put $\alpha = \dfrac{1}{2\sqrt{-1}} \sum\limits_{j,k=1}^{n} h_{jk} \omega_j \wedge \bar{\omega}_k$. If α is closed, then $\iota(X_j)\alpha=0$ for $j=r+1, \cdots, n$ by (3.8).

Since $(\iota(X_j)\alpha)(\bar{X}_k) = \alpha(X_j, \bar{X}_k) = \dfrac{1}{2\sqrt{-1}} h_{jk}$ and $H=(h_{jk})$ is a hermitian matrix, we have $h_{jk}=0$ for $j=r+1, \cdots, n$; $k=1, \cdots, n$ and $j=1, \cdots, n$; $k=r+1, \cdots, n$, so that $\alpha = \dfrac{1}{2\sqrt{-1}} \sum\limits_{j,k=1}^{r} h_{jk} \omega_j \wedge \bar{\omega}_k$.   q.e.d.

Consider a holomorphic line bundle L on G/Γ. Let $C(L)$ denote the Chern class of L. Then we have $C(L) \in H^{1,1}(G/\Gamma, \boldsymbol{R})$ ([15], Chapter V, n°4.).

Proposition 3.5. *Let G be a simply connected complex solvable Lie group and Γ be a lattice of G satisfying the condition (M) and such that $H_{d,r}^{0,1}(G/\Gamma) \cong H^1(\mathfrak{g}^-)$ (canonically). Let L be a holomorphic line bundle on G/Γ. Then there is a unique real invariant form $\alpha \in \Lambda^2 \mathfrak{g}^*$ of type $(1, 1)$ in $C(L)$, and this is a curvature form of a connection η of type $(1, 0)$.*

Proof. It is easy to see that there is a real closed 2-form β of type $(1, 1)$ in

$C(L)$ which is a curvature form of a connection ω of type $(1,0)$ ([15], Chapter V, n°4).

According to Proposition 3.3, we have $\beta=\alpha+d\gamma$ where is γ a real 1-form on G/Γ. Decompose $\gamma=\gamma^{1,0}+\gamma^{0,1}$ where $\gamma^{1,0}$ (resp. $\gamma^{0,1}$) is the component of type $(1, 0)$ (resp. $(0, 1)$) of γ. Then we have $d''\gamma^{0,1}=0$, since β and α are of type $(1, 1)$. By the assumption (2), there is a right invariant 1-form θ of type $(0, 1)$ such that $\gamma^{0,1}-\theta=d''f$ where $f\in C^\infty(G/\Gamma, \boldsymbol{C})$.

We can write $\theta=\sum\limits_{j=1}^{r} a_j\bar\omega_j$, $a_j\in\boldsymbol{C}$ $(j=1, \cdots, r)$, where $\{\omega_1, \cdots, \omega_n\}$ is the same as before, since $H^1(\mathfrak{g}^-)=(\mathfrak{g}^-/[\mathfrak{g}^-, \mathfrak{g}^-])^*$. We then have $d\theta=\sum\limits_{j=1}^{r} a_j d\bar\omega_j=0$, so that $\beta=\alpha+d'\gamma^{0,1}+d''\gamma^{1,0}=\alpha+d'\gamma^{0,1}+\overline{d'\gamma^{0,1}}=\alpha+d'(\theta+d''f)+\overline{d'(\theta+d''f)}$ $=\alpha+d'd''(f-\bar f)$. Put $\psi=d'(\bar f-f)$. We then have $\beta=\alpha+dd'(\bar f-f)=\alpha+d\psi$. Since β is a curvature from ω of a connection of type $(1, 0)$ by definition and ψ is of type $(1, 0)$, α is a curvature form of a connection $\eta=\omega-\psi$ of type $(1, 0)$.

                                                                        q.e.d.

From now on we always assume that G and Γ satisfies the assumptions of Proposition 3.5.

Consider a holomorphic line bundle L on G/Γ. We fix a (sufficiently fine) simple covering $\{U_i\}$ on G/Γ and choose a connected component U_{i_0} of $p^{-1}(U_i)$ for each i, $p: G\to G/\Gamma$ being the canonical map; let $U_{i\gamma}$ denote the image of U_{i_0} under the right translation $R_\gamma(g)=g\gamma$ for $\gamma\in\Gamma$. Then $p^{-1}(U_i)=\bigcup\limits_{\gamma\in\Gamma} U_{i\gamma}$ is a disjoint union and p maps each $U_{i\gamma}$ biholomorphically to U_i.

We may consider a holomorphic line bundle L on G/Γ is given by a system of transition functions $\{g_{jk}\}$ relative to the covering $\{U_i\}$ of G/Γ. Let $C(L)$ be the Chern class of L and α be the unique real right invariant form of type $(1, 1)$ in $C(L)$. By Proposition 3.5, α is a curvature form of a connection η of type $(1, 0)$, so that there is an element $\eta_j\in A^{1,0}(U_j)$ for each j satisfying $\eta_k-\eta_j=\dfrac{\sqrt{-1}}{2\pi}d\log g_{jk}$ on $U_j\cap U_k\neq\phi$ and $\alpha=d\eta_j$ on U_j.

Proposition 3.6. *Identify $\mathfrak{g}^+$ to the complex Lie algebra $(\mathfrak{g}, I)$. Then we can take a basis $\{X_1, \cdots, X_n\}$ of $\mathfrak{g}^+$ such that a map $\psi: \mathfrak{g}^+\to G$ defined by*

$$\psi(\sum_{i=1}^{n} z_i X_i) = (\exp z_1 X_1)\cdots(\exp z_n X_n)$$

is biholomorphic. In particular, G is biholomorphic to $\boldsymbol{C}^n$. Moreover G has a system of coordinates $(z_1, \cdots, z_n)$ such that, for $j=1, \cdots, r$, $z_j(gg')=z_j(g)+z_j(g')$ for any $g, g'\in G$, where $r=\dim \mathfrak{g}^+/[\mathfrak{g}^+, \mathfrak{g}^+]$.

Proof. We prove this proposition by induction on the dimension n of $\mathfrak{g}^+$. Assume that it has been proved for all dimensions $<n$. Since $\mathfrak{g}^+$ is solva-

ble, it has an abelian ideal $\mathfrak{a}^+$ of dimension >0. Let A be the connected complex abelian subgroup of G whose Lie algebra is $\mathfrak{a}^+$; A is simply connected and G/A is a simply connected complex solvable Lie group of complex dimension $<n$. Applying our proposition to G/A, we get a basis $\{X_1^*, \cdots, X_m^*\}$ of $^+\mathfrak{g}/\mathfrak{a}^+$ such that a map $\psi^*: \mathfrak{g}^+/\mathfrak{a}^+ \to G/A$ defined by

$$\psi^*(\sum_{i=1}^{m} z_i X_i^*) = (\exp z_1 X_1^*) \cdots (\exp z_m X_m^*)$$

is biholomorphic. Take elements $X_1, \cdots, X_m \in \mathfrak{g}^+$ such that $\pi_*(X_i) = X_i^*$ where $\pi_*: \mathfrak{g}^+ \to \mathfrak{g}^+/\mathfrak{a}^+$ is a projection. Choose also a basis $\{X_{m+1}, \cdots, X_n\}$ of $\mathfrak{a}^+$. Then every element of A can be written uniquely in the form $(\exp z_{m+1} X_{m+1}) \cdots (\exp z_n X_n)$. Let g be any element of G and $g^* = \pi(g)$ where $\pi: G \to G/A$ is a projection. Then we can write uniquely g^* in the form $(\exp z_1 X_1^*) \cdots (\exp z_m X_m^*)$. Hence, we have $g = (\exp z_1 X_1) \cdots (\exp z_m X_m) a$ $(a \in A)$ and a can be written in the form $(\exp z_{m+1} X_{m+1}) \cdots (\exp z_n X_n)$, which proves that g is in the form $(\exp z_1 X_1)(\exp z_2 X_2) \cdots (\exp z_n X_n)$. Moreover $z_1, \cdots, z_m$ are uniquely determined by $\pi(g)$ (and a fortiori by g); hence a is determined by g and $z_{m+1}, \cdots, z_n$ are uniquely determined by g. Since exp is holomorphic, z_j $(j=1, \cdots, n)$ are holomorphic functions on G and $\psi: \mathfrak{g}^+ \to G$ is biholomorphic.

Since we can choose a basis $\{X_1, \cdots, X_n\}$ of $\mathfrak{g}^+$ in such a way that $\{X_{r+1}, \cdots, X_n\}$ is a basis of $[\mathfrak{g}^+, \mathfrak{g}^+]$ and $\psi: \mathfrak{g}^+ \to G$ is biholomorphic, the last assertion follows from the Campbell-Hausdorff formula ([4] p. 170).   q.e.d.

We may assume that $\omega_j = dz_j$ for $j=1, \cdots, r$ by changing a basis of $\mathfrak{g}^+$ if necessary. Then by Proposition 3.4, we get

$$\alpha = \frac{1}{2\sqrt{-1}} \sum_{j,k=1}^{r} h_{jk} \omega_j \wedge \overline{\omega}_k = \frac{1}{2\sqrt{-1}} \sum_{j,k=1}^{r} h_{jk} dz_j \wedge d\overline{z_k}$$

where (h_{jk}) is a hermitian matrix.

4. Divisors on a compact complex parallelisable solvmanifold

Let M and N be complex manifolds and $\Phi: M \to N$ be a surjective holomorphic map. For a divisor $\tilde{D}$ on N, $\Phi^*(\tilde{D})$ denotes the divisor on M defined by $\Phi_x^{-1}(\tilde{D}_{\Phi(x)})$ for all $x \in M$ ([15] Appendice n°7). We call this divisor $\Phi^*(\tilde{D})$ on M the pull back of the divisor $\tilde{D}$ on N. In this section we prove the following theorem.

Theorem 2. *Let G be a simply connected complex solvable Lie group. Let Γ be a lattice of G. Assume that Γ satisfies the condition (M) and that $H_{\bar{\partial}}^{0,1}(G/\Gamma) \cong H^1(\mathfrak{g}^-)$ (canonically). Then, for each positive divisor D on G/Γ, there exists a positive divisor $\tilde{D}$ on the complex torus T such that the divisor D is the pull back of the divisor $\tilde{D}$ on T by the projection $\pi: G/\Gamma \to T$, i.e., $D = \pi^* \tilde{D}$.*

If G is nilpotent, the condition (M) is always satisfied ([11] Theorem 2.1). Moreover, by Theorem 1 in the section 2, $H^{0,1}_{\bar{\partial}}(G/\Gamma)\cong H^1(\mathfrak{g}^-)$. Thus we get:

Corollary. *Let G be a simply connected complex nilpotent Lie group and Γ be a lattice of G. Then the conclusion of Theorem 2 holds.*

Let D denote a positive divisor on G/Γ. Take a representative $\{(U_i, f_i)\}$ of D, where $f_i: U_i \to C$ is a holomorphic function. Let $L=\{D\}$ denote the holomorphic line bundle corresponding to the divisor D. ([15] Chapter V, n° 6). Let $\{g_{jk}\}$ denote the system of transition functions of $L=\{D\}$ with respect to $\{(U_i, f_i)\}$. We then have $f_j=g_{jk}f_k$ on $U_j\cap U_k\neq\phi$ by definition.

Let M be a complex manifold, $\tilde{M}$ be the universal covering of M and $p: \tilde{M}\to M$ be the covering map. Let Π denote the fundamental group $\pi_1(M)$ of M.

A map $j: \Pi\times\tilde{M}\to C^*$ is said to be *an automorphic factor* if

(1) the function $z\to j(\sigma, z)$ is holomorphic for any $\sigma\in\Pi$, and

(2) $j(\sigma\tau, z)=j(\sigma, \tau(z))\cdot j(\tau, z)$ for any $\sigma, \tau\in\Pi$ and any $z\in\tilde{M}$.

Let f be a holomorphic function on $\tilde{M}$ which is not identically zero. f is said to be *automorphic of type j* if

$$f(\sigma(z)) = j(\sigma, z)f(z) \qquad \text{for } z\in\tilde{M} \text{ and } \sigma\in\Pi.$$

Proposition 4.1. *Let D be a positive divisor of G/Γ. Then D is the divisor of a holomorphic automorphic function θ on G, for which the automorphic factor $j(\gamma, g): \Gamma\times G\to C^*$ is given by*

$$j(\gamma, g) = \exp 2\pi\sqrt{-1}\left(\frac{1}{2\sqrt{-1}}\sum_{k,l=1}^r h_{kl}z_k(g)\bar{z}_l(\gamma)+C(\gamma)\right),$$

where $H=(h_{jk})$ is a hermitian matrix determined by the form α in the Chern class $C(L)=C(\{D\})$:

$$\alpha = \frac{1}{2\sqrt{-1}}\sum_{k,l=1}^r h_{kl}\,dz_k\wedge d\bar{z}_l,$$

and $C(\gamma)\in C$ is a constant depending only on $\gamma\in\Gamma$.

Proof. Let us define $\varphi_{i\gamma}(g)$ for $g\in U_{i\gamma}$ by

$$\varphi_{i\gamma}(g) = \eta_i(p(g))+\frac{1}{2\sqrt{-1}}\sum_{k,l=1}^r h_{kl}\bar{z}_l(g\gamma^{-1})dz_k,$$

where η_i is the component of the connection introduced before. Then $\varphi_{i\gamma}$ is an element of $A^{1,0}(U_{i\gamma})$ satisfying $d\varphi_{i\gamma}=0$. Since $U_{i\gamma}$ is simply connected, there is a holomorphic function $\psi_{i\gamma}$ satisfying $d\psi_{i\gamma}=\varphi_{i\gamma}$. Define $\theta_{i\gamma}(g)$ for $g\in U_{i\gamma}$ by

$$\theta_{i\gamma}(g) = f_i(p(g)) \exp 2\pi\sqrt{-1}(\psi_{i\gamma}(g)).$$

We then have

$$\theta_{i\gamma}(g) = \theta_{j\delta}(g) \exp 2\pi\sqrt{-1} \left(\frac{1}{2\sqrt{-1}} \sum_{k,l=1}^{r} h_{kl} z_k(g) \bar{z}_l(\gamma\delta^{-1}) + C_{i\gamma,j\delta} \right)$$

on $U_{i\gamma} \cap U_{j\delta}$, where $C_{i\gamma,j\delta} \in C$ is a constant.   Applying Proposition 3.6, we get

$$\frac{\sqrt{-1}}{2} d \log g_{ij}(p(g)) + \varphi_{i\gamma}(g) - \varphi_{j\delta}(g) = \frac{1}{2\sqrt{-1}} \sum_{k,l=1}^{r} h_{kl}(\bar{z}_l(\delta) - \bar{z}_l(\gamma)) dz_k .$$

Put $a_{i\gamma,j\delta} = \exp 2\pi\sqrt{-1} C_{i\gamma,j\delta}$. $\{a_{i\gamma,j\delta}\}$ satisfies relations

(4.1) $a_{i\gamma,j\delta} \cdot a_{j\delta,k\nu} = a_{i\gamma,k\nu}$ on $U_{i\gamma} \cap U_{j\delta} \cap U_{k\nu} \neq \phi$,

since

$$a_{i\gamma,j\delta} = \exp 2\pi\sqrt{-1} C_{i\gamma,j\delta}$$
$$= g_{ij}^{-1}(p(g)) \exp 2\pi\sqrt{-1} \left\{ (\psi_{i\gamma} - \psi_{j\delta}) + \frac{1}{2\sqrt{-1}} \sum_{k,l=1}^{r} h_{kl}(\bar{z}_l(\gamma) - z_l(\delta)) \right\} .$$

By the principal of monodromy ([15], Chapter V, n°1), there is a system of constant functions $\{b_{i\gamma}\}$ such that

$$a_{i\gamma,j\delta} = b_{i\gamma}^{-1} \cdot b_{j\delta} ,$$

since G is simply connected and $\{U_i\}$ is an open covering of G.   We define a holomorphic function θ on G by

$$\theta(g) = \theta_{i\gamma}(g) \exp 2\pi\sqrt{-1} \left(\frac{1}{2\sqrt{-1}} \sum_{k,l=1}^{r} h_{kl} z_k(g) \bar{z}_l(\gamma) + b_{i\gamma} \right)$$

on $g \in U_{i\gamma}$.

We can see easily that θ is well defined and θ is different from zero.

Note that

$$\theta_{i\gamma}(g\gamma) = \theta_{i0}(g) \exp 2\pi\sqrt{-1} d_{i\gamma}$$

for $g \in U_{i0}$, where $d_{i\gamma}$ is a constant.   In fact, we have

$$d(R_\gamma^* \psi_{i\gamma}) - d\psi_{0i} = R_\gamma^* \varphi_{i\gamma} - \varphi_{i0} = 0   \text{   on   } U_{i0} ,$$

and

$$\psi_{i\gamma}(g\gamma) - \psi_{i0}(g) = d_{i\gamma}   \text{   on   } U_{i0} .$$

We now show that

$$\theta(g\gamma) = \theta(g) \cdot \exp 2\pi\sqrt{-1} \left(\frac{1}{2\sqrt{-1}} \sum_{j,k=1}^{r} h_{jk} z_j(g) \bar{z}_k(\gamma) + C(\gamma) \right)$$

for $g \in G$ and $\gamma \in \Gamma$, where $C(\gamma)$ is a constant.   For $g \in U_{i0}$, we have

$$\theta(g\gamma) = \theta_{i\gamma}(g\gamma) \cdot \exp 2\pi\sqrt{-1}\Big(\frac{1}{2\sqrt{-1}} \sum_{k,l=1}^{r} h_{kl} z_k(g\gamma)\bar{z}_l(\gamma) + b_{i\gamma}\Big)$$

$$= \theta_{i_0}(g) \cdot \exp 2\pi\sqrt{-1}\Big\{d_{i\gamma} + \frac{1}{2\sqrt{-1}} \sum_{k,l=1}^{r} h_{kl} z_k(g\gamma)\bar{z}_l(\gamma) + b_{i\gamma}\Big\}.$$

Since $\theta(g) = \theta_{i_0}(g) \exp 2\pi\sqrt{-1}\, b_{i_0}$ on U_{i_0}, and since $z_k(g\gamma) = z_k(g) + z_k(\gamma)$ by Proposition 3.6,

$$\theta(g\gamma) = \theta(g) \exp 2\pi\sqrt{-1}\Big\{\frac{1}{2\sqrt{-1}} \sum_{k,l=1}^{r} h_{kl}\, z_k(g)\bar{z}_l(\gamma) + C_i(\gamma)\Big\}$$

for $g \in U_{i_0}$, where $C_i(\gamma)$ is a constant.   Since $\theta(g\gamma)$ and

$$\theta(g) \exp 2\pi\sqrt{-1}\Big\{\frac{1}{2\sqrt{-1}} \sum_{k,l=1}^{r} h_{kl}\, z_k(g)\bar{z}_l(\gamma) + C_i(\gamma)\Big\}$$

are holomorphic functions on G, we have

$$\theta(g\gamma) = \theta(g) \exp 2\pi\sqrt{-1}\Big(\frac{1}{2\sqrt{-1}} \sum_{k,l=1}^{r} h_{kl}\, z_k(g)\bar{z}_l(\gamma) + C(\gamma)\Big)$$

for $g \in G$ and $\gamma \in \Gamma$.   By the definition of θ, we have $p^*D = \mathrm{div}(\theta)$.   q.e.d.

From now on, let e denote $\exp 2\pi\sqrt{-1}$ and $H(g_1, g_2) = \sum_{k,l=1}^{r} h_{kl} z_k(g_1)\bar{z}_l(g_2)$. Then $j(\gamma, g) = e\Big(\frac{1}{2\sqrt{-1}} H(g, \gamma) + C(\gamma)\Big)$ for $g \in G$ and $\gamma \in \Gamma$.

Since $j(\gamma_1\gamma_2, g) = j(\gamma_1, g)\, j(\gamma_2, g\gamma_1)$, we get

$$C(\gamma_1\gamma_2) \equiv C(\gamma_1) + C(\gamma_2) + \frac{1}{2\sqrt{-1}} H(\gamma_1, \gamma_2) \qquad (\mathrm{mod}\ 1).$$

In particular, $C(e) \in \mathbf{Z}$ and

$$C(\gamma^{-1}) \equiv -C(\gamma) + \frac{1}{2\sqrt{-1}} H(\gamma, \gamma) \qquad \text{for}\ \ \gamma \in \Gamma.$$

Lemma 4.2. $C(\gamma) \in \mathbf{R}$ for $\gamma \in [\Gamma, \Gamma]$.

Proof.   Since $[\Gamma, \Gamma] \subset [G, G]$, $H(g, \gamma) = 0$ for $\gamma \in [\Gamma, \Gamma]$ and $g \in G$.   It is enough to show that $C(\gamma) \in \mathbf{R}$ for $\gamma = \gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}$, $\gamma_1, \gamma_2 \in \Gamma$.   In this case,

$$C(\gamma) \equiv C(\gamma_1\gamma_2) + C(\gamma_1^{-1}\gamma_2^{-1}) + \frac{1}{2\sqrt{-1}} H(\gamma_1\gamma_2, \gamma_1^{-1}\gamma_2^{-1})$$

$$\equiv C(\gamma_1) + C(\gamma_2) + \frac{1}{2\sqrt{-1}} H(\gamma_1, \gamma_2) + C(\gamma_1^{-1}) + C(\gamma_2^{-1}) + \frac{1}{2\sqrt{-1}} H(\gamma_1, \gamma_2)$$

$$+ \frac{1}{2\sqrt{-1}} \{-H(\gamma_1, \gamma_1) - H(\gamma_2, \gamma_2) - H(\gamma_2, \gamma_1) - H(\gamma_1, \gamma_2)\}$$

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$$\equiv \frac{1}{2\sqrt{-1}}(H(\gamma_1, \gamma_2)-H(\gamma_2, \gamma_1)) = \frac{1}{2\sqrt{-1}}(H(\gamma_1, \gamma_2)-\overline{H(\gamma_1, \gamma_2)}) \in \boldsymbol{R}.$$

<div align="right">q.e.d.</div>

Proposition 4.3. $[\Gamma, \Gamma]$ *is a lattice of* $[G, G]$ *and* $[\Gamma, \Gamma]$ *is a subgroup of finite index of* $\Gamma \cap [G, G]$.

Proof. It follows from Porposition 3.2 that $\Gamma[G, G]/[G, G]$ is a lattice of $G/[G, G]$. Since $G/[G, G]$ is a vector group of dimension $2r=\dim_R \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$, $\Gamma[G, G]/[G,G]=\Gamma/\Gamma \cap [G, G]$ is a free abelian group of rank $2r$. On the other hand, since G is simply connected, $\pi_1(G/\Gamma)=\Gamma$ and is Γ finitely generated. It follow that $H_1(G/\Gamma, \boldsymbol{Z})\cong\Gamma/[\Gamma, \Gamma]$. Since $\dim H^1(G/\Gamma, \boldsymbol{R})=\dim H^1(\mathfrak{g}, \boldsymbol{R})=\dim_R \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]=2r$ by a theorem of Mostow (cf. [8], [11] Corollary 7.29.), $\Gamma/[\Gamma, \Gamma]$ is then direct sum of a free abelian group of rank $2r$ and a finite group. The group $(\Gamma \cap [G, G])/[\Gamma, \Gamma]$ is finite, because $\Gamma/\Gamma \cap [G, G]\approx(\Gamma/[\Gamma, \Gamma])/(\Gamma \cap [G, G]/[\Gamma, \Gamma])$ is a free abelian group of rank $2r$. Since $[G, G]/\Gamma \cap [G, G]$ is compact by Proposition 3.2, $[G, G]/[\Gamma, \Gamma]$ is compact   q.e.d.

Proposition 4.4. $C(\gamma)\in \boldsymbol{Z}$ *for* $\gamma \in [\Gamma, \Gamma]$.

Proof. Let θ be a holomorphic automorphic function on G of type $j(\gamma, g)$. We then have

$$\theta(g\gamma_1) = \theta(g)e(C(\gamma_1)) \qquad \text{for } g\in G \text{ and } \gamma_1\in [\Gamma, \Gamma].$$

Since θ is not identically zero, there is a point $g_0\in G$ such that $\theta(g_0)\neq0$.

Define a holomorphic function $F: [G, G]\to\boldsymbol{C}$ by $F(g_1)=\theta(g_0g_1)$. Then F is different from zero and satisfies $F(g_1\gamma_1)=\theta(g_0g_1\gamma_1)=\theta(g_0g_1)e(C(\gamma_1))=F(g_1)e(C(\gamma_1))$ for $g_1\in [G, G]$ and $\gamma_1\in [\Gamma, \Gamma]$ and $F(e)\neq0$.

Let $f: [G, G]\to\boldsymbol{R}$ denote C^∞-function $|F(g_1)|$. Then $f(g_1\gamma_1)=f(g_1)$ for $\gamma_1\in [\Gamma, \Gamma]$ since $C(\gamma_1)\in\boldsymbol{R}$ by Lemma 4.2.

We also denote by f the function on $[G, G]/[\Gamma, \Gamma]$ induced by $f: [G, G]\to\boldsymbol{R}$. Since $[\Gamma, \Gamma]$ is a lattice of $[G, G]$, $[G, G]/[\Gamma, \Gamma]$ is a compact complex manifold. Hence, $f: [G, G]/[\Gamma, \Gamma]\to\boldsymbol{R}$ is bounded:

$$|F(g_1)| = f(g_1) = f(p(g_1))\leq c$$

for some constant $c>0$.

Since $[G, G]$ is biholomorphic onto $\boldsymbol{C}^m$, a holomorphic bounded function $F: [G, G]\to\boldsymbol{C}$ is constant. Since $F(\gamma_1)=F(e)e(C(\gamma_1))$, $C(\gamma_1)\in\boldsymbol{Z}$.   q.e.d.

Let $A(g_1, g_2)=\frac{1}{2\sqrt{-1}}(H(g_1, g_2)-\overline{H(g_1, g_2)})$. We then get

$$A(\gamma_1, \gamma_2) = \frac{1}{2\sqrt{-1}}(H(\gamma_1, \gamma_2)-\overline{H(\gamma_1, \gamma_2)})$$

$$\equiv C(\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1})\equiv0 \qquad (\text{mod } 1).$$

We put $d(\gamma)=C(\gamma)-\dfrac{1}{4\sqrt{-1}}H(\gamma, \gamma)$ for $\gamma\in\Gamma$. We have then

$$d(\gamma\delta)\equiv\frac{1}{2}A(\gamma, \delta)+d(\gamma)+d(\delta) \quad(\text{mod } 1)\quad\text{for } \gamma, \delta\in\Gamma .$$

Let $\rho(\gamma)$ be the imaginary part of $d(\gamma)$. We see that $\rho(\gamma\delta)=\rho(\gamma)+\rho(\delta)$ for $\gamma, \delta\in\Gamma$, that is, $\rho\colon \Gamma\to R$ is a homomorphism. It is clear that $\text{Ker }\rho\supset[\Gamma, \Gamma]$. Moreover we have $\text{Ker }\rho\supset\Gamma\cap[G, G]$, since $[\Gamma, \Gamma]$ is a subgroup of finite index of $\Gamma\cap[G, G]$. Hence ρ induces a homomorphism $\bar\rho\colon \Gamma/\Gamma\cap[G, G]\to R$.

Since $\pi(\Gamma)=\Gamma\cdot[G, G]/[G, G]\approx\Gamma/\Gamma\cap[G, G]$ and $\pi(\Gamma)$ is a lattice of $G/[G, G]$, $\bar\rho$ can be extended to a homomorphism from $G/[G, G]$ to R, so that $\rho\colon \Gamma\to R$ can be extended to a homomorphism $\rho\colon G\to R$.

Consider now the biholomorphic map $\Phi\colon G\to C^n$ given by $\Phi(g)=(z_1(g), \cdots, z_n(g))$. Let $z_j(g)=x_j(g)+\sqrt{-1}\,y_j(g)$ for $j=1, \cdots, r$. Note that $\Phi\colon G\to C^n$ induces a map from $G/[G, G]$ onto C^r given by $\pi(g)\to(z_1(g), \cdots, z_r(g))$. We can write $\rho\colon G\to R$ as

$$\rho(g) = \sum_{j=1}^{r} a_j x_j(g)+\sum_{j=1}^{r} b_j y_j(g)$$

for $g\in G$, where $a_j, b_j\in R, j=1, \cdots, r$.

Define $l\colon G\to C$ by

$$l(g) = = \sqrt{-1}\cdot\sum_{j=1}^{r} a_j z_j(g)+\sum_{j=1}^{r} b_j z_j(g) .$$

We have $\text{Im } l(g)=\rho(g)$ and $d(\gamma)-l(\gamma)\in R$ for $\gamma\in\Gamma$.
Note that $l\colon G\to C$ is a holomorphic homomorphism.

Since we can regard $A(g_1, g_2)$ as an alternating form on a vector group $G/[G, G]$ such that $A(g_1, g_2)$ takes integers on the lattice $\pi(\Gamma)$, there is a R-bilinear form B which is Z-valued on the lattice $\pi(\Gamma)$ and $A(g_1, g_2)=B(g_1, g_2)-B(g_2, g_1)$ ([15] Chapter VI, n°2).

Define $\chi\colon \Gamma\to\{z\in C\mid |z|=1\}$ by

$$\chi(\gamma) = e\Big(d(\gamma)-l(\gamma)-\frac{1}{2}B(\gamma, \gamma)\Big) .$$

χ is a character of Γ, since $A(\gamma_1, \gamma_2)\in Z$ for $\gamma_1, \gamma_2\in\Gamma$. Put

$$\psi(\gamma) = \chi(\gamma)e\Big(\frac{1}{2}B(\gamma, \gamma)\Big)\quad\text{for } \gamma\in\Gamma .$$

We get

$$j(\gamma, g) = e\Big(\frac{1}{2\sqrt{-1}}H(g, \gamma)+\frac{1}{4\sqrt{-1}}H(\gamma, \gamma)+l(\gamma)\Big)\cdot\psi(\gamma)$$

for $\gamma\in\Gamma$ and $g\in G$.

Since $l(g)$: $G \to C$ is a holomorphic map which satisfies $l(g\gamma) = l(g) + l(\gamma)$ for $g \in G$ and $\gamma \in \Gamma$, $j(\gamma, g)$ is equivalent to the automorphic factor

$$e\left(\frac{1}{2\sqrt{-1}}H(g, \gamma) + \frac{1}{4\sqrt{-1}}H(\gamma, \gamma)\right)\psi(\gamma).$$

We need the following proposition to show that $\psi | \Gamma \cap [G, G] = id$.

Proposition 4.5. *Let θ be a holomorphic automorphic function on G of type*

$$j(\gamma, g) = e\left(\frac{1}{2\sqrt{-1}}H(g, \gamma) + \frac{1}{4\sqrt{-1}}H(\gamma, \gamma)\right)\cdot\psi(\gamma).$$

Then the hermitian form $H = (h_{jk})$ is non-negative. Moreover $\theta(g \cdot g_0) = \theta(g)$ for $g \in G$, if $g_0 \in G$ satisfies $H(g_0, g_0) = 0$.

Proof.   Let f: $G \to R$ denote the function defined by

$$f(g) = |\theta(g)|^2 e\left(\frac{-1}{2\sqrt{-1}}H(g, g)\right) = |\theta(g)|^2 \exp\left(-\pi H(g, g)\right).$$

We have $f(g\gamma) = f(g)$ for $\gamma \in \Gamma$, so that f induces a function F: $G/\Gamma \to R$.   Since G/Γ is compact, there is a constant $c > 0$ such that $0 \leq F(p(g)) \leq c$ for $g \in G$. Therefore we get

$$f(g) = |\theta(g)|^2 \exp\left(-\pi H(g, g)\right) \leq c \qquad \text{for } g \in G.$$

Thus we have

$$|\theta(g)|^2 \leq c \exp \pi H(g, g) \qquad \text{for } g \in G.$$

Suppose that $H(g_1, g_1) < 0$ for some $g_1 \in G$.   Define $g(\tau) \in G$ $(\tau \in C)$ by

$$g(\tau) = \Phi^{-1}(\tau z_1(g_1) + z_1(g), \cdots, \tau z_n(g_1) + z_n(g)).$$

Then we have $g(0) = g$ and

$$|\theta(g(\tau))|^2 \leq c \exp \pi H(g(\tau), g(\tau)).$$

Put $\rho = H(g(\tau), g(\tau))$.

$$\begin{aligned}
\rho &= \sum_{j,k=1}^{r} h_{jk}(\tau z_j(g_1) + z_j(g))\cdot\overline{(\tau z_k(g_1) + z_k(g))} \\
&= |\tau|^2 \cdot \sum_{j,k=1}^{r} h_{jk} z_j(g_1)\bar{z}_k(g_1) + 2 \operatorname{Re}\left(\tau H(g_1, g)\right) + H(g, g) \\
&= |\tau|^2 H(g_1, g_1) + 2 \operatorname{Re}\left(\tau H(g_1, g)\right) + H(g, g).
\end{aligned}$$

For any $\varepsilon > 0$, there is $R > 0$ such that $\pi\rho \leq \log \varepsilon$ for every τ satisfying $|\tau| \geq R$. Fix $g_1, g \in G$, and we have

$$|\theta(g(\tau))|^2 \leq c\varepsilon \qquad \text{for } |\tau| \geq R.$$

Therefore $\theta(g(\tau))$ is a bounded holomorphic function on C. Hence $\theta(g(\tau))$ is constant with respect to $\tau \in C$. Tending $\varepsilon \to 0$, we get $|\theta(g(\tau))|^2 = 0$. In particular,

$$|\theta(g)|^2 = |\theta(g(0))|^2 = 0.$$

Hence $\theta \equiv 0$ on G, since g can be any element of G. This is a contradiction. Therefore $H = (h_{jk})$ is a non-negative hermitian form.

Take an element $g_0 \in G$ satisfying $H(g_0, g_0) = 0$. Then we have $H(g, g_0) = 0$ for any $g \in G$ since $H(g, g) \geq 0$ for any $g \in G$. Put

$$g_0(\tau) = \Phi^{-1}(\tau z_1(g_0), \cdots, \tau z_n(g_0)) \in G$$

for $\tau \in C$. Then we have

$$|\theta(g \cdot g_0(\tau))|^2 \leq c \cdot \exp \tau H(g \cdot g_0(\tau), g \cdot g_0(\tau))$$
$$= c \cdot \exp \pi(H(g, g) + 2 \operatorname{Re} \tau H(g, g_0) + |\tau|^2 H(g_0, g_0))$$
$$= c \cdot \exp \pi H(g, g).$$

This shows that $\theta(g \cdot g_0((\tau))$ is a bounded holomorphic function with respect to $\tau \in C$. Hence $\theta(g \cdot g_0(\tau))$ is constant with respect to $\tau \in C$. In particular, $\theta(g) = \theta(g \cdot g_0(0)) = \theta(g \cdot g_0(1)) = \theta(g \cdot g_0)$.   q.e.d.

Take an element $g_1 \in G$ satisfying $\theta(g_1) \neq 0$. Since $H(g_0, g_0) = 0$ for $g_0 \in [G, G]$, $\theta(g g_0) = \theta(g)$ for $g \in G$. In particular, $\theta(g \cdot \gamma) = \theta(g)$ for $\gamma \in \Gamma \cap [G, G]$. Put $g = g_1 \gamma^{-1}$. Then $0 \neq \theta(g_1) = \theta(g \cdot \gamma) = \theta(g)$. Since

$$\theta(g \cdot \gamma) = \theta(g) \cdot e\left(\frac{1}{2\sqrt{-1}} H(g, \gamma) + \frac{1}{4\sqrt{-1}} H(\gamma, \gamma)\right) \psi(\gamma) = \theta(g) \psi(\gamma),$$

$\psi(\gamma) = 1$, for $\gamma \in \Gamma \cap [G, G]$. Note that $B(g, g) = 0$ for $g \in [G, G]$. Hence, $\chi \colon \Gamma \to \{z \in C \mid |z| = 1\}$ satisfies that

$$\chi \mid \Gamma \cap [G, G] \equiv 1.$$

Since $\pi(\Gamma) \simeq \Gamma / \Gamma \cap [G, G]$, χ induces a character

$$\tilde{\chi} \colon \pi(\Gamma) \to \{z \in C \mid |z| = 1\}.$$

Let $\Theta \colon G/[G, G] \to C$ denote the holomorphic function on $G/[G, G]$ induced by $\theta \colon G \to C$ and $\tilde{j} \colon \pi(\Gamma) \times G/[G, G] \to C^*$ the automorphic factor induced by $j \colon \Gamma \times G \to C^*$.

Denote $\tilde{D}$ the divisor on $(G/[G, G])/\pi(\Gamma)$ denfied by the holomorphic automorphic function Θ on $G/[G, G]$. We then get $D = \pi^* \tilde{D}$. Therefore we have proved Theorem 2.

Let D be a divisor on G/Γ. Then there exist positive divisors D^+, D^- on G/Γ such that D^+ and D^- are relatively prime and $D=D^+-D^-$ ([15], Appendix n°6). By Theorem 2, there are holomorphic theta functions Θ_1, Θ_2 on the complex torus T such that $D^+=\pi^*(\mathrm{div}\,\Theta_1)$ and $D^-=\pi^*(\mathrm{div}\,\Theta_2)$. Since $\pi\colon G/\Gamma\to T$ is onto holomorphic,

$$D = D^+-D^- = \pi^*(\mathrm{div}\,\Theta_1)-\pi^*(\mathrm{div}\,\Theta_2)$$

$$= \pi^*\,\mathrm{div}\!\left(\frac{\Theta_1}{\Theta_2}\right).$$

Note that $\dfrac{\Theta_1}{\Theta_2}$ is a memorphic theta function on the complex torus T.

It is easy to see that if the divisor $D=0$ the corresponding automorphic function θ is trivial.

Take a meromorphic function ψ on G/Γ. Let $D=\mathrm{div}(\psi)$. Since $D=\pi^*\mathrm{div}\!\left(\dfrac{\Theta_1}{\Theta_2}\right)$, we get that $\psi=\dfrac{\Theta_1\circ\pi}{\Theta_2\circ\pi}$. Since $\psi(g\gamma)=\psi(\gamma)$ for $g\in G$ are $\gamma\in\Gamma$,

$\dfrac{\Theta_1\circ\pi(g\gamma)}{\Theta_2\circ\pi(g\gamma)}=\dfrac{\Theta_1\circ\pi(g)}{\Theta_2\circ\pi(g)}$, hence $\dfrac{\Theta_1}{\Theta_2}$ is a meromorphic function on T. Thus we get that if ψ is a meromorphic function on G/Γ, there is a meromorphic function $\tilde{\psi}$ on the torus T such that $\psi=\pi^*\tilde{\psi}$.

Let $K(G/\Gamma)$ (resp. $K(T)$) denote the field of all meromorphic functions on G/Γ (resp. on T).

We now get the following corollary of Theorem 2.

Corollary *Under the assumptions of Theorem 2, there is a canonical isomorphism $\pi^*\colon K(T)\to K(G/\Gamma)$. In particular, the transcendence degree of $K(G/\Gamma)$ over C is not more than the complex dimention of complex torus T.*

5. Remarks and examples of compact complex parallelisable nilmanifolds

Proposition 5.1. *Let M be a compact complex parallelisable manifold of complex dimension 2. Then M is a complex torus.*

Proof. By a theorem of Wang [14], $M=G/\Gamma$ where G is a simply connected complex Lie group of dimension 2 and Γ is a lattice of G. Let $\{X_1, X_2\}$ be a basis of $\mathfrak{g}^+$ and $\{\omega_1, \omega_2\}$ be the dual basis of $(\mathfrak{g}^+)^*$. We may consider ω_1, ω_2 as holomorphic 1-forms on G/Γ. Since G/Γ is 2 dimensional, ω_1, ω_2 are d-closed; $d\omega_1=d\omega_2=0$. Thus $[X_1, X_2]=0$. Hence, G is abelian and G/Γ is a complex torus.   q.e.d.

Now we shall give some examples of compact complex parallelisable nilmanifolds.

(1)   Let G be a simply connected complex nilpotent Lie group defined by

$$
G = \left\{ \begin{pmatrix} 1 & z_{12} & \cdots\cdots & \cdots z_{1n} \\ & 1 & \ddots & & z_{2n} \\ & & \ddots & \ddots & \vdots \\ & 0 & & 1 & z_{n-1n} \\ & & & & 1 \end{pmatrix} \middle| z_{ij} \in C, i < j \right\}
$$

and Γ be a lattice of G defined by

$$
\Gamma = \left\{ \begin{pmatrix} 1 & a_{12} & \cdots\cdots & \cdot a_{1n} \\ & 1 & \ddots & & a_{2n} \\ & & \ddots & \ddots & \vdots \\ & 0 & & 1 & a_{n-1n} \\ & & & & 1 \end{pmatrix} \middle| a_{ij} \in Z + \sqrt{-1}\, Z, \ i < j \right\}.
$$

Then G/Γ is a compact complex parallelisable nilmanifold.   In this case, we see that the transcendence degree of $K(G/\Gamma)$ over C is $n-1$.

(2)   Let G be a simply connected complex nilpotent Lie group defined by

$$
G = \left\{ \begin{pmatrix} 1 & z_1 & z_2 \cdots\cdots\cdots z_{n-1} & w \\ & 1 & 0 \cdots\cdots\cdots 0 & y_{n-1} \\ & & 1 \ddots \ \vdots & \vdots \\ & & \ddots \ \ddots \ \vdots & \vdots \\ & & 1 \ \ 0 & y_2 \\ & 0 & 1 & y_1 \\ & & & 1 \end{pmatrix} \middle| \begin{array}{l} z_j, y_j, w \in C \\ j = 1, 2, \cdots, n-1 \end{array} \right\}
$$

for $n \geq 2$, and Γ be a lattice of G defined by

$$
\Gamma = \left\{ \begin{pmatrix} 1 & a_1 & a_2 \cdots\cdots\cdots a_{n-1} & c \\ & 1 & 0 \cdots\cdots\cdots 0 & b_{n-1} \\ & & 1 \ddots \ \vdots & \vdots \\ & & \ddots \ \ddots \ \vdots & \vdots \\ & & 1 \ \ 0 & b_2 \\ & 0 & 1 & b_1 \\ & & & 1 \end{pmatrix} \middle| \begin{array}{l} a_j, b_j, c \in Z + \sqrt{-1}\, Z \\ j = 1, 2, \cdots, n-1 \end{array} \right\}.
$$

Then G/Γ is a compact complex parallelisable nilmanifold. In this case, we see that the transcendence degree of $K(G/\Gamma)$ over C is $2(n-1)$.

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