<table>
<thead>
<tr>
<th><strong>Title</strong></th>
<th>On compact complex parallelisable solvmanifolds</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Author(s)</strong></td>
<td>Sakane, Yusuke</td>
</tr>
<tr>
<td><strong>Citation</strong></td>
<td>Osaka Journal of Mathematics. 13(1) P.187-P.212</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>1976</td>
</tr>
<tr>
<td><strong>Text Version</strong></td>
<td>publisher</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="https://doi.org/10.18910/5964">https://doi.org/10.18910/5964</a></td>
</tr>
<tr>
<td><strong>DOI</strong></td>
<td>10.18910/5964</td>
</tr>
<tr>
<td><strong>rights</strong></td>
<td></td>
</tr>
</tbody>
</table>
ON COMPACT COMPLEX PARALLELISABLE SOLVMANIFOLDS

Dedicated to the memory of Taira Honda

YUSUKE SAKANE

(Received December 4, 1974)

1. Introduction

This paper deals with compact complex solvmanifolds. Our main purpose is to generalize the theory on the divisor group of a complex torus to these manifolds. By a solvmanifold we mean a homogeneous space of solvable Lie group. Let $G$ be a simply connected complex solvable Lie group and $\Gamma$ be a lattice of $G$, that is, a discrete subgroup of $G$ such that $G/\Gamma$ is compact. The de Rham cohomology group and the Dolbeault cohomology group of a compact complex manifold $G/\Gamma$ play an important role in studying the divisor group of a complex manifold $G/\Gamma$. The de Rham cohomology group of a compact solvmanifold $G/\Gamma$ has been discussed by Matsushima [7], Nomizu [10] and Mostow [8].

Let $M$ be a compact connected complex manifold and $H^{p,q}(M)$ denote the Dolbeault cohomology group of $M$ of type $(p, q)$. Let $g$ be a complex Lie algebra and $J$ be the canonical complex structure of $g$. Then $g^c = g^+ \oplus g^-$, where $g^\pm = \{X \in g^c | IX = \pm \sqrt{-1} X \}$. In section 2, we prove:

Theorem 1. Let $G$ be a simply connected complex nilpotent Lie group and $\Gamma$ be a lattice of $G$. Then there is a canonical isomorphism

$$H^{p,q}_c(G/\Gamma) \cong H^q(g^-) \otimes \Lambda^p(g^+)^*$$

where $H^q(g^-)$ denotes the Lie algebra cohomology group of $g^-$ and $(g^+)^*$ denotes the dual vector space of $g^+$.

Let $G$ be a simply connected complex solvable Lie group and $\Gamma$ be a lattice of $G$ which has the following property:

$(M) \ Ad(G)$ and $Ad(\Gamma)$ have the same Zariski closure in the group $Aut(g^c)$.

This condition has been used by Mostow in his study of lattices of solvable
Lie group [8]. Denote by \([G, G]\) the commutator group of \(G\) and let \(\pi: G \to G/[G, G]\) be the projection. Then \(\Gamma \cap [G, G]\) is a lattice of \([G, G]\), so that \(\pi(\Gamma)\) is a lattice of \(G/[G, G]\) and \((G/\Gamma, \pi, ([G, G]/\pi(\Gamma), [G, G]/([G, G] \cap \Gamma))\) is a homomomorphic fiber bundle. Let \(T\) denote the complex torus \((G/[G, G])/\pi(\Gamma)\).

In section 3, we study Chern classes of holomorphic line bundles over these compact complex solvmanifolds.

Let \(M\) and \(N\) be complex manifolds and \(\phi: M \to N\) be a surjective holomorphic map. For a divisor \(\bar{D}\) on \(N\) let \(\phi^*(\bar{D})\) denote the divisor on \(M\) defined by \(\phi^*X_{\phi(x)}\) for all \(x \in M\). We call the divisor \(\phi^*(\bar{D})\) on \(M\) the pull back of the divisor \(\bar{D}\) on \(N\) [15]. In section 4, we prove:

**Theorem 2.** Let \(G\) be a simply connected complex solvable Lie group and \(\Gamma\) be a lattice of \(G\). Assume that \(\Gamma\) satisfies the condition (M) and that \(H^q_{d\bar{\partial}}(G/\Gamma) \simeq H^q(\mathbb{C}^\infty)\) canonically. Then, under the notation introduced above, for each positive divisor \(D\) on \(G/\Gamma\), there exists a positive divisor \(\bar{D}\) on the complex torus \(T\) such that the divisor \(D\) is the pull back of the divisor \(\bar{D}\) on \(T\) by the projection \(\pi: G/\Gamma \to T\), i.e., \(D = \pi^\ast \bar{D}\).

Note that our assumption in Theorem 2 is always satisfied if \(G\) is a simply connected complex nilpotent Lie group and \(\Gamma\) is a lattice of \(G\).

If \(M\) is a compact connected complex manifold, \(K(M)\) will denote the field of all meromorphic functions on \(M\).

**Corollary.** Under the condition of Theorem 2, there is a canonical isomorphism

\[\pi^\ast: K(T) \simeq K(G/\Gamma).\]

In particular, the transcendence degree of \(K(G/\Gamma)\) over \(\mathbb{C}\) is not larger than the complex dimension of the complex torus \(T\).

The author would like to express his deep appreciation to Professor Yozo Matsushima for his thoughtfull guidence and encouragement given during the completion of this paper.

### 2. Dolbeault cohomology groups of compact complex nilmanifolds

Let \(M\) be a complex manifold and \(H^{p,q}_\mathbb{C}(M)\) denote the Dolbeault cohomology of \(M\) of type \((p, q)\). Let \(G\) be a simply connected complex Lie group and \(\Gamma\) be a uniform lattice of \(G\). Let \(\mathfrak{g}\) denote the Lie algebra of all right invariant vector fields on \(G\), \(I\) denote the complex structure of \(\mathfrak{g}\) and \(\mathfrak{g}^+\) (resp. \(\mathfrak{g}^-\)) denote the vector space of the \(\sqrt{-1}\) (resp. \(-\sqrt{-1}\)) eigenvectors of \(I\) in the complexification \(\mathfrak{g}^\mathbb{C}\) of \(\mathfrak{g}\). We identify \(\mathfrak{g}^+\) to the Lie algebra of all right invariant holomorphic vector fields on \(G\) and the dual space \((\mathfrak{g}^+)^*\) to the space of all right invariant holomorphic 1-forms on \(G\). Moreover we may identify an element of
$g^+$ (resp. $(g^+)^*$) to a holomorphic vector field (resp. a holomorphic 1-form) on $G/\Gamma$. Let $\Lambda^p T^*(G/\Gamma)$ be the $p$-th exterior product bundle of the holomorphic cotangent bundle $T^*(G/\Gamma)$ of $G/\Gamma$. Since $G/\Gamma$ is a compact complex parallelizable manifold, the holomorphic vector bundle $\Lambda^p T^*(G/\Gamma)$ on $G/\Gamma$ is the trivial vector bundle $G/\Gamma \times \Lambda^p(g^+)^*$. Thus we have an isomorphism

\[(2.1) \quad H^{q\sharp}(G/\Gamma) \cong H^{q\sharp}(G/\Gamma)^* \otimes \Lambda(p^+)^* .\]

**Theorem 1.** Let $G$ be a simply connected complex nilpotent Lie group and $\Gamma$ be a lattice of $G$. Then we have a canonical isomorphism

\[H^{q\sharp}(G/\Gamma) \cong H^{q}(g^-)^* \otimes \Lambda(p^+)^* \]

where $H^q(g^-)$ denoted the $q$-th Lie algebra cohomology of with the trivial representation $\rho_0: g^- \rightarrow \mathbb{C}$.

We need some preparations to prove Theorem 1. Consider the descending central series $\{C^k(G)\}$ of $G$, where $C^k(G) = [G, C^{k-1}(G)]$ and $C^0(G) = G$. Since $G$ is nilpotent, there is an integer $m \in \mathbb{N}$ such that $C^m(G) = (e)$ and $C^{m+1}(G) = (e)$. Let $A$ denote the group $C^m(G)$. Then $A$ is contained in the center $Z(G)$ of $G$. Since $G$ is a simply connected nilpotent Lie group and $A$ is connected, $A$ is a simply connected closed Lie subgroup. Let $\Gamma$ be a lattice of $G$. Then $A \cap \Gamma$ is a lattice of $A$ ([11] p. 31 Corollary 1) and $A \Gamma$ is closed in $G$ ([11] p. 23 Theorem 1.13). Let $\pi: G \rightarrow G/A$ be the canonical map. Then $\pi(\Gamma)$ is a lattice of $G/A$. Since $A/(A \cap \Gamma) \cong A \Gamma/\Gamma$ is a complex torus, we have a holomorphic principal fiber bundle $(G/\Gamma, (G/\Gamma)/\pi(\Gamma), \pi, (G/\Gamma)/\pi(\Gamma))$.

Let $C^\infty(G, \mathbb{C})$ be the vector space of all complex valued $C^\infty$-functions on $G$. Define the subspaces $\mathcal{C}$ and $\mathcal{C}'$ of $C^\infty(G, \mathbb{C})$ by

\[\mathcal{C} = \{ f \in C^\infty(G, \mathbb{C}) \mid f(g\gamma) = f(g) \quad \text{for all} \quad \gamma \in \Gamma \}\]

and

\[\mathcal{C}' = \{ f \in \mathcal{C} \mid f(ga) = f(g) \quad \text{for all} \quad a \in A \} .\]

For a right invariant vector field $X \in g$ and $f \in C^\infty(G, \mathbb{C})$, put

\[(Xf)(g) = \frac{d}{dt} f(a(t)g) \big|_{t=0} \]

where $a(t)$ is the one parameter subgroup corresponding to $X$. Then $C^\infty(G, \mathbb{C})$ is a $g$-module, and hence $\mathcal{C}$ and $\mathcal{C}'$ are $g^c$-submodules of $C^\infty(G, \mathbb{C})$.

Let $\alpha$ be the Lie subalgebra of $g$ corresponding to the complex Lie subgroup $A$ of $G$. Then $\alpha^c$ has the decomposition $\alpha^c = \alpha^c + \alpha^-$ with respect to the complex structure $I$, and $\mathcal{C}$ and $\mathcal{C}'$ are $\alpha^-$-modules. Let $\{A^q(\alpha^-, \mathcal{C}), d\}$ (resp. $\{A^q(\alpha^-, \mathcal{C}')$, $d\}$ denote the cochain complex of $\alpha^-$-module $\mathcal{C}$ (resp. $\mathcal{C}'$) and $H^*(\alpha^-, \mathcal{C})$ (resp. $H^*(\alpha^-, \mathcal{C}')$) denote the Lie algebra cohomology of $\alpha^-$-module.
Since \( \alpha^- \) is an ideal of \( \mathfrak{g}^- \), \( A^q(\alpha^-, \mathbb{C}) \) (resp. \( A^q(\alpha^-, \mathbb{C}') \)) is \( \mathfrak{g}^- \)-module by

\[
(L_X \omega(X_1, \cdots, X_d) = X(\omega(X_1, \cdots, X_d)) - \sum_{j=1}^{d} \omega(X_1, \cdots, [X, X_j], \cdots, X_d)
\]

where \( X \in \mathfrak{g}^- \), \( \omega \in A^q(\alpha^-, \mathbb{C}) \) (resp. \( \omega \in A^q(\alpha^-, \mathbb{C}') \)) and \( X_1, \cdots, X_d \in \alpha^- \). Moreover \( L_X d = d \circ L_X \) for all \( X \in \mathfrak{g}^- \). Thus \( H^*(\alpha^-, \mathbb{C}) \) and \( H^*(\alpha^-, \mathbb{C}') \) are \( \mathfrak{g}^- \)-modules.

**Proposition 2.1.** The inclusion map \( \iota_0 : \mathbb{C}' \to \mathbb{C} \) induces an isomorphism \( \iota_0^* \) of \( \mathfrak{g}^- \)-modules

\[
\iota_0^*: H^q(\alpha^-, \mathbb{C}') \to H^q(\alpha^-, \mathbb{C}).
\]

This follows from Kodaira and Spencer [6] §2, but we shall give an elementary proof (cf. [11] VII §4).

Let \( \{X_1, \cdots, X_l\} \) be a basis of \( \alpha^+ \) and \( \{\omega_1, \cdots, \omega_l\} \) be the dual basis. We regard \( \omega_j \) \( (j=1, \cdots, l) \) as the holomorphic invariant 1-forms on the complex torus \( A/(\mathbb{A} \cap \Gamma) \). Define an invariant hermitian metric \( h \) on \( A/(\mathbb{A} \cap \Gamma) \) by

\[
h = \sum \omega_j \omega_j,
\]

and \( \frac{1}{l!} \Omega^l \) defines a Haar measure \( da \) on \( A/\mathbb{A} \cap \Gamma \). We may assume that

\[
\int_{A/\mathbb{A} \cap \Gamma} \frac{1}{l!} \Omega^l = 1
\]

by changing the choice of a basis of \( \alpha^+ \) if necessary. For \( f \in \mathbb{C} \) and \( x \in \mathbb{C} \), let \( f_x(a) = f(xa) \) for \( a \in \mathbb{A} \). Then we can define a \( \mathfrak{g}^c \)-module homomorphism \( H : \mathbb{C} \to \mathbb{C}' \) by

\[
H(f)(x) = \int_{A/\mathbb{A} \cap \Gamma} f_x(a) \frac{\Omega^l}{l!} = \int_{A/\mathbb{A} \cap \Gamma} f(xa) da .
\]

Let \( Y_j = \frac{1}{2} (X_j + X_j) \) and \( Y_{j+1} = \frac{\sqrt{-1}}{2} (X_j - X_j) \) for \( j = 1, \cdots, l \). Then \( \{Y_1, \cdots, Y_{2l}\} \) is a basis of \( \alpha \). Let \( \{\theta_1, \cdots, \theta_{2l}\} \) be its dual basis. Let \( A^r(\alpha, \mathbb{C}) \) denote the vector space of all \( \mathbb{C} \)-valued \( r \)-forms on \( A/\mathbb{A} \cap \Gamma \). Note that each element \( \omega \in A^r(\alpha, \mathbb{C}) \) can be written uniquely as

\[
\omega = \sum_{k_1, \cdots, k_r} f_{k_1, \cdots, k_r} \theta_{k_1} \wedge \cdots \wedge \theta_{k_r} \quad \text{where} \quad f_{k_1, \cdots, k_r} \in \mathbb{C}.
\]

For simplicity, let \( \theta_K = \theta_{k_1} \wedge \cdots \wedge \theta_{k_r} \) and \( f_K = f_{k_1, \cdots, k_r} \) for \( K = (k_1, \cdots, k_r) \) \( (1 \leq k_1 < \cdots < k_r \leq 2l) \). Then \( \omega = \sum_K f_K \theta_K \).

Let \( A^{p,q}(\alpha, \mathbb{C}) \) denote the vector space of all \( \mathbb{C} \)-valued forms of type \((p, q)\) on \( A/\mathbb{A} \cap \Gamma \). Each element \( \omega \in A^{p,q}(\alpha, \mathbb{C}) \) can be written uniquely as
where \( I = (i_1, \ldots, i_p) \) (1 \( \leq i_1 < \cdots < i_p \leq l \)), \( J = (j_1, \ldots, j_q) \) (1 \( \leq j_1 < \cdots < j_q \leq l \)), \( f_{ij} \in \mathbb{C} \), \( \omega_j = \omega_1 \wedge \cdots \wedge \omega_{i_j} \), and \( \overline{\omega}_j = \overline{\omega}_1 \wedge \cdots \wedge \overline{\omega}_{i_j} \).

Define operators \( d: A^r(a, \mathbb{C}) \to A^{r+1}(a, \mathbb{C}) \) by

\[
d\omega = \sum_{j=1}^{2^l} f_j \theta_j \wedge \overline{\theta}_j
\]

for \( \omega = \sum_{k=1}^{2^l} f_k \theta_k \in A^r(a, \mathbb{C}) \), \( d': A^{p,q}(a, \mathbb{C}) \to A^{p,q+1}(a, \mathbb{C}) \) by

\[
d'\omega = \sum_{j=1}^{2^l} (\sum_{k=1}^{2^l} X_k f_j) \omega_k \wedge \omega_j \wedge \overline{\omega}_j
\]

for \( \omega = \sum_{d_j} f_j \omega_j \wedge \overline{\omega}_j \in A^{p,q}(a, \mathbb{C}) \) and \( d'': A^{p,q}(a, \mathbb{C}) \to A^{p,q+1}(a, \mathbb{C}) \) by

\[
d''\omega = \sum_{j=1}^{2^l} (\sum_{k=1}^{2^l} \overline{X}_k f_j) \overline{\omega}_k \wedge \omega_j \wedge \overline{\omega}_j
\]

for \( \omega = \sum_{d_j} f_j \omega_j \wedge \overline{\omega}_j \in A^{p,q}(a, \mathbb{C}) \). Then \( d d'' = d'' d' = d' d'' = 0 \).

Define \( \langle \omega, \eta \rangle \in \mathbb{C}' \) for \( \omega, \eta \in A^{p,q}(a, \mathbb{C}) \) by

\[
\langle \omega, \eta \rangle(x) = \sum_{j=1}^{2^l} \int_{A/A \cap \Gamma} f_j(xa) g_j(xa) da = \int_{A/A \cap \Gamma} \omega \wedge \overline{\eta},
\]

where \( \omega = \sum_{d_j} f_j \omega_j \wedge \overline{\omega}_j \), \( \eta = \sum_{d_j} g_j \omega_j \wedge \overline{\omega}_j \) and \( * \) is the operation defined by the natural orientation of \( A/A \cap \Gamma \) and the metric \( h \) on \( A/A \cap \Gamma \).

Let \( f \in C^\infty(G/A \Gamma, \mathbb{C}) \) denote the function corresponding to \( f \in \mathbb{C}' \). Define a hermitian inner product \( (\ , \ ) \) on \( A^{p,q}(a, \mathbb{C}) \) by

\[
(\omega, \eta) = \int_{G/A \Gamma} \langle \omega, \eta \rangle(x) dx
\]

where \( dx \) denotes an invariant measure on \( G/A \Gamma \).

Define \( (\omega, \eta) = 0 \) if \( \omega \in A^{p,q}(a, \mathbb{C}) \), \( \eta \in A^{p',q'}(a, \mathbb{C}) \) for \( (p, q) \neq (p', q') \).

Since \( A^r(a, \mathbb{C}) = \sum_{p+q=r} A^{p,q}(a, \mathbb{C}) \), we have thus an hermitian inner product \( (\ , \ ) \) on \( A^r(a, \mathbb{C}) \).

Now define the adjoint operators \( \delta, \delta', \delta'' \) of \( d, d', d'' \) by \( \delta = -*d* \), \( \delta' = -*d''* \), \( \delta'' = -*d'* \) respectively. We then have

\[
\begin{align*}
(d\omega, \eta) &= (\omega, \delta\eta) \quad \text{for} \quad \omega \in A^r(a, \mathbb{C}) \quad \text{and} \quad \eta \in A^{r+1}(a, \mathbb{C}), \\
(d'\omega, \eta) &= (\omega, \delta'\eta) \quad \text{for} \quad \omega \in A^{p,q}(a, \mathbb{C}) \quad \text{and} \quad \eta \in A^{p+1,q}(a, \mathbb{C}), \\
(d''\omega, \eta) &= (\omega, \delta''\eta) \quad \text{for} \quad \omega \in A^{p,q}(a, \mathbb{C}) \quad \text{and} \quad \eta \in A^{p,q+1}(a, \mathbb{C}).
\end{align*}
\]

with respect to the hermitian inner product \( (\ , \ ) \).

Define Laplacians \( \Delta, \Box', \Box'' \) by

\[
\Delta = d\delta + \delta d, \quad \Box' = d\delta' + \delta' d, \quad \Box'' = d\delta'' + \delta'' d''
\]
Then, by a direct computation we get

$$\Delta \omega = -\sum_{k} \left( \sum_{j} Y_{j} \phi_{K} \right) \theta_{K}$$

for $$\omega = \sum_{k} f_{K} \theta_{K}$$, and

$$\Box' \omega = \Box'' \omega = -\sum_{j} \left( \sum_{i} X_{j} X_{j} \right) f_{j} \omega_{j} \wedge \bar{\omega}_{j}$$

for $$\omega = \sum_{j} f_{j} \omega_{j} \wedge \bar{\omega}_{j}$$.

Since $$X_{j} X_{j} f = (Y_{j} + Y_{j}^{2}) f$$ for each $$f \in \mathcal{C}$$, we see $$\Delta = \Box' = \Box''$$.

Since $$A$$ is abelian and simply connected, we may identify $$A$$ (resp. the lattice $$A \cap \Gamma$$ of $$A$$) with Euclidean space $$(\mathbb{R}^{n}, \langle \cdot, \cdot \rangle)$$ (resp. a lattice $$D$$ in $$\mathbb{R}^{n}$$). For a fixed $$x \in G$$ and $$f \in \mathcal{C}$$, $$f_{x}$$ can be regarded as a function on the torus $$\mathbb{R}^{n}/D$$.

Consider the Fourier expansion of $$f_{x}$$,

$$f_{x}(a) = f(\chi a) = \sum_{\alpha \in D'} C_{\alpha}(x) \exp \frac{2\pi \sqrt{-1} \langle \alpha, a \rangle}{\langle \alpha, \alpha \rangle}$$

where $$D' = \{ \alpha \in \mathbb{R}^{n} | \langle \alpha, d \rangle \in \mathbb{Z} \text{ for any } d \in D \}$$ and $$C_{\alpha}(x) = \int_{A/A \cap \Gamma} f(\chi a) \exp -2\pi \sqrt{-1} \langle \alpha, a \rangle da$$ for $$\alpha \in D'$$. Note that $$H(f)(x) = C_{\alpha}(x) = \int_{A/A \cap \Gamma} f(\chi a) da$$.

For $$Y \in A$$, $$f \in \mathcal{C}$$ and $$x \in G$$, we have

$$(Y f)(x) = \frac{d}{dt} f(a(t)x) \big|_{t=0}$$

where $$a(t)$$ is the one parameter subgroup corresponding to $$Y$$. Since $$A$$ is contained in the center of $$G$$,

$$(Y f)(x) = \frac{d}{dt} \big|_{t=0} f(\chi a(t) x)$$

$$= \frac{d}{dt} \big|_{t=0} \{ \sum_{\alpha \in D'} C_{\alpha}(x) \exp \frac{2\pi \sqrt{-1} \langle \alpha, a(t) \rangle}{\langle \alpha, \alpha \rangle} \}$$

$$= \frac{d}{dt} \big|_{t=0} \{ \sum_{\alpha \in D'} C_{\alpha}(x) \exp \frac{2\pi \sqrt{-1} \langle \alpha, a \rangle + \langle \alpha, a(t) \rangle}{\langle \alpha, \alpha \rangle} \}$$

$$= 2\pi \sqrt{-1} \sum_{\alpha \in D'} C_{\alpha}(x) \langle \alpha, Y \rangle \exp 2\pi \sqrt{-1} \langle \alpha, a \rangle$$.

Since $$\langle Y_{j}, Y_{k} \rangle = \frac{1}{4} \delta_{jk}$$ for $$j, k = 1, \ldots, 2l$$, it follows that

$$4(\Delta f)(x) = -4 \sum_{j=1}^{2l} \langle Y_{j} f(x), Y_{j}^{2} f(x) \rangle = (2\pi)^{2} \sum_{\alpha \in D'} C_{\alpha}(x) \| \alpha \|^{2} \exp 2\pi \sqrt{-1} \langle \alpha, a \rangle$$

where $$\| \alpha \|^{2} = \langle \alpha, \alpha \rangle$$.

Define an operator $$G : \mathcal{C} \to \mathcal{C}$$ by

$$G(f)(x) = \frac{1}{(2\pi)^{2}} \sum_{\alpha \in D'} C_{\alpha}(x) \exp \frac{2\pi \sqrt{-1} \langle \alpha, a \rangle}{\| \alpha \|^{2}}$$

for $$x \in G$$ and $$f \in \mathcal{C}$$. We can show that $$G(f)(xa) = G(f)(yb)$$ if $$xa = yb$$ where $$a, b \in A$$ ([11] p. 118). Thus $$G(f) \in C^{\omega}(G, \mathcal{C})$$. We also have $$G(f)(x\gamma) = G(f)(x)$$ for any $$\gamma \in \Gamma$$. Hence, $$G(f) \in \mathcal{C}$$. It is obvious that
4\Delta G(f) = 4G\Delta(f) = f \quad \text{if} \quad H(f) = 0, \\
\text{and} \quad G \circ H(f) = H \circ G(f) = 0 \quad \text{for any} \quad f \in \mathcal{C}. \quad \text{Therefore} \\
f = H(f) + 4\Delta G(f) = H(f) + 4G\Delta(f) \quad \text{for any} \quad f \in \mathcal{C}. \\
\text{Define} \quad H: A^p(q)(\alpha, \mathcal{C}) \to A^p(q)(\alpha, \mathcal{C}') \quad \text{and} \quad G: A^p(q)(\alpha, \mathcal{C}) \to A^p(q)(\alpha, \mathcal{C}) \quad \text{by} \\
H(\omega) = \sum_{I,J} H(f_{ij})\omega_I \wedge \omega_J \quad \text{for} \quad \omega = \sum_{I,J} f_{ij}\omega_I \wedge \omega_J \\
\text{and} \\
G(\omega) = \sum_{I,J} G(f_{ij})\omega_I \wedge \omega_J \quad \text{for} \quad \omega = \sum_{I,J} f_{ij}\omega_I \wedge \omega_J. \\
\text{Then we have} \\
\omega = H(\omega) + 4G\Delta(\omega) = H(\omega) + 4\Delta G(\omega) \\
\text{and} \\
\omega = H(\omega) + 4G\Box''(\omega) = H(\omega) + 4\Box''G(\omega). \\
\text{Obviously} \quad d'' \circ H = d' \circ H = 0. \quad \text{Since} \quad \int_{(X_f)(\mathcal{C}')} \omega da = \int_{(X_f)(\mathcal{C})} \omega da = 0 \quad \text{for} \quad j = 1, \ldots, l \quad \text{and} \quad f \in \mathcal{C}, \quad H \circ d'' = H \circ d' = 0. \quad \text{By the definition of} \quad H, \quad \text{it is obvious that} \quad \star \circ H = H \circ \star, \quad \text{so that} \quad \delta'' \circ H = H \circ \delta'' = 0. \\
\text{Let} \quad A^\epsilon(\alpha, \mathcal{C}) = \sum_{\alpha} A^{p,\epsilon}(\alpha, \mathcal{C}). \\
\textbf{Lemma 4.2.} \quad \text{Let} \quad F: A^\epsilon(\alpha, \mathcal{C}) \to A^\epsilon(\alpha, \mathcal{C}) \quad \text{be an additive operator which commutes with} \quad \Box''. \quad \text{Then} \quad F \quad \text{commutes with} \quad H \quad \text{and} \quad G. \quad \text{In particular,} \quad G \quad \text{commutes with} \quad d'' \quad \text{and} \quad \delta''. \\
\textbf{Proof.} \quad \text{See [15] Chapter IV lemma 3.} \\
\text{Proof of Proposition 2.1.} \quad \text{Note that the cochain complex} \quad \{A^{p,\epsilon}(\alpha, \mathcal{C}), d''\} \quad \text{is exactly the cochain complex of} \quad \alpha^{-}\text{-module} \quad \mathcal{C}. \quad \text{The inclusion map} \quad \iota_\epsilon: \mathcal{C}' \to \mathcal{C} \quad \text{induces a cochain map} \quad \iota_\epsilon^\epsilon: A^\epsilon(\alpha', \mathcal{C}') \to A^\epsilon(\alpha, \mathcal{C}). \quad \text{In particular, the following diagram commutes} \\
A^{p,\epsilon}(\alpha, \mathcal{C}) \xrightarrow{\iota_\epsilon^\epsilon} A^{p,\epsilon}(\alpha, \mathcal{C}) \xrightarrow{d''} A^{p,\epsilon+1}(\alpha, \mathcal{C}). \\
\text{Since} \quad d''(\omega) = 0 \quad \text{for any} \quad \omega \in A^{p,\epsilon}(\alpha, \mathcal{C}'), \quad H^\epsilon(\alpha^-, \mathcal{C}') = A^{p,\epsilon}(\alpha, \mathcal{C}'). \\
\text{Let} \quad \iota_\epsilon^\epsilon: H^\epsilon(\alpha^-, \mathcal{C}') \to H^\epsilon(\alpha^-, \mathcal{C}) \quad \text{denote the map induced from the cochain map} \quad \iota_\epsilon^\epsilon: A^\epsilon(\alpha^-, \mathcal{C}') \to A^\epsilon(\alpha^-, \mathcal{C}). \quad \text{Since} \quad H \circ d'' = d'' \circ H, \quad H: A^{p,\epsilon}(\alpha, \mathcal{C}) \to A^{p,\epsilon}(\alpha, \mathcal{C}') \quad \text{induces a linear map} \quad H: H^\epsilon(\alpha^-, \mathcal{C}) \to H^\epsilon(\alpha^-, \mathcal{C}'). \\
\text{We claim that} \quad \iota_\epsilon^\epsilon \circ H = \text{id} \quad \text{and} \quad H \circ \iota_\epsilon^\epsilon = \text{id}. \quad \text{By definition} \quad H \circ \iota_\epsilon^\epsilon[\omega] = [\omega] \quad \text{for} \quad [\omega] \in H^\epsilon(\alpha^-, \mathcal{C}'). \quad \text{Since} \quad \omega = H(\omega) + 4G\Box''(\omega) = H(\omega) + 4Gd''\delta''\omega = H(\omega) + 4d''G\delta''\omega \quad \text{for any} \quad \omega \in A^{p,\epsilon}(\alpha, \mathcal{C}) \quad \text{such that} \quad d''\omega = 0, \quad \iota_\epsilon^\epsilon H[\omega] = [\omega] \quad \text{for any} \quad [\omega] \in H^\epsilon(\alpha^-, \mathcal{C}). \quad \text{It is now obvious that} \quad \iota_\epsilon^\epsilon \text{is a} \quad \mathcal{G}^{-}\text{-module homomorphism.} \quad \text{q.e.d.}
Proof of Theorem 1. Let \( A_{g^q} \) be the space of all \( \mathbb{C} \)-valued \( C^\infty \)-differential forms on \( G/\Gamma \) of type \((0, q)\). Take a basis \( \{X_1, \ldots, X_n\} \) of \( g^+ \) and let \( \{\omega_1, \ldots, \omega_n\} \) be the dual basis of \( (g^+)^* \). We regard an element \( \omega \in (g^+)^* \) as a holomorphic 1-form on \( G/\Gamma \). Then any element \( \omega \in A_{g^q} \) can be written as \( \omega = \sum f_j \omega_j \) where \( \omega_j = \bar{\omega}_{j_1} \wedge \cdots \wedge \bar{\omega}_{j_q} \), \( J = (j_1, \ldots, j_q) \), \( 1 \leq j_1 < \cdots < j_q \leq n \) and \( f_j \in \mathbb{C} \).

The operator \( d'' : A_{g^q} \rightarrow A_{g^{q+1}} \) can be written as
\[
d'' \omega = \sum \left( \sum_{k=1}^n X_k f_j \right) \omega_k \wedge \bar{\omega}_J + f_j d\omega_J
\]
for \( \omega = \sum f_j \omega_j \).

Therefore the Dolbeault cohomology group \( H_{q^*} \) can be regarded as the Lie algebra cohomology \( H_q(g^-) \) of \( g^- \)-module \( C \).

\[
H_{q^*}(G/\Gamma) \cong H_q(g^-, C)
\]

(2.2) Regarding \( \mathbb{C} \) as constant functions on \( G \), we have the inclusion map \( \iota : \mathbb{C} \rightarrow C \) of \( g^- \)-modules. Now by (2.1), Theorem 1 is equivalent to assert that \( \iota \) induces an isomorphism on the cohomology groups
\[
\iota^* : H^q(g^-) \rightarrow H^q(g^-, C)
\]

We prove the isomorphism \( \iota^* : H^q(g^-) \rightarrow H^q(g^-, C) \) by the induction on the dimension of \( G/\Gamma \). If \( G \) is abelian, \( G/\Gamma \) is a complex torus and our claim is well-known. As before, let \( A \) be the normal subgroup of \( G \) contained in the center of \( G \) and \( \alpha \) be the ideal in \( g \) corresponding to \( A \). Consider the Hochschild and Serre spectral sequences for \( g^- \)-modules \( C \) and \( C \), and a homomorphism of these spectral sequences induced by the inclusion map \( \iota : C \rightarrow C \);

\[
E_2(\iota) : H^t(g^-/\alpha^-) \rightarrow H^t(g^-(\alpha^-), C)
\]

for \( t, s = 0, 1, 2, \ldots \).

Consider also the \( g^- \)-module \( C' \). Then we have a commutative diagram of \( g^- \)-modules
\[
\begin{array}{ccc}
C & \xrightarrow{\iota} & C \\
\downarrow f & & \downarrow t_0 \\
C' & \xrightarrow{t_0} & C'.
\end{array}
\]

This commutative diagram induces the corresponding commutative diagram of spectral sequences.
By proposition 2.1, we have an isomorphism of $\mathfrak{g}^-$-modules $\iota_\#: H^s(\alpha^-, C) \to H^s(\alpha^-, C)$. Hence,

$$E_s(\iota): H^s(\mathfrak{g}^-/\alpha^-, H^s(\alpha^-, C)) \to H^s(\mathfrak{g}^-/\alpha^-, H^s(\alpha^-, C)).$$

By proposition 2.1, we have an isomorphism of $\mathfrak{g}^-$-modules $\iota_\#: H^s(\alpha^-, C') \to H^s(\alpha^-, C)$. Hence,

$$E_s(\iota_\#: H^s(\alpha^-, C') \to H^s(\alpha^-, C))$$

is an isomorphism.

We shall show that $E_s(j)$ is an isomorphism. Since $\alpha^-$ is contained in the center of $\mathfrak{g}^-$, $\mathfrak{g}^-$ acts trivially on $\mathcal{g}^s(\alpha^-, C) = \mathcal{A}^s(\alpha^-, C)$. Hence,

$$E_s(\iota): H^s(\mathfrak{g}^-/\alpha^-, H^s(\alpha^-, C)) \to H^s(\mathfrak{g}^-/\alpha^-, \mathcal{A}^s(\alpha^-, C)).$$

Since $\alpha^-$ acts trivially on $\mathcal{C}'$, $H^s(\alpha^-, \mathcal{C}) = \mathcal{A}^s(\alpha^-, \mathcal{C})$. Consider the action of $\mathfrak{g}^-$ on $H^s(\alpha, \mathcal{C}')$. For an $s$-cochain $\omega = \sum_{j} f_j \omega_j \in \mathcal{A}^s(\alpha^-, \mathcal{C}')$ and $P \in \mathfrak{g}^-$,

$$L_\omega \omega = \sum_{j} \sum_{j} f_j \omega_j \sigma_j, \quad \text{since } \alpha^- \text{ is contained in the center of } \mathfrak{g}^-.$$

Hence, $H^s(\alpha^-, \mathcal{C}')$ and $\mathcal{C}' \otimes H^s(\alpha, \mathcal{C})$ are isomorphic as $\mathfrak{g}^-$-modules. Hence, we have

$$H^s(\mathfrak{g}^-/\alpha^-, H^s(\alpha^-, \mathcal{C})) \approx H^s(\mathfrak{g}^-/\alpha^-, \mathcal{C} \otimes H^s(\alpha, \mathcal{C})).$$

We now regard $\mathcal{C}'$ as the vector space of all $C$-valued $C^\infty$-functions on $(G/A)_{\pi(\Gamma)}$. It is easy to see that this identification is compatible with $\mathfrak{g}^-/\alpha^-$-module structure. Thus we have

$$H^s(\mathfrak{g}^-/\alpha^-, C^\infty((G/A)_{\pi(\Gamma)}, C)) = H^s(\mathfrak{g}^-/\alpha^-, \mathcal{C}').$$

By the assumption of the induction, we get

$$H^s(\mathfrak{g}^-/\alpha^-, C^\infty((G/A)_{\pi(\Gamma)}, C)) = H^s(\mathfrak{g}^-/\alpha^-, C).$$

Hence, we have an isomorphism

$$E_s(j): H^s(\mathfrak{g}^-/\alpha^-, H^s(\alpha^-, C)) \to H^s(\mathfrak{g}^-/\alpha^-, H^s(\alpha^-, C')).$$

Thus $E_s(\iota): H^q(\mathfrak{g}^-/\alpha^-, H^s(\alpha^-, C)) \to H^q(\mathfrak{g}^-/\alpha^-, H^s(\alpha^-, C))$ is an isomorphism. By a theorem on spectral sequence ([13] Chapter 9, §1 Theorem 3), this implies an existence of an isomorphism

$$\iota^*: H^q(\mathfrak{g}^-, C) \approx H^q(\mathfrak{g}^-, C).$$

Combining this (2.1) and (2.2), we get

$$H^q_\mathfrak{g}(G/\Gamma) \approx H^q(\mathfrak{g}^-) \otimes \Lambda^q(\mathfrak{g}^*).$$

q.e.d.
Corollary 1 (Kodaira [9]). Let \( r \) be the dimension of the vector space of all closed holomorphic 1-forms on a compact complex parallelisable nilmanifold \( G/\Gamma \). Then \( \dim H^0_\mathbb{C}(G/\Gamma) = r \).

Proof. Let \( \omega \) be a closed holomorphic 1-form on \( G/\Gamma \). Then \( \omega = \sum_{j=1}^{n} f_j \phi_j \) where \( \{\phi_1, \ldots, \phi_n\} \) is a basis of \( (g^+)^* \) and \( f_j \) (\( j=1, \ldots, n \)) are holomorphic functions on \( G/\Gamma \). Since \( G/\Gamma \) is compact, \( f_j \) are constant. Hence, \( \omega \in (g^+)^* \). Moreover \( d\omega = 0 \) if and only if \( \omega([g^+, g^+]) = 0 \). Thus \( r = \dim (g^+/[g^+, g^+]) \). Since \( \dim H^r(g^-) = \dim (g^-/[g^-, g^-]) = \dim (g^+/[g^+, g^+]) \), we have \( r = \dim H^0_\mathbb{C}(G/\Gamma) \) by Theorem 1. \( \Box \)

Let \( M \) be a compact connected complex manifold. Let \( b_r \) (resp. \( h^{p,q} \)) denote \( \dim \mathbb{R} H^r(M, \mathbb{R}) \) (resp. \( \dim_c H^{p,q}(M) \)).

Corollary 2. If \( M \) is a compact complex parallelisable nilmanifold \( G/\Gamma \),
\[
b_{2k+1} = 2(2^{k}+h_{2k}^{0,1}+\cdots+h_{k+1}^{0,1})
\]
\[
b_{2k} = 2(2^{k}+h_{2k}^{0,1}+\cdots+h_{k+1}^{0,1}+h_{k}^{0,0})
\]
for \( 2k+1, 2k \leq n = \dim_c G \).

Proof. By a theorem of Nomizu [10] (See [11] Corollary 7.28.), \( H^r(G/\Gamma, \mathbb{R}) \approx H^r(g, \mathbb{R}) \). Thus \( H^r(G/\Gamma, \mathbb{C}) \approx H^r(g^+, \mathbb{C}) \approx H^r(g^-) \). Since \( g^c = g^+ \oplus g^- \) and \( [g^+, g^-] = 0 \), \( H^r(g^+) \approx \sum_{j \geq 0} H^j(g^+) \otimes H^{r-j}(g^-) \). Since \( \dim H^j(g^+) = \dim H^j(g^-) \)\( k^{0,q} \) and \( \dim H^r(g^+) = b_r \), \( b_r = \sum_{j \geq 0} H^j(g^+) \). \( \Box \)

Example. Let \( G \) be a nilpotent Lie group defined by
\[
G = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \mid z_1, z_2, z_3 \in \mathbb{C} \right\}
\]
Let \( \Gamma \) be a lattice in \( G \), for example,
\[
\Gamma = \left\{ \begin{pmatrix} 1 & a_1 & a_2 \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \end{pmatrix} \mid a_1, a_2, a_3 \in \mathbb{Z} + \sqrt{-1} \mathbb{Z} \right\}
\]
We can take a basis \( \{X_1, X_2, X_3\} \) of \( g^+ \) such that
\[
[X_1, X_2] = X_3, \quad [X_2, X_3] = [X_3, X_1] = 0.
\]
Then the dual basis \( \{\omega_1, \omega_2, \omega_3\} \) satisfies that
\[
d\omega_3 = -\omega_1 \wedge \omega_2, \quad d\omega_1 = d\omega_2 = 0.
\]
Now it follows easily from Theorem 1 that $h^{0,1}=h^{0,2}=2$. Note that $h^{1,0}=3$. By corollary 2, we get
\[ b_0 = b_4 = 1, \quad b_1 = b_2 = 4, \quad b_2 = b_4 = 8 \quad \text{and} \quad b_2 = 10. \]

3. Chern classes of holomorphic line bundles over a compact complex parallelisable solvmanifold

Let $G$ be a simply connected complex solvable Lie group and $\Gamma$ be a lattice of $G$. We assume the following condition:

(M) $Ad(G)$ and $Ad(\Gamma)$ have the same Zariski closure in $Aut(g^c)$.

Lemma 3.1. If $G$ is non-abelian, we have $\Gamma \cap [G, G] \neq \{e\}$.

Proof. Suppose that $\Gamma \cap [G, G] = \{e\}$. Since $[\Gamma, \Gamma] \subset \Gamma \cap [G, G]$, $\Gamma$ is abelian, so is $Ad(\Gamma)$. Since $Ad(\Gamma)$ and $Ad(G)$ have the same Zariski closure, $Ad([G, G]^*) = [Ad(G), Ad(G)^*] = [Ad(G)^*, Ad(G)^*] = [Ad(\Gamma)^*, Ad(\Gamma)^*] = X^*$, where $X^*$ denotes the Zariski closure of $X$ in $Aut(g^c)$. Hence, $[G, G]$ is contained in the center $Z$ of $G$. Thus $G$ is nilpotent. Since $\Gamma$ is abelian, $G$ is abelian [11]. This is a contradiction. q.e.d.

Proposition 3.2. $\Gamma_1 = \Gamma \cap [G, G]$ is a lattice of $[G, G]$.

Proof. At first note the following:

If $m$ is an ideal of $g$ and $\rho_1$ (resp. $\rho_2$) is the representation on $m^c$ (resp. $g^c/m^c$) induced by the adjoint representation $Ad: G \rightarrow Aut(g^c)$, $\rho_1(G)$ and $\rho_1(\Gamma)$ (resp. $\rho_2(G)$ and $\rho_2(\Gamma)$) have the same Zariski closure in $Aut(m^c)$ (resp. $Aut(g/m^c)$).

Now $[G, G]$ is a simply connected nilpotent closed Lie subgroup of $G$ and $\Gamma_1$ is a discrete subgroup of $[G, G]$. Let $H$ be the connected closed subgroup of $[G, G]$ such that $H/\Gamma_1$ is compact ([11] Proposition 2.5.). We claim that $H$ is a normal subgroup of $G$. Let $exp: [g, g] \rightarrow [G, G]$ be the exponential map. Then $exp^{-1}(\Gamma_1) = I$ is a lattice in the Lie algebra $\mathfrak{h}$ of $H$ and $I \otimes R = \mathfrak{h}$ ([11] Theorem 2.12). Since $\Gamma_1 = \Gamma \cap [G, G]$ is a normal subgroup of $\Gamma$, $exp Ad(\gamma)L = (\exp L)\gamma^{-1} \in \Gamma_1$ for any $L \in I = exp^{-1}(\Gamma_1)$ and $\gamma \in \Gamma$. Hence, $Ad(\gamma)I \subset I$ and $Ad(\gamma)\mathfrak{h} = \mathfrak{h}$ for any $\gamma \in \Gamma$. Since $Ad(G)$ and $Ad(\Gamma)$ have the same Zariski closure in $Aut(g^c)$, $Ad(G)\mathfrak{h} = \mathfrak{h}$. Hence, $\mathfrak{h}$ is an ideal in $g$. Thus $H$ is a normal subgroup of $G$.

Since $H \subset [G, G]$ and $\Gamma_1 \subset H$, $H \cap \Gamma_1 = H \cap [G, G]$ and $\Gamma_1 = H \cap \Gamma_1 = \Gamma_1$. Thus $H/\Gamma_1$ is compact and $H \cdot \Gamma_1$ is closed in $G$ ([11] Theorem 1.13). Hence, $H \cdot \Gamma/H$ is a lattice of $G/H$. We claim that $\Gamma \cap H \cap [G, H] = \{e\}$. Let $a \in \Gamma \cap H \cap [G, H]$. Since $[G/H, G/H] = [G, G]H/H = [G, G]/H$, $a = \gamma H = g \in H$ for some $\gamma \in \Gamma$ and $g \in [G, G]$, that is, $a = \gamma H = g \in H$ for some $h \in H$. Since $H \subset [G, G]$, $\gamma \in [G, G] \cap \Gamma_1 \subset H$. Hence, $a = \gamma H = H$. 
Since $\text{Ad}(G/H)$ and $\text{Ad}(\Gamma H/H)$ have the same Zariski closure in $\text{Aut}(g^\Gamma/[h^\Gamma])$, $G/H$ is abelian by Lemma 3.1. Hence $H \supseteq [G, G]$. Thus $H=[G, G]$ and is $\Gamma$, a lattice of $[G, G]$. q.e.d.


Now we denote by $A^{1,1}(G/\Gamma, R)$ the vector space of all real differential forms of type $(1, 1)$ on $G/\Gamma$. Let $H^{1,1}(G/\Gamma, R)$ be the vector space

$$\left\{ \omega \in A^{1,1}(G/\Gamma, R) | d\omega = 0 \right\} \quad \left\{ \omega \in A^{1,1}(G/\Gamma, R) | \omega = d\theta, \theta \text{ is a real 1-form} \right\}.$$

We shall characterize $H^{1,1}(G/\Gamma, R)$ in terms of the Lie algebra $\mathfrak{g}$ of $G$.

**Proposition 3.3.** Suppose that a lattice $\Gamma$ of $G$ satisfies the condition $(M)$. Then, for any real closed form $\alpha$ of type $(1, 1)$ on $G/\Gamma$, there is a unique real right invariant closed form $\beta \in \Lambda^2(\mathfrak{g}^\ast)$ of type $(1, 1)$ on $G$ such that $\alpha = \beta + d\eta$ on $G/\Gamma$ where $\eta$ is a real 1-form on $G/\Gamma$.

Proof. According to a theorem of Mostow ([8], [11]), for a given real closed 2-form $\alpha$, there is a real right invariant closed 2-form $\beta \in \Lambda^2(\mathfrak{g}^\ast)$ such that

$$\alpha = \beta + d\gamma$$

where $\gamma$ is a real 1-form on $G/\Gamma$. Let $\beta = \beta_{b^0} + \beta_{b^1} + \beta_{b^0^1}$ where $\beta_{b^0^1}$ is the component of $\beta$ of type $(p, q)$. Since $\beta$ is a real form, $\beta_{b^0^1} = \beta_{b^0^2}$ and $\beta_{b^1^1}$ is a real form. Let $\gamma = \gamma^{1,0} + \eta^{0,1}$, $\eta^{1,0} = \gamma^{0,1}$. Taking a basis $\{X_1, \cdots, X_n\}$ of $\mathfrak{g}^\ast$, let $\{\omega_1, \cdots, \omega_n\}$ be its dual basis of $(\mathfrak{g}^\ast)^\ast$. We identify $\omega_j (j=1, \cdots, n)$ as holomorphic 1-forms on $G/\Gamma$. We then have

$$\gamma^{b_{1,1}} = \sum_{j=1}^n f_j \omega_j$$

where $f_j \in C^\infty(G/\Gamma, C)$ for $j=1, \cdots, n$ and

$$\beta_{b^0^2} = \sum_{j<k} a_{jk} \omega_j \wedge \omega_k$$

where $a_{jk} \in C$. Since $\alpha$ is of type $(1,1)$, we get

$$\beta_{b^0^2} + d''\gamma^{b_{1,1}} = 0$$

and

$$\alpha = \beta_{b^1^1} + d''\eta^{1,0} + d''\gamma^{1,0}$$

by comparing the type of forms of both hands. We now have
\[ d''\gamma^{\alpha \beta} = d'' (\sum_{f=1}^{n} f_j \tilde{\omega}_j) = \sum_{f=1}^{n} (d'' f_j \wedge \tilde{\omega}_j + f_j d\tilde{\omega}_j) \]
\[ = \sum_{j=1}^{n} X_k f_j \tilde{\omega}_k \wedge \tilde{\omega}_j - \sum_{j=1}^{n} \sum_{k<l} f_j \tilde{C}_{kl} \tilde{\omega}_k \wedge \tilde{\omega}_l \]

where \( \tilde{C}_{kl} \) are the structure constant of Lie algebra \( g^+ \) with respect to the basis \( \{X_1, \ldots, X_n\} \). By (3.2), we get the equalities

\[ (3.3) \quad a_{kl} = X_k f_i - X_l f_k - \sum_{f=1}^{n} f_j \tilde{C}_{kl} \quad \text{for} \quad 1 \leq k < l \leq n. \]

Integrating (3.3) on \( G/\Gamma \), we have

\[ (3.4) \quad \int_{G/\Gamma} a_{kl} \, dg = \int_{G/\Gamma} (X_k f_i) \, dg - \int_{G/\Gamma} (X_l f_k) \, dg - \sum_{f=1}^{n} \int_{G/\Gamma} f_j \tilde{C}_{kl} \, dg \]

where \( dg \) is an invariant measure on \( G/\Gamma \). Since \( G \) is unimodular, \( \int_{G/\Gamma} (X_k f_i) \, dg = \int_{G/\Gamma} (X_l f_k) \, dg = 0 \), and we get

\[ (3.5) \quad a_{kl} \int_{G/\Gamma} \, dg = - \sum_{f=1}^{n} \tilde{C}_{kl} \int_{G/\Gamma} f_j \, dg. \]

Let \( b_j \in C \) denote \( \int_{G/\Gamma} f_j \, dg/\int_{G/\Gamma} \, dg \). Then (3.5) can be written as

\[ (3.6) \quad a_{kl} = - \sum_{f=1}^{n} b_j \tilde{C}_{kl}. \]

\[ \beta^{0,2} = \sum_{f=1}^{n} a_{kl} \omega_k \wedge \omega_l = - \sum_{f=1}^{n} \sum_{j<k} b_j \tilde{C}_{kl} \omega_k \wedge \omega_l \]

\[ = \sum_{j=1}^{n} b_j (\sum_{k<l} \tilde{C}_{kl} \omega_k \wedge \omega_l) = \sum_{j=1}^{n} b_j (d\tilde{\omega}_j) = d(\sum_{j=1}^{n} b_j \tilde{\omega}_j). \]

Put \( \eta = \sum_{j=1}^{n} b_j \tilde{\omega}_j \). We then see that \( \eta \) is of type \((0, 1)\), \( \beta^{0,2} = d\eta \) and \( \beta^{2,0} = d\eta \).

By (3.1), we get

\[ \alpha = \beta^{1,1} + d(\eta + \gamma) + d\gamma = \beta^{1,1} + d\theta \]

where \( \theta = \eta + \gamma \) is a real 1-form on \( G/\Gamma \).

It remains to show the uniqueness of \( \beta^{1,1} \). It is sufficient to see that if \( \beta^{1,1} = d\theta, \theta \) is a real 1-form, then \( \beta^{1,1} = 0 \). Put \( \beta^{1,1} = \sum_{j=1}^{n} a_{jk} \omega_j \wedge \omega_k \) and \( \theta = \theta^{0,1} + \theta^{1,0} \) where \( \theta^{0,1} = \sum_{j=1}^{n} g_j \tilde{\omega}_j, \quad g_j \in C^\infty(G/\Gamma, C) \) \((j = 1, \ldots, n)\). Since \( d''\theta^{0,1} = \sum_{j=1}^{n} X_k g_j \omega_k \wedge \omega_j \) and \( d''\tilde{\theta}^{0,1} = d\tilde{\theta}^{0,1} = \sum_{j=1}^{n} X_k \tilde{g}_j \omega_k \wedge \omega_j \), we get

\[ (3.7) \quad a_{jk} = X_j \tilde{g}_k - \tilde{X}_k g_j. \]

Integrating (3.7) on \( G/\Gamma \), we have
200 Y. SAKANE

Hence, \(a_{jk}=0\) for \(j, k=1, \ldots, n\) and \(\beta^{1}\equiv0\).

We now determine real closed right invariant forms of type \((1, 1)\) on \(G/\Gamma\). Take a basis \(\{X_1, \ldots, X_n\}\) of \(g^+\) such that \(\{X_{r+1}, \ldots, X_n\}\) is a basis of \([g^+, g^+]\). Let \(\{\omega_1, \ldots, \omega_n\}\) be its dual basis of \((g^+)^*\).

Proposition 3.4. Let \(\alpha\) be a right invariant real 2-form of type \((1, 1)\) on \(G\). Then \(d\alpha=0\) if and only if \(\alpha(e)=\frac{1}{2\sqrt{-1}} \sum_{j=1}^{r} h_{jk} \omega_j \wedge \bar{\omega}_k\) where \(H=(h_{jk}) \subseteq M(r, \mathbb{C})\) is a hermitian matrix, and 
\(r=\dim g^+/[g^+, g^+]\).

Proof. Since \(\alpha\) is a right invariant form on \(G\), \(\alpha\) defines a bilinear form on \(g^+ \times g^-\). Now \(d\alpha=0\) if and only if
\[\alpha([X, Y], Z) = 0 \quad \text{for} \quad X, Y, Z \in g^+,
\]
and since \((d\alpha)(X, Y, Z) = -\alpha([X, Y], Z) + \alpha([X, Z], Y) - \alpha([Y, Z], X)\) for \(X, Y, Z \in g^+\) and since \([g^+, g^-]\) = \(0\). In particular, for a real form \(\alpha\) of type \((1, 1)\), we get
\[d\alpha = 0 \quad \text{if and only if} \quad \iota([X, Y])\alpha = 0 \quad \text{for} \quad X, Y \in g^+.
\]
Note that \(d\omega_j=0\) for \(j=1, \ldots, r\). Therefore, if \(\alpha(e) = \frac{1}{2\sqrt{-1}} \sum_{j=1}^{r} h_{jk} \omega_j \wedge \bar{\omega}_k\) then \(d\alpha=0\). Conversely, put \(\alpha(e) = \frac{1}{2\sqrt{-1}} \sum_{j=1}^{r} h_{jk} \omega_j \wedge \bar{\omega}_k\). If \(\alpha\) is closed, then \(\iota(X_j)\alpha=0\) for \(j=r+1, \ldots, n\) by (3.8).

Since \((\iota(X_j)\alpha)(X_k) = \alpha(X_j, X_k) = 0\) and \(H=(h_{jk})\) is a hermitian matrix, we have \(h_{jk}=0\) for \(j=r+1, \ldots, n; k=1, \ldots, n\) and \(j=1, \ldots, n; k=r+1, \ldots, n\), so that \(\alpha(e) = \frac{1}{2\sqrt{-1}} \sum_{j=1}^{r} h_{jk} \omega_j \wedge \bar{\omega}_k\).

Consider a holomorphic line bundle \(L\) on \(G/\Gamma\). Let \(C(L)\) denote the Chern class of \(L\). Then we have \(C(L) \in H^*(G/\Gamma, \mathbb{R})\) ([15], Chapter V, n°4.).

Proposition 3.5. Let \(G\) be a simply connected complex solvable Lie group and \(\Gamma\) be a lattice of \(G\) satisfying the condition (M) and such that \(H^2;\mathbb{Z}(G/\Gamma) \cong H^1(g^-)\) (canonically). Let \(L\) be a holomorphic line bundle on \(G/\Gamma\). Then there is a unique real invariant form \(\alpha \in \Lambda^2 g^*\) of type \((1, 1)\) in \(C(L)\), and this is a curvature form of a connection \(\gamma\) of type \((1, 0)\).

Proof. It is easy to see that there is a real closed 2-form \(\beta\) of type \((1, 1)\) in
According to Proposition 3.3, we have $\beta = \alpha + d\gamma$ where $\gamma$ is a real 1-form on $G/\Gamma$. Decompose $\gamma = \gamma^{1,0} + \gamma^{0,1}$ where $\gamma^{1,0}$ (resp. $\gamma^{0,1}$) is the component of type $(1,0)$ (resp. $(0,1)$) of $\gamma$. Then we have $d''\gamma^{0,1} = 0$, since $\beta$ and $\alpha$ are of type $(1,1)$. By the assumption (2), there is a right invariant 1-form $\theta$ of type $(0,1)$ such that $\gamma^{0,1} - \theta = d''f$ where $f \in C^\infty(G/\Gamma, \mathbb{C})$.

We can write $\theta = \sum_i a_i \omega_i$, $a_i \in \mathbb{C}$ ($i = 1, \ldots, r$), where $\{\omega_1, \ldots, \omega_r\}$ is the same as before, since $H^1(g^-) = (g^-/[g^-, g^-])^*$. We then have $d\theta = \sum_i a_i d\omega_i = 0$, so that $\beta = \alpha + d'\gamma^{0,1} + d''\gamma^{1,0} = \alpha + d'(\theta + d''f) + d'(\theta + d''f) = \alpha + d''(f - \bar{f})$. Put $\psi = d'(f - \bar{f})$. We then have $\beta = \alpha + d\psi + d\bar{\psi}$. Since $\beta$ is a curvature from $\omega$ of a connection of type $(1,0)$ by definition and $\psi$ is of type $(1,0)$, $\alpha$ is a curvature form of a connection $\eta = \omega - \psi$ of type $(1,0)$.

q.e.d.

From now on we always assume that $G$ and $\Gamma$ satisfies the assumptions of Proposition 3.5.

Consider a holomorphic line bundle $L$ on $G/\Gamma$. We fix a (sufficiently fine) simple covering $\{U_i\}$ on $G/\Gamma$ and choose a connected component $U_i\gamma$ of $p^{-1}(U_i)$ for each $i$, $\gamma: G \to G/\Gamma$ being the canonical map; let $U_i\gamma$ denote the image of $U_i\gamma$ under the right translation $R_\gamma(g) = g\gamma$ for $\gamma \in \Gamma$. Then $p^{-1}(U_i) = \bigcup_{\gamma \in \Gamma} U_i\gamma$ is a disjoint union and $p$ maps each $U_i\gamma$ biholomorphically to $U_i$.

We may consider a holomorphic line bundle $L$ on $G/\Gamma$ is given by a system of transition functions $\{g_{ij}\}$ relative to the covering $\{U_i\}$ of $G/\Gamma$. Let $C(L)$ be the Chern class of $L$ and $\alpha$ be the unique real right invariant form of type $(1,1)$ in $C(L)$. By Proposition 3.5, $\alpha$ is a curvature form of a connection $\eta$ of type $(1,0)$, so that there is an element $\eta_j \in A^{1,0}(U_j)$ for each $j$ satisfying $\eta_k - \eta_j = \frac{\sqrt{-1}}{2\pi} d \log g_{jk}$ on $U_j \cap U_k + \phi$ and $\alpha = d\eta_j$ on $U_j$.

**Proposition 3.6.** Identify $g^+$ to the complex Lie algebra $(\mathfrak{g}, I)$. Then we can take a basis $\{X_1, \ldots, X_n\}$ of $g^+$ such that a map $\psi: g^+ \to G$ defined by

$$\psi\left(\sum_{i=1}^n z_i X_i\right) = (\exp z_1 X_1) \cdots (\exp z_n X_n)$$

is biholomorphic. In particular, $G$ is biholomorphic to $C^n$. Moreover $G$ has a system of coordinates $(z_1, \ldots, z_n)$ such that, for $j = 1, \ldots, r$, $z_j(gg') = z_j(g) + z_j(g')$ for any $g, g' \in G$, where $r = \dim g^+/[g^+, g^+]$.

Proof. We prove this proposition by induction on the dimension $n$ of $g^+$. Assume that it has been proved for all dimensions $< n$. Since $g^+$ is solva-
202 Y. SAKANE

ble, it has an abelian ideal $\alpha^+$ of dimension $> 0$. Let $A$ be the connected complex abelian subgroup of $G$ whose Lie algebra is $\alpha^+$; $A$ is simply connected and $G/A$ is a simply connected complex solvable Lie group of complex dimension $< n$. Applying our proposition to $G/A$, we get a basis \{\(X_1^+, \ldots, X_m^+\) of $^+g/\alpha^+$ such that a map $\psi^*: g^+/\alpha^+ \rightarrow G/A$ defined by

$$\psi^*(\sum_{i=1}^{m} z_i X_i^+) = (\exp z_1 X_1^+) \cdots (\exp z_m X_m^+)$$

is biholomorphic. Take elements $X_1, \ldots, X_m \in g^+$ such that $\pi^*(X_i) = X_i^+$ where $\pi^*: g^+ \rightarrow g^+/\alpha^+$ is a projection. Choose also a basis \{\(X_{m+1}, \ldots, X_n\) of $\alpha^+$. Then every element of $A$ can be written uniquely in the form $\exp z_1 X_1^+ \cdots (\exp z_n X_n)$. Let $g$ be any element of $G$ and $g^* = \pi(g)$ where $\pi: G \rightarrow G/A$ is a projection. Then we can write uniquely $g^*$ in the form $\exp z_1 X_1^+ \cdots (\exp z_m X_m^+)$. Hence, we have $g = (\exp z_1 X_1^+ \cdots (\exp z_m X_m^+)) a (a \in A)$ and $a$ can be written in the form $\exp z_{m+1} X_{m+1} \cdots (\exp z_n X_n)$, which proves that $g$ is in the form $\exp z_1 X_1^+ \cdots (\exp z_m X_m^+)$. Moreover $z_1, \ldots, z_m$ are uniquely determined by $\pi(g)$ (and a fortiori by $g$); hence $a$ is determined by $g$ and $z_{m+1}, \ldots, z_n$ are uniquely determined by $g$. Since exp is holomorphic, $z_j (j = 1, \ldots, n)$ are holomorphic functions on $G$ and $\psi^*: g^+ \rightarrow G$ is biholomorphic.

Since we can choose a basis \{\(X_1, \ldots, X_n\) of $q^+$ in such a way that \{\(X_{r+1}, \ldots, X_n\) is a basis of $[g^+, g^+]$ and $\psi^*: g^+ \rightarrow G$ is biholomorphic, the last assertion follows from the Campbell-Hausdorff formula ([4] p. 170). q.e.d.

We may assume that $\omega_j = dz_j$ for $j = 1, \ldots, r$ by changing a basis of $q^+$ if necessary. Then by Proposition 3.4, we get

$$\alpha = \frac{1}{2\sqrt{-1}} \sum_{j,k=1}^{r} h_{jk} \omega_j \wedge \omega_k = \frac{1}{2\sqrt{-1}} \sum_{j,k=1}^{r} h_{jk} dz_j \wedge d\bar{z}_k$$

where $(h_{jk})$ is a hermitian matrix.

4. Divisors on a compact complex parallelisable solvmanifold

Let $M$ and $N$ be complex manifolds and $\Phi: M \rightarrow N$ be a surjective holomorphic map. For a divisor $D$ on $N$, $\Phi^*(D)$ denotes the divisor on $M$ defined by $\Phi^{-1}(D)$ for all $x \in M$ ([15] Appendix n°7). We call this divisor $\Phi^*(D)$ on $M$ the pull back of the divisor $D$ on $N$. In this section we prove the following theorem.

**Theorem 2.** Let $G$ be a simply connected complex solvable Lie group. Let $\Gamma$ be a lattice of $G$. Assume that $\Gamma$ satisfies the condition (M) and that $H^2_{\text{dR}}(G/\Gamma) \cong H^1(\Gamma^-)$ (canonically). Then, for each positive divisor $D$ on $G/\Gamma$, there exists a positive divisor $\tilde{D}$ on the complex torus $T$ such that the divisor $\tilde{D}$ is the pull back of the divisor $D$ on $T$ by the projection $\pi: G/\Gamma \rightarrow T$, i.e., $D = \pi^* \tilde{D}$. 

If $G$ is nilpotent, the condition $(M)$ is always satisfied ([11] Theorem 2.1). Moreover, by Theorem 1 in the section 2, $H^{0,1}_{\mathcal{A}}(G/\Gamma)\cong H^1(G^-)$. Thus we get:

**Corollary.** Let $G$ be a simply connected complex nilpotent Lie group and $\Gamma$ be a lattice of $G$. Then the conclusion of Theorem 2 holds.

Let $D$ denote a positive divisor on $G/\Gamma$. Take a representative $\{(U_i, f_i)\}$ of $D$, where $f_i: U_i \to \mathbb{C}$ is a holomorphic function. Let $L = \{D\}$ denote the holomorphic line bundle corresponding to the divisor $D$. ([15] Chapter V, n° 6). Let $\{g_{jk}\}$ denote the system of transition functions of $L = \{D\}$ with respect to $\{(U_i, f_i)\}$. We then have $f_j = g_{jk}f_k$ on $U_j \cap U_k + \phi$ by definition.

Let $M$ be a complex manifold, $\tilde{M}$ be the universal covering of $M$ and $p: \tilde{M} \to M$ be the covering map. Let $\Pi$ denote the fundamental group $\pi_1(M)$ of $M$.

A map $j: \Pi \times \tilde{M} \to \mathbb{C}^*$ is said to be an automatic factor if

1. the function $z \mapsto j(\sigma, z)$ is holomorphic for any $\sigma \in \Pi$, and
2. $j(\sigma \tau, z) = j(\sigma, \tau(z)) \cdot j(\tau, z)$ for any $\sigma, \tau \in \Pi$ and any $z \in \tilde{M}$.

Let $f$ be a holomorphic function on $\tilde{M}$ which is not identically zero. $f$ is said to be automatic of type $j$ if

$$f(\sigma(z)) = j(\sigma, z)f(z) \quad \text{for } z \in \tilde{M} \text{ and } \sigma \in \Pi.$$

**Proposition 4.1.** Let $D$ be a positive divisor of $G/\Gamma$. Then $D$ is the divisor of a holomorphic automatic function $\theta$ on $G$, for which the automatic factor $j(\gamma, g): \Gamma \times G \to \mathbb{C}^*$ is given by

$$j(\gamma, g) = \exp 2\pi \sqrt{-1} \left( \frac{1}{2\sqrt{-1}} \sum_{k,l=1}^{r_1} h_{kl} \bar{z}_k(g) \bar{z}_l(\gamma) + C(\gamma) \right),$$

where $H = (h_{jk})$ is a hermitian matrix determined by the form $\alpha$ in the Chern class $C(L) = C(\{D\})$:

$$\alpha = \frac{1}{2\sqrt{-1}} \sum_{k,l=1}^{r_1} h_{kl} dz_k \wedge d\bar{z}_l,$$

and $C(\gamma) \in \mathbb{C}$ is a constant depending only on $\gamma \in \Gamma$.

**Proof.** Let us define $\varphi_{\gamma}(g)$ for $g \in U_{\gamma}$ by

$$\varphi_{\gamma}(g) = \eta_\gamma(p(g)) + \frac{1}{2\sqrt{-1}} \sum_{k,l=1}^{r_1} h_{kl} \bar{z}_k(g) \bar{z}_l(\gamma^{-1})$$

where $\eta_\gamma$ is the component of the connection introduced before. Then $\varphi_{\gamma}$ is an element of $A^{1,0}(U_{\gamma})$ satisfying $d\varphi_{\gamma} = 0$. Since $U_{\gamma}$ is simply connected, there is a holomorphic function $\psi_{\gamma}$ satisfying $d\psi_{\gamma} = \varphi_{\gamma}$. Define $\theta_{\gamma}(g)$ for $g \in U_{\gamma}$ by

$$\theta_{\gamma}(g) = f_i(p(g)) \exp 2\pi \sqrt{-1}(\psi_{\gamma}(g)).$$
We then have
\[ \theta_{r}(g) = \theta_{j}(g) \exp 2\pi \sqrt{-1} \left( \frac{1}{2\sqrt{-1}} \sum_{k} h_{kl} \bar{z}_{k}(g) \bar{z}_{l}(\gamma^{-1}) + C_{r,l,j} \right) \]
on $U_{i} \cap U_{j}$, where $C_{r,l,j} \in C$ is a constant. Applying Proposition 3.6, we get
\[ \frac{\sqrt{-1}}{2} d \log g_{ij}(p(g)) + \varphi_{r}(g) - \varphi_{j}(g) = \frac{1}{2\sqrt{-1}} \sum_{k} h_{kl}(\bar{z}_{l}(\delta) - \bar{z}_{l}(\gamma)) dz_{k}. \]
Put $a_{r,l,j} = \exp 2\pi \sqrt{-1} C_{r,l,j}$. \{a_{r,l,j}\} satisfies relations
\[ (4.1) \quad a_{r,l,j} \cdot a_{r,l,k} = a_{r,l,k} \quad \text{on} \quad U_{r} \cap U_{j} \cap U_{k} \neq \phi, \]
since
\[ a_{r,l,j} = \exp 2\pi \sqrt{-1} C_{r,l,j} \]
\[ g_{i,j}(p(g)) \exp 2\pi \sqrt{-1} \left( \left( \varphi_{r} - \varphi_{j} \right) + \frac{1}{2\sqrt{-1}} \sum_{k} h_{kl}(\bar{z}_{l}(\gamma) - \bar{z}_{l}(\delta)) \right). \]

By the principal of monodromy ([15], Chapter V, n°1), there is a system of constant functions \{b_{r}\} such that
\[ a_{r,l,j} = b_{r}^{-1} \cdot b_{j}, \]
since $G$ is simply connected and \{\{U_{i}\}\} is an open covering of $G$. We define a holomorphic function $\theta$ on $G$ by
\[ \theta(g) = \theta_{r}(g) \exp 2\pi \sqrt{-1} \left( \frac{1}{2\sqrt{-1}} \sum_{k} h_{kl} \bar{z}_{k}(g) \bar{z}_{l}(\gamma) + C_{r} \right) \]
on $g \in U_{r}$. We can see easily that $\theta$ is well defined and $\theta$ is different from zero.

Note that
\[ \theta_{r}(g\gamma) = \theta_{r}(g) \exp 2\pi \sqrt{-1} d_{r} \]
for $g \in U_{r}$, where $d_{r}$ is a constant. In fact, we have
\[ d(R_{r}^{*} \varphi_{r}) - d\varphi_{0} = R_{r}^{*} \varphi_{r} - \varphi_{0} = 0 \quad \text{on} \quad U_{r}, \]
and
\[ \varphi_{r}(g\gamma) - \varphi_{r}(g) = d_{r} \quad \text{on} \quad U_{r}. \]

We now show that
\[ \theta(g\gamma) = \theta(g) \cdot \exp 2\pi \sqrt{-1} \left( \frac{1}{2\sqrt{-1}} \sum_{k} h_{j,k} \bar{z}_{j}(g) \bar{z}_{k}(\gamma) + C(\gamma) \right) \]
for $g \in G$ and $\gamma \in \Gamma$, where $C(\gamma)$ is a constant. For $g \in U_{i}$, we have
\[ \theta(g\gamma) = \theta_i(g\gamma) \cdot \exp 2\pi\sqrt{-1} \left( \frac{1}{2\sqrt{-1}} \sum_{i,t=1}^{k} h_{kt} z_k(g\gamma) \bar{z}_i(\gamma) + b_{ir} \right) \]

\[ = \theta_i(g) \cdot \exp 2\pi\sqrt{-1} \left\{ \frac{1}{2\sqrt{-1}} \sum_{i,t=1}^{k} h_{kt} z_k(g\gamma) \bar{z}_i(\gamma) + b_{ir} \right\} . \]

Since \( \theta(g) = \theta_i(g) \) exp \( 2\pi\sqrt{-1} b_{ir} \) on \( U_{io} \), and since \( z_k(g\gamma) = z_k(g) + z_k(\gamma) \) by Proposition 3.6,

\[ \theta(g\gamma) = \theta(g) \exp 2\pi\sqrt{-1} \left\{ \frac{1}{2\sqrt{-1}} \sum_{i,t=1}^{k} h_{kt} z_k(g) \bar{z}_i(\gamma) + C_i(\gamma) \right\} \]

for \( g \in U_{io} \), where \( C_i(\gamma) \) is a constant. Since \( \theta(g\gamma) \) and

\[ \theta(g) \exp 2\pi\sqrt{-1} \left\{ \frac{1}{2\sqrt{-1}} \sum_{i,t=1}^{k} h_{kt} z_k(g) \bar{z}_i(\gamma) + C_i(\gamma) \right\} \]

are holomorphic functions on \( G \), we have

\[ \theta(g\gamma) = \theta(g) \exp 2\pi\sqrt{-1} \left( \frac{1}{2\sqrt{-1}} \sum_{i,t=1}^{k} h_{kt} z_k(g) \bar{z}_i(\gamma) + C(\gamma) \right) \]

for \( g \in G \) and \( \gamma \in \Gamma \). By the definition of \( \theta \), we have \( p^*D = \text{div}(\theta) \). q.e.d.

From now on, let \( e \) denote \( \exp 2\pi\sqrt{-1} \) and \( H(g, \gamma) = \sum \limits_{k,i=1} h_{kt} z_k(g) \bar{z}_i(\gamma) \).

Then \( j(\gamma, g) = \theta \left( \frac{1}{2\sqrt{-1}} H(g, \gamma) + C(\gamma) \right) \) for \( g \in G \) and \( \gamma \in \Gamma \).

Since \( j(\gamma_1, \gamma_2, g) = j(\gamma_1, g) j(\gamma_2, g) \), we get

\[ C(\gamma_1, \gamma_2) \equiv C(\gamma_1) + C(\gamma_2) + \frac{1}{2\sqrt{-1}} H(\gamma_1, \gamma_2) \mod 1. \]

In particular, \( C(e) \in \mathbb{Z} \) and

\[ C(\gamma^{-1}) \equiv -C(\gamma) + \frac{1}{2\sqrt{-1}} H(\gamma, \gamma) \quad \text{for} \quad \gamma \in \Gamma. \]

**Lemma 4.2.** \( C(\gamma) \in \mathbb{R} \) for \( \gamma \in \Gamma, \Gamma \).

**Proof.** Since \( [\Gamma, \Gamma] \subset [G, G], H(g, \gamma) = 0 \) for \( \gamma \in [\Gamma, \Gamma] \) and \( g \in G \). It is enough to show that \( C(\gamma) \in \mathbb{R} \) for \( \gamma = \gamma_1, \gamma_1^{-1} \gamma_2, \gamma_1, \gamma_2 \in \Gamma \). In this case,

\[ C(\gamma) \equiv C(\gamma_1, \gamma_2) + C(\gamma_1^{-1} \gamma_2^{-1}) + \frac{1}{2\sqrt{-1}} H(\gamma_1, \gamma_2, \gamma_1^{-1} \gamma_2^{-1}) \]

\[ \equiv C(\gamma_1) + C(\gamma_2) + \frac{1}{2\sqrt{-1}} H(\gamma_1, \gamma_2) + C(\gamma_1^{-1}) + C(\gamma_2^{-1}) + \frac{1}{2\sqrt{-1}} H(\gamma_1, \gamma_2) \]

\[ + \frac{1}{2\sqrt{-1}} \{ -H(\gamma_1, \gamma_1) - H(\gamma_2, \gamma_2) - H(\gamma_2, \gamma_1) - H(\gamma_1, \gamma_2) \} \]
Proposition 4.3. \([\Gamma, \Gamma]\) is a lattice of \([G, G]\) and \([\Gamma, \Gamma]\) is a subgroup of finite index of \(\Gamma \cap [G, G]\).

Proof. It follows from Proposition 3.2 that \(\Gamma'\Gamma G, G]/[G, G]\) is a lattice of \(G/[G, G]\). Since \(G/[G, G]\) is a vector group of dimension \(2r = \dim_R \mathbb{G}/[\mathbb{g}, \mathbb{g}]\), \(\Gamma'\Gamma G, G]/[G, G]=\Gamma/\Gamma \cap [G, G]\) is a free abelian group of rank \(2r\). On the other hand, since \(G\) is simply connected, \(\pi_1(G/\Gamma)=T\) and is \(\Gamma\) finitely generated. It follow that \(H^G/T C, \mathbb{Z})^{\Gamma}/[\Gamma, \Gamma]\). Since \(\dim H'(G/\Gamma, \mathbb{R})=\dim H'(\mathbb{g}, \mathbb{R})=\dim_R \mathbb{g}/[\mathbb{g}, \mathbb{g}]=2r\) by a theorem of Mostow (cf. [8], [11] Corollary 7.29.), \(\Gamma'[\Gamma, \Gamma]\) is then direct sum of a free abelian group of rank \(2r\) and a finite group. The group \((\Gamma \cap [G, G])/[\Gamma, \Gamma]\) is finite, because \(\Gamma/\Gamma \cap [G, G]\cong(\Gamma/[\Gamma, \Gamma])/[\Gamma, \Gamma]\) is a free abelian group of rank \(2r\). Since \([G, G]/\Gamma \cap [G, G]\) is compact by Proposition 3.2, \([G, G]/[\Gamma, \Gamma]\) is compact q.e.d.

Proposition 4.4. \(C(\gamma)\in \mathbb{Z}\) for \(\gamma \in [\Gamma, \Gamma]\).

Proof. Let \(\theta\) be a holomorphic automorphic function on \(G\) of type \(j(\gamma, g)\). We then have

\[
\theta(g\gamma_1) = \theta(g)e(C(\gamma_1)) \quad \text{for } g \in G \text{ and } \gamma_1 \in [\Gamma, \Gamma].
\]

Since \(\theta\) is not identically zero, there is a point \(g_0 \in G\) such that \(\theta(g_0)\neq 0\).

Define a holomorphic function \(F: [G, G] \to \mathbb{C}\) by \(F(g_0) = \theta(g_0g_1)\). Then \(F\) is different from zero and satisfies \(F(g, \gamma_1) = \theta(g_0g, \gamma_1) = \theta(g_0g_1)e(C(\gamma_1)) = F(g_1)e(C(\gamma_1))\) for \(g_1 \in [G, G]\) and \(\gamma_1 \in [\Gamma, \Gamma]\) and \(F(e)\neq 0\).

Let \(f: [G, G] \to \mathbb{R}\) denote \(C^\infty\)-function \(|F(g)|\). Then \(f(g, \gamma_1) = f(g_1)\) for \(\gamma_1 \in [\Gamma, \Gamma]\) since \(C(\gamma_1) \in \mathbb{R}\) by Lemma 4.2.

We also denote by \(f\) the function on \([G, G]/[\Gamma, \Gamma]\) induced by \(f: [G, G] \to \mathbb{R}\).

Since \([\Gamma, \Gamma]\) is a lattice of \([G, G]\), \([G, G]/[\Gamma, \Gamma]\) is a compact complex manifold. Hence, \(f: [G, G]/[\Gamma, \Gamma] \to \mathbb{R}\) is bounded:

\[
|F(g_1)| = f(g_1) = f(p(g_1)) \leq \epsilon
\]

for some constant \(\epsilon > 0\).

Since \([G, G]\) is biholomorphic onto \(\mathbb{C}^m\), a holomorphic bounded function \(F: [G, G] \to \mathbb{C}\) is constant. Since \(F(\gamma_1) = F(e)e(C(\gamma_1))\), \(C(\gamma_1) \in \mathbb{Z}\). q.e.d.

Let \(A(g_1, g_2) = \frac{1}{2\sqrt{-1}}(H(g_1, g_2) - \overline{H(g_1, g_2)})\). We then get

\[
A(\gamma_1, \gamma_2) = \frac{1}{2\sqrt{-1}}(H(\gamma_1, \gamma_2) - \overline{H(\gamma_1, \gamma_2)})
\]

\[
\equiv C(\gamma_1 \gamma_2 \gamma_1^{-1} \gamma_2^{-1}) \equiv 0 \quad (\text{mod } 1).
\]
We put \( d(\gamma) = C(\gamma) - \frac{1}{4\sqrt{-1}} H(\gamma, \gamma) \) for \( \gamma \in \Gamma \). We have then

\[
d(\gamma \delta) \equiv \frac{1}{2} A(\gamma, \delta) + d(\gamma) + d(\delta) \pmod{1} \quad \text{for } \gamma, \delta \in \Gamma.
\]

Let \( \rho(\gamma) \) be the imaginary part of \( d(\gamma) \). We see that \( \rho(\gamma \delta) = \rho(\gamma) + \rho(\delta) \) for \( \gamma, \delta \in \Gamma \), that is, \( \rho: \Gamma \to \mathbb{R} \) is a homomorphism. It is clear that \( \text{Ker} \rho \supseteq [\Gamma, \Gamma] \). Moreover we have \( \text{Ker} \rho \supseteq [\Gamma, [G, G]] \), since \([\Gamma, \Gamma]\) is a subgroup of finite index of \( \Gamma \cap [G, G] \). Hence \( \rho \) induces a homomorphism \( \rho: \Gamma/[\Gamma, [G, G]] \to \mathbb{R} \).

Since \( \pi(\Gamma) = \Gamma \cdot [G, G]/[G, G] \approx \Gamma \cap [G, G] \) and \( \pi(\Gamma) \) is a lattice of \([G, G] \), \( \rho \) can be extended to a homomorphism from \([G, G] \) to \( \mathbb{R} \), so that \( \rho: \Gamma \to \mathbb{R} \) can be extended to a homomorphism \( \rho: G \to \mathbb{R} \).

Consider now the biholomorphic map \( \Phi: G \to \mathbb{C}^r \) given by \( \Phi(\gamma) = (x_1(\gamma), \ldots, x_r(\gamma)) \). Let \( z_j(\gamma) = x_j(\gamma) + \sqrt{-1} y_j(\gamma) \) for \( j = 1, \ldots, r \). Note that \( \Phi: G \to \mathbb{C}^r \) induces a homomorphism \( \pi(\Gamma) \to \mathbb{C} \) given by \( \pi(\gamma) \mapsto (x_1(\gamma), \ldots, x_r(\gamma)) \). We can write \( \rho: G \to \mathbb{R} \) as

\[
\rho(g) = \sum_{j=1}^{r} a_j x_j(g) + \sum_{j=1}^{r} b_j y_j(g)
\]

for \( g \in G \), where \( a_j, b_j \in \mathbb{R} \), \( j = 1, \ldots, r \).

Define \( l: G \to \mathbb{C} \) by

\[
l(g) = \sqrt{-1} \sum_{j=1}^{r} a_j z_j(g) + \sum_{j=1}^{r} b_j z_j(g).
\]

We have \( \text{Im} l(g) = \rho(g) \) and \( d(\gamma) - l(\gamma) \in \mathbb{R} \) for \( \gamma \in \Gamma \).

Note that \( l: G \to \mathbb{C} \) is a holomorphic homomorphism.

Since we can regard \( A(g_1, g_2) \) as an alternating form on a vector group \( G/[G, G] \) such that \( A(g, g) \) takes integers on the lattice \( \pi(\Gamma) \), there is a \( \mathbb{R} \)-bilinear form \( B \) which is \( \mathbb{Z} \)-valued on the lattice \( \pi(\Gamma) \) and \( A(g_1, g_2) = B(g_1, g_2) - B(g_2, g_1) \) ([15] Chapter VI, n°2).

Define \( \chi: \Gamma \to \{ \xi \in \mathbb{C} \mid |\xi| = 1 \} \) by

\[
\chi(\gamma) = e \left( d(\gamma) - l(\gamma) - \frac{1}{2} B(\gamma, \gamma) \right).
\]

\( \chi \) is a character of \( \Gamma \), since \( A(\gamma_1, \gamma_2) \in \mathbb{Z} \) for \( \gamma_1, \gamma_2 \in \Gamma \). Put

\[
\psi(\gamma) = \chi(\gamma) e \left( \frac{1}{2} B(\gamma, \gamma) \right)
\]

for \( \gamma \in \Gamma \).

We get

\[
j(\gamma, g) = e \left( \frac{1}{2\sqrt{-1}} H(g, \gamma) + \frac{1}{4\sqrt{-1}} H(\gamma, \gamma) + l(\gamma) \right) \psi(\gamma)
\]

for \( \gamma \in \Gamma \) and \( g \in G \).
Since \( l(g) : G \rightarrow C \) is a holomorphic map which satisfies \( l(g\gamma) = l(g) + l(\gamma) \) for \( g \in G \) and \( \gamma \in \Gamma \), \( j(\gamma, g) \) is equivalent to the automorphic factor

\[
e\left(\frac{1}{2\sqrt{-1}} H(g, \gamma) + \frac{1}{4\sqrt{-1}} H(\gamma, \gamma)\right) \psi(\gamma).
\]

We need the following proposition to show that \( \psi | \Gamma \cap [G, G] = \text{id.} \)

**Proposition 4.5.** Let \( \theta \) be a holomorphic automorphic function on \( G \) of type

\[
j(\gamma, g) = e\left(\frac{1}{2\sqrt{-1}} H(g, \gamma) + \frac{1}{4\sqrt{-1}} H(\gamma, \gamma)\right) \psi(\gamma).
\]

Then the hermitian form \( H = (h_{jk}) \) is non-negative. Moreover \( \theta(g \cdot g_0) = \theta(g) \) for \( g \in G \), if \( g_0 \in G \) satisfies \( H(g_0, g_0) = 0 \).

**Proof.** Let \( f : G \rightarrow \mathbb{R} \) denote the function defined by

\[
f(g) = |\theta(g)|^2 e\left(\frac{-1}{2\sqrt{-1}} H(g, g)\right) = |\theta(g)|^2 \exp(-\pi H(g, g)).
\]

We have \( f(g\gamma) = f(g) \) for \( \gamma \in \Gamma \), so that \( f \) induces a function \( F : G/\Gamma \rightarrow \mathbb{R} \). Since \( G/\Gamma \) is compact, there is a constant \( c > 0 \) such that \( 0 \leq F(p(g)) \leq c \) for \( g \in G \). Therefore we get

\[
f(g) = |\theta(g)|^2 \exp(-\pi H(g, g)) \leq c \quad \text{for } g \in G.
\]

Thus we have

\[
|\theta(g)|^2 \leq c \exp \pi H(g, g) \quad \text{for } g \in G.
\]

Suppose that \( H(g_1, g_1) < 0 \) for some \( g_1 \in G \). Define \( g(\tau) \in G \) (\( \tau \in \mathbb{C} \)) by

\[
g(\tau) = \Phi^{-1}(\tau z_1(g_1) + z_1(g), \ldots, \tau z_n(g_1) + z_n(g)).
\]

Then we have \( g(0) = g \) and

\[
|\theta(g(\tau))|^2 \leq c \exp \pi H(g(\tau), g(\tau)).
\]

Put \( \rho = H(g(\tau), g(\tau)) \).

\[
\rho = \sum_{j,k} h_{jk}(\tau z_j(g_1) + z_j(g)) \cdot (\tau z_k(g_1) + z_k(g))
\]

\[
= |\tau|^2 \sum_{j,k} h_{jk} \sigma_j(g_1) \sigma_k(g_1) + 2 \text{Re}(\tau H(g, g)) + H(g, g)
\]

\[
= |\tau|^2 \pi H(g_1, g_1) + 2 \text{Re}(\tau H(g_1, g)) + H(g, g).
\]

For any \( \varepsilon > 0 \), there is \( R > 0 \) such that \( \pi \rho \leq \log \varepsilon \) for every \( \tau \) satisfying \( |\tau| \geq R \). Fix \( g_1, g \in G \), and we have
Therefore $\theta(g(\tau))$ is a bounded holomorphic function on $\mathbb{C}$. Hence $\theta(g(\tau))$ is constant with respect to $\tau \in \mathbb{C}$. Tending $\varepsilon \to 0$, we get $|\theta(g(\tau))|^2 = 0$. In particular,

$$|\theta(g)|^2 = |\theta(g(0))|^2 = 0.$$ 

Hence $\theta \equiv 0$ on $G$, since $g$ can be any element of $G$. This is a contradiction. Therefore $H=(h_{jk})$ is a non-negative hermitian form.

Take an element $g_0 \in G$ satisfying $H(g_0, g_0)=0$. Then we have $H(g, g_0)=0$ for any $g \in G$ since $H(g, g) \geq 0$ for any $g \in G$. Put

$$g_0(\tau) = \Phi^{-1}(\tau z_1(g_0), \ldots, \tau z_n(g_0)) \in G$$

for $\tau \in \mathbb{C}$. Then we have

$$|\theta(g \cdot g_0(\tau))|^2 \leq c \cdot \exp \tau H(g, g_0(\tau), g \cdot g_0(\tau))$$

$$= c \cdot \exp \pi H(g, g) + 2 \text{Re} \tau H(g, g_0) + |\tau|^2 H(g_0, g_0)$$

$$= c \cdot \exp \pi H(g, g).$$

This shows that $\theta(g \cdot g_0(\tau))$ is a bounded holomorphic function with respect to $\tau \in \mathbb{C}$. Hence $\theta(g \cdot g_0(\tau))$ is constant with respect to $\tau \in \mathbb{C}$. In particular, $\theta(g)=\theta(g \cdot g_0(0))=\theta(g \cdot g_0(1))=\theta(g \cdot g_0)$. q.e.d.

Take an element $g_1 \in G$ satisfying $\theta(g_1) \neq 0$. Since $H(g_0, g_0)=0$ for $g_0 \in [G, G]$, $\theta(g_0)=\theta(g)$ for $g \in G$. In particular, $\theta(g \cdot \gamma) = \theta(g)$ for $\gamma \in \Gamma \cap [G, G]$. Put $g_1=g_0^{-1}$. Then $0 \neq \theta(g_1)=\theta(g \cdot \gamma)-\theta(g)$. Since

$$\theta(g \cdot \gamma) = \theta(g) \cdot \exp \left( \frac{1}{2\sqrt{-1}} H(g, \gamma) + \frac{1}{4\sqrt{-1}} H(\gamma, \gamma) \right) \psi(\gamma) = \theta(g) \psi(\gamma),$$

$\psi(\gamma)=1$, for $\gamma \in \Gamma \cap [G, G]$. Note that $B(g, g)=0$ for $g \in [G, G]$. Hence, $\chi: \Gamma \to \{z \in \mathbb{C} \mid |z|=1\}$ satisfies that

$$\chi|\Gamma \cap [G, G] \equiv 1.$$ 

Since $\pi(\Gamma) \approx \Gamma / \Gamma \cap [G, G]$, $\chi$ induces a character

$$\hat{\chi}: \pi(\Gamma) \to \{z \in \mathbb{C} \mid |z|=1\}.$$ 

Let $\Theta: G/[G, G] \to \mathbb{C}$ denote the holomorphic function on $G/[G, G]$ induced by $\theta: G \to \mathbb{C}$ and $\hat{f}: \pi(\Gamma) \times G/[G, G] \to \mathbb{C}^*$ the automorphic factor induced by $j: \Gamma \times G \to \mathbb{C}^*$. Denote $\tilde{D}$ the divisor on $(G/[G, G]) / \pi(\Gamma)$ denoted by the holomorphic automorphic function $\Theta$ on $G/[G, G]$. We then get $D = \pi^* \tilde{D}$. Therefore we have proved Theorem 2.
Let $D$ be a divisor on $G/\Gamma$. Then there exist positive divisors $D^+, D^-$ on $G/\Gamma$ such that $D^+ \text{ and } D^-$ are relatively prime and $D = D^+ - D^- \ ([15], \text{Appendix n}°6)$. By Theorem 2, there are holomorphic theta functions $\Theta_1, \Theta_2$ on the complex torus $T$ such that $D^+ = \pi^*(\text{div } \Theta_1)$ and $D^- = \pi^*(\text{div } \Theta_2)$. Since $\pi: G/\Gamma \rightarrow T$ is onto holomorphic,

$$D = D^+ - D^- = \pi^*(\text{div } \Theta_1) - \pi^*(\text{div } \Theta_2)$$

$$= \pi^* \text{div} \left( \frac{\Theta_1}{\Theta_2} \right).$$

Note that $\Theta_1$ is a meromorphic theta function on the complex torus $T$.

It is easy to see that if the divisor $D=0$ the corresponding automorphic function $\theta$ is trivial.

Take a meromorphic function $\psi$ on $G/\Gamma$. Let $D = \text{div} (\psi)$. Since $D = \pi^*(\text{div} \Theta)$, we get that $\psi = \frac{\Theta_1 \circ \pi}{\Theta_2 \circ \pi}$. Since $\psi(g\gamma) = \psi(\gamma)$ for $g \in G$ are $\gamma \in \Gamma$, $\Theta_1 \circ \pi(g\gamma) = \Theta_2 \circ \pi(g)$, hence $\Theta_1$ is a meromorphic function on $T$. Thus we get that if $\psi$ is a meromorphic function on $G/\Gamma$, there is a meromorphic function $\tilde{\psi}$ on the torus $T$ such that $\tilde{\psi} = \pi^* \psi$.

Let $K(G/\Gamma)$ (resp. $K(T)$) denote the field of all meromorphic functions on $G/\Gamma$ (resp. on $T$).

We now get the following corollary of Theorem 2.

**Corollary** Under the assumptions of Theorem 2, there is a canonical isomorphism $\pi^*: K(T) \rightarrow K(G/\Gamma)$. In particular, the transcendence degree of $K(G/\Gamma)$ over $C$ is not more than the complex dimension of complex torus $T$.

5. Remarks and examples of compact complex parallelisable nilmanifolds

**Proposition 5.1.** Let $M$ be a compact complex parallelisable manifold of complex dimension 2. Then $M$ is a complex torus.

Proof. By a theorem of Wang [14], $M = G/\Gamma$ where $G$ is a simply connected complex Lie group of dimension 2 and $\Gamma$ is a lattice of $G$. Let $\{X_1, X_2\}$ be a basis of $\mathfrak{g}^+$ and $\{\omega_1, \omega_2\}$ be the dual basis of $(\mathfrak{g}^+)^*$. We may consider $\omega_1, \omega_2$ as holomorphic 1-forms on $G/\Gamma$. Since $G/\Gamma$ is 2 dimensional, $\omega_1, \omega_2$ are 1-closed; $d\omega_1 = d\omega_2 = 0$. Thus $[X_1, X_2] = 0$. Hence, $G$ is abelian and $G/\Gamma$ is a complex torus.

Now we shall give some examples of compact complex parallelisable nilmanifolds.
(1) Let $G$ be a simply connected complex nilpotent Lie group defined by

$$G = \begin{pmatrix} \begin{array}{c} z_{12} & \cdots & z_{1n} \\ \vdots & \ddots & \vdots \\ 1 & \cdots & z_{mn} \\ 0 & \cdots & 1 \\ 0 & \cdots & 0 \\ z_{ij} \in \mathbb{C}, i < j \end{array} \end{pmatrix}$$

and $\Gamma$ be a lattice of $G$ defined by

$$\Gamma = \begin{pmatrix} \begin{array}{c} a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ 1 & \cdots & a_{nm} \\ 0 & \cdots & 1 \\ 0 & \cdots & 0 \\ a_{ij} \in \mathbb{Z} + \sqrt{-1} \mathbb{Z}, i < j \end{array} \end{pmatrix} \subseteq \mathbb{Z}^n.$$  

Then $G/\Gamma$ is a compact complex parallelisable nilmanifold. In this case, we see that the transcendence degree of $K(G/\Gamma)$ over $\mathbb{C}$ is $n-1$.

(2) Let $G$ be a simply connected complex nilpotent Lie group defined by

$$G = \begin{pmatrix} \begin{array}{c} \begin{array}{c} z_1 \\ 1 \\ 1 \\ 0 \end{array} \\ 1 \\ 1 \\ 0 \end{array} \end{pmatrix} \begin{pmatrix} \begin{array}{c} z_{2} \\ \vdots \\ 1 \\ 1 \\ 0 \end{array} \end{pmatrix} \begin{pmatrix} \begin{array}{c} z_{n-1} \\ \vdots \\ 1 \\ 0 \end{array} \end{pmatrix} \begin{pmatrix} \begin{array}{c} w \\ y_{n-1} \\ y_2 \\ y_1 \end{array} \end{pmatrix} \begin{pmatrix} \begin{array}{c} z_j, y_j, w \in \mathbb{C} \\ j = 1, 2, \ldots, n-1 \end{array} \end{pmatrix}$$

for $n \geq 2$, and $\Gamma$ be a lattice of $G$ defined by

$$\Gamma = \begin{pmatrix} \begin{array}{c} \begin{array}{c} a_1 \\ 1 \\ 1 \\ 0 \end{array} \\ 1 \\ 1 \\ 0 \end{array} \end{pmatrix} \begin{pmatrix} \begin{array}{c} a_2 \\ \vdots \\ 1 \\ 0 \end{array} \end{pmatrix} \begin{pmatrix} \begin{array}{c} a_{n-1} \\ \vdots \\ 1 \\ 0 \end{array} \end{pmatrix} \begin{pmatrix} \begin{array}{c} c \\ b_{n-1} \\ b_2 \\ b_1 \end{array} \end{pmatrix} \begin{pmatrix} \begin{array}{c} a_j, b_j, c \in \mathbb{Z} + \sqrt{-1} \mathbb{Z} \\ j = 1, 2, \ldots, n-1 \end{array} \end{pmatrix}.$$
Then $G/\Gamma$ is a compact complex parallelisable nilmanifold. In this case, we see that the transcendence degree of $K(G/\Gamma)$ over $\mathbb{C}$ is $2(n-1)$.

References