Araki, S. and Iriye, K.
Osaka J. Math.
19 (1982), 1-55

EQUIVARIANT STABLE HOMOTOPY GROUPS OF SPHERES WITH INVOLUTIONS, I

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(Received Apıil 28, 1980)

Introduction. Equivariant stable homotopy groups of spheres with linear involutions were first discussed by Bredon [4, 5] and then by Landweber [10]. But the precise results of computation are not published except a few number of examples even though it seems that they computed these groups to a certain extent. In the present series of works we try to compute these groups systematically.

We use the notations $\pi_{p,q}^S$ of Landweber [10] to denote these groups which are denoted by $\pi_{p+q,p}^S$ in Bredon [4, 5]. As is well-known there are two types of homomorphisms; the one is the *forgetful homomorphism* ψ: $\pi_{p,q}^S \to \pi_{p+q}^S$ and the other is the *fixed-point homomorphism* ϕ: $\pi_{p,q}^S \to \pi_q^S$. There are also exact sequences involving these homomorphisms, i.e.

$$\cdots \to \pi_{p+q}^S \to \pi_{p,q}^S \xrightarrow{\chi} \pi_{p-1,q}^S \xrightarrow{\psi} \pi_{p+q-1}^S \to \cdots$$

and

$$\cdots \to \pi_{q+1}^S \to \lambda_{p,q}^S \to \pi_{p,q}^S \xrightarrow{\phi} \pi_q^S \to \cdots,$$

which are called *forgetful* and *fixed-point exact sequences*. These were certainly two of basic tools by Bredon and Landweber, and we also use these as a part of our basic tools.

Here we present two isomorphisms; the one is

$$\phi\colon \pi_{p,q}^S \approx \pi_q^S \qquad \text{for } p+q<0\,,$$

the other is that the fixed-point exact sequence splits in a large part and gives the isomorphism

$$\pi_{p,q}^S \approx \lambda_{p,q}^S \oplus \pi_q^S \qquad \text{for } p<q \text{ or } q<-1\,.$$

The first isomorphism reduces the computation of $\pi_{p,q}^S$ for $p+q<0$ to that of ordinary stable stems. The second isomorphism reduces the computation of

*) Supported by Grant-in-Aid for Scientific Research, No. 454017.

$\pi_{p,q}^{S}$ for $p<q$ or $q<-1$ to that of ordinary stable stems as soon as the restricted equivariant homotopy group $\lambda_{p,q}^{S}$ is computed. Our computation will stop at this stage, i.e. we will not go into computation of ordinary stable stems any more.

It is known [10] that $\lambda_{p,q}^{S}$ is isomorphic to a certain stable homotopy group of a stunted projective space, and hence to a certain homotopy group of Stiefel manifold in the stable range. Thus we can read $\lambda_{p,q}^{S}$ as groups from Hoo-Mahowald [7] in the range $p+q\leqslant 13$. But in the range $p\geqslant q\geqslant 1$ we meet with technical problems to decide extensions in fixed-point exact sequences, so we need to know generators of $\lambda_{p,q}^{S}$ from the view-point of equivariant homotopy theory and can not use the table of [7]. We compute also $\lambda_{p,q}^{S}$ from our point of view.

There holds the isomorphism

$$\lambda_{p,q}^{S}=\operatorname*{colim}_{r}\pi_{S}^{r-p,-q-1}(S_{+}^{r,0})\,.$$

We reach to $\lambda_{p,q}^{S}$ by step-wise computation of equivariant stable cohomotopy groups $\pi_{S}^{r-p,-q-1}(S_{+}^{r,0})$ recursively on r. For this purpose we use mainly the exact sequence associated with the equivariant cofibration $S_{+}^{1,0}\subset S_{+}^{r+1,0}\to S_{+}^{r+1,0}/S_{+}^{1,0}\approx_{T}\Sigma^{1,0}(S_{+}^{r,0})$ (occasionally we use the others for the sake of simplicity).

Clifford modules give certain periodic isomorphisms among equivariant stable cohomotopy groups of $S_{+}^{r,0}$ which give rise to periodic isomorphisms among $\lambda_{p,q}^{S}$ as the above mentioned colimit is stable. These periodic isomorphisms among $\lambda_{p,q}^{S}$ are reflected on James periodicity [8] through the above mentioned isomophisms between $\lambda_{p,q}^{S}$ and stable homotopy groups of stunted projective spaces.

These periodic isomorphisms are given by multiplication with certain periodicity elements. Observation of behaviors of these periodicity elements in relevant exact sequences is crucial in our computation.

As it was observed by Segal [13], $\pi_{0,0}^{S}$ is identified with the Burnside ring of $\boldsymbol{Z}/2$ (in our case) and hence $\pi_{0,0}^{S}=\boldsymbol{Z}[\rho]/(1-\rho^{2})$. Observation of actions of the generator ρ in equivariant homotopy groups is particularly important, which helped us to decide non-trivial extensions in many places.

At the moment we have finished computation of $\pi_{p,q}^{S}$ for $p+q\leqslant 13$. In the present paper we present computation for $p+q\leqslant 8$. The rests will be presented in subsequent papers.

§1 is a preparatory section. In §2 we discuss several commutativity relations of used homomorphisms. In §3 basic properties of periodicity elements coming from irreducible Clifford modules are discussed. We regard an often used construction of Bredon [5] as a kind of squaring operation. Its basic adding formula (Proposition 4.2) is discussed in §4, which gives rise to the above mentioned splitting of the fixed-point exact sequencces. In §5 we define an elementary homomorphism $\theta\colon \pi_n^S \to \pi_{0,n}^S$. The combination of θ and the external squaring operation gives an interpretation of Kahn-Priddy theorem [9] for $\boldsymbol{Z}/2$ from view-point of equivariant homotopy theory. In §6 we define equivariant stable Toda brackets. §§7–15 are devoted to our actual computation.

Notations and elementary results of [1, 2] are used freely. A cyclic groups generated by x is denoted by $\boldsymbol{Z}\cdot x$ or $\boldsymbol{Z}/\boldsymbol{n}\cdot x$ according as x is of order infinite or $\boldsymbol{n}$.

1. Stable τ-cohomotopy theory

Let X be a finite pointed τ-complex, p, q, r, s, k and l be integers, and $k \geqslant p$, $l \geqslant q$. We define a Λ-homomorphism

$$\tilde{\mathcal{E}}_{r,s}\colon [\Sigma^{k-p,l-q}X, \Sigma^{k,l}]^\tau \to [\Sigma^{r+k-p,s+l-q}X, \Sigma^{r+k,s+l}]^\tau \tag{1.1}$$

as the composition of the following sequence:

$$[\Sigma^{k-p,l-q}X, \Sigma^{k,l}]^\tau \xrightarrow{\Sigma_*^{r,s}} [\Sigma^{r,s}\Sigma^{k-p,l-q}X, \Sigma^{r,s}\Sigma^{k,l}]^\tau$$
$$\xrightarrow{\rho^{ps}\cdot(T^{-1}\wedge 1)^*\circ T_*} [\Sigma^{r+k-p,s+l-q}X, \Sigma^{r+k,s+l}]^\tau,$$

where ρ is the homomorphism defined in [2], (1.2), and $T\colon \Sigma^{p,q}\Sigma^{r,s} \approx \Sigma^{p+r,q+s}$ is τ-homeomorphism of [2], p. 381.

For fixed p and q, $\{[\Sigma^{k-p,l-q}X, \Sigma^{k,l}]^\tau, \tilde{\mathcal{E}}_{r,s}\}$ forms a direct system of abelian groups. Put

$$\pi_S^{p,q}(X) = \operatorname*{colim}_{k,l} [\Sigma^{k-p,l-q}X, \Sigma^{k,l}]^\tau, \tag{1.2}$$

and call it the (p,q)-th *stable τ-cohomotopy group* of X. The group

$$\pi_{p,q}^S = \pi_S^{-p,-q}(\Sigma^{0,0})$$

is the (p,q)-th equivariant stable homotopy group of sphere with involution [4, 5, 10], which we call the (p,q)-th *stable τ-homotopy group* of $\Sigma^{0,0}$ in consistency with our terminologies. The groups $\pi_{p,k-p}^S$, $p \in \boldsymbol{Z}$, are called the stable τ-homotopy groups of k stem.

Next we define a suspension isomorphism

$$\sigma^{r,s}\colon \pi_S^{p,q}(X) \approx \pi_S^{p+r,q+s}(\Sigma^{r,s}X) \tag{1.3}$$

in the same way as [2], (7.5). Then by the parallel argument to [2], §§7 and 8, and defining the multiplications by smash products, we obtain

Proposition 1.4. *$\pi_S^{*,*} = \{\pi_S^{p,q}(\ \ ), \sigma, \bar{\sigma};\ (p, q) \in \boldsymbol{Z}\times\boldsymbol{Z}\}$ is a reduced τ-cohomology theory with the commutative multiplication. The forgetful and fixed-point cohomology theories associated with $\pi_S^{*,*}$ are both the stable cohomotopy theory.*

We remark that the definition of the τ-spectrum $\boldsymbol{SR}$ in [2], p. 386, is not convenient for the present paper. Here we modify it as follows: take $\Sigma^{n,n}$ as the n-th term and the composition

$$\varepsilon_n = (J^n \wedge 1)\circ T\colon \Sigma^{1,1}\Sigma^{n,n} \to \Sigma^{n+1,n+1}$$

as the n-th structure map, where J is the involution of $\Sigma^{1,0}$ in [2], p. 385. Then restricting the double colimit (1.2) to the cofinal diagonal one, one can easily identify $\pi_S^{*,*}$ with $\widetilde{\boldsymbol{SR}}^{*,*}$. Hence as to τ-cohomotopy theory $\pi_S^{*,*}$ we can use the results of [2].

The forgetful exact sequence [2], (5.1), of $\pi_S^{*,*}$ gives rise to the following exact sequence

$$\text{(1.5)} \qquad \cdots \to \pi_{p+q}^S \to \pi_{p,q}^S \xrightarrow{\chi} \pi_{p-1,q}^S \xrightarrow{\psi} \pi_{p+q-1}^S \to \cdots$$

in case $X=\Sigma^{0,0}$, which is already used by Bredon [4, 5].

Next we define the fixed-point exact sequence of $\pi_S^{*,*}$. For our computation it is as important as the forgetful exact sequence (1.5). For a finite pointed τ-complex X and each $(p,q)\in \boldsymbol{Z}\times\boldsymbol{Z}$, we put

$$\text{(1.6)} \qquad \lambda_S^{p,q}(X) = \operatorname*{colim}_r \{\pi_S^{r+p,q-1}(S_+^{r,0}\wedge X),\ \rho^r\xi_{r+1,r}^*\}$$

where $\xi_{r+1,r}^*$ is the Λ-homomorphism defined in [1], §2. Landweber [10], p. 126, defined the restricted equivariant homotopy group $\lambda_{p,q}^S$, which we can identify with $\lambda_S^{-p,-q}(\Sigma^{0,0})$ as follows: since

$$\begin{aligned}
&\Sigma^{k+p,l+q}/\Sigma^{k+p-r,l+q} = S^{k+p,l+q+1}/S^{k+p-r,l+q+1}\\
&\simeq_\tau (S^{r,0}\times B^{k+p-r,l+q+1}\cup B^{r,0}\times S^{k+p-r,l+q+1})/B^{r,0}\times S^{k+p-r,l+q+1}\\
&= S_+^{r,0}\wedge (B^{k+p-r,l+q+1}/S^{k+p-r,l+q+1})\\
&= \Sigma^{k+p-r,l+q+1}S_+^{r,0}\,,
\end{aligned}$$

we have

$$\begin{aligned}
\pi_S^{r-p,-q-1}(S_+^{r,0}) &= \operatorname*{colim}_{k,l}\,[\Sigma^{k+p-r,l+q+1}S_+^{r,0},\ \Sigma^{k,l}]^\tau\\
&= \operatorname*{colim}_{k,l}\,[\Sigma^{k+p,l+q}/\Sigma^{k+p-r,l+q},\ \Sigma^{k,l}]^\tau\,,
\end{aligned}$$

hence

$$\begin{aligned}
\lambda_S^{-p,-q}(\Sigma^{0,0}) &= \operatorname*{colim}_{k,l,r}\,[\Sigma^{k+p,l+q}/\Sigma^{k+p-r,l+q},\ \Sigma^{k,l}]^\tau\\
&= \operatorname*{colim}_{k,l}\,[\Sigma^{k+p,l+q}/\Sigma^{0,l+q},\ \Sigma^{k,l}]^\tau\\
&= \lambda_{p,q}^S\,.
\end{aligned}$$

Consider the following commutative diagram:

$$\begin{array}{ccccccc}
\cdots \to \pi_S^{p,q-1}(X) & \xrightarrow{\alpha_r} & \pi_S^{p+r,q-1}(X) & \xrightarrow{\beta_r} & \pi_S^{p+r,q-1}(S_+^{r,0}\wedge X) & \xrightarrow{\delta_r} & \cdots \\
\| & & \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \rho^r\xi^*_{r+1,r}} & & \\
\cdots \to \pi_S^{p,q-1}(X) & \xrightarrow{\alpha_{r+1}} & \pi_S^{p+r+1,q-1}(X) & \xrightarrow{\beta_{r+1}} & \pi_S^{p+r+1,q-1}(S_+^{r+1,0}\wedge X) & \xrightarrow{\delta_{r+1}} & \cdots .
\end{array}$$

The commutativity of this diagram will be shown in Proposition 2.1. Since the fixed-point homomorphism ϕ: $\pi_S^{p+q,q-1}(X) \to \pi_S^{q-1}(\phi X)$ is an isomorphism for $r \geqslant \dim X - p - q + 2$ by [2], Proposition 5.4, passing to the colimit of the above diagram, we get the following exact sequence:

$$(1.7) \qquad \cdots \to \pi_S^{q-1}(\phi X) \xrightarrow{\partial_p} \lambda_S^{p,q}(X) \to \pi_S^{p,q}(X) \xrightarrow{\phi} \pi_S^q(\phi X) \to \cdots .$$

We call this exact sequence the *fixed-point exact sequence* of $\pi_S^{*,*}$. In a sense we may regard this exact sequence as a special case of the exact sequence of tom Dieck [6]. In case $X = \Sigma^{0,0}$ the sequence (1.7) is reduced to the following exact sequence defined by Landweber [10], p. 126:

$$(1.8) \qquad \cdots \to \pi_{q+1}^S \xrightarrow{\partial_p} \lambda_{p,q}^S \to \pi_{p,q}^S \xrightarrow{\phi} \pi_q^S \xrightarrow{\partial_p} \lambda_{p,q-1}^S \to \cdots .$$

The τ-cofibration $S_+^{1,0} \subset S_+^{r+1,0} \to S_+^{r+1,0}/S_+^{1,0} \approx_\tau \Sigma^{1,0} S_+^{r,0}$ implies the following exact sequence:

$$(1.9) \qquad \cdots \to \pi_S^{r+1-p,-q-2}(S_+^{1,0}) \xrightarrow{\delta_{r,1}} \pi_S^{r-p,-q-1}(S_+^{r,0}) \xrightarrow{\xi^*_{r+1,r}}$$

$$\pi_S^{r+1-p,-q-1}(S_+^{r+1,0}) \xrightarrow{\eta^*_{1,r+1}} \pi_S^{r+1-p,-q-1}(S_+^{1,0}) \to \cdots$$

for $r \geqslant 1$ and $(p,q) \in \boldsymbol{Z} \times \boldsymbol{Z}$. Since $\pi_S^{p,q}(S_+^{1,0}) \approx \pi_{-p-q}^S$, we obtain

Proposition 1.10. $\xi^*_{r+1,r}$: $\pi_S^{r-p,-q-1}(S_+^{r,0}) \to \pi_S^{r+1-p,-q-1}(S_+^{r+1,0})$ *is isomorphic for* $r \geqslant p+q+2$ *and epimorphic for* $r \geqslant p+q+1$.

Corollary 1.11. $\lambda_{p,q}^S \approx \pi_S^{r-p,-q-1}(S_+^{r,0})$ *for* $r \geqslant p+q+2$.

Because of Corollary 1.11 we often identify the fixed-point exact sequence (1.8) with the exact sequence

$$(1.12) \qquad \cdots \to \pi_{p,q+1}^S \xrightarrow{\chi^r} \pi_{p-r,q+1}^S \xrightarrow{\beta_r} \pi_S^{r-p,-q-1}(S_+^{r,0}) \xrightarrow{\delta_r} \pi_{p,q}^S \to \cdots$$

(associated with the τ-cofibration $S_+^{r,0} \subset B_+^{r,0} \to \Sigma^{r,0}$) in the range $r \geqslant p+q+2$.

2. Relations between various homomorphisms

In this section we will give the relations between the various homomorphisms defined in [1], §2. Throughout this section X is assumed to be a finite τ-complex.

Proposition 2.1. *There hold the following relations* (*for positive integers* r, s *and* t);

i) $\alpha_{r+s}=\alpha_r\circ\alpha_s$,

ii) $\eta^*_{r,r+s+t}=\eta^*_{r,r+s}\circ\eta^*_{r+s,r+s+t}$,

iii) $\xi^*_{r+s+t,t}=\xi^*_{r+s+t,s+t}\circ\xi^*_{s+t,t}$,

iv) $\eta^*_{r,r+s}\circ\beta_{r+s}=\beta_r$,

v) $\beta_s\circ\delta_r=\delta_{s,t}$,

vi) $\delta_{r+s}\circ\xi^*_{r+s,s}=\rho^r\delta_s$,

vii) $\delta_{t,r+s}\circ\xi^*_{r+s,s}=\rho^r\delta_{t,s}$,

viii) $\xi^*_{r+s,s}\circ\eta^*_{s,r+s}=\alpha_r$,

ix) $\beta_{r+s}\circ\alpha_r=\xi^*_{r+s,s}\circ\beta_s$,

x) $\delta_{s+t,r}\circ\eta^*_{r,r+s}=\xi^*_{s+t,t}\circ\delta_{t,r+s}$,

xi) $\delta_r\circ\eta^*_{r,r+s}=\alpha_s\circ\delta_{r+s}$,

xii) $\eta^*_{r+s,r+s+t}\circ\xi^*_{r+s+t,s+t}=\xi^*_{r+s,s}\circ\eta^*_{s,s+t}$.

Proof. Relations i), ii), iv), v), ix) and xi) are proved already in [1, 2]. Here we prove the rests. We will prove the relations only for the special case $X=pt$ to avoid too big diagrams. The proofs of general cases are entirely parallel to the special case and will be left to readers.

Proof of iii). The commutative diagram

$$\begin{tikzcd}
S_+^{r+s+t,0} \arrow[r, "\xi_{r+s+t,s+t}"] \arrow[dr, "\xi_{r+s+t,t}"'] & \Sigma^{r,0}S_+^{s+t,0} \arrow[d, "\Sigma^{r,0}\xi_{s+t,t}"] \\
 & \Sigma^{r+s,0}S_+^{t,0}
\end{tikzcd}$$

induces the following commutative diagram

$$\begin{tikzcd}
\pi_S^{p-r,q+1}(S_+^{s+t,0}) \arrow[r, "\sigma^{r,0}"] & \pi_S^{p,q+1}(\Sigma^{r,0}S_+^{s+t,0}) \arrow[r, "(\xi_{r+s+t,s+t})^*"] & \pi_S^{p,q+1}(S_+^{r+s+t,0}) \\
\pi_S^{p-r,q+1}(\Sigma^{s,0}S_+^{t,0}) \arrow[u, "(\xi_{s+t,t})^*"] \arrow[r, "\sigma^{r,0}"] & \pi_S^{p,q+1}(\Sigma^{r+s,0}S_+^{t,0}) \arrow[u, "(\Sigma^{r,0}\xi_{s+t,t})^*"] \arrow[r, "(\xi_{r+s+t,t})^*"] & \pi_S^{p,q+1}(S_+^{r+s+t,0}) \arrow[u, equal] \\
\pi_S^{p-r-s,q+1}(S_+^{t,0}) \arrow[u, "\sigma^{s,0}"] \arrow[ur, "\sigma^{r+s}"'] & &
\end{tikzcd}$$

which proves iii).

Proof of vi). The collapsing map $B_+^{r+s,0}\approx_\tau(B^{r,0}\times B^{s,0})_+\rightarrow(B^{r,0}\times B^{s,0})/(S^{r,0}\times B^{s,0})=\Sigma^{r,0}B_+^{s,0}$ induces the following commutative diagam

$$\begin{tikzcd}
S_+^{r+s,0} \arrow[r, "\iota_{r+s}"] \arrow[d, "\xi_{r+s,s}"] & B_+^{r+s,0} \arrow[r, "\pi_{r+s}"] \arrow[d] & \Sigma^{r+s,0} \arrow[d, equal] \\
\Sigma^{r,0}S_+^{s,0} \arrow[r, "\Sigma^{r,0}\iota_s"] & \Sigma^{r,0}B_+^{s,0} \arrow[r, "\Sigma^{r,0}\pi_s"] & \Sigma^{r+s,0},
\end{tikzcd}$$

which induces in turn the following commutative diagram:

$$\begin{tikzcd}
\pi_S^{p,q}(S_+^{r+s,0}) \arrow[r, "\delta^*"] & \pi_S^{p,q+1}(\Sigma^{r+s,0}) & \pi_S^{p-r-s,q+1}(\Sigma^{0,0}) \arrow[l, "\sigma^{r+s,0}"'] \arrow[ddl, "\sigma^{s,0}"] \\
\pi_S^{p,q}(\Sigma^{r,0}S_+^{s,0}) \arrow[u, "(\xi_{r+s,s})^*"] \arrow[r, "\delta^*"] & \pi_S^{p,q+1}(\Sigma^{r+s,0}) \arrow[u, equal] & \\
\pi_S^{p-r,q}(S_+^{s,0}) \arrow[u, "\sigma^{r,0}"] \arrow[r, "\rho^r\delta^*"] & \pi_S^{p-r,q+1}(\Sigma^{s,0}) \arrow[u, "\sigma^{r,0}"] &
\end{tikzcd}$$

This diagram proves vi).

Proof of vii). We have $\delta_{t,r+s}\circ\xi^*_{r+s,s}=\beta_t\circ\delta_{r+s}\circ\xi^*_{r+s,s}=\beta_t\circ(\rho^r\delta_s)=\rho^r\delta_{t,s}$, by v) and vi).

Proof of viii). In the following commutative diagram

$$\begin{tikzcd}
S_+^{r+s,0} \arrow[r, "\xi_{r+s,s}"] \arrow[d, "\eta_{r+s,r+s+r}"] & \Sigma^{r,0}S_+^{s,0} \arrow[d, "\Sigma^{r,0}\eta_{s,s+r}"] \\
S_+^{r+s+r,0} \arrow[r, "\xi_{r+s+r\ s+r}"] & \Sigma^{r,0}S_+^{s+r,0}
\end{tikzcd}$$

the map $\eta_{r+s,r+s+r}$ is τ-homotopic to the map $\eta'_{r+s,r+s+r}$, $\eta'_{r+s,r+s+r}(x_1,\cdots,x_{r+s})=(0,\cdots,0,x_1,\cdots,x_{r+s})$ for $(x_1,\cdots,x_{r+s})\in S^{r+s,0}$, since $r>0$. Then $\xi_{r+s+r,s+r}\circ\eta'_{r+s,r+s+r}=\zeta_{0,r}\wedge 1$, where $\zeta_{0,r}\colon\Sigma^{0,0}\subset\Sigma^{r,0}$ is a canonical τ-inclusion. Now it follows the relation viii) from the τ-homotopy $\Sigma^{r,0}\eta_{s,s+t}\circ\xi_{r+s,s}\simeq_\tau\zeta_{0,r}\wedge 1$.

Proof of x). We have $\delta_{s+t,r}\circ\eta^*_{r,r+s}=\beta_{s+t}\circ\delta_r\circ\eta^*_{r,r+s}=\beta_{s+t}\circ\alpha_s\circ\delta_{r+s}=\xi^*_{s+t,t}\circ\beta_t\circ\delta_{r+s}=\xi^*_{s+t,t}\circ\delta_{t,r+s}$, by v), ix) and xi).

Proof of xii). The following commutative diagram

$$\begin{tikzcd}
S_+^{r+s,0} \arrow[r, "\eta_{r+s,r+s+t}"] \arrow[d, "\xi_{r+s,s}"] & S_+^{r+s+t,0} \arrow[d, "\xi_{r+s+t,s+t}"] \\
\Sigma^{r,0}S_+^{s,0} \arrow[r, "\Sigma^{r,0}\eta_{s,s+t}"] & \Sigma^{r,0}S_+^{s+t,0}
\end{tikzcd}$$

induces the relation xii). □

Proposition 2.2. *For positive integers r, s and t, the homomorphism α_r*

commutes with the homomorphisms β_s, $\eta^*_{s,s+t}$, $\xi^*_{s,t}$, δ_s *and* $\delta_{s,t}$.

Proof. This proposition is routine if we pay attention to the fact that $\rho=1$ on Im α_r for $r>0$. $\square$

Next we define the internal product

$$\pi_S^{p,q}(X_+)\otimes_\Lambda\pi_S^{r,s}(X_+)\to\pi_S^{p+r,q+s}(X_+) \tag{2.3}$$

by $xy=d^*(x\wedge y)$, where $x\in\pi_S^{p,q}(X_+)$, $y\in\pi_S^{r,s}(X_+)$ and $d\colon X_+\to X_+\wedge X_+$ is the diagonal map. Then $\pi_S^{*,*}(X_+)=\sum_{p,q}\pi_S^{p,q}(X_+)$ is a bigraded Λ-algebra with unit $1=\pi^*1$, $\pi\colon X\to pt$, which is commutative in the sense of [2], §6.

Proposition 2.4. *For* $u\in\pi_S^{p,q}(X_+)$, $u_1, u_2\in\pi_S^{*,*}(X_+)$ *and* $v\in\pi_S^{*,*}(S_+^{r,0}\wedge X_+)$, *there hold the relations*:

i) $\alpha_r(u_1u_2)=\alpha_r(u_1)u_2=u_1\alpha_r(u_2)$,

ii) $\beta_r(u_1u_2)=\beta_r(u_1)\beta_r(u_2)$,

iii) $\delta_r(v\beta_r(u))=\delta_r(v)u$,

iv) $\delta_r(\beta_r(u)v)=\rho^{(p+q)r+p}(-1)^q u\delta_r(v)$.

Proof. Relations i), ii) and iii) are easily seen by routine arguments of generalized cohomology theories. The relation iv) follows from the relation iii) and the commutativity of τ-cohomotopy theory. $\square$

Proposition 2.5. *For* u, u_1, $u_2\in\pi_S^{*,*}(S_+^{r+s,0}\wedge X_+)$, $v\in\pi_S^{*,*}(S_+^{s,0}\wedge X_+)$ *and* $w\in\pi_S^{*,*}(S_+^{r,0}\wedge X_+)$, *we have*

i) $\xi^*_{r+s,s}(v\eta^*_{s,s+r}(u))=\xi^*_{r+s,s}(v)u$,

ii) $\eta^*_{r,r+s}(u_1u_2)=\eta^*_{r,r+s}(u_1)\eta^*_{r,r+s}(u_2)$,

iii) $\delta_{s,r}(w\eta^*_{r,r+s}(u))=\delta_{s,r}(w)\eta^*_{s,r+s}(u)$.

Proof. The relation ii) is obvious. We will prove the relations i) and iii) only for the special case $X=pt$ to avoid too big diagrams. The proofs of general cases are entirely parallel to the special case and will be left to readers.

Consider the following diagrams:

$$\begin{array}{ccc}
\pi_S^{p,q}(S_+^{s,0}\wedge S_+^{r+s,0}) & \xrightarrow{d'^*} & \pi_S^{p,q}(S_+^{s,0}) \\
\Big\downarrow{\scriptstyle \sigma^{r,0}} & & \Big\downarrow{\scriptstyle \sigma^{r,0}} \\
\pi_S^{r+p,q}(\Sigma^{r,0}S_+^{s,0}\wedge S_+^{r+s,0}) & \xrightarrow{(1\wedge d')^*} & \pi_S^{r+p,q}(\Sigma^{r,0}S_+^{s,0}) \\
\Big\downarrow{\scriptstyle (\xi_{r+s,s}\wedge 1)^*} & & \Big\downarrow{\scriptstyle (\xi_{r+s,s})^*} \\
\pi_S^{r+p,q}(S_+^{r+s,0}\wedge S_+^{r+s,0}) & \xrightarrow{d^*} & \pi_S^{r+p,q}(S_+^{r+s,0}),
\end{array} \tag{2.6}$$

where $d'=(1\wedge\eta_{s,r+s})\circ d$, and

$$
(2.7)\qquad
\begin{tikzcd}
\pi_S^{p,q}(S_+^{r,0}\wedge S_+^{r+s,0}) \arrow[r, "d''^*"] \arrow[d, "\sigma^{0,1}"] & \pi_S^{p,q}(S_+^{r,0}) \arrow[d, "\sigma^{0,1}"] \\
\pi_S^{p,1+q}(\Sigma^{0,1}S_+^{r,0}\wedge S_+^{r+s,0}) \arrow[r, "(1\wedge d'')^*"] \arrow[d, "(\delta\wedge 1)^*"] & \pi_S^{p,1+q}(\Sigma^{0,1}S_+^{r,0}) \arrow[d, "\delta^*"] \\
\pi_S^{p,1+q}(\Sigma^{r,0}S_+^{s,0}\wedge S_+^{r+s,0}) \arrow[r, "(1\wedge d''')^*"] & \pi_S^{p,1+q}(\Sigma^{r,0}S_+^{s,0}) \\
\pi_S^{p-r,1+q}(S_+^{s,0}\wedge S_+^{r+s,0}) \arrow[r, "d'''^*"] \arrow[u, "\sigma^{r,0}"] & \pi_S^{p-r,1+q}(S_+^{s,0}) \arrow[u, "\sigma^{r,0}"]
\end{tikzcd},
$$

where $d''=(1\wedge\eta_{r,r+s})\circ d$ and $d'''=(1\wedge\eta_{s,s+r})\circ d$. Since $d'^*(v\wedge u)=v\eta^*_{s,r+s}(u)$, $d''^*(w\wedge u)=w\eta^*_{r,r+s}(u)$ and $d'''^*(\delta_{s,r}(w)\wedge u)=\delta_{s,r}(w)\eta^*_{s,s+r}(u)$, we get the relations i) and iii) if we prove that these two diagrams are commutative.

Observe the following two diagrams:

$$
(2.8)\qquad
\begin{tikzcd}
\Sigma^{r,0}S_+^{s,0}\wedge S_+^{r+s,0} & \Sigma^{r,0}S_+^{s,0} \arrow[l, "1\wedge d'"'] \\
S_+^{r+s,0}\wedge S_+^{r+s,0} \arrow[u, "\xi_{r+s,s}\wedge 1"] & S_+^{r+s,0} \arrow[l, "d"'] \arrow[u, "\xi_{r+s,s}"]
\end{tikzcd},
$$

$$
(2.9)\qquad
\begin{tikzcd}
\Sigma^{0,1}S_+^{r,0}\wedge S_+^{r+s,0} & \Sigma^{0,1}S_+^{r,0} \arrow[l, "1\wedge d''"'] \\
\Sigma^{r,0}S_+^{s,0}\wedge S_+^{r+s,0} \arrow[u, "\delta\wedge 1"] & \Sigma^{r,0}S_+^{s,0} \arrow[l, "1\wedge d'''"'] \arrow[u, "\delta"]
\end{tikzcd}.
$$

Obviously (2.8) is commutative up to τ-homotopy. The τ-homotopy commutativity of (2.9) is verfied by the following τ-homotopy commutative diagram:

$$
\begin{tikzcd}
\Sigma^{0,1}S_+^{r,0}\wedge S_+^{r+s,0} & \Sigma^{0,1}S_+^{r,0} \arrow[l, "1\wedge d''"'] \\
(S_+^{r+s,0}\cup CS_+^{r,0})\wedge S_+^{r+s,0} \arrow[u] \arrow[d] & S_+^{r+s,0}\cup CS_+^{r,0} \arrow[l] \arrow[u] \arrow[d] \\
(S_+^{r+s,0}/S_+^{r,0})\wedge S_+^{r+s,0} & S_+^{r+s,0}/S_+^{r,0} \arrow[l] \\
\Sigma^{r,0}S_+^{s,0}\wedge S_+^{r+s,0} \arrow[u, phantom, "\wr"] & \Sigma^{r,0}S_+^{s,0} \arrow[l, "1\wedge d'''"'] \arrow[u, phantom, "\wr"]
\end{tikzcd}
$$

Now the τ-homotopy commutativity of (2.8) and (2.9) implies the commutativity of (2.6) and (2.7). $\square$

3. Periodicity elements

As we discussed in [1], §3, an orthogonal multiplication μ: $\boldsymbol{R}^k \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ induces the *periodic* isomorphism

$$\omega_{k,n}^*: \pi_S^{p,q}(S_+^{k,0} \wedge X_+) \approx \pi_S^{p-n,q+n}(S_+^{k,0} \wedge X_+) \tag{3.1}$$

for any $(p, q) \in \boldsymbol{Z} \times \boldsymbol{Z}$ and a finite τ-complex X. Put

$$\omega_{k,n} = \omega_{k,n}^{*-1}(1) \in \pi_S^{n,-n}(S_+^{k,0} \wedge X_+) \tag{3.2}$$

(denoted by the same letter as [1], (3.2)). We see easily that $\omega_{k,n}$ is an invertible element and that $\omega_{k,n}^*(u) = \omega_{k,n}^{-1} u$ for $u \in \pi_S^{*,*}(S_+^{k,0} \wedge X_+)$. We call $\omega_{k,n}$ a *periodicity element* of type (k, n).

The periodic isomorphism $\omega_{k,n}^*$ depends on the choice of orthogonal multiplications and is not defined uniquely.

We say that two orthogonal multiplications μ and μ': $\boldsymbol{R}^k \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ are *isomorphic* iff there exist two orthogonal matrices A and B in $O(n)$ such that $\mu' = B \circ \mu \circ (1 \times A)$. Let $\omega_{k,n}^*$ and $\omega_{k,n}'^*$ be the induced ones from μ and μ' respectively. By definition of $\omega_{k,n}^*$ and Λ-actions in τ-cohomology theories [1, 2] we see easily that

$$\begin{aligned} \omega_{k,n}'^* &= (\det A) \cdot \omega_{k,n}^* \quad &&\text{when } \det B = 1\,, \\ \omega_{k,n}'^* &= \rho(\det A) \cdot \omega_{k,n}^* \quad &&\text{when } \det B = -1\,. \end{aligned} \tag{3.3}$$

Thus, the periodicity element changes only by signs in τ-cohomology theories as far as the inducing orthogonal multiplication changes with in the same isomorphism class.

Let μ: $\boldsymbol{R}^k \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ be an orthogonal multiplication. We define an orthogonal multiplication $\tilde{\mu}$: $\boldsymbol{R}^k \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ by $\tilde{\mu}(x, y) = \mu(x, A^{-1}y)$, $x \in \boldsymbol{R}^k$ and $y \in \boldsymbol{R}^n$, where $A \in O(n)$ defined by $Ay = \mu(e_1, y)$, $e_1 = (1, 0, \cdots, 0)$. $\tilde{\mu}$ is a *normalized* orthogonal multiplication in the sense that $\tilde{\mu}(e_1, y) = y$ for any $y \in \boldsymbol{R}^n$. We call $\tilde{\mu}$ the *normalization* of μ. Clearly any orthogonal multiplication is isomorphic to its normalization.

As is well-known, a normalized orthogonal multiplication μ: $\boldsymbol{R}^k \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ gives rise to a C_{k-1}-module $\boldsymbol{R}^n$ by $e_i \cdot y = \mu(e_{i+1}, y)$, $1 \leqslant i \leqslant n-1$, $e_i = (0, \cdots, 0, 1, 0, \cdots, 0)$ and $y \in \boldsymbol{R}^n$, vice versa. Hence we may identify normalized orthogonal multiplications with their corresponding Clifford modules.

We regard for each orthogonal multiplication μ its normalization $\tilde{\mu}$ as the Clifford module. As is easily seen, the correspondence "$\mu \to \tilde{\mu}$" gives rise to a bijection between isomorphism classes of orthogonal multiplications and of Clifford modules.

In the following we restrict ourselves to the case $n=a_k$ ($a_k=2^{\varphi(k-1)}$, [3]), and abbreviate $\omega^*_{k,a_k}=\omega^*_k$, $\omega_{k,a_k}=\omega_k$, and so on. For every orthogonal multiplication $\mu\colon \boldsymbol{R}^k\times\boldsymbol{R}^n\to\boldsymbol{R}^n$, $n=a_k$, its corresponding C_{k-1}-module is irreducible [3].

In case $k\not\equiv 0 \pmod 4$, there exists exactly one isomorphism class of irreducible C_{k-1}-modules [3]. Thus ω^*_k is defined uniquely up to signs in τ-cohomology theories by (3.3).

In case $k\equiv 0 \pmod 4$, there exist exactly two isomorphism classes of irreducible C_{k-1}-modules [3]. Correspondingly we have two isomorphism classes of orthogonal multiplications: $\boldsymbol{R}^k\times\boldsymbol{R}^n\to\boldsymbol{R}^n$. Choose μ and $\bar{\mu}\colon \boldsymbol{R}^k\times\boldsymbol{R}^n\to\boldsymbol{R}^n$ from each isomorphism class and denote by ω^*_k and $\bar{\omega}^*_k$ the induced periodic isomorphism respectively. These are again determined uniquely up to signes in τ-cohomology theories.

Put $\omega_k=\omega^{*-1}_k(1)$ (and $\bar{\omega}_k=\bar{\omega}^{*-1}_k(1)$ in case $k\equiv 0 \pmod 4$).

Proposition 3.4. *There hold the relations*

$$\eta^*_{k,k+1}\omega_{k+1}=\begin{cases}\omega_k & \text{when } k\equiv 3,5,6 \text{ or } 7 \pmod 8,\\ \omega_k^2 & \text{when } k\equiv 1 \text{ or } 2 \pmod 8,\\ \bar{\omega}_k\omega_k & \text{when } k\equiv 0 \pmod 4,\end{cases}$$

up to signs.

Proof. For two orthogonal multiplications μ and $\mu'\colon \boldsymbol{R}^k\times\boldsymbol{R}^n\to\boldsymbol{R}^n$ we define an orthogonal multiplication $\mu+\mu'\colon \boldsymbol{R}^k\times\boldsymbol{R}^{2n}\to\boldsymbol{R}^{2n}$ by

$$(\mu+\mu')(x,y_1,y_2)=(\mu(x,y_1),\ \mu'(x,y_2))$$

for $x\in\boldsymbol{R}^k$ and $y_1,y_2\in\boldsymbol{R}^n$. Let $\omega_{k,n}$, $\omega'_{k,n}$ and $\omega_{k,2n}$ be the τ-homeomorphisms induced by μ, μ' and $\mu+\mu'$ respectively. The τ-homotopy commutative diagram

$$\begin{array}{ccccc} S^{k,0}_+\wedge\Sigma^{2n,0} & \xrightarrow{d\wedge 1} & S^{k,0}_+\wedge S^{k,0}_+\wedge\Sigma^{n,0}\wedge\Sigma^{n,0} & \xrightarrow{1\wedge T'\wedge 1} & S^{k,0}_+\wedge\Sigma^{n,0}\wedge S^{k,0}_+\wedge\Sigma^{n,0}\\ \downarrow{\scriptstyle \omega_{k,2n}} & & & & \downarrow{\scriptstyle \omega_{k,n}\wedge\omega'_{k,n}}\\ S^{k,0}_+\wedge\Sigma^{0,2n} & \xrightarrow{d\wedge 1} & S^{k,0}_+\wedge S^{k,0}_+\wedge\Sigma^{0,n}\wedge\Sigma^{0,n} & \xrightarrow{1\wedge T'\wedge 1} & S^{k,0}_+\wedge\Sigma^{0,n}\wedge S^{k,0}_+\wedge\Sigma^{0,n}\end{array}$$

yields the relation

$$\omega^*_{k,2n}=\rho^n\omega^*_{k,n}\circ\omega'^*_{k,n}\,.$$

Let $i\colon C_{k-1}\to C_k$ be the inclusion which extends the inclusion $\boldsymbol{R}^{k-1}\to\boldsymbol{R}^k$. For any irreducible C_k-module M, by [3], §5, we see that i^*M is also irreducible when $k\equiv 3$, 5, 6 or 7 (mod 8), that $i^*M\cong 2M'$ for some irreducible C_{k-1}-module M' when $k\equiv 1$ or 2 (mod 8), and that $i^*M\cong M'\oplus M''$ for the representatives M' and M'' of the two distinct isomorphism classes of irreducible C_{k-1}-modules when $k\equiv 0$ (mod 4). Thus we obtain the desired relations. □

The following theorem is a key to compute $\pi_S^{*,*}(S_*^{r,0})$, $r>1$.

Theorem 3.5. *$\psi(\delta_1\omega_1)=\pm 2$ and $\psi(\delta_k\omega_k)$ $(\in\pi_{k-1}^S)$, $k>1$, is the J-image of the generator of $\pi_{k-1}^S(O)$.*

Proof. By definition, ω_k is represented by the composition of the following sequence:

$$S_+^{k,0}\wedge\Sigma^{0,n}\xrightarrow{\omega_{k,n}^{-1}}S_+^{k,0}\wedge\Sigma^{n,0}\xrightarrow{p\wedge 1}\Sigma^{n,0},$$

where $\omega_{k,n}$ is the τ-homeomorphism defined by [1], (3.2), and $p: S_+^{k,0}\to\Sigma^{0,0}$ is a τ-map to collapse $S^{k,0}$ to a point. $\delta_k\omega_k$ is represented by the composition of the following sequence:

$$\begin{aligned}\Sigma^{k,n-1}&\simeq_\tau(B_+^{k,0}\cup CS_+^{k,0})\wedge\Sigma^{0,n-1}\to S_+^{k,0}\wedge\Sigma^{0,1}\Sigma^{0,n-1}\\ &=S_+^{k,0}\wedge\Sigma^{0,n}\xrightarrow{\omega_{k,n}^{-1}}S_+^{k,0}\wedge\Sigma^{n,0}\xrightarrow{p\wedge 1}\Sigma^{n,0}.\end{aligned}$$

Thus the composition

$$\Sigma^{k+n-1}\simeq(B_+^k\cup CS_+^{k-1})\wedge\Sigma^{n-1}\to S_+^{k-1}\wedge\Sigma^n\xrightarrow{\psi(\omega_{k,n}^{-1})}S_+^{k-1}\wedge\Sigma^n\xrightarrow{p\wedge 1}\Sigma^n \tag{3.6}$$

represents $\psi(\delta_k\omega_k)$, where B^k is the unit ball in $\boldsymbol{R}^k$ and S^{k-1} is the unit sphere in $\boldsymbol{R}^k$.

Consider the following diagram

$$\begin{tikzcd}
\Sigma^k\wedge\Sigma^{n-1}\approx S^{k+n-1}\approx S^{k-1}*S^{n-1} \arrow[r, "H(\tilde\mu)"] \arrow[d, "f"] & \Sigma S^{n-1}\approx\Sigma^n \\
(B_+^k\cup CS_+^{k-1})\wedge\Sigma^{n-1} \arrow[u, "h\wedge 1"] \arrow[r, "\delta\wedge 1"'] \arrow[ur, phantom, "g"] & S_+^{k-1}\wedge\Sigma^n \arrow[r, "\psi(\omega_{k,n}^{-1})"'] & S_+^{k-1}\wedge\Sigma^n. \arrow[ul, "p\wedge 1"']
\end{tikzcd} \tag{3.7}$$

Let μ be the normalized orthogonal multiplication which gives rise to the τ-homeomorphism $\omega_{k,n}$. Let $\tilde\mu: S^{k-1}\times S^{n-1}\to S^{n-1}$ be given by $\tilde\mu(x,y)=A_x^{-1}(y)$, where $x\in S^{k-1}$, $y\in S^{n-1}$ and $A_x(y)=\mu(x,y)$. For a map a: $A\times B\to C$ we denote the Hopf construction on it by $H(a)$: $A*B\to\Sigma C$. f: $S^{k-1}*S^{n-1}\to S_+^{k-1}\wedge\Sigma^1\wedge\Sigma^{n-1}=S_+^{k-1}\wedge\Sigma^n$ is defined by $f(x,t,y)=x\wedge t\wedge\phi(y)$, where $(x,t,y)\in S^{k-1}*S^{n-1}$ and ϕ: $S^{n-1}\to\Sigma^{n-1}$ is a homeomorphism. g is given by the composition of the following sequence:

$$\Sigma^k\wedge\Sigma^{n-1}\approx SS^{k-1}\wedge\Sigma^{n-1}\xrightarrow{p'\wedge 1}S_+^{k-1}\wedge\Sigma^1\wedge\Sigma^{n-1}\approx S_+^{k-1}\wedge\Sigma^n,$$

where SS^{k-1} is an unreduced suspension of S^{k-1} and $p'\colon SS^{k-1}\to\Sigma S_+^{k-1}$ is a map which collapses the two poles of the sphere to a base point of ΣS_+^{k-1}. $h\colon B_+^k\cup CS_+^{k-1}\to\Sigma^k$ is the canonical homotopy equivalence. $\delta\colon B_+^k\cup CS_+^{k-1}\to\Sigma S_+^{k-1}$ is the canonical map which collapses B_+^k to a point. It is routine to see that $H(\tilde{\mu})$ is homotopic to $(p\wedge 1)\circ\psi(\omega_{k,n}^{-1})\circ f$, that f is identified with g through the canonical homeomorphisms, and that $g\circ(h\wedge 1)$ is homotopic to $-\delta\wedge 1$. Thus the diagram is homotopy commutative up to sign, which implies that $\psi(\delta_k\omega_k)=\pm H(\tilde{\mu})$. Since $H(\mu)=-H(\tilde{\mu})$, we have $\psi(\delta_k\omega_k)=\pm H(\mu)$. When $k=1$, this yields $\psi(\delta_1\omega_1)=\pm 2$.

Now consider the case $k>1$. Atiyah, Bott and Shapiro [3] gave the generators of $\widetilde{KO}(S^k)$ in terms of $\boldsymbol{Z}/2$-graded Clifford modules. Let $M=M^0\oplus M^1$ be an irreducible $\boldsymbol{Z}/2$-graded C_k-module and let

$$\sigma\colon B^k\times M^1\to B^k\times M^0$$

be given by $\sigma(v, e)=(v, -ve)$ for $v\in B^k$ and $e\in M^1$. The complex of vector bundles

$$0\to B^k\times M^1\xrightarrow{\sigma} B^k\times M^0\to 0 \tag{3.8}$$

represents the generator of $\widetilde{KO}(S^k)\approx KO(B^k, S^{k-1})$, [3], §11. Let

$$\sigma'\colon B^k\times M^0\to B^k\times M^0$$

be defined by $\sigma'(v, e)=(v, -v(e_ke))$ for $v\in B^k$ and $e\in M^0$. The complex of vector bundles

$$0\to B^k\times M^0\xrightarrow{\sigma'} B^k\times M^0\to 0$$

is isomorphic to the complex (3.8). Thus, through the isomorphism $\widetilde{KO}(S^k)\approx\pi_{k-1}^S(O)$, we see that orthogonal multiplication corresponding to the irreducible C_{k-1}-module M^0 gives the generator of $\pi_{k-1}^S(O)$. As J-homomorphisms are given by the Hopf construction, the above arguments establish the theorem. □

The periodicity elements ω_k (and $\bar{\omega}_k$ in case $k\equiv 0 \pmod 4$) have been so far discussed up to signs. Hereafter throughout the present work we fix the periodicity elements ω_k, $k\geqslant 1$, so that they satisfy the following conditions:

(3.9. i) $\psi(\delta_1\omega_1)=2$,

(3.9. ii) $\eta_{k,k+1}^*\omega_{k+1}=\omega_k$ when $k\equiv 3, 5, 6$ or $7 \pmod 8$,

(3.9. iii) $\eta_{k,k+1}^*\omega_{k+1}=\omega_k^2$ when $k\equiv 1$ or $2 \pmod 8$,

(3.9. iv) $\eta_{k,k+1}^*\omega_{k+1}=\bar{\omega}_k\omega_k$ when $k\equiv 0 \pmod 4$.

Such a choice of the periodicity elements ω_k, $k\geqslant 1$, is certainly possible by (3.3) and Proposition 3.4.

Put

$$c_{4k}=\overline{\omega}_{4k}\omega_{4k}^{-1}\in\pi_S^{0,0}(S_+^{4k,0}).$$

Then

(3.9.iv′) $\eta_{k,k+1}^{*}\omega_{k+1}=c_{4k}\omega_{4k}^{2}$,

c_{4k} is a unit of $\pi_S^{0,0}(S_+^{4k,0})$ and $\eta_{4k-1,4k}^{*}c_{4k}=1$.

As for the elements of stable homotopy groups we use customary notations. By Theorem 3.5 we may identify $\psi(\delta_2\omega_2)=\eta$, $\psi(\delta_4\omega_4)=\nu$ and $\psi(\delta_8\omega_8)=\sigma$ and so on. Thus η, ν and σ are the generators of cyclic groups of order 2, 24 and 240, respectively (not those of their 2-primary components !). Then we have relations $\psi(\delta_3\omega_3)=\psi(\delta_5\omega_5)=\psi(\delta_6\omega_6)=\psi(\delta_7\omega_7)=0$, $\psi(\delta_9\omega_9)=\eta\sigma$ and $\psi(\delta_{10}\omega_{10})=\eta^2\sigma$, as is well-known.

4. External squaring operation

Bredon [5] constructed an element $b(x)\in\pi_{n,n}^S$ for any $x\in\pi_n^S$ such that $\psi(b(x))=x^2$ and $\phi(b(x))=x$. We want to regard this construction as a kind of external squaring operation.

A τ-subspace $B_{p,q}$ of $\boldsymbol{R}^{p,q}$ is defined by

$$B_{p,q}=\{(u_1,\cdots,u_p,v_1,\cdots,v_q)\in\boldsymbol{R}^{p,q}\mid |u_i|\leqslant 1,\ |v_j|\leqslant 1\}.$$

Through a standard τ-homeomorphism $(B^{p,q},S^{p,q})\approx_\tau(B_{p,q},\partial B_{p,q})$ we identify $\Sigma^{p,q}=B_{p,q}/\partial B_{p,q}$. A subspace B_n of $\boldsymbol{R}^n$ is defined by

$$B_n=\{(t_1,\cdots,t_n)\in\boldsymbol{R}^n\mid |t_i|\leqslant 1\}$$

and we identify $\Sigma^n=B_n/\partial B_n$. We regard $\Sigma^n\wedge\Sigma^n$ as a τ-space with the involution τ defined by $\tau(x\wedge y)=y\wedge x$ for $x,y\in\Sigma^n$, and a space

$$\tilde{B}_{2n}=\{(u_1,\cdots,u_n,v_1,\cdots,v_n)\in\boldsymbol{R}^{2n}\mid |u_i-v_i|\leqslant 2,\ |u_i+v_i|\leqslant 2\}$$

as a τ-space with the involution τ defined by $\tau(u,v)=(v,u)$ for $(u,v)\in\tilde{B}_{2n}$. Define a τ-homeomorphism

$$\tilde{\alpha}_n\colon B_{n,n}/\partial B_{n,n}\to\tilde{B}_{2n}/\partial\tilde{B}_{2n}$$

by $\tilde{\alpha}_n(u,v)=(u+v,-u+v)$ for $(u,v)\in B_{n,n}$. Using a function $q\colon\boldsymbol{R}\to[-1,1]$ defined by $q(x)=1$ for $x\geqslant 1$, $=x$ for $|x|\leqslant 1$, $=-1$ for $x\leqslant -1$, we define a τ-homotopy equivalence $p_n\colon\tilde{B}_{2n}/\partial\tilde{B}_{2n}\to B_{2n}/\partial B_{2n}=\Sigma^n\wedge\Sigma^n$ by $p_n(u_1,\cdots,u_n,v_1,\cdots,v_n)=(q(u_1),\cdots,q(u_n),q(v_1),\cdots,q(v_n))$ for $(u_1,\cdots,u_n,v_1,\cdots,v_n)\in\tilde{B}_{2n}$. Thus we obtain a τ-homotopy equivalence

$$\alpha_n=p_n\circ\tilde{\alpha}_n\colon\Sigma^{n,n}\to\Sigma^n\wedge\Sigma^n.$$

Let $\beta_n\colon \Sigma^n\wedge\Sigma^n\to\Sigma^{n,n}$ be a τ-homotopy inverse to α_n, and define a map

$$sq\colon [\Sigma^{n+k}, \Sigma^k] \to [\Sigma^{n+k,n+k}, \Sigma^{k,k}]^\tau$$

by $sq([f])=\rho^{nk}[\beta_k\circ(f\wedge f)\circ\alpha_{n+k}]^\tau$ for $[f]\in[\Sigma^{n+k}, \Sigma^k]$. Here we show the formula

$$sq([\Sigma f]) = \tilde{\varepsilon}_{1,1}(sq([f]))\,. \tag{4.1}$$

Proof. Let $T_n\colon \Sigma^{1,1}\Sigma^n\wedge\Sigma^n\to\Sigma^{n+1}\wedge\Sigma^{n+1}$ be the composition of the following sequence:

$$\Sigma^{1,1}\Sigma^n\wedge\Sigma^n \xrightarrow{\alpha_1\wedge 1} \Sigma^1\wedge\Sigma^1\wedge\Sigma^n\wedge\Sigma^n \xrightarrow{T'} \Sigma^{n+1}\wedge\Sigma^{n+1}$$

where T' is the τ-homeomorphism defined by $T'(u_1\wedge u_2\wedge v_1\wedge v_2)=u_1\wedge v_1\wedge u_2\wedge v_2$ for $u_1, u_2\in\Sigma^1$ and $v_1, v_2\in\Sigma^n$. Then the following diagram is commutative:

$$\begin{tikzcd}
\Sigma^{1,1}\Sigma^{n+k,n+k} \arrow[r, "1\wedge\alpha_{n+k}"] \arrow[d, "T"] & \Sigma^{1,1}\Sigma^{n+k}\wedge\Sigma^{n+k} \arrow[r, "\Sigma^{1,1}f\wedge f"] \arrow[d, "T_{n+k}"] & \Sigma^{1,1}\Sigma^k\wedge\Sigma^k \arrow[d, "T_k"] \\
\Sigma^{n+k+1,n+k+1} \arrow[r, "\alpha_{n+k+1}"] & \Sigma^{n+k+1}\wedge\Sigma^{n+k+1} \arrow[r, "\Sigma f\wedge\Sigma f"] & \Sigma^{k+1}\wedge\Sigma^{k+1}
\end{tikzcd},$$

which implies (4.1). □

By (4.1) we obtain a map

$$Sq;\ \pi_n^S \to \pi_{n,n}^S$$

passing to the colimit of $sq\colon [\Sigma^{n+k}, \Sigma^k]\to[\Sigma^{n+k,n+k}, \Sigma^{k,k}]^\tau$.

Proposition 4.2. *For $x, y\in\pi_n^S$ there holds the relation*

$$Sq(x+y) = Sq(x)+Sq(y)+\delta_1\xi^{-1}(xy)\,,$$

where $\xi\colon \pi_S^{-n+1,-n-1}(S_+^{1,0})\approx\pi_{2n}^S$ is the canonical isomorphism. Moreover

$$\phi\circ Sq = id\colon \pi_n^S \to \pi_n^S$$

and $\psi\circ Sq(x)=x^2$ for $x\in\pi_n^S$.

Proof. Let $f\colon \Sigma^{n+k}\to\Sigma^k$ and $g\colon \Sigma^{n+k}\to\Sigma^k$ be maps representing x and y respectively. Let $\delta'\colon \Sigma^{n+k,n+k}\to\Sigma^{n+k-1,n+k+1}\wedge S_+^{1,0}$ be the τ-map defined by

$$\delta'(s_1, \cdots, s_{n+k}, t_1, \cdots, t_{n+k}) = \begin{cases}(s_1, \cdots, s_{n+k-1}, 2s_{n+k}+1, t_1, \cdots, t_{n+k}, -1), & -1\leqslant s_{n+k}\leqslant 0\,, \\ (s_1, \cdots, s_{n+k-1}, 1-2s_{n+k}, t_1, \cdots, t_{n+k}, +1), & 0\leqslant s_{n+k}\leqslant 1\,.\end{cases}$$

Let $F=(f\wedge g)\circ\delta'$, where we regard $f\wedge g\colon \Sigma^{2n+2k} \to \Sigma^{2k}$ as an element of $[\Sigma^{n+k-1,n+k+1}\wedge S_+^{1,0}, \Sigma^{k,k}]^\tau$ through the canonical isomorphism $[\Sigma^{n+k-1,n+k+1}\wedge S_+^{1,0},$

$\Sigma^{k,k}]^\tau \approx [\Sigma^{2n+2k}, \Sigma^{2k}]$ given by restriction to $\Sigma^{n+k-1,n+k+1}\times\{-1\}$. Since $\rho=-1$ on $\pi_{\xi}^{*,*}(S_+^{1,0})$ and $f\wedge g$ is a representative of $(-1)^{nk}\xi^{-1}(xy)$, we see that $[F]^\tau$ is a representative of $\rho^{nk}\delta_1\xi^{-1}(xy)$ by means of [2], (12.2). Hence it remains only to prove the formula

$$\begin{aligned}(4.3)\qquad &[(f+g)\wedge(f+g)\circ\alpha_{n+k}]^\tau\\ &=[(f\wedge f)\circ\alpha_{n+k}]^\tau+[(g\wedge g)\circ\alpha_{n+k}]^\tau+[\alpha_k\circ F]^\tau .\end{aligned}$$

Since $(f+g)$ is given by

$$(f+g)\,(s,u)=\begin{cases} f(4s+2,u), & -3/4\leqslant s\leqslant -1/4,\\ g(4s-2,u), & 1/4\leqslant s\leqslant 3/4,\\ *, & \text{otherwise,}\end{cases}$$

for $(s,u)=(s,u_1,\cdots,u_{n+k-1})\in\Sigma^{n+k}$, $(f+g)\wedge(f+g)$ is given by

$$\begin{aligned}&(f+g)\wedge(f+g)\,(s,u,t,v)\\ &=\begin{cases} f(4s+2,u)\wedge g(4t+2,v), & -3/4\leqslant s\leqslant -1/4,\ -3/4\leqslant t\leqslant -1/4,\\ f(4s+2,u)\wedge g(4t-2,v), & -3/4\leqslant s\leqslant -1/4,\ \ 1/4\leqslant t\leqslant 3/4,\\ f(4s-2,u)\wedge g(4t+2,v), & 1/4\leqslant s\leqslant 3/4,\ -3/4\leqslant t\leqslant -1/4,\\ f(4s-2,u)\wedge g(4t-2,v), & 1/4\leqslant s\leqslant 3/4,\ \ 1/4\leqslant t\leqslant 3/4,\\ *, & \text{otherwise.}\end{cases}\end{aligned}$$

For $-1\leqslant a<b\leqslant 1$ and $-1\leqslant c<d\leqslant 1$, we put $K(a,b,c,d)=\{(s_1,\cdots,s_{n+k},t_1,\cdots,t_{n+k})\in B_{n+k,n+k}\,|\,a\leqslant s_1+t_1\leqslant b,\ c\leqslant -s_1+t_1\leqslant d,\ |s_i+t_i|\leqslant 1,\ |-s_i+t_i|\leqslant 1,\ 2\leqslant i\leqslant n+k\}\subset B_{n+k,n+k}$. Then,

$$\begin{aligned}&(f+g)\wedge(f+g)\circ\alpha_{n+k}(s,u,t,v)\\ &=\begin{cases} f(4(s+t)+2,u+v)\wedge f(4(-s+t)+2,-u+v) & \text{on } K(-3/4,-1/4,-3/4,-1/4),\\ f(4(s+t)+2,u+v)\wedge g(4(-s+t)-2,-u+v) & \text{on } K(-3/4,-1/4,\ 1/4,\ 3/4),\\ g(4(s+t)-2,u+v)\wedge f(4(-s+t)+2,-u+v) & \text{on } K(\ 1/4,\ 3/4,-3/4,-1/4),\\ g(4(s+t)-2,u+v)\wedge g(4(-s+t)-2,-u+v) & \text{on } K(\ 1/4,\ 3/4,\ 1/4,\ 3/4),\\ *, & \text{otherwise.}\end{cases}\end{aligned}$$

It represents $[f\wedge f\circ\alpha_{n+k}]^\tau$ on $K(-3/4,-1/4,-3/4,-1/4)$ and $[g\wedge g\circ\alpha_{n+k}]^\tau$ on $K(1/4,3/4,1/4,3/4)$ respectively. Hence if we define the map

$$F'\colon \Sigma^{n+k}\wedge\Sigma^{n+k}\to\Sigma^k\wedge\Sigma^k$$

by

$$F'((s,u),(t,v))=\begin{cases} f(2s+1,u)\wedge g(2t-1,v), & -1\leqslant s\leqslant 0,\quad 0\leqslant t\leqslant 1,\\ g(2s-1,u)\wedge f(2t+1,v), & 0\leqslant s\leqslant 1,\ -1\leqslant t\leqslant 0,\\ *, & \text{otherwise,}\end{cases}$$

(4.3) is induced from the τ-homotopy commutativity of the following diagram:

$$(4.4)\qquad \begin{tikzcd} \Sigma^{n+k,n+k} \arrow[r, "\alpha_{n+k}"] \arrow[d, "F"] & \Sigma^{n+k}\wedge\Sigma^{n+k} \arrow[d, "F'"] \\ \Sigma^{k,k} \arrow[r, "\alpha_k"] & \Sigma^k\wedge\Sigma^k . \end{tikzcd}$$

Let $\Sigma^1\Sigma^{n+k-1}\wedge\Sigma^1\Sigma^{n+k-1}\wedge S_+^{1,0}$ be a τ-space with the involution τ defined by $\tau(s, u, t, v, \varepsilon)=(s, v, t, u, -\varepsilon)$, and condider the τ-map

$$\delta'' : \Sigma^{n+k}\wedge\Sigma^{n+k} \to \Sigma^1\Sigma^{n+k-1}\wedge\Sigma^1\Sigma^{n+k-1}\wedge S_+^{1,0}$$

defined by

$$\delta''(s, u, t, v) = \begin{cases} (2s+1, u, 2t-1, v, -1), & -1\leqslant s\leqslant 0, \quad 0\leqslant t\leqslant 1, \\ (2t+1, u, 2s-1, v, +1), & 0\leqslant s\leqslant 1, \quad -1\leqslant t\leqslant 0, \\ *, & \text{otherwise.} \end{cases}$$

Since $F'=(f\wedge g)\circ\delta''$, the τ-homotopy commutativity of (4.4) is induced from that of the following diagram:

$$(4.5)\qquad \begin{tikzcd} \Sigma^{n+k,n+k} \arrow[r, "\alpha_{n+k}"] \arrow[d, "\delta'"] & \Sigma^{n+k}\wedge\Sigma^{n+k} \arrow[d, "\delta''"] \\ \Sigma^{n+k-1,n+k+1}S_+^{1,0} \arrow[r, "\alpha"] & \Sigma^1\Sigma^{n+k-1}\wedge\Sigma^1\Sigma^{n+k-1}\wedge S_+^{1,0} , \end{tikzcd}$$

where α is the composition of the following τ-homotopy equivalences

$$\begin{aligned} \Sigma^{n+k-1,n+k+1}S_+^{1,0} &\approx_\tau \Sigma^{n+k-1,n+k-1}\Sigma^{0,2}S_+^{1,0} \\ &\simeq_\tau (\Sigma^{n+k-1}\wedge\Sigma^{n+k-1})\wedge\Sigma^{0,2}\wedge S_+^{1,0} \\ &\approx_\tau \Sigma^1\Sigma^{n+k-1}\wedge\Sigma^1\Sigma^{n+k-1}\wedge S_+^{1,0} . \end{aligned}$$

Now the τ-homotopy commutativity of (4.5) comes from the following obviously τ-homotopy commutative diagram

$$\begin{tikzcd} \Sigma^{1,1} \arrow[r, "\alpha_1"] \arrow[d, "\delta'"] & \Sigma^1\wedge\Sigma^1 \arrow[d, "\delta''"] \\ \Sigma^{0,2}S_+^{1,0} \arrow[r, phantom, "="] & \Sigma^2 S_+^{1,0} . \end{tikzcd}$$

Thus the main formula is proved.

The relations between Sq and ϕ or ψ are obvious. $\square$

Corollary 4.6. *$\chi\circ Sq$: $\pi_n^S\to\pi_{n-1,n}^S$ is a homomorhism for any $n\in\boldsymbol{Z}$.*

Corollary 4.7. *For $x \in \pi_n^S$ and an integer $m \geqslant 1$, there holds*

$$Sq(mx) = mSq(x) + m(m-1)/2 \cdot \delta_1 \xi^{-1}(x^2) .$$

Proposition 4.8. *We have the direct sum decomposition*

$$\pi_{p,q}^S \approx \lambda_{p,q}^S \oplus \pi_q^S \quad \textit{for } p<q \textit{ or } q<-1 .$$

Proof. Consider the fixed-point exact sequence (1.8). In case $p<q$ the homomorphism $\chi^{q-p} \circ Sq \colon \pi_q^S \to \pi_{p,q}^S$ gives the desired splitting. In case $q<-1$, since $\pi_{q+1}^S \approx \pi_q^S = 0$, we have $\pi_{p,q}^S \approx \lambda_{p,q}^S \oplus \pi_q^S$. $\square$

The multiplicative formula of external squaring operation is also easily obtained by routine arguments.

Proposition 4.9. *Let $x \in \pi_p^S$ and $y \in \pi_q^S$, then*

$$Sq(xy) = Sq(x) \cdot Sq(y) .$$

5. Homomorphism θ

Since

$$\begin{aligned} \pi_n^S &= \operatorname*{colim}_k \, [\Sigma^{n+k}, \Sigma^k] \\ &= \operatorname*{colim}_k \, \{[\Sigma^{0,n+k}, \Sigma^{0,k}]^T, \tilde{\varepsilon}_{0,s}\} \end{aligned}$$

and $\{[\Sigma^{0,n+k}, \Sigma^{0,k}]^T, \tilde{\varepsilon}_{0,s}\}$ is a direct subsystem of $\{[\Sigma^{l,n+k}, \Sigma^{l,k}]^T, \tilde{\varepsilon}_{r,s}\}$, inclusions

$$[\Sigma^{0,n+k}, \Sigma^{0,k}]^T \xrightarrow{\Sigma_*^{l,0}} [\Sigma^{l,n+k}, \Sigma^{l,k}]^T$$

induce a map

$$\theta \colon \pi_n^S \to \pi_{0,n}^S \tag{5.1}$$

passing to the colimit. Obviously we obtain

Proposition 5.2. *$\theta \colon \pi_n^S \to \pi_{0,n}^S$ is a ring homomorphism and satisfies the relation*

$$\psi \circ \theta = \phi \circ \theta = id \colon \pi_n^S \to \pi_n^S$$

for any integer.

By this Proposition, we see that a part of the forgetful exact sequence (1.5) splits to the following:

$$0 \to \pi_{1,n}^S \xrightarrow{\chi} \pi_{0,n}^S \underset{\theta}{\overset{\psi}{\rightleftarrows}} \pi_n^S \to 0 \ ,$$

and we obtain

Proposition 5.3. *For any $n\in \boldsymbol{Z}$ we have the direct sum decomposition*

$$\pi_{0,n}^S \approx \pi_{1,n}^S \oplus \pi_n^S ,$$

and ρ acts as 1 on $\pi_{1,n}^S$.

We identify $x\in\pi_n^S$ with $\theta(x)\in\pi_{0,n}^S$ and regard π_n^S as the direct summand of $\pi_{0,n}^S$ so that $\pi_S^{*,*}(X)$ becomes a π_*^S-module.

Next we consider the short exact sequence

$$0 \to \lambda_{0,n}^S \to \pi_{0,n}^S \xrightarrow{\phi} \pi_n^S \to 0 \tag{5.4}$$

for $n>0$. In this short exact sequence, ϕ has two left inverses, i.e. θ and $\chi^n\circ Sq$. Since $\psi\circ\theta=id$ and $\psi\circ\chi^n\circ Sq=0$, $\pi_{0,n}^S$ contains $\pi_n^S\oplus\pi_n^S$ as a direct summand (the one is the θ-image and the other is the $\chi^n\circ Sq$-image). Moreover, since $\phi(\theta-\chi^n\circ Sq)=0$, we get a homomorphism

$$\tilde{\theta} = \theta-\chi^n\circ Sq\colon \pi_n^S \to \lambda_{0,n}^S$$

by restricting the image, and there holds

$$\tilde{\psi}\circ\tilde{\theta} = id\colon \pi_n^S \to \pi_n^S ,$$

where $\tilde{\psi}=\psi\,|\,\lambda_{0,n}^S\colon \lambda_{0,n}^S\to\pi_n^S$. Landweber [10] showed that $\lambda_{0,n}^S$ is isomorphic to $\pi_n^S(P_+^\infty)$. Thus we obtain the split epimorphism

$$\pi_n^S(P_+^\infty) \approx \lambda_{0,n}^S \underset{\tilde{\theta}}{\overset{\tilde{\psi}}{\rightleftarrows}} \pi_n^S \to 0 \tag{5.5}$$

for $n>0$. This is the Kahn-Priddy theorem [9] in case $\boldsymbol{Z}/2$. We owe this remark to H. Minami.

6. Equivariant Toda brackets

Let W, X, Y and Z be finite pointed τ-complexes. For fixed integers p and q, we put

$$\{X, Y\}^{p,q} = \operatorname*{colim}_{k,l} \{[\Sigma^{k-p,l-q}X, \Sigma^{k,l}Y]^\tau, \tilde{\varepsilon}_{r,s}\} \tag{6.1}$$

in the same way as (1.2). Obviously $\{X, Y\}^{p,q}$ is a Λ-module. For $[u]^\tau\in[\Sigma^{k,l}X, \Sigma^{p,q}Y]^\tau$ and $[v]^\tau\in[\Sigma^{m,n}Y, \Sigma^{r,s}Z]^\tau$ we consider

$$\tilde{\varepsilon}_{p,q}(v)\circ\tilde{\varepsilon}_{m,n}(u)\in[\Sigma^{k+m,l+n}X, \Sigma^{p+r,q+s}Z]^\tau . \tag{6.2}$$

As is easily seen

$$\tilde{\varepsilon}_{a,b}(\tilde{\varepsilon}_{p,q}(v)\circ\tilde{\varepsilon}_{m,n}(u)) = \tilde{\varepsilon}_{a+p,b+q}(v)\circ\tilde{\varepsilon}_{a+m,b+n}(u) .$$

Thus, passing to the colimit, we get the composition

(6.3) $$\{Y, Z\}^{r,s}\times\{X, Y\}^{p,q}\to\{X, Z\}^{p+r,q+s}.$$

For $x\in\{Y, Z\}^{r,s}$ and $y\in\{X, Y\}^{p,q}$ we denote their composition by $x\circ y$. Obviously the composition (6.3) is a bilinear Λ-module homomorphism. Moreover for $x\in\pi_S^{r,s}$ and $y\in\pi_S^{p,q}(X)$ we have

(6.4) $$x\circ y=y\wedge x\in\pi_S^{p+r,q+s}(X).$$

Now we discuss Toda brackets in τ-homotopy theory, which is defined in the same way as [14], Chapter I, and the propositions parallel to it are valid.

Suppose the 3-elements $[a]^\tau\in[\Sigma^{k,l}Y, \Sigma^{m,n}Z]^\tau$, $[b]^\tau\in[\Sigma^{r,s}X, \Sigma^{k,l}Y]^\tau$ and $[c]^\tau\in[\Sigma^{p,q}W, \Sigma^{r,s}X]^\tau$ satisfy the relations $[a]^\tau\circ[b]^\tau=[b]^\tau\circ[c]^\tau=0$. Let H: $\Sigma^{0,1}\Sigma^{p,q}W\to\Sigma^{m,n}Z$ be a τ-map belonging to the Toda bracket $\{[a]^\tau, [b]^\tau, [c]^\tau\}$. We define $\bar{H}$: $\Sigma^{p,q+1}W\to\Sigma^{m,n}Z$ by $\bar{H}=H\circ(T'\wedge 1)$, where T': $\Sigma^{p,q+1}=\Sigma^{p,q}\Sigma^{0,1}\to\Sigma^{0,1}\Sigma^{p,q}$ is the switching map, and define the *equivariant Toda bracket*

$$\{[a]^\tau, [b]^\tau, [c]^\tau\}^\tau\in[\Sigma^{p,q+1}W, \Sigma^{m,n}Z]^\tau$$

as the set of all the τ-homotopy classes of mapping given as above.

Obviously we have

Lemma 6.5. *Let* $X\xrightarrow{f}Y\xrightarrow{i}Y\cup_fCX\xrightarrow{j}\Sigma^{0,1}X$ *be a τ-cofibration sequence. For each p and q, we can construct the following τ-homotopy commutative diagram*:

$$\begin{tikzcd}
\Sigma^{p,q}Y \arrow[r, "\Sigma^{p,q}i"] \arrow[d, equal] & \Sigma^{p,q}(Y\cup_fCX) \arrow[r, "\Sigma^{p,q}j"] \arrow[d] & \Sigma^{p,q+1}X \arrow[d, "T'\wedge 1"] \\
\Sigma^{p,q}Y \arrow[r] & \Sigma^{p,q}Y\cup_{\Sigma^{p,q}f}C\Sigma^{p,q}X \arrow[r] & \Sigma^{0,1}\Sigma^{p,q}X,
\end{tikzcd}$$

where T': $\Sigma^{p,q+1}=\Sigma^{p,q}\Sigma^{0,1}\to\Sigma^{0,1}\Sigma^{p,q}$ *is the switching map.*

Because of this Lemma, we obtain easily the following

Lemma 6.6. *For each non-negative integers p and q, we have*

$$\tilde{\varepsilon}_{p,q}\{[a]^\tau, [b]^\tau, [c]^\tau\}^\tau\subset\{\tilde{\varepsilon}_{p,q}([a]^\tau), \tilde{\varepsilon}_{p,q}([b]^\tau), \tilde{\varepsilon}_{p,q}([c]^\tau)\}^\tau.$$

Next consider $\alpha\in\{Y, Z\}^{p,q}$, $\beta\in\{X, Y\}^{k,l}$ and $\gamma\in\{W, X\}^{m,n}$ satisfying $\alpha\circ\beta=\beta\circ\gamma=0$. Let a: $\Sigma^{r-p,s-q}Y\to\Sigma^{r,s}Z$, b: $\Sigma^{r-p-k,s-q-l}X\to\Sigma^{r-p,s-q}Y$ and c: $\Sigma^{r-p-k-m,s-q-l-n}W\to\Sigma^{r-p-k,s-q-l}X$ be representatives of α, β and γ respectively, and assume that $a\circ b$ and $b\circ c$ are τ-homotopic to zero. In virtue of Lemma 6.6, $\{\{[a]^\tau, [b]^\tau, [c]^\tau\}^\tau, \tilde{\varepsilon}_{m,n}\}$ forms a direct system. Put

$$\langle\alpha, \beta, \gamma\rangle^\tau=\operatorname{colim}\ \{[a]^\tau, [b]^\tau, [c]^\tau\}^\tau\subset\{W, Z\}^{p+k+m,q+l+n-1}$$

and call it the *stable equivariant Toda bracket* of α, β and γ. Then $\langle\alpha, \beta, \gamma\rangle^\tau$ is defined as a coset module $\{X, Z\}^{p+k,q+l-1}\circ\gamma+\alpha\circ\{W, Y\}^{k+m,l+n-1}$.

By parallel arguments to [14] we obtain

Proposition 6.7.

i) $\langle\alpha, \beta, \gamma\rangle^{\tau}\equiv 0$ *if one of* α, β *and* γ *is* 0.

ii) $\langle\alpha, \beta, \gamma\rangle^{\tau}\circ\delta\subset\langle\alpha, \beta, \gamma\circ\delta\rangle^{\tau}$.

iii) $\langle\alpha, \beta, \gamma\circ\delta\rangle^{\tau}\subset\langle\alpha, \beta\circ\gamma, \delta\rangle^{\tau}$.

iv) $\langle\alpha\circ\beta, \gamma, \delta\rangle^{\tau}\subset\langle\alpha, \beta\circ\gamma, \delta\rangle^{\tau}$.

v) $\alpha\circ\langle\beta, \gamma, \delta\rangle^{\tau}\subset\rho^{p}(-1)^{q}\langle\alpha\circ\beta, \gamma, \delta\rangle^{\tau}$, $\alpha\in\{Y, Z\}^{p,q}$.

vi) $\langle\alpha, \beta, \gamma\rangle^{\tau}\circ\delta = \rho^{p}(-1)^{q+1}\alpha\circ\langle\beta, \gamma, \delta\rangle^{\tau}$, $\alpha\in\{Y, Z\}^{p,q}$.

vii) $\langle\alpha+\alpha', \beta, \gamma\rangle^{\tau}\subset\langle\alpha, \beta, \gamma\rangle^{\tau}+\langle\alpha', \beta, \gamma\rangle^{\tau}$.

viii) $\langle\alpha, \beta+\beta', \gamma\rangle^{\tau} = \langle\alpha, \beta, \gamma\rangle^{\tau}+\langle\alpha, \beta', \gamma\rangle^{\tau}$.

ix) $\langle\alpha, \beta, \gamma+\gamma'\rangle^{\tau}\subset\langle\alpha, \beta, \gamma\rangle^{\tau}+\langle\alpha, \beta, \gamma'\rangle^{\tau}$.

x) $\sigma^{p,q}\langle\alpha, \beta, \gamma\rangle^{\tau} = \rho^{p}(-1)^{q}\langle\alpha, \beta, \sigma^{p,q}\gamma\rangle^{\tau}$.

xi) $\psi(\langle\alpha, \beta, \gamma\rangle^{\tau})\subset\langle\psi(\alpha), \psi(\beta), \psi(\gamma)\rangle$.

xii) $\phi(\langle\alpha, \beta, \gamma\rangle^{\tau})\subset\langle\phi(\alpha), \phi(\beta), \phi(\gamma)\rangle$.

Proposition 6.8. *Let* $F \xrightarrow{i} E \xrightarrow{j} B \xrightarrow{\delta} \Sigma^{0,1}F$ *be a* τ*-cofibration sequence.*

i) *Assume that* $\alpha\in\{Y, Z\}^{*,*}$, $\beta\in\{X, Y\}^{*,*}$ *and* $\gamma\in\{B, X\}^{*,*}$ *satisy* $\alpha\circ\beta=\beta\circ j^{*}\gamma=0$. *Then*

$$i^{*}\langle\alpha, \beta, j^{*}\gamma\rangle^{\tau} = \alpha\circ\delta^{*-1}(\beta\circ\gamma)\,.$$

ii) *Assume that* $\alpha\in\{Y, Z\}^{*,*}$, $\beta\in\{X, Y\}^{*,*}$ *and* $\gamma\in\{E, X\}^{*,*}$ *satisfy* $\alpha\circ\beta=\beta\circ i^{*}\gamma=0$. *Then*

$$\delta^{*}\langle\alpha, \beta, i^{*}\gamma\rangle^{\tau} = -\alpha\circ j^{*-1}(\beta\circ\gamma)\,.$$

iii) *Assume that* $\alpha\in\{Y, Z\}^{*,*}$, $\beta\in\{X, Y\}^{*,*}$ *and* $\gamma\in\{F, X\}^{*,*}$ *satisfy* $\alpha\circ\beta=\beta\circ\delta^{*}\gamma=0$. *Then*

$$j^{*}\langle\alpha, \beta, \delta^{*}\gamma\rangle^{\tau} = -\alpha\circ i^{*-1}(\beta\circ\gamma)\,.$$

Proof. Case i). Let $[a]^{\tau}$, $[b]^{\tau}$ and $[c]^{\tau}$ be the representatives of α, β and γ respectively which satisfy $a\circ b\simeq_{\tau}0$ and $j^{*}(b\circ c)\simeq_{\tau}0$. And let g be an element of $\delta^{*-1}(b\circ c)$. Then we obtain the following τ-homotopy commutative diagram:

(6.9)
$$\begin{tikzcd}
\Sigma^{*,*}B \arrow[r, "1\wedge\delta"] \arrow[d, equal] & \Sigma^{*,1+*}F \arrow[r, "-1\wedge i"] \arrow[d, "T^{-1}\wedge 1"] & \Sigma^{*,1+*}E \arrow[r, "-1\wedge j"] \arrow[d, "T^{-1}\wedge 1"] & \Sigma^{*,1+*}B \arrow[d, "T^{-1}\wedge 1"'] \\
\Sigma^{*,*}B \arrow[r, "\delta"] \arrow[d, "c"'] & \Sigma^{0,1}\Sigma^{*,*}F \arrow[r, "-1\wedge 1\wedge i"] \arrow[d, dashed, "g'"] & \Sigma^{0,1}\Sigma^{*,*}E \arrow[r, "-1\wedge 1\wedge j"] \arrow[d, dashed, "f"] & \Sigma^{0,1}\Sigma^{*,*}B \arrow[d, "1\wedge c"'] \\
\Sigma^{*,*}X \arrow[r, "b"] & \Sigma^{*,*}Y \arrow[r] \arrow[d, equal] & C(b) \arrow[r, "\delta'"] \arrow[d, dashed, "f'"] & \Sigma^{0,1}\Sigma^{*,*}X \\
 & \Sigma^{*,*}Y \arrow[r, "a"] & \Sigma^{*,*}Z &
\end{tikzcd}$$

where $g=g'\circ(T^{-1}\wedge 1)$. Since

$$\begin{aligned} a\circ g &= f'\circ f\circ(T^{-1}\wedge 1)\circ(-1\wedge i) \\ &= f'\circ(-f)\circ(T^{-1}\wedge 1)\circ(1\wedge i)\,, \\ (-f)\circ\delta' &= (1\wedge c)\circ(1\wedge 1\wedge j)\,, \end{aligned}$$

$f'\circ(-f)\circ(T^{-1}\wedge 1)$ is a representative of $\langle\alpha, \beta, j^*\gamma\rangle^\tau$, and $f'\circ(-f)\circ(T^{-1}\wedge 1)\circ(1\wedge i)$ is a representative of $i^*\langle\alpha, \beta, j^*\gamma\rangle^\tau$. Hence we have

$$\alpha\circ\delta^{*-1}(\beta\circ\gamma)\subset i^*\langle\alpha, \beta, j^*\gamma\rangle^\tau .$$

Since we are working in the stable range, for a given $f'\circ(-f)\circ(T^{-1}\wedge 1)$, a representative of $\langle\alpha, \beta, j^*\gamma\rangle^\tau$, we may suppose that there exists g and the diagram (6.9) is commutative up to τ-homotopy. Hence we obtain

$$\alpha\circ\delta^{*-1}(\beta\circ\gamma)\supset i^*\langle\alpha, \beta, j^*\gamma\rangle^\tau .$$

Thus the relation i) is proved.

The proofs of cases ii) and iii) are parallel to the case i). □

7. Negative and 0 stems

Since the τ-spectrum $\boldsymbol{SR}$ is $(-1, -1)$-connected, from [2], Proposition 5.4, it follows

Proposition 7.0. *The fixed-point homomorphism gives the isomorphism*

$$\phi\colon \pi^S_{p,q} \approx \pi^S_q \qquad \textit{for } p+q<0 .$$

Thus the computation of the stable τ-homotopy groups of negative stems is completely reduced to the computation of the ordinary stable homotopy groups of spheres.

We identify π^S_q with $\pi_S^{0,-q}(S^{1,0}_+)$ by the isomorphism $\beta_1\circ\theta\colon \pi^S_q\to\pi_S^{0,-q}(S^{1,0}_+)$. Then we obtain

Proposition 7.1.

$$\bigoplus_{p,q}\pi_S^{p,q}(S^{1,0}_+) = \bigoplus_r \pi^S_r\otimes \boldsymbol{Z}[\omega_1, \omega_1^{-1}] .$$

And ρ acts as -1 on $\pi_S^{p,q}(S^{1,0}_+)$.

The isomorphism ξ in Proposition 4.2 is given by $\xi^{-1}(z)=z\omega_1^{1-n}$ for $z\in\pi^S_{2n}$. Thus Proposition 4.2 implies

Corollary 7.2. *For $x, y\in\pi^S_n$ there holds the relation*

$$Sq(x+y) = Sq(x)+Sq(y)+\delta_1(xy\omega_1^{1-n}) .$$

In particular, for $x \in \pi_n^S$ *and an integer* $m \geqslant 1$ *there holds the relation*

$$mSq(x) = Sq(mx) - m(m-1)/2 \cdot \delta_1(x^2 \omega_1^{1-n}) .$$

In the exact sequence (1.9) for $r=1$, we have $\delta_{1,1}\omega_1^{2n} = \delta_{1,1}\eta_{1,2}^* \omega_2^n = 0$ and $\delta_{1,1}\omega_1^{2n+1} = \delta_{1,1}(\omega_1 \eta_{1,2}^* \omega_2^n) = 2\omega_1^{2n}$ by Propositions 2.1, 2.5 and (3.9). Hence, for any $\alpha \in \pi_*^S$, we have

$$\delta_{1,1}(\alpha\omega_1^{2n}) = 0 \text{ and } \delta_{1,1}(\alpha\omega_1^{2n+1}) = 2\alpha\omega_1^{2n} . \tag{7.3}$$

Thus we get a short exact sequence

$$\begin{aligned} 0 \to \pi_r^S \otimes \boldsymbol{Z}[\omega_1^2, \omega_1^{-2}]\omega_1 \oplus \pi_r^S \otimes \boldsymbol{Z}/2[\omega_1^2, \omega_1^{-2}] &\xrightarrow{\xi_{2,1}^*} \\ \bigoplus_{p+q=r} \pi_S^{2-p,-q-1}(S_+^{2,0}) \xrightarrow{\eta_{1,2}^*} \mathrm{Tor}(\pi_{r-1}^S, \boldsymbol{Z}/2) &\otimes \boldsymbol{Z}[\omega_1^2, \omega_1^{-2}]\omega_1 \\ &\oplus \pi_{r-1}^S \otimes \boldsymbol{Z}[\omega_1^2, \omega_1^{-2}] \to 0 . \end{aligned} \tag{7.4}$$

for any integei r.

Since $\xi_{2,1}^* \omega_1^{2n} = \xi_{2,1}^* \eta_{1,2}^* \omega_2^n = \chi\omega_2^n$ by Proposition 2.2 and (3.9), (7.4) implies

Proposition 7.5. $(\pi_S^{2-p,-q-1}(S_+^{2,0}), p+q=0)$

i) $\pi_S^{2n,-2n+1}(S_+^{2,0}) = \boldsymbol{Z} \cdot \xi_{2,1}^* \omega_1^{2n-1}$,

ii) $\pi_S^{2n+1,-2n}(S_+^{2,0}) = \boldsymbol{Z}/2 \cdot \chi\omega_2^n$

for any integer n.

Now compute $\pi_{p,q}^S$ for $p+q=0$. Since $\lambda_{p,q}^S \approx \pi_S^{2-p,-q-1}(S_+^{2,0})$ for $p+q=0$ by Corollary 1.11, Proposition 7.5 describes $\lambda_{p,q}^S$ for $p+q=0$. By Proposition 4.8 the groups $\pi_{p,q}^S$ for $p+q=0$ are determined except $\pi_{0,0}^S$ and $\pi_{1,-1}^S$.

By [2], Proposition 12.5, we know that $\pi_{0,0}^S = \boldsymbol{Z} \cdot 1 \oplus \boldsymbol{Z} \cdot \rho$ and $\delta_1\omega_1 = 1-\rho$. Since $\pi_{0,-1}^S \approx \pi_{-1}^S = 0$, Proposition 5.3 implies that $\pi_{1,-1}^S = 0$. Thus we obtain

Theorem 7.6. $(\pi_{p,q}^S, p+q=0)$

i) $\pi_{0,0}^S = \boldsymbol{Z} \cdot 1 \oplus \boldsymbol{Z} \cdot \rho = \Lambda$.

ii) $\pi_{1,-1}^S = 0$.

iii) $\pi_{2n,-2n}^S \approx \boldsymbol{Z} \cdot \delta_1 \omega_1^{1-2n} \oplus \pi_{-2n}^S$ *for any integer* n.

iv) $\pi_{2n+1,-2n-1}^S \approx \boldsymbol{Z}/2 \cdot \chi\delta_2\omega_2^{-n} \oplus \pi_{-2n-1}^S$ *for* $n \neq 0$.

v) *There holds the relation*

$$\delta_1 \omega_1 = 1 - \rho .$$

8. 1 stem

Regard the complex numbers $\boldsymbol{C}$ as a τ-space by conjugation and identify $\boldsymbol{C} = \boldsymbol{R}^{1,1}$ as usual. The product of the complex numbers $m_1 : \boldsymbol{C} \times \boldsymbol{C} \to \boldsymbol{C}$ gives rise to a τ-map m_1: $\boldsymbol{R}^{1,1}\times\boldsymbol{R}^{1,1}\to\boldsymbol{R}^{1,1}$. Restricting this τ-map to $S^{1,1}\times S^{1,1}$, we get a τ-map $\bar{m}_1$: $S^{1,1}\times S^{1,1}\to S^{1,1}$. By the Hopf construction on $\bar{m}_1$ we obtain a τ-map $H(\bar{m}_1)$: $S^{1,1}*S^{1,1}\to\Sigma^{0,1}S^{1,1}$. Since $S^{1,1}*S^{1,1}\approx_\tau S^{2,2}\approx_\tau\Sigma^{2,1}$ and $\Sigma^{0,1}S^{1,1}\approx_\tau \Sigma^{0,1}\Sigma^{1,0}\approx_\tau\Sigma^{1,1}$, we have a τ-map $\mathring{\eta}$: $\Sigma^{2,1}\to\Sigma^{1,1}$. The element of $\pi^S_{1,0}$ represented by $\mathring{\eta}$ is also denoted by the same letter. It is obvious that $\psi(\mathring{\eta})=\eta$, hence $\beta_1(\mathring{\eta})=\eta\omega_1^{-1}$. $\phi(\mathring{\eta})$ is represented by the Hopf construction on $\phi(\bar{m}_1)$: $S^{0,1}\times S^{0,1}\to S^{0,1}$, the restriction of $\bar{m}_1$ to the fixed points, which implies that $\phi(\mathring{\eta})=2$.

Next consider the short exact sequence (Proposition 5.3):

$$\begin{array}{c} 0\to\pi^S_{1,0}\xrightarrow{\chi}\pi^S_{0,0}\xrightarrow{\psi}\pi^S_0\to 0\,. \\ \| \\ \boldsymbol{Z}\cdot 1\oplus\boldsymbol{Z}\cdot\rho \end{array}$$

Since $\psi(1)=1$ and $\psi(\rho)=-1$, we see that χ: $\pi^S_{1,0}\to\boldsymbol{Z}\cdot(1+\rho)$ is an isomorphism. Since $\phi\circ\chi=\phi$: $\pi^S_{1,0}\to\pi^S_0$ and $\phi(1+\rho)=2$, the map ϕ: $\pi^S_{1,0}\to\pi^S_0$ is isomorphic. Thus we obtain

(8.1) $$\pi^S_{1,0}=\boldsymbol{Z}\cdot\mathring{\eta}\ \text{ and }\ \chi\mathring{\eta}=1+\rho\,.$$

Put

(8.2) $$\mathring{\eta}_{4n}=\delta_2\omega_2^{1-2n}\ \text{ and }\ \mathring{\eta}=\mathring{\eta}_0=\delta_2\omega_2\,.$$

Then $\beta_1(\mathring{\eta}_{4n})=\eta\omega_1^{-4n}$ and $\beta_1(\mathring{\eta})=\beta_1(\eta)=\eta$ by Propositions 2.1, 2.5, Theorem 3.5 and (3.9).

Proposition 8.3. $(\pi_S^{2-p,-q-1}(S^{2,0}_+),\ p+q=1)$

i) $\pi_S^{2n,-2n}(S^{2,0}_+)=\boldsymbol{Z}\cdot\omega_2^n\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot(1+\rho)\omega_2^n$,

ii) $\pi_S^{4n+1,-4n-1}(S^{2,0}_+)=\boldsymbol{Z}/\boldsymbol{2}\cdot\beta_2(\chi\mathring{\eta}_{-4n})$,

iii) $\pi_S^{4n+3,-4n-3}(S^{2,0}_+)=\boldsymbol{Z}/\boldsymbol{2}\cdot\chi\eta\omega_2^{1+2n}$

for any integer n.

Proof. Observe the short exact sequence (7.4) for $r=1$:

$$0\to\boldsymbol{Z}[\omega_1,\omega_1^{-1}]\otimes\boldsymbol{Z}/\boldsymbol{2}\cdot\eta\xrightarrow{\xi^*_{2,1}}\bigoplus_{p+q=1}\pi_S^{2-p,-q-1}(S^{2,0}_+)$$
$$\xrightarrow{\eta^*_{1,2}}\boldsymbol{Z}[\omega_1^2,\omega_1^{-2}]\to 0\,.$$

Since $\xi^*_{2,1}(\eta\omega_1^{2n-1})=\xi^*_{2,1}\eta^*_{1,2}(\mathring{\eta}\omega_2^n)=\chi\mathring{\eta}\omega_2^n=(1+\rho)\omega_2^n$ and $\eta^*_{1,2}\omega_2^n=\omega_1^{2n}$ by Proposition 2.1 and (3.9), we obtain the identity i). Since $\xi^*_{2,1}(\eta\omega_1^{4n})=\xi^*_{2,1}\beta_1(\mathring{\eta}_{-4n})=\beta_2(\chi\mathring{\eta}_{-4n})$ and $\xi^*_{2,1}(\eta\omega_1^{4n+2})=\xi^*_{2,1}\eta^*_{1,2}(\eta\omega_2^{1+2n})=\chi\eta\omega_2^{1+2n}$, we obtain the identities ii) and iii). □

Next, using the exact sequence (1.9) for $r=2$, we compute $\pi_S^{3-p,-q-1}(S^{3,0}_+)$ for $p+q=1$.

Proposition 8.4. i) $\delta_{2,1}\omega_1^{4n}=0$, ii) $\delta_{2,1}\omega_1^{4n+1}=2\omega_2^{2n}-(1+\rho)\omega_2^{2n}$, iii) $\delta_{2,1}\omega_1^{4n+2}=\beta_2(\chi\mathring{\eta}_{-4n})$ *and* iv) $\delta_{2,1}\omega_1^{4n+3}=2\omega_2^{2n+1}$ *for any integer n.*

Proof. i) $\delta_{2,1}\omega_1^{4n}=\delta_{2,1}\eta_{1,3}^*\omega_3^n=0$, ii) $\delta_{2,1}\omega_1^{4n+1}=\delta_{2,1}(\omega_1\eta_{1,3}^*\omega_3^n)=\delta_{2,1}(\omega_1)\omega_2^{2n}$ $=(1-\rho)\omega_2^{2n}=2\omega_2^{2n}-(1+\rho)\omega_2^{2n}$ since $\delta_{2,1}\omega_1=\beta_2\delta_1\omega_1=\beta_2(1-\rho)$, iii) $\delta_{2,1}\omega_1^{4n+2}=$ $\beta_2\delta_1\eta_{1,2}^*\omega_2^{2n+1}=\beta_2(\chi\delta_2\omega_2^{2n+1})=\beta_2(\chi\mathring{\eta}_{-4n})$. iv) Since $\eta_{1,2}^*\delta_{2,1}\omega_1^{4n+3}=\delta_{1,1}\omega_1^{4n+3}=2\omega_1^{4n+2}$ by (7.3) and $\eta_{1,2}^*\omega_2^{2n+1}=\omega_1^{4n+2}$, we see that $\delta_{2,1}\omega_1^{4n+3}=2\omega_2^{2n+1}+a\cdot\xi_{2,1}^*(\eta\omega_1^{4n+1})$ for some $a\in \boldsymbol{Z}/2$. On the other hand, $\delta_{2,1}(\eta\omega_1^{4n+3})=\delta_{2,1}\eta_{1,3}^*(\hat{\eta}\omega_3^{n+1})=0$ and $\delta_{2,1}(\eta\omega_1^{4n+3})$ $=\delta_{2,1}(\omega_1^{4n+3}\eta_{1,3}^*\beta_3(\eta))=(\delta_{2,1}\omega_1^{4n+3})\eta=a\cdot\xi_{2,1}^*(\eta^2\omega_1^{4n+1})$, i.e. $a\cdot\xi_{2,1}^*(\eta^2\omega_1^{4n+1})=0$. By (7.4) $\xi_{2,1}^*(\eta^2\omega_1^{4n+1})\neq 0$, hence $a=0$, which implies iv). $\square$

The above proposition implies that

$$
\begin{aligned}
(8.5)\qquad & \operatorname{Ker}\Big[\delta_{2,1}\colon \bigoplus_{p+q=2}\pi_S^{3-p,-q-1}(S_+^{1,0}) \to \bigoplus_{p+q=2}\pi_S^{2-p,-q}(S_+^{2,0})\Big] \\
&= \boldsymbol{Z}[\omega_1^4, \omega_1^{-4}]\otimes\{\boldsymbol{Z}\cdot 1\oplus \boldsymbol{Z}\cdot 2\omega_1^2\}\ ,
\end{aligned}
$$

and we have the isomorphism

$$
\xi_{3,2}^*\colon \bigoplus_{p+q=1}\pi_S^{2-p,-q-1}(S_+^{2,0})/\operatorname{Im}\delta_{2,1}\approx \bigoplus_{p+q=1}\pi_S^{3-p,-q-1}(S_+^{3,0})
$$

by Proposition 1.10. Since Proposition 8.4 gives $\operatorname{Im}\delta_{2,1}$, we obtain

Proposition 8.6. $(\pi_S^{3-p,-q-1}(S_+^{3,0}),\ p+q=1)$

i) $\pi_S^{4n+1,-4n}(S_+^{3,0})=\boldsymbol{Z}/4\cdot\chi\omega_3^n$,

ii) $\pi_S^{4n+2,-4n-1}(S_+^{3,0})=0$,

iii) $\pi_S^{4n+3,-4n-2}(S_+^{3,0})=\boldsymbol{Z}/2\cdot\xi_{3,2}^*\omega_2^{2n+1}\oplus\boldsymbol{Z}/2\cdot\rho\xi_{3,2}^*\omega_2^{2n+1}$,

iv) $\pi_S^{4n+4,-4n-3}(S_+^{3,0})=\boldsymbol{Z}/2\cdot\chi\eta\xi_{3,2}^*\omega_2^{2n+1}$

for any integer n.

By the fixed-point exact sequence (1.12) for $r=3$, we have $\pi_{2,-1}^S=0$.

Proposition 8.6 describes $\lambda_{p,q}^S$ for $p+q=1$. By Proposition 4.8 the groups $\pi_{p,q}^S$ for $p+q=1$ are determined except $\pi_{1,0}^S$ and $\pi_{2,-1}^S$. On the other hand we have already seen that $\pi_{1,0}^S=\boldsymbol{Z}\cdot\hat{\eta}$ and $\pi_{2,-1}^S=0$. Thus we obtain the following theorems.

Theorem 8.7. $(\pi_{p,q}^S,\ p+q=1)$

i) $\pi_{1,0}^S=\boldsymbol{Z}\cdot\hat{\eta}$.

ii) $\pi_{2,-1}^S=0$.

iii) $\pi_{4n,-4n+1}^S\approx\boldsymbol{Z}/2\cdot\mathring{\eta}_{4n}\oplus\boldsymbol{Z}/2\cdot\rho\mathring{\eta}_{4n}\oplus\pi_{-4n+1}^S$ *for any integer n.*

iv) $\pi_{4n+1,-4n}^S\approx\pi_{-4n}^S$ *for* $n\neq 0$.

v) $\pi_{4n+2,-4n-1}^S\approx\boldsymbol{Z}/4\cdot\chi\delta_3\omega_3^{-n}\oplus\pi_{-4n-1}^S$ *for* $n\neq 0$.

vi) $\pi_{4n+3,-4n-2}^S\approx\boldsymbol{Z}/2\cdot\chi\eta\mathring{\eta}_{4n+4}\oplus\pi_{-4n-2}^S$ *for any integer n.*

Theorem 8.8. *There hold the following relations*:

i) $(1+\rho)\mathring{\eta}=(1+\rho)\eta$ $(=\delta_1(\eta\omega_1))$ *in* $\pi^S_{0,1}$.

ii) $\beta_1(\hat{\eta})=\eta\omega_1^{-1}$, $\phi(\hat{\eta})=2$, $\rho\hat{\eta}=\hat{\eta}$ *and* $\chi\hat{\eta}=1+\rho$.

iii) $\chi^2\mathring{\eta}_{4n}=0$ *for any integer n.*

REMARK. $\chi\mathring{\eta}_{4n}$ is non-zero, which is the generator of the first direct summand of $\pi^S_{4n-1,-4n+1}$ (Theorem 7.6). But $\chi^2\mathring{\eta}_{4n}=\chi^2\delta_2\omega_2^{1-2n}=0$.

9. 2 stem

Computation of $\pi_S^{2-p,-q-1}(S_+^{2,0})$ for $p+q=2$. Observe the short exact sequence (7.4) for $r=2$. $\eta^*_{1,2}\beta_2(\mathring{\eta}_{-4n})=\eta\omega_1^{4n}$, $\beta_2(\mathring{\eta}_{-4n})$ is of order 2, $\eta^*_{1,2}(\eta\omega_2^{2n+1})=\eta\omega_1^{4n+2}$, $\eta\omega_1^{4n+2}$ is of order 2, $\eta^*_{1,2}(\hat{\eta}\omega_2^n)=\eta\omega_1^{2n-1}$ and $2\hat{\eta}\omega_2^n=\chi\hat{\eta}^2\omega_2^n=\xi^*_{2,1}(\hat{\eta}^2\omega_1^{2n})=\xi^*_{2,1}(\eta^2\omega_1^{2n-2})$. Thus we obtain

Proposition 9.1. $(\pi_S^{2-p,-q-1}(S_+^{2,0}), p+q=2)$

i) $\pi_S^{4n,-4n-1}(S_+^{2,0})=\boldsymbol{Z}/2\cdot\beta_2(\mathring{\eta}_{-4n})\oplus\boldsymbol{Z}/2\cdot\rho\beta_2(\mathring{\eta}_{-4n})$,

ii) $\pi_S^{4n+2,-4n-3}(S_+^{2,0})=\boldsymbol{Z}/2\cdot\eta\omega_2^{2n+1}\oplus\boldsymbol{Z}/2\cdot\rho\eta\omega_2^{2n+1}$,

iii) $\pi_S^{2n-1,-2n}(S_+^{2,0})=\boldsymbol{Z}/4\cdot\hat{\eta}\omega_2^n$

for any integer n.

Computation of $\pi_S^{3-p,-q-1}(S_+^{3,0})$ for $p+q=2$. $\delta_{2,1}(\eta\omega_1^{4n})=\delta_{2,1}\eta^*_{1,3}(\eta\omega_3^n)=0$, $\delta_{2,1}(\eta\omega_1^{4n+1})=\delta_{2,1}(\omega_1\eta^*_{1,3}\beta_3(\mathring{\eta}_{-4n}))=(1+\rho)\beta_2(\mathring{\eta}_{-4n})$ of order 2, $\delta_{2,1}(\eta\omega_1^{4n+2})=\delta_{2,1}(\omega_1^{4n+3}\eta^*_{1,3}\beta_3(\hat{\eta}))=2\hat{\eta}\omega_2^{2n+1}$ of order 2 by Proposition 8.4, iv), and $\delta_{2,1}(\eta\omega_1^{4n+3})=\delta_{2,1}\eta^*_{1,3}(\hat{\eta}\omega_3^{n+1})=0$. Thus we have

$$\text{(9.2)}\qquad \begin{aligned}&\operatorname{Ker}[\delta_{2,1}\colon \bigoplus_{p+q=3}\pi_S^{3-p,-q-1}(S_+^{1,0})\to\bigoplus_{p+q=3}\pi_S^{2-p,-q}(S_+^{2,0})]\\ &=\boldsymbol{Z}[\omega_1^4,\omega_1^{-4}]\otimes\{\boldsymbol{Z}/2\cdot\eta\oplus\boldsymbol{Z}/2\cdot\eta\omega_1^{-1}\}\,.\end{aligned}$$

The exact sequence (1.9) for $r=2$ and (8.5) give the short exact sequence

$$\begin{aligned}0\to\bigoplus_{p+q=2}\pi_S^{2-p,-q-1}(S_+^{2,0})/\operatorname{Im}\delta_{2,1}&\xrightarrow{\xi^*_{3,2}}\bigoplus_{p+q=2}\pi_S^{3-p,-q-1}(S_+^{3,0})\\ &\xrightarrow{\eta^*_{1,3}}\boldsymbol{Z}[\omega_1^4,\omega_1^{-4}]\otimes\{\boldsymbol{Z}\cdot1\oplus\boldsymbol{Z}\cdot2\omega_1^2\}\to0\,.\end{aligned}$$

Since we have computed

$$\operatorname{Im}[\delta_{2,1}\colon\bigoplus_{p+q=2}\pi_S^{3-p,-q-2}(S_+^{1,0})\to\bigoplus_{p+q=2}\pi_S^{2-p,-q-1}(S_+^{2,0})]$$

just above (9.2), remarking that $\xi^*_{3,2}(\hat{\eta}\omega_2^{2n})=\hat{\eta}\xi^*_{3,2}\omega_2^{2n}=\hat{\eta}\xi^*_{3,2}\eta^*_{2,3}\omega_3^n=\chi\hat{\eta}\omega_3^n=(1+\rho)\omega_3^n$ we obtain

Proposition 9.3. $(\pi_S^{3-p,-q-1}(S_+^{3,0}), p+q=2)$

i) $\pi_S^{4n,-4n}(S_+^{3,0})=\boldsymbol{Z}\cdot\omega_3^n\oplus\boldsymbol{Z}/4\cdot(1+\rho)\omega_3^n$,

ii) $\pi_S^{4n+1,-4n-1}(S_+^{3,0})=\boldsymbol{Z}/2\cdot\beta_3(\chi\mathring{\eta}_{-4n})$,

iii) $\pi_S^{4n+2,-4n-2}(S_+^{3,0})=\boldsymbol{Z}\cdot\delta_{3,1}\omega_1^{4n+3}\oplus\boldsymbol{Z}/2\cdot\hat{\eta}\xi_{3,2}^*\omega_2^{2n+1}$,

iv) $\pi_S^{4n+3,-4n-3}(S_+^{3,0})=\boldsymbol{Z}/2\cdot\eta\xi_{3,2}^*\omega_2^{2n+1}\oplus\boldsymbol{Z}/2\cdot\rho\eta\xi_{3,2}^*\omega_2^{2n+1}$

for any integer n.

Computation of $\pi_S^{4-p,-q-1}(S_+^{4,0})$ for $p+q=2$. $\delta_{3,1}\omega_1^{4n}=\delta_{3,1}\eta_{1,4}^*\omega_4^n=0$, $\delta_{3,1}\omega_1^{4n+1}=\delta_{3,1}(\omega_1\eta_{1,4}^*\omega_4^n)=(1-\rho)\omega_3^n=2\omega_3^n-(1+\rho)\,\omega_3^n$ of order ∞, $\delta_{3,1}\omega_1^{4n+2}=\delta_{3,1}\eta_{1,2}^*\omega_2^{2n+1}=\beta_3(\chi\delta_2\omega_2^{2n+1})=\beta_3(\chi\mathring{\eta}_{-4n})$ of order 2 and $\delta_{3,1}\omega_1^{4n+3}$ is of order ∞. Thus we have

$$(9.4)\qquad \mathrm{Ker}\,[\delta_{3,1}\colon \bigoplus_{p+q=3}\pi_S^{4-p,-q-1}(S_+^{1,0})\to\bigoplus_{p+q=3}\pi_S^{3-p,-q}(S_+^{3,0})]$$
$$=\boldsymbol{Z}[\omega_1^4,\omega_1^{-4}]\otimes\{\boldsymbol{Z}\cdot 1\oplus\boldsymbol{Z}\cdot 2\omega_1^2\}\,.$$

The exact sequence (1.9) for $r=3$ and Proposition 1.10 give the isomorphism

$$\xi_{4,3}^*\colon \bigoplus_{p+q=2}\pi_S^{3-p,-q-1}(S_+^{3,0})/\mathrm{Im}\,\delta_{3,1}\approx\bigoplus_{p+q=2}\pi_S^{4-p,-q-1}(S_+^{4,0})\,.$$

Since we have computed Im $\delta_{3,1}$ just above (9.4), remarking that $\xi_{4,3}^*\omega_3^n=\xi_{4,3}^*\eta_{3,4}^*\omega_4^n=\chi\omega_4^n$ we obtain

Proposition 9.5. $(\pi_S^{4-p,-q-1}(S_+^{4,0}),\,p+q=2)$

i) $\pi_S^{4n+1,-4n}(S_+^{4,0})=\boldsymbol{Z}/8\cdot\chi\omega_4^n$,

ii) $\pi_S^{4n+2,-4n-1}(S_+^{4,0})=0$,

iii) $\pi_S^{4n+3,-4n-2}(S_+^{4,0})=\boldsymbol{Z}/2\cdot\hat{\eta}\xi_{4,2}^*\omega_2^{2n+1}$,

iv) $\pi_S^{4n+4,-4n-3}(S_+^{4,0})=\boldsymbol{Z}/2\cdot\eta\xi_{4,2}^*\omega_2^{2n+1}\oplus\boldsymbol{Z}/2\cdot\rho\eta\xi_{4,2}^*\omega_2^{2n+1}$

for any integer n.

Proposition 9.5 describes $\lambda_{p,q}^S$ for $p+q=2$. By Proposition 4.8 the groups $\pi_{p,q}^S$ for $p+q=2$ are determined except $\pi_{1,1}^S$, $\pi_{2,0}^S$ and $\pi_{3,-1}^S$.

Since $Sq\eta$ is of order 2 by Corollary 7.2, $Sq\colon\pi_1^S\to\pi_{1,1}^S$ is a homomorphism. Hence we have $\pi_{1,1}^S=\lambda_{1,1}^S\oplus\pi_1^S=\boldsymbol{Z}/2\cdot\hat{\eta}\mathring{\eta}\oplus\boldsymbol{Z}/2\cdot Sq\eta$. Since $\chi\colon\pi_{1,1}^S\to\pi_{0,1}^S$ is a monomorphism by Proposition 5.3, we see that $\hat{\eta}\mathring{\eta}=\hat{\eta}\eta$ (by Theorem 8.8, i)), $\rho\cdot Sq\eta=Sq\eta$ and $\chi\cdot Sq\eta=\eta+\mathring{\eta}$ or $\eta+\rho\mathring{\eta}$.

By the forgetful exact sequence and $\pi_{1,0}^S=\boldsymbol{Z}\cdot\hat{\eta}$ we see that $\pi_{2,0}^S=\boldsymbol{Z}\cdot\hat{\eta}^2$.

By the fixed-point exact sequence (1.12) for $r=4$ and $\pi_S^{1,0}(S_+^{4,0})=\boldsymbol{Z}/8\cdot\beta_4(\chi)$ we see that $\pi_{3,-1}^S=0$.

Summarizing the aboves we get the following theorems.

Theorem 9.6. $(\pi_{p,q}^S,\,p+q=2)$

i) $\pi_{2,0}^S=\boldsymbol{Z}\cdot\hat{\eta}^2$.

ii) $\pi_{3,-1}^S=0$.

iii) $\pi_{4n,-4n+2}^S\approx\boldsymbol{Z}/2\cdot\eta\mathring{\eta}_{4n}\oplus\boldsymbol{Z}/2\cdot\rho\eta\mathring{\eta}_{4n}\oplus\pi_{-4n+2}^S$ *for any interger n.*

iv) $\pi^S_{4n+1,-4n+1} \approx \boldsymbol{Z}/2 \cdot \hat{\eta}\mathring{\eta}_{4n} \oplus \pi^S_{-4n+1}$ *for any integer n.*

v) $\pi^S_{4n+2,-4n} \approx \pi^S_{-4n}$ *for* $n \neq 0$.

vi) $\pi^S_{4n+3,-4n-1} \approx \boldsymbol{Z}/8 \cdot \chi\delta_4\omega_4^{-n} \oplus \pi^S_{-4n-1}$ *for* $n \neq 0$.

Theorem 9.7. i) $\hat{\eta}\mathring{\eta} = \hat{\eta}\eta$, ii) $\rho \cdot Sq\eta = Sq\eta$ *and* iii) $\chi \cdot Sq\eta = \eta + \mathring{\eta}$ *or* $\eta + \rho\mathring{\eta}$.

10. 3 stem

Proposition 10.1. *There is an element* $\hat{\nu} \in \pi^S_{2,1}$ *satisfying the following conditions*:

i) $\psi(\hat{\nu}) = \nu$ *and* $\phi(\hat{\nu}) = \eta$.

ii) $\beta_1(\hat{\nu}) = \nu\omega_1^{-2}$ *and* $\beta_2(\hat{\nu}) = \nu\omega_2^{-1}$.

Proof. Identify the quaternions $\boldsymbol{H}$ with $\boldsymbol{R}^{2,2}$ by the involution $\tau(q) = -iqi$ for $q \in \boldsymbol{H}$. The product of the quaternions $m_2: \boldsymbol{H} \times \boldsymbol{H} \to \boldsymbol{H}$ gives rise to a τ-map $m_2: \boldsymbol{R}^{2,2} \times \boldsymbol{R}^{2,2} \to \boldsymbol{R}^{2,2}$. In the same way as the case of $\hat{\eta}$, we have a τ-map $\hat{\nu}: \Sigma^{4,3} \to \Sigma^{2,2}$ by the Hopf construction. The element of $\pi^S_{2,1}$ represented by $\hat{\nu}$ is also denoted by the same letter. Obviously $\psi(\hat{\nu}) = \pm\nu$. Replace $\hat{\nu}$ by $-\hat{\nu}$ when $\psi(\hat{\nu}) = -\nu$. Then we see that $\psi(\hat{\nu}) = \nu$, hence $\beta_1(\hat{\nu}) = \nu\omega_1^{-2}$.

The fixed-point set of the involution $\tau: \boldsymbol{H} \to \boldsymbol{H}$ is the complex numbers, whence $\phi(\hat{\nu}) = \eta$.

$\eta^*_{1,2}\beta_2(\hat{\nu}) = \beta_1(\hat{\nu}) = \nu\omega_1^{-2}$. It follows that $\beta_2(\hat{\nu}) - \nu\omega_2^{-1} \in \operatorname{Ker} \eta^*_{1,2} = \operatorname{Im}[\xi^*_{2,1}: \pi_S^{-3,-1}(S^{1,0}_+) \to \pi_S^{-2,-1}(S^{2,0}_+)] = 0$, since $\pi_S^{-3,-1}(S^{1,0}_+) \approx \pi^S_4 = 0$, i.e. $\beta_2(\hat{\nu}) = \nu\omega_2^{-1}$. □

Computation of $\pi_S^{2-p,-q-1}(S^{2,0}_+)$ for $p+q=3$. Observe the short exact sequence (7.4) for $r=3$. $\eta^*_{1,2}\beta_2(\eta\mathring{\eta}_{-4n}) = \eta^2\omega_1^{4n}$, $\beta_2(\eta\mathring{\eta}_{-4n})$ is of order 2, $\eta^*_{1,2}\beta_2(\hat{\eta}\mathring{\eta}_{-4n}) = \eta^2\omega_1^{4n-1}$, $\beta_2(\hat{\eta}\mathring{\eta}_{-4n})$ is of order 2, $\eta^*_{1,2}(\hat{\eta}\eta\omega_2^{2n+1}) = \eta^2\omega_1^{4n+1}$, $\hat{\eta}\eta\omega_2^{2n+1}$ is of order 2, $\eta^*_{1,2}(\hat{\eta}^2\omega_2^{2n}) = \eta^2\omega_1^{4n-2}$ and $2\hat{\eta}^2\omega_2^{2n} = \chi\hat{\eta}^3\omega_2^{2n} = \xi^*_{2,1}(\beta_1(\hat{\eta}^3)\eta^*_{1,3}\omega_3^n) = \xi^*_{2,1}(\eta^3\omega_1^{4n-3}) = 12\xi^*_{2,1}(\nu\omega_1^{4n-3})$. Thus we obtain

Proposition 10.2. $(\pi_S^{2-p,-q-1}(S^{2,0}_+), p+q=3)$

i) $\pi_S^{4n,-4n-2}(S^{2,0}_+) = \boldsymbol{Z}/2 \cdot \beta_2(\eta\mathring{\eta}_{-4n}) \oplus \boldsymbol{Z}/24 \cdot \xi^*_{2,1}(\nu\omega_1^{4n-1})$,

ii) $\pi_S^{4n+1,-4n-3}(S^{2,0}_+) = \boldsymbol{Z}/2 \cdot \hat{\eta}\eta\omega_2^{2n+1} \oplus \boldsymbol{Z}/2 \cdot \xi^*_{2,1}(\nu\omega_1^{4n})$,

iii) $\pi_S^{4n-2,-4n}(S^{2,0}_+) = \boldsymbol{Z}/2 \cdot (\hat{\eta}^2\omega_2^{2n} - 6\xi^*_{2,1}(\nu\omega_1^{4n-3})) \oplus \boldsymbol{Z}/24 \cdot \xi^*_{2,1}(\nu\omega_1^{4n-3})$,

iv) $\pi_S^{4n-1,-4n-1}(S^{2,0}_+) = \boldsymbol{Z}/2 \cdot \beta_2(\hat{\eta}\mathring{\eta}_{-4n}) \oplus \boldsymbol{Z}/2 \cdot \xi^*_{2,1}(\nu\omega_1^{4n-2})$

for any integer n.

Computation of $\pi_S^{3-p,-q-1}(S^{3,0}_+)$ for $p+q=3$. $\delta_{2,1}(\eta^2\omega_1^{4n}) = \delta_{2,1}\eta^*_{1,3}(\eta^2\omega_3^n) = 0$, $\delta_{2,1}(\eta^2\omega_1^{4n+1}) = \delta_{2,1}\eta^*_{1,2}(\hat{\eta}\eta\omega_2^{2n+1}) = \xi^*_{2,1}\delta_{1,2}(\hat{\eta}\eta\omega_2^{2n+1}) = \xi^*_{2,1}\beta_1(\hat{\eta}\eta\mathring{\eta}_{-4n}) = \xi^*_{2,1}(\eta^3\omega_1^{4n-1}) = 12\xi^*_{2,1}(\nu\omega_1^{4n-1})$ of order 2, $\delta_{2,1}(\eta^2\omega_1^{4n+2}) = \delta_{2,1}\eta^*_{1,3}(\hat{\eta}^2\omega_3^{n+1}) = 0$ and $\delta_{2,1}(\eta^2\omega_1^{4n+3}) = \delta_{2,1}\eta^*_{1,3}(\hat{\eta}\eta\omega_3^{n+1}) = 0$. Thus we have

(10.3)
$$\begin{aligned}&\operatorname{Ker}\,[\delta_{2,1}\colon \bigoplus_{p+q=4}\pi_S^{3-p,-q-1}(S_+^{1,0})\to\bigoplus_{p+q=4}\pi_S^{2-p,-q}(S_+^{2,0})]\\&=\boldsymbol{Z}[\omega_1^4,\omega_1^{-4}]\otimes\{\boldsymbol{Z}/2\cdot\eta^2\oplus\boldsymbol{Z}/2\cdot\eta^2\omega_1^{-1}\oplus\boldsymbol{Z}/2\cdot\eta^2\omega_1^{-2}\}\,.\end{aligned}$$

The exact sequence (1.9) for $r=2$ and (9.2) give the short exact sequence

$$\begin{aligned}0\to\bigoplus_{p+q=3}\pi_S^{2-p,-q-1}(S_+^{2,0})/\operatorname{Im}\delta_{2,1}&\xrightarrow{\xi_{3,2}^*}\bigoplus_{p+q=3}\pi_S^{3-p,-q-1}(S_+^{3,0})\\&\xrightarrow{\eta_{1,3}^*}\boldsymbol{Z}[\omega_1^4,\omega_1^{-4}]\otimes\{\boldsymbol{Z}/2\cdot\eta\oplus\boldsymbol{Z}/2\cdot\eta\omega_1^{-1}\}\to 0\,.\end{aligned}$$

Since $\eta_{1,3}^*\beta_3(\mathring{\eta}_{-4n})=\eta\omega_1^{4n}$, $\beta_3(\mathring{\eta}_{-4n})$ is of order 2, $\eta_{1,3}^*(\hat{\eta}\omega_3^n)=\eta\omega_1^{4n-1}$ and $2\hat{\eta}\omega_3^n=\xi_{3,2}^*(\hat{\eta}^2\omega_2^{2n})$, by the computation of $\operatorname{Im}\delta_{2,1}$ above (10.3) we obtain

Proposition 10.4. $(\pi_S^{3-p,-q-1}(S_+^{3,0}),\ p+q=3)$

i) $\pi_S^{4n,-4n-1}(S_+^{3,0})=\boldsymbol{Z}/2\cdot\beta_3(\mathring{\eta}_{-4n})\oplus\boldsymbol{Z}/2\cdot\rho\beta_3(\mathring{\eta}_{-4n})\oplus\boldsymbol{Z}/2\cdot\xi_{3,1}^*(\nu\omega_1^{4n-2})$,

ii) $\pi_S^{4n+1,-4n-2}(S_+^{3,0})=\boldsymbol{Z}/2\cdot\beta_3(\chi\eta\mathring{\eta}_{-4n})\oplus\boldsymbol{Z}/12\cdot\xi_{3,1}^*(\nu\omega_1^{4n-1})$,

iii) $\pi_S^{4n+2,-4n-3}(S_+^{3,0})=\boldsymbol{Z}/2\cdot\hat{\eta}\eta\xi_{3,2}^*\omega_2^{2n+1}\oplus\boldsymbol{Z}/2\cdot\xi_{3,1}^*(\nu\omega_1^{4n})$,

iv) $\pi_S^{4n-1,-4n}(S_+^{3,0})=\boldsymbol{Z}/4\cdot(\hat{\eta}\omega_3^n-3\xi_{3,1}^*(\nu\omega_1^{4n-3}))\oplus\boldsymbol{Z}/24\cdot\xi_{3,1}^*(\nu\omega_1^{4n-3})$
$=\boldsymbol{Z}/8\cdot\hat{\eta}\omega_3^n\oplus\boldsymbol{Z}/12\cdot(\hat{\eta}\omega_3^n-\xi_{3,1}^*(\nu\omega_1^{4n-3}))$

for any integer n.

Computation of $\pi_S^{4-p,-q-1}(S_+^{4,0})$ for $p+q=3$. $\delta_{3,1}(\eta\omega_1^{4n})=\delta_{3,1}\eta_{1,4}^*(\eta\omega_4^n)=0$, $\delta_{3,1}(\eta\omega_1^{4n+1})=\delta_{3,1}\eta_{1,2}^*(\hat{\eta}\omega_2^{2n+1})=\beta_3(\chi\hat{\eta}\mathring{\eta}_{-4n})=(1+\rho)\beta_3(\mathring{\eta}_{-4n})$ of order 2, $\delta_{3,1}(\eta\omega_1^{4n+2})=\delta_{3,1}\eta_{1,2}^*(\eta\omega_2^{2n+1})=\beta_3(\chi\eta\mathring{\eta}_{-4n})$ of order 2 and $\delta_{3,1}(\eta\omega_1^{4n+3})=\delta_{3,1}\eta_{1,4}^*(\hat{\eta}\omega_4^{n+1})=0$. Thus we have

(10.5)
$$\begin{aligned}&\operatorname{Ker}\,[\delta_{3,1}\colon \bigoplus_{p+q=4}\pi_S^{4-p,-q-1}(S_+^{1,0})\to\bigoplus_{p+q=4}\pi_S^{3-p,-q}(S_+^{3,0})]\\&=\boldsymbol{Z}[\omega_1^4,\omega_1^{-4}]\otimes\{\boldsymbol{Z}/2\cdot\eta\oplus\boldsymbol{Z}/2\cdot\eta\omega_1^{-1}\}\,.\end{aligned}$$

The exact sequence (1.9) for $r=3$ and (9.4) give the short exact sequence

$$\begin{aligned}0\to\bigoplus_{p+q=3}\pi_S^{3-p,-q-1}(S_+^{3,0})/\operatorname{Im}\delta_{3,1}&\xrightarrow{\xi_{4,3}^*}\bigoplus_{p+q=3}\pi_S^{4-p,-q-1}(S_+^{4,0})\\&\xrightarrow{\eta_{1,4}^*}\boldsymbol{Z}[\omega_1^4,\omega_1^{-4}]\otimes\{\boldsymbol{Z}\cdot 1\oplus\boldsymbol{Z}\cdot 2\omega_1^2\}\to 0\,.\end{aligned}$$

Thus we obtain

Proposition 10.6. $(\pi_S^{4-p,-q-1}(S_+^{4,0}),\ p+q=3)$

i) $\pi_S^{4n,-4n}(S_+^{4,0})=\boldsymbol{Z}\cdot\omega_4^n\oplus\boldsymbol{Z}/8\cdot(1+\rho)\omega_4^n\oplus\boldsymbol{Z}/12\cdot\{(1+\rho)\omega_4^n-\xi_{4,1}^*(\nu\omega_1^{4n-3})\}$,

ii) $\pi_S^{4n+1,-4n-1}(S_+^{4,0})=\boldsymbol{Z}/2\cdot\beta_4(\chi\mathring{\eta}_{-4n})\oplus\boldsymbol{Z}/2\cdot\xi_{4,1}^*(\nu\omega_1^{4n-2})$,

iii) $\pi_S^{4n+2,-4n-2}(S_+^{4,0})=\boldsymbol{Z}\cdot\delta_{4,1}\omega_1^{4n+3}\oplus\boldsymbol{Z}/12\cdot\xi_{4,1}^*(\nu\omega_1^{4n-1})$,

iv) $\pi_S^{4n+3,-4n-3}(S_+^{4,0})=\boldsymbol{Z}/2\cdot\hat{\eta}\eta\xi_{4,2}^*\omega_2^{2n+1}\oplus\boldsymbol{Z}/2\cdot\xi_{4,1}^*(\nu\omega_1^{4n})$

for any integer n.

Computation of $\pi_S^{5-p,-q-1}(S_+^{5,0})$ for $p+q=3$.

Proposition 10.7. i) $\delta_{4,1}\omega_1^{8n}=0$, ii) $\delta_{4,1}\omega_1^{8n+4}=\xi_{4,1}^*(\nu\omega_1^{8n})$, iii) $\delta_{4,1}\omega_1^{4n+2}=\beta_4(\chi\mathring{\eta}_{-4n})$, iv) $\delta_{4,1}\omega_1^{8n+1}=2\omega_4^n\bar{\omega}_4^n-(1+\rho)\omega_4^n\bar{\omega}_4^n$ *and* v) $\delta_{4,1}\omega_1^{8n+5}=2\omega_4^{n+1}\bar{\omega}_4^n-\{(1+\rho)\omega_4^{n+1}\bar{\omega}_4^n-\xi_{4,1}^*(\nu\omega_1^{8n+1})\}$ *for any integer n.*

Proof. i) $\delta_{4,1}\omega_1^{8n}=\delta_{4,1}\eta_{1,5}^*\omega_5^n=0$. ii) $\delta_{4,1}\omega_1^{8n+4}=\delta_{4,1}\eta_{1,4}^*\omega_4^{n+1}\bar{\omega}_4^n=\beta_4(\chi^3\delta_4\omega_4^{n+1}\bar{\omega}_4^n)=\xi_{4,1}^*\beta_1(\delta_4\omega_4^{n+1}\bar{\omega}_4^n)=\xi_{4,1}^*(\nu\omega_1^{8n})$ since $\psi(\delta_4\omega_4)=\nu$ by Theorem 3.5. iii) $\delta_{4,1}\omega_1^{4n+2}=\delta_{4,1}\eta_{1,2}^*\omega_2^{2n+1}=\beta_4(\chi\delta_2\omega_2^{2n+1})=\beta_4(\chi\mathring{\eta}_{-4n})$. iv) $\delta_{4,1}\omega_1^{8n+1}=\delta_{4,1}(\omega_1\eta_{1,5}^*\omega_5^n)=(1-\rho)\omega_4^n\bar{\omega}_4^n=2\omega_4^n\bar{\omega}_4^n-(1+\rho)\omega_4^n\bar{\omega}_4^n$ by (3.9). v) Observe the exact sequence (1.12) for $r=4$. Since $\delta_4\xi_{4,1}^*(\nu\omega_1^{8n+1})=\rho^3\delta_1(\nu\omega_1^{8n+1})=-\delta_1(\omega_1\beta_1(\delta_4\omega_4^{n+1}\bar{\omega}_4^n))=(\rho-1)\delta_4\omega_4^{n+1}\bar{\omega}_4^n$, we see that $\delta_4\{2\omega_4^{n+1}\bar{\omega}_4^n-((1+\rho)\omega_4^{n+1}\bar{\omega}_4^n-\xi_{4,1}^*(\nu\omega_1^{8n+1}))\}=0$. By Theorem 7.6, iii), $\pi_{-8n-4,8n+4}^S=\boldsymbol{Z}\cdot\delta_1\omega_1^{8n+5}\oplus\chi^{8n+4}\cdot\pi_{8n+4}^S$. Since $\beta_4(\chi^{8n+4}\cdot\pi_{8n+4}^S)=0$, there exists an integer a satisfying

$$a\cdot\delta_{4,1}\omega_1^{8n+5}=2\omega_4^{n+1}\bar{\omega}_4^n-\{(1+\rho)\omega_4^{n+1}\bar{\omega}_4^n-\xi_{4,1}^*(\nu\omega_1^{8n+1})\}\ .$$

Apply $\eta_{1,4}^*$ to both sides of this equality. Since $\eta_{1,4}^*\delta_{4,1}\omega_1^{8n+5}=\delta_{1,1}\omega_1^{8n+5}=2\omega_1^{8n+4}$, $\eta_{1,4}^*\omega_4^{n+1}\bar{\omega}_4^n=\omega_1^{8n+4}$ and $\eta_{1,4}^*\xi_{1,4}^*(\nu\omega_1^{8n+1})=0$, we see that $2a\cdot\omega_1^{8n+4}=2\omega_1^{8n+4}$, hence $a=1$. Thus we obtain the relation v). □

Propositions 10.6 and 10.7 imply that

$$\text{(10.8)}\qquad \begin{aligned}&\mathrm{Ker}\,[\delta_{4,1}\colon \bigoplus_{p+q=4}\pi_S^{5-p,-q-1}(S_+^{1,0})\to\bigoplus_{p+q=4}\pi_S^{4-p,-q}(S_+^{4,0})]\\&=\boldsymbol{Z}[\omega_1^8,\omega_1^{-8}]\otimes\{\boldsymbol{Z}\cdot1\oplus\sum_{i=1}^{3}\boldsymbol{Z}\cdot2\omega_1^{2i}\}\ ,\end{aligned}$$

and we have the isomorphism

$$\xi_{5,4}^*\colon\bigoplus_{p+q=3}\pi_S^{4-p,-q-1}(S_+^{4,0})/\mathrm{Im}\,\delta_{4,1}\approx\bigoplus_{p+q=3}\pi_S^{5-p,-q-1}(S_+^{5,0})$$

by Proposition 1.10. Thus we obtain

Proposition 10.9. $(\pi_S^{5-p,-q-1}(S_+^{5,0}),\ p+q=3)$

i) $\pi_S^{8n+1,-8n}(S_+^{5,0})=\boldsymbol{Z}/\mathbf{16}\cdot\chi\omega_5^n\oplus\boldsymbol{Z}/\mathbf{12}\cdot\{2\chi\omega_5^n-\xi_{5,1}^*(\nu\omega_1^{8n-3})\}$,

ii) $\pi_S^{4n+2,-4n-1}(S_+^{5,0})=\boldsymbol{Z}/\mathbf{2}\cdot\xi_{5,1}^*(\nu\omega_1^{4n-2})$,

iii) $\pi_S^{4n+3,-4n-2}(S_+^{5,0})=\boldsymbol{Z}/\mathbf{12}\cdot\xi_{5,1}^*(\nu\omega_1^{4n-1})$,

iv) $\pi_S^{8n+4,-8n-3}(S_+^{5,0})=\boldsymbol{Z}/\mathbf{2}\cdot\mathring{\eta}\eta\xi_{5,2}^*\omega_2^{4n+1}$,

v) $\pi_S^{8n+5,-8n-4}(S_+^{5,0})=\boldsymbol{Z}/\mathbf{24}\cdot\xi_{5,4}^*\omega_4^{n+1}\bar{\omega}_4^n\oplus\boldsymbol{Z}/\mathbf{8}\cdot(1+\rho)\xi_{5,4}^*\omega_4^{n+1}\bar{\omega}_4^n$,

vi) $\pi_S^{8n+8,-8n-7}(S_+^{5,0})=\boldsymbol{Z}/\mathbf{2}\cdot\xi_{5,1}^*(\nu\omega_1^{8n+4})\oplus\boldsymbol{Z}/\mathbf{2}\cdot\mathring{\eta}\eta\xi_{5,2}^*\omega_2^{4n+3}$

for any integer n.

Proposition 10.9 describes $\lambda_{p,q}^S$ for $p+q=3$. By Proposition 4.8 the groups $\pi_{p,q}^S$ for $p+q=3$ are determined except $\pi_{2,1}^S$, $\pi_{3,0}^S$ and $\pi_{4,-1}^S$.

The forgetful exact sequence

$$\pi_{1,2}^S \xrightarrow{\beta_1} \pi_S^{-1,-2}(S_+^{1,0}) \xrightarrow{\delta_1} \pi_{2,1}^S \xrightarrow{\chi} \pi_{1,1}^S \xrightarrow{\beta_1} \pi_S^{-1,-1}(S_+^{1,0})$$

induces a short exact sequence

$$0 \to \boldsymbol{Z}/\boldsymbol{12}\cdot\nu\delta_1\omega_1^{-1} \to \pi_{2,1}^S \xrightarrow{\chi} \boldsymbol{Z}/\boldsymbol{2}\cdot(\hat{\eta}\eta+Sq\eta) \to 0\,.$$

By Proposition 10.1 we have $\pi_{2,1}^S=\boldsymbol{Z}/\boldsymbol{24}\cdot\hat{\nu}$, $2\hat{\nu}=\nu\delta_1\omega_1^{-1}$ and $\chi\hat{\nu}=\hat{\eta}\eta+Sq\eta$. Since $\beta_1\colon \pi_{2,1}^S\to\pi_S^{-2,-1}(S_+^{1,0})$ is isomorphic, $\rho\cdot\hat{\nu}=-\hat{\nu}$.

By the forgetful exact sequence and $\pi_{2,0}^S=\boldsymbol{Z}\cdot\hat{\eta}^2$ we see that $\pi_{3,0}^S=\boldsymbol{Z}\cdot\hat{\eta}^3$.

By the fixed-point exact sequence (1.12) for $r=5$ and $\pi_S^{1,0}(S_+^{5,0})=\boldsymbol{Z}/\boldsymbol{16}\cdot\beta_5(\chi)\oplus\boldsymbol{Z}/\boldsymbol{12}\cdot\{2\beta_5(\chi)-\xi_{5,1}^*(\nu\omega_1^{-3})\}$ we see that $\pi_{4,-1}^S=\boldsymbol{Z}/\boldsymbol{12}\cdot\nu\delta_1\omega_1^{-3}$.

Put

$$\text{(10.10)} \qquad \mathring{\nu}_{8n} = \delta_4\omega_4^{1-n}\varpi_4^{-n} \ \text{ and } \ \mathring{\nu} = \mathring{\nu}_0 = \delta_4\omega_4\,.$$

Then $\beta_1(\mathring{\nu}_{8n})=\nu\omega_1^{-8n}$ and $\beta_1(\mathring{\nu})=\beta_1(\nu)=\nu$ by Theorem 3.5.

Summarizing the aboves we get the following theorems.

Theorem 10.11. $(\pi_{p,q}^S,\ p+q=3)$

i) $\pi_{2,1}^S=\boldsymbol{Z}/\boldsymbol{24}\cdot\hat{\nu}$.

ii) $\pi_{3,0}^S=\boldsymbol{Z}\cdot\hat{\eta}^3$.

iii) $\pi_{4,-1}^S=\boldsymbol{Z}/\boldsymbol{12}\cdot\nu\delta_1\omega_1^{-3}$.

iv) $\pi_{8n,-8n+3}^S\approx\boldsymbol{Z}/\boldsymbol{24}\cdot\mathring{\nu}_{8n}\oplus\boldsymbol{Z}/\boldsymbol{8}\cdot(1+\rho)\mathring{\nu}_{8n}\oplus\pi_{-8n+3}^S$ *for any integer* n.

v) $\pi_{8n+1,-8n+2}^S\approx\boldsymbol{Z}/\boldsymbol{2}\cdot\hat{\eta}\eta\mathring{\eta}_{8n}\oplus\pi_{-8n+2}^S$ *for any integer* n.

vi) $\pi_{4n+2,-4n+1}^S\approx\boldsymbol{Z}/\boldsymbol{12}\cdot\nu\delta_1\omega_1^{-4n-1}\oplus\pi_{-4n+1}^S$ *for* $n\neq0$.

vii) $\pi_{4n+3,-4n}^S\approx\boldsymbol{Z}/\boldsymbol{2}\cdot\chi^3\hat{\nu}\delta_4\omega_4^{-n}\oplus\pi_{-4n}^S$ *for* $n\neq0$.

viii) $\pi_{8n+4,-8n-1}^S\approx\boldsymbol{Z}/\boldsymbol{16}\cdot\chi\delta_5\omega_5^{-n}\oplus\boldsymbol{Z}/\boldsymbol{12}\cdot(2\chi\delta_5\omega_5^{-n}-\nu\delta_1\omega_1^{-8n-3})\oplus\pi_{-8n-1}^S$ *for* $n\neq0$.

ix) $\pi_{8n+5,-8n-2}^S\approx\boldsymbol{Z}/\boldsymbol{2}\cdot\chi^3\hat{\nu}\mathring{\nu}_{8n+8}\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\hat{\eta}\eta\mathring{\eta}_{8n+4}\oplus\pi_{-8n-2}^S$ *for any integer* n.

Theorem 10.12. i) $(1-\rho)\mathring{\nu}=(1-\rho)\nu$ $(=\delta_1(\nu\omega_1))$, ii) $\rho\cdot\hat{\nu}=-\hat{\nu}$ *and* iii) $\nu\delta_1\omega_1^{-1}=2\hat{\nu}$ *and* $\chi\hat{\nu}=\hat{\eta}\eta+Sq\eta$.

REMARK. $\nu\delta_1\omega_1^{-8n-3}$, $n\neq0$, is of order 24 and $12\nu\delta_1\omega_1^{-8n-3}=8\chi^4\delta_8\omega_8^{-n}$.

11. 4 stem

Observe the following exact sequence associated with the τ-cofibration $S_+^{5,0}\subset S_+^{6,0}\to\Sigma^{5,0}S_+^{1,0}$:

$$\cdots \to \pi_S^{1-p,-q-1}(S_+^{1,0}) \xrightarrow{\xi_{6,1}^*} \pi_S^{6-p,-q-1}(S_+^{6,0}) \xrightarrow{\eta_{5,6}^*} \pi_S^{6-p,-q-1}(S_+^{5,0})$$
$$\xrightarrow{\delta_{1,5}} \pi_S^{1-p,-q}(S_+^{1,0}) \to \cdots\,.$$

Since $\pi_S^{1-p,-q-1}(S_+^{1,0})\approx\pi_4^S=0$ for $p+q=4$, the above exact sequence implies that

$$(11.1)\qquad \eta^*_{5,6}\colon \bigoplus_{p+q=4} \pi_S^{6-p,-q-1}(S_+^{6,0}) \approx \mathrm{Ker}\,[\delta_{1,5}\colon \bigoplus_{p+q=4} \pi_S^{6-p,-q-1}(S_+^{5,0}) \to \bigoplus_{p+p=4} \pi_S^{1-p,-q}(S_+^{1,0})]\,.$$

Proposition 10.9 and (11.1) imply the following

Proposition 11.2. ($\pi_S^{6-p,-q-1}(S_+^{6,0})$, $p+q=4$)

i) $\pi_S^{8n+1,-8n}(S_+^{6,0}) = \boldsymbol{Z}/\mathbf{16}\cdot\chi\omega_6^n$,

ii) $\pi_S^{4n+2,-4n-1}(S_+^{6,0}) = \boldsymbol{Z}/\mathbf{2}\cdot\chi^2\hat{\nu}\xi^*_{6,4}\omega_4^n$,

iii) $\pi_S^{4n+3,-4n-2}(S_+^{6,0}) = 0$,

iv) $\pi_S^{8n+4,-8n-3}(S_+^{6,0}) = 0$,

v) $\pi_S^{8n+5,-8n-4}(S_+^{6,0}) = \boldsymbol{Z}/\mathbf{8}\cdot\hat{\eta}\xi^*_{6,4}\omega_4^{n+1}\bar{\omega}_4^n$,

vi) $\pi_S^{8n+8,-8n-7}(S_+^{6,0}) = \boldsymbol{Z}/\mathbf{2}\cdot\chi^2\mathring{\nu}\xi^*_{6,4}\omega_4^{n+1}\bar{\omega}_4^n$,

for any integer n.

Proposition 11.2 describes $\lambda_{p,q}^S$ for $p+q=4$. By Proposition 4.8 the groups $\pi_{p,q}^S$ for $p+q=4$ are determined except $\pi_{2,2}^S$, $\pi_{3,1}^S$, $\pi_{4,0}^S$ and $\pi_{5,-1}^S$.

The forgetful exact sequence (1.5) induces the exact sequence

$$0 \to \pi_{p,q}^S \xrightarrow{\chi} \pi_{p-1,q}^S \xrightarrow{\psi} \pi_3^S \quad \text{for } p+q=4\,.$$

This exact sequence together with Theorem 10.11 gives that $\pi_{2,2}^S = \boldsymbol{Z}/\mathbf{2}\cdot Sq\eta^2$, $\pi_{3,1}^S = 0$, $\pi_{4,0}^S = \boldsymbol{Z}\cdot\hat{\eta}^4$ and $\pi_{5,-1}^S = 0$. Thus we obtain

Theorem 11.3. ($\pi_{p,q}^S$, $p+q=4$)

i) $\pi_{3,1}^S = 0$.

ii) $\pi_{4,0}^S = \boldsymbol{Z}\cdot\hat{\eta}^4$.

iii) $\pi_{5,-1}^S = 0$.

iv) $\pi_{4n,-4n+4}^S \approx \boldsymbol{Z}/\mathbf{2}\cdot\chi^2\hat{\nu}\delta_4\omega_4^{1-n} \oplus \pi_{-4n+4}^S$ *for* $n \neq 1$.

v) $\pi_{8n+1,-8n+3}^S \approx \boldsymbol{Z}/\mathbf{8}\cdot\hat{\eta}\mathring{\nu}_{8n} \oplus \pi_{-8n+3}^S$ *for any integer n.*

vi) $\pi_{8n+2,-8n+2}^S \approx \pi_{-8n+2}^S$ *for any integer n.*

vii) $\pi_{4n+3,-4n+1}^S \approx \pi_{-4n+1}^S$ *for* $n \neq 0$.

viii) $\pi_{8n+5,-8n-1}^S \approx \boldsymbol{Z}/\mathbf{16}\cdot\chi\delta_6\omega_6^{-n} \oplus \pi_{-8n-1}^S$ *for* $n \neq 0$.

ix) $\pi_{8n+6,-8n-2}^S \approx \boldsymbol{Z}/\mathbf{2}\cdot\chi^2\mathring{\nu}\mathring{\nu}_{8n+8} \oplus \pi_{-8n-2}^S$ *for any integer n.*

Theorem 11.4. i) $\hat{\eta}\hat{\nu} = 0$, ii) $\hat{\nu}\mathring{\eta}_{8n} = 0$ *for any integer n.* iii) $\hat{\nu}\mathring{\eta}_{8n+4} = \chi^2\mathring{\nu}\mathring{\nu}_{8n+8} \neq 0$ *in* $\pi_{8n+6,-8n-2}^S$ *for any integer n.*

Proof. i) $\hat{\eta}\hat{\nu} \in \pi_{3,1}^S = 0$, hence $\hat{\eta}\hat{\nu} = 0$. ii) $\hat{\nu}\mathring{\eta}_{8n} = \delta_2(\hat{\nu}\omega_2^{-4n+1}) = \delta_2(\nu\omega_2^{-4n}) = \delta_2\beta_2(\mathring{\nu}_{8n}) = 0$. iii) Since $\chi^3\mathring{\nu}\xi^*_{5,4}\omega_4^{n+1}\bar{\omega}_4^n = \xi^*_{5,1}(\nu\omega_1^{8n+4}) = \xi^*_{5,1}(\hat{\nu}\omega_1^{8n+6}) = \chi\hat{\nu}\xi^*_{5,2}\omega_2^{4n+3}$, we have $\eta^*_{5,6}(\hat{\nu}\xi^*_{6,2}\omega_2^{4n+3}) = \chi\hat{\nu}\xi^*_{5,2}\omega_2^{4n+3} = \chi^3\mathring{\eta}\xi^*_{5,4}\omega_4^{n+1}\bar{\omega}_4^n = \eta^*_{5,6}(\chi^2\mathring{\nu}\xi^*_{6,4}\omega_4^{n+1}\bar{\omega}_4^n)$. By (11.1) we see that $\hat{\nu}\xi^*_{6,2}\omega_2^{4n+3} = \chi^2\mathring{\nu}\xi^*_{6,4}\omega_4^{n+1}\bar{\omega}_4^n$. Applying δ_6 to both sides of this equality, we get $\hat{\nu}\delta_2\omega_2^{4n+3}=\chi^2\mathring{\nu}\delta_4\omega_4^{n+1}\overline{\omega}_4^n$, i.e. $\hat{\nu}\mathring{\eta}_{-8n-4}=\chi^2\mathring{\nu}\mathring{\nu}_{-8n}$. □

12. 5 stem

Since $\pi_4^S\approx\pi_5^S=0$, the exact sequence associated with the τ-cofibration $S_+^{6,0}\subset S_+^{7,0}\to\Sigma^{6,0}S_+^{1,0}$ implies that

$$\eta_{6,7}^*: \bigoplus_{p+q=5}\pi_S^{7-p,-q-1}(S_+^{7,0})\approx\bigoplus_{p+q=5}\pi_S^{7-p,-q-1}(S_+^{6,0}). \tag{12.1}$$

Since $\hat{\nu}\xi_{6,2}^*\omega_2^{4n+1}=\xi_{6,2}^*(\hat{\nu}\omega_2^{4n+1})=\xi_{6,2}^*(\nu\omega_2^{4n})=\xi_{6,2}^*\eta_{2,8}^*(\nu\omega_8^n)=0$ and $\hat{\eta}\hat{\nu}=0$ by Thoerem 11.4, the stable equivariant Toda bracket

$$\langle\hat{\eta},\ \hat{\nu},\ \xi_{6,2}^*\omega_2^{4n+1}\rangle^\tau\subset\pi_S^{8n+3,-8n-4}(S_+^{6,0})$$

is well-defined. By Proposition 6.8, i), we see that $\eta_{4,6}^*\langle\hat{\eta},\ \hat{\nu},\ \xi_{6,2}^*\omega_2^{4n+1}\rangle^\tau=\hat{\eta}\circ\delta_{2,4}^{-1}(\hat{\nu}\omega_2^{4n+1})=\hat{\eta}\circ\delta_{2,4}^{-1}(\nu\omega_2^{4n})\ni\hat{\eta}\omega_4^{n+1}\overline{\omega}_4^n$ as $\delta_{2,4}\omega_4^{n+1}\overline{\omega}_4^n=\beta_2(\mathring{\nu}_{-8n})=\nu\omega_2^{4n}$. Let

$$[\eta\omega_1^{8n+3}]_6\in\langle\hat{\eta},\ \hat{\nu},\ \xi_{6,2}^*\omega_2^{4n+1}\rangle^\tau\subset\pi_S^{8n+3,-8n-4}(S_+^{6,0})$$

be the element such that $\eta_{4,6}^*[\eta\omega_1^{8n+3}]_6=\hat{\eta}\omega_4^{n+1}\overline{\omega}_4^n$. Then $\eta_{6,7}^*(\chi\xi_{7,6}^*[\eta\omega_1^{8n+3}]_6)=\xi_{6,4}^*\eta_{4,6}^*[\eta\omega_1^{8n+3}]_6=\hat{\eta}\xi_{6,4}^*\omega_4^{n+1}\overline{\omega}_4^n$. Thus from Proposition 11.2 and (12.1) we obtain

Proposition 12.2 $(\pi_S^{7-p,-q-1}(S_+^{7,0}),\ p+q=5)$

i) $\pi_S^{8n+1,-8n}(S_+^{7,0})=\boldsymbol{Z}/\mathbf{16}\cdot\chi\omega_7^n$,

ii) $\pi_S^{4n+2,-4n-1}(S_+^{7,0})=\boldsymbol{Z}/\mathbf{2}\cdot\chi\hat{\nu}\xi_{7,4}^*\omega_4^n$,

iii) $\pi_S^{4n+3,-4n-2}(S_+^{7,0})=0$,

iv) $\pi_S^{8n+4,-8n-3}(S_+^{7,0})=0$,

v) $\pi_S^{8n+5,-8n-4}(S_+^{7,0})=\boldsymbol{Z}/\mathbf{8}\cdot\chi\xi_{7,6}^*[\eta\omega_1^{8n+3}]_6$,

vi) $\pi_S^{8n+8,-8n-7}(S_+^{7,0})=\boldsymbol{Z}/\mathbf{2}\cdot\chi\mathring{\nu}\xi_{7,4}^*\omega_4^{n+1}\overline{\omega}_4^n$

for any integer n.

Proposition 12.2 describes $\lambda_{p,q}^S$ for $p+q=5$. By Proposition 4.8 the groups $\pi_{p,q}^S$ for $p+q=5$ are determined except $\pi_{3,2}^S$, $\pi_{4,1}^S$, $\pi_{5,0}^S$ and $\pi_{6,-1}^S$.

The forgetful exact sequence (1.5) implies that

$$\chi\colon \pi_{p,q}^S\to\pi_{p-1,q}^S$$

is isomorphic for $p+q=5$. Hence by Theorem 11.3 we have $\pi_{3,2}^S=\boldsymbol{Z}/\mathbf{2}\cdot\chi\hat{\nu}^2$, $\pi_{4,1}^S=0$, $\pi_{5,0}^S=\boldsymbol{Z}\cdot\chi^{-1}\hat{\eta}^4$ and $\pi_{6,-1}^S=0$. By [10], Theorem 2.2, there exists an element $y_7\in\pi_{7,0}^S$ such that $\phi(y_7)=2^4$. Since $\phi\colon\pi_{4,0}^S\to\pi_0^S$ is a monomorphism, $\chi^3y_7=\hat{\eta}^4$, hence $\pi_{5,0}^S=\boldsymbol{Z}\cdot\chi^2y_7$.

Put

$$\tilde{\nu}_{8n+3}=\delta_6[\eta\omega_1^{-8n+3}]_6\in\langle\hat{\eta},\ \hat{\nu},\ \mathring{\eta}_{8n}\rangle^\tau\subset\pi_{8n+3,-8n+3}^S. \tag{12.3}$$

Then we obtain

Theorem 12.4. $(\pi^S_{p,q},\ p+q=5)$

i) $\pi^S_{3,2}=\boldsymbol{Z}/\boldsymbol{2}\cdot\chi\hat{\nu}^2$.

ii) $\pi^S_{4,1}=0$.

iii) $\pi^S_{5,0}=\boldsymbol{Z}\cdot\chi^2 y_7$.

iv) $\pi^S_{6,-1}=0$.

v) $\pi^S_{4n,-4n+5}\approx\pi^S_{-4n+5}$ *for any integer n.*

vi) $\pi^S_{4n+1,-4n+4}\approx\boldsymbol{Z}/\boldsymbol{2}\cdot\chi\hat{\nu}\delta_4\omega_4^{1-n}\oplus\pi^S_{-4n+4}$ *for* $n\neq 1$.

vii) $\pi^S_{8n+2,-8n+3}\approx\boldsymbol{Z}/\boldsymbol{8}\cdot\chi\tilde{\nu}_{8n+3}\oplus\pi^S_{-8n+3}$ *for any integer n.*

viii) $\pi^S_{8n+3,-8n+2}\approx\pi^S_{-8n+2}$ *for* $n\neq 0$.

ix) $\pi^S_{8n+6,-8n-1}\approx\boldsymbol{Z}/\boldsymbol{16}\cdot\chi\delta_7\omega_7^{-n}\oplus\pi^S_{-8n-1}$ *for* $n\neq 0$.

x) $\pi^S_{8n+7,-8n-2}\approx\boldsymbol{Z}/\boldsymbol{2}\cdot\chi\mathring{\nu}\mathring{\nu}_{8n+8}\oplus\pi^S_{-8n-2}$ *for any integer n.*

Proposition 12.5. i) $\pi_S^{8n+3,-8n-4}(S_+^{6,0})=\boldsymbol{Z}/\boldsymbol{8}\cdot[\eta\omega_1^{8n+3}]_6$ *and* $\rho[\eta\omega_1^{8n+3}]_6=[\eta\omega_1^{8n+3}]_6$. ii) $\tilde{\nu}_{8n+3}$ *is of order* 8 *and* $\rho\tilde{\nu}_{8n+3}=\tilde{\nu}_{8n+3}$. iii) $\psi(\tilde{\nu}_{8n+3})=\nu^2$ *and* $\chi^2\tilde{\nu}_{8n+3}=\hat{\eta}\mathring{\nu}_{8n}$.

Proof. i) Consider the exact sequence

$$\pi_S^{8n-2,-8n-4}(S_+^{1,0})\xrightarrow{\xi^*_{6,1}}\pi_S^{8n+3,-8n-4}(S_+^{6,0})\xrightarrow{\eta^*_{5,6}}\pi_S^{8n+3,-8n-4}(S_+^{5,0})\to 0\,.$$

Since $\xi^*_{6,1}(\nu^2\omega_1^{8n-2})=\xi^*_{6,1}\eta^*_{1,6}\beta_6(\hat{\nu}\mathring{\nu}_{-8n})=\beta_6(\chi^5\hat{\nu}\mathring{\nu}_{-8n})=0$, $\eta^*_{5,6}\colon\pi_S^{8n+3,-8n-4}(S_+^{6,0})\to\pi_S^{8n+3,-8n-4}(S_+^{5,0})$ is isomorphic. The exact sequence associated with the τ-cofibration $S_+^{4,0}\subset S_+^{5,0}\to\Sigma^{4,0}S_+^{1,0}$ implies that $\eta^*_{4,5}\colon\pi_S^{8n+3,-8n-4}(S_+^{5,0})\to\pi_S^{8n+3,-8n-4}(S_+^{4,0})$ is isomorphic, hence $\eta^*_{4,6}\colon\pi_S^{8n+3,-8n-4}(S_+^{6,0})\to\pi_S^{8n+3,-8n-4}(S_+^{4,0})$ is isomorphic.

Observe the exact sequence

$$0\to\pi_S^{8n+3,-8n-4}(S_+^{4,0})\xrightarrow{\eta^*_{3,4}}\pi_S^{8n+3,-8n-4}(S_+^{3,0})\xrightarrow{\delta_{1,3}}\pi_S^{8n,-8n-3}(S_+^{1,0})\,.$$

Since $\delta_{1,3}(\hat{\eta}\omega_3^{2n+1})=\delta_{1,3}\eta^*_{3,4}(\hat{\eta}\omega_4^{n+1}\bar{\omega}_4^n)=0$ and $\delta_{1,3}(\hat{\eta}\omega_3^{2n+1}-\xi^*_{3,1}(\nu\omega_1^{8n+3}))=-\delta_{1,1}(\nu\omega_1^{8n+3})=-2\nu\omega_1^{8n+2}$ of order 12, we have $\pi_S^{8n+3,-8n-4}(S_+^{4,0})=\boldsymbol{Z}/\boldsymbol{8}\cdot\hat{\eta}\omega_4^{n+1}\bar{\omega}_4^n$. The above arguments prove i).

ii) is obtained from i) and Theorem 12.4, vii).

iii). $\psi(\tilde{\nu}_{8n+3})\in\psi(\delta_6\langle\hat{\eta},\hat{\nu},\xi^*_{6,2}\omega_2^{-4n+1}\rangle^\tau)\subset\psi(\langle\hat{\eta},\hat{\nu},\mathring{\eta}_{8n}\rangle^\tau)\subset\langle\eta,\nu,\eta\rangle=\nu^2$ by Proposition 6.7 and [14], Lemma 5.12, i.e. $\psi(\tilde{\nu}_{8n+3})=\nu^2$. And $\chi^2\tilde{\nu}_{8n+3}=\chi^2\delta_6[\eta\omega_1^{-8n+3}]_6=\delta_4\eta^*_{4,6}[\eta\omega_1^{-8n+3}]_6=\delta_4(\hat{\eta}\omega_4^{1-n}\bar{\omega}_4^{-n})=\hat{\eta}\mathring{\nu}_{8n}$. □

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The exact sequence (1.9) with $\pi_4^S\approx\pi_5^S=0$ implies that

$$\xi^*_{2,1}\colon\boldsymbol{Z}[\omega_1,\omega_1^{-1}]\otimes\boldsymbol{Z}/\boldsymbol{2}\cdot\nu^2\to\bigoplus_{p+q=6}\pi_S^{2-p,-q-1}(S_+^{2,0})\text{ and}$$

$$\xi^*_{3,2}\colon\bigoplus_{p+q=6}\pi_S^{2-p,-q-1}(S_+^{2,0})\to\bigoplus_{p+q=6}\pi_S^{3-p,-q-1}(S_+^{3,0})\text{ are isomorphic.}$$

Thus we obtain

Proposition 13.1. $(\pi_S^{3-p,-q-1}(S_+^{3,0}), p+q=6)$

$$\bigoplus_{p+q=6} \pi_S^{3-p,-q-1}(S_+^{3,0}) = \bigoplus_n \boldsymbol{Z}/2\cdot\xi_{3,1}^*(\nu^2\omega_1^n).$$

By a routine argument using the exact sequence (1.9) we have

$$\begin{aligned}(13.2)\qquad &\operatorname{Ker}[\delta_{r-1,1}\colon \bigoplus_{p+q=6}\pi_S^{r-p,-q-1}(S_+^{1,0}) \to \bigoplus_{p+q=6}\pi_S^{r-1-p,-q}(S_+^{r-1,0})]\\ &= \operatorname{Ker}[\delta_{r-3,1}\colon \bigoplus_{p+q=4}\pi_S^{r-2-p,-q-1}(S_+^{1,0}) \to \bigoplus_{p+q=4}\pi_S^{r-3-p,-q}(S_+^{r-3,0})].\end{aligned}$$

Computation of $\pi_S^{4-p,-q-1}(S_+^{4,0})$ for $p+q=6$. The exact sequence (1.9) for $r=3$, (13.2) and (7.3) induce the short exact sequence

$$\begin{aligned}0 \to \bigoplus_{p+q=6}\pi_S^{3-p,-q-1}(S_+^{3,0}) &\xrightarrow{\xi_{4,3}^*} \bigoplus_{p+q=6}\pi_S^{4-p,-q-1}(S_+^{4,0})\\ &\xrightarrow{\eta_{1,4}^*} \boldsymbol{Z}[\omega_1^2,\omega_1^{-2}]\otimes\{\boldsymbol{Z}/24\cdot\nu\oplus\boldsymbol{Z}/2\cdot\eta^3\omega_1^{-1}\} \to 0.\end{aligned}$$

$\eta_{1,4}^*\beta_4(\mathring{\nu}_{-8n})=\nu\omega_1^{8n}$, $\beta_4(\mathring{\nu}_{-8n})$ is of order 24, $\eta_{1,4}^*(\mathring{\nu}\omega_4^{n+1}\overline{\omega}_4^n)=\nu\omega_1^{8n+4}$, $\mathring{\nu}\omega_4^{n+1}\overline{\omega}_4^n$ is of order 24, $\eta_{1,4}^*(\mathring{\nu}\omega_4^n)=\nu\omega_1^{4n-2}$, $\mathring{\nu}\omega_4^n$ is of order 24, $\eta_{1,4}^*\beta_4(\mathring{\eta}\eta\mathring{\eta}_{-4n})=\eta^3\omega_1^{4n-1}$, $\beta_4(\mathring{\eta}\eta\mathring{\eta}_{-4n})$ is of order 2, $\eta_{1,4}^*(\mathring{\eta}^3\omega_4^n)=\eta^3\omega_1^{4n-3}$ and $2\mathring{\eta}^3\omega_4^n=\chi\mathring{\eta}^4\omega_4^n=\chi^4y_7\omega_4^n=0$, i.e. $\mathring{\eta}^3\omega_4^n$ is of order 2. Thus we obtain

Proposition 13.3. $(\pi_S^{4-p,-q-1}(S_+^{4,0}), p+q=6)$

i) $\pi_S^{4n-3,-4n}(S_+^{4,0})=\boldsymbol{Z}/2\cdot\mathring{\eta}^3\omega_4^n\oplus\boldsymbol{Z}/2\cdot\xi_{4,1}^*(\nu^2\omega_1^{4n-6})$,

ii) $\pi_S^{4n-2,-4n-1}(S_+^{4,0})=\boldsymbol{Z}/24\cdot\mathring{\nu}\omega_4^n\oplus\boldsymbol{Z}/2\cdot\xi_{4,1}^*(\nu^2\omega_1^{4n-5})$,

iii) $\pi_S^{4n-1,-4n-2}(S_+^{4,0})=\boldsymbol{Z}/2\cdot\beta_4(\mathring{\eta}\eta\mathring{\eta}_{-4n})\oplus\boldsymbol{Z}/2\cdot\xi_{4,1}^*(\nu^2\omega_1^{4n-4})$,

iv) $\pi_S^{8n,-8n-3}(S_+^{4,0})=\boldsymbol{Z}/24\cdot\beta_4(\mathring{\nu}_{-8n})\oplus\boldsymbol{Z}/2\cdot\xi_{4,1}^*(\nu^2\omega_1^{8n-3})$,

v) $\pi_S^{8n+4,-8n-7}(S_+^{4,0})=\boldsymbol{Z}/24\cdot\mathring{\nu}\omega_4^{n+1}\overline{\omega}_4^n\oplus\boldsymbol{Z}/2\cdot\xi_{4,1}^*(\nu^2\omega_1^{8n+1})$

for any integer n.

Computation of $\pi_S^{5-p,-q-1}(S_+^{5,0})$ for $p+q=6$. $\delta_{4,1}(\nu\omega_1^{4n})=0$, $\delta_{4,1}(\nu\omega_1^{8n+1}) =\delta_{4,1}(\omega_1\mathring{\nu}_{-8n})=2\beta_4(\mathring{\nu}_{-8n})-(1+\rho)\beta_4(\mathring{\nu}_{-8n})=2\beta_4(\mathring{\nu}_{-8n})-\beta_4(\chi\mathring{\eta}\mathring{\nu}_{-8n})=2\beta_4(\mathring{\nu}_{-8n})-\beta_4(\chi^3\bar{\nu}_{-8n+3})=2\beta_4(\mathring{\nu}_{-8n})-\xi_{4,1}^*(\nu^2\omega_1^{8n-3})$ of order 12 by Proposition 12.5, $\delta_{4,1}(\nu\omega_1^{8n+2}) =\delta_{4,1}(\mathring{\nu}\omega_1^{8n+4})=\beta_4(\chi^3\mathring{\nu}\mathring{\nu}_{-8n})=\xi_{4,1}^*(\nu^2\omega_1^{8n-2})$ of order 2, $\delta_{4,1}(\nu\omega_1^{8n+3})=\delta_{4,1}(\mathring{\nu}\omega_1^{8n+5})= 2\mathring{\nu}\omega_4^{n+1}\overline{\omega}_4^n+\xi_{4,1}^*(\nu^2\omega_1^{8n-1})$ of order 12 by Proposition 10.7, $\delta_{4,1}(\nu\omega_1^{8n+4})=\beta_4(\chi^3\nu\mathring{\nu}_{-8n}) =\xi_{4,1}^*(\nu^2\omega_1^{8n})$ of order 2, $\delta_{4,1}(\nu\omega_1^{8n+5})=2\mathring{\nu}\omega_4^{n+1}\overline{\omega}_4^n-\{(1+\rho)\mathring{\nu}\omega_4^{n+1}\overline{\omega}_4^n-\xi_{4,1}^*(\nu^2\omega_1^{8n+1})\} =2\mathring{\nu}\omega_4^{n+1}\overline{\omega}_4^n$ of order 12, $\delta_{4,1}(\nu\omega_1^{8n+6})=\delta_{4,1}(\mathring{\nu}\omega_1^{8n+8})=0$ and $\delta_{4,1}(\nu\omega_1^{8n+7})=\delta_{4,1}(\mathring{\nu}\omega_1^{8n+9}) =2\mathring{\nu}\omega_4^{n+1}\overline{\omega}_4^{n+1}$ of order 12. Thus we have

$$\begin{aligned}(13.4)\qquad &\operatorname{Ker}[\delta_{4,1}\colon \bigoplus_{p+q=7}\pi_S^{5-p,-q-1}(S_+^{1,0})\to\bigoplus_{p+q=7}\pi_S^{4-p,-q}(S_+^{4,0})]\\ &= \boldsymbol{Z}[\omega_1^8,\omega_1^{-8}]\otimes\{\boldsymbol{Z}/24\cdot\nu\omega_1^{-2}\oplus\boldsymbol{Z}/24\cdot\nu\oplus\boldsymbol{Z}/12\cdot 2\nu\omega_1^2\oplus\boldsymbol{Z}/12\cdot 2\nu\omega_1^4\}\\ &\oplus \boldsymbol{Z}[\omega_1^4,\omega_1^{-4}]\otimes\{\boldsymbol{Z}/2\cdot\eta^3\omega_1^{-1}\oplus\boldsymbol{Z}/2\cdot\eta^3\omega_1\}.\end{aligned}$$

The exact sequence (1.9) for $r=4$, (13.2) and (10.3) give the short exact sequence

$$0 \to \bigoplus_{p+q=6} \pi_S^{4-p,-q-1}(S_+^{4,0})/\mathrm{Im}\,\delta_{4,1} \xrightarrow{\xi^*_{5,4}} \bigoplus_{p+q=6} \pi_S^{5-p,-q-1}(S_+^{5,0})$$
$$\xrightarrow{\eta^*_{1,5}} \boldsymbol{Z}[\omega_1^4, \omega_1^{-4}] \otimes \{\boldsymbol{Z}/2\cdot\eta^2 \oplus \boldsymbol{Z}/2\cdot\eta^2\omega_1^{-1} \oplus \boldsymbol{Z}/2\cdot\eta^2\omega_1^{-2}\} \to 0\,.$$

Since $\eta^*_{1,5}(\hat{\eta}^2\omega_5^n)=\eta^2\omega_1^{8n-2}$, $2\hat{\eta}^2\omega_5^n=\chi\hat{\eta}^3\omega_5^n=\xi^*_{5,4}(\hat{\eta}^3\omega_4^n\bar{\omega}_4^n)$, $\eta^*_{1,5}\beta_5(\hat{\eta}\mathring{\eta}_{-4n})=\eta^2\omega_1^{4n-1}$, $\beta_5(\hat{\eta}\mathring{\eta}_{-4n})$ is of order 2, $\eta^*_{1,5}\beta_5(\eta\mathring{\eta}_{-4n})=\eta^2\omega_1^{4n}$, $\beta_5(\eta\mathring{\eta}_{-4n})$ is of order 2, $\eta^*_{1,5}(\hat{\eta}\eta^*_{5,6}[\eta\omega_1^{8n+3}]_6)=\eta^2\omega_1^{8n+2}$ and $2\hat{\eta}\eta^*_{5,6}[\eta\omega_1^{8n+3}]_6=\hat{\eta}^3\xi^*_{5,4}\omega_4^{n+1}\bar{\omega}_4^n$, by the computation of Im $\delta_{4,1}$ above (13.4) we obtain

Proposition 13.5. $(\pi_S^{5-p,-q-1}(S_+^{5,0}),\ p+q=6)$

i) $\pi_S^{8n-2,-8n}(S_+^{5,0})=\boldsymbol{Z}/4\cdot\hat{\eta}^2\omega_5^n\oplus\boldsymbol{Z}/2\cdot\xi^*_{5,1}(\nu^2\omega_1^{8n-6})$,

ii) $\pi_S^{8n-1,-8n-1}(S_+^{5,0})=\boldsymbol{Z}/2\cdot\beta_5(\hat{\eta}\mathring{\eta}_{-8n})\oplus\boldsymbol{Z}/2\cdot\chi\hat{\nu}\omega_5^n\oplus\boldsymbol{Z}/2\cdot\xi^*_{5,1}(\nu^2\omega_1^{8n-5})$,

iii) $\pi_S^{8n,-8n-2}(S_+^{5,0})=\boldsymbol{Z}/2\cdot\beta_5(\eta\mathring{\eta}_{-8n})\oplus\boldsymbol{Z}/2\cdot\rho\beta_5(\eta\mathring{\eta}_{-8n})\oplus\boldsymbol{Z}/2\cdot\xi^*_{5,1}(\nu^2\omega_1^{8n-4})$,

iv) $\pi_S^{8n+1,-8n-3}(S_+^{5,0})=\boldsymbol{Z}/4\cdot\beta_5(\chi\mathring{\nu}_{-8n})$,

v) $\pi_S^{8n+2,-8n-4}(S_+^{5,0})=\boldsymbol{Z}/4\cdot\hat{\eta}\eta^*_{5,6}[\eta\omega_1^{8n+3}]_6$,

vi) $\pi_S^{8n+3,-8n-5}(S_+^{5,0})=\boldsymbol{Z}/2\cdot\beta_5(\hat{\eta}\mathring{\eta}_{-8n-4})\oplus\boldsymbol{Z}/4\cdot\hat{\nu}\xi^*_{5,4}\omega_4^{n+1}\bar{\omega}_4^n$,

vii) $\pi_S^{8n+4,-8n-6}(S_+^{5,0})=\boldsymbol{Z}/2\cdot\beta_5(\eta\mathring{\eta}_{-8n-4})\oplus\boldsymbol{Z}/2\cdot\rho\beta_5(\eta\mathring{\eta}_{-8n-4})$,

viii) $\pi_S^{8n+5,-8n-7}(S_+^{5,0})=\boldsymbol{Z}/2\cdot\mathring{\nu}\xi^*_{5,4}\omega_4^{n+1}\bar{\omega}_4^n\oplus\boldsymbol{Z}/2\cdot\xi^*_{5,1}(\nu^2\omega_1^{8n+1})$

for any integer n.

Computation of $\pi_S^{6-p,-q-1}(S_+^{6,0})$ for $p+q=6$. $\delta_{5,1}(\eta^2\omega_1^{4n})=\delta_{5,1}(\eta^2\omega_1^{4n-1})=\delta_{5,1}(\eta^2\omega_1^{8n+6})=0$, $\delta_{5,1}(\eta^2\omega_1^{4n+1})=(1+\rho)\beta_5(\eta\mathring{\eta}_{-4n})$ of order 2 and $\delta_{5,1}(\eta^2\omega_1^{8n+2})=\delta_{5,1}\eta^*_{1,4}(\hat{\eta}^2\omega_4^{n+1}\bar{\omega}_4^n)=\beta_5(\chi^3\hat{\eta}^2\mathring{\nu}_{-8n})=4\beta_5(\chi\mathring{\nu}_{-8n})=0$. Thus we have

$$\text{(13.6)}\qquad \mathrm{Ker}\,[\delta_{5,1}\colon \bigoplus_{p+q=7}\pi_S^{6-p,-q-1}(S_+^{1,0}) \to \bigoplus_{p+q=7}\pi_S^{5-p,-q}(S_+^{5,0})]$$
$$= \boldsymbol{Z}[\omega_1^4, \omega_1^{-4}] \otimes \{\boldsymbol{Z}/2\cdot\eta^2 \oplus \boldsymbol{Z}/2\cdot\eta^2\omega_1^{-1} \oplus \boldsymbol{Z}/2\cdot\eta^2\omega_1^{-2}\}\,.$$

The exact sequence (1.9) for $r=5$, (13.2) and (10.5) give the short exact sequence

$$0 \to \bigoplus_{p+q=6} \pi_S^{5-p,-q-1}(S_+^{5,0})/\mathrm{Im}\,\delta_{5,1} \xrightarrow{\xi^*_{6,5}} \bigoplus_{p+q=6} \pi_S^{6-p,-q-1}(S_+^{6,0})$$
$$\xrightarrow{\eta^*_{1,6}} \boldsymbol{Z}[\omega_1^4, \omega_1^{-4}] \otimes \{\boldsymbol{Z}/2\cdot\eta \oplus \boldsymbol{Z}/2\cdot\eta\omega_1^{-1}\} \to 0\,.$$

Since $\eta^*_{1,6}\beta_6(\mathring{\eta}_{-4n})=\eta\omega_1^{4n}$, $\beta_6(\mathring{\eta}_{-4n})$ is of order 2, $\eta^*_{1,6}(\hat{\eta}\omega_6^n)=\eta\omega_1^{8n-1}$, $2\hat{\eta}\omega_6^n=\xi^*_{6,5}(\hat{\eta}^2\omega_5^n)$, $\eta^*_{1,6}[\eta\omega_1^{8n+3}]_6=\eta\omega_1^{8n+3}$, $[\eta\omega_1^{8n+3}]_6$ is of order 8 and Im $\delta_{5,1}$ is computed above (13.6), we obtain

Proposition 13.7. $(\pi_S^{6-p,-q-1}(S_+^{6,0}),\ p+q=6)$

i) $\pi_S^{8n-1,-8n}(S_+^{6,0})=\boldsymbol{Z}/8\cdot\hat{\eta}\omega_6^n\oplus\boldsymbol{Z}/2\cdot\xi^*_{6,1}(\nu^2\omega_1^{8n-6})$,

ii) $\pi_S^{8n,-8n-1}(S_+^{6,0})=\boldsymbol{Z}/2\cdot\beta_6(\mathring{\eta}_{-8n})\oplus\boldsymbol{Z}/2\cdot\rho\beta_6(\mathring{\eta}_{-8n})\oplus\boldsymbol{Z}/2\cdot\chi^2\hat{\nu}\omega_6^n$

$\oplus \boldsymbol{Z}/2\cdot\xi^*_{6,1}(\nu^2\omega_1^{8n-5})$,

iii) $\pi_S^{8n+1,-8n-2}(S_+^{6,0})=\boldsymbol{Z}/2\cdot\beta_6(\chi\eta\mathring{\eta}_{-8n})\oplus\boldsymbol{Z}/2\cdot\xi^*_{6,1}(\nu^2\omega_1^{8n-4})$,

iv) $\pi_S^{8n+2,-8n-3}(S_+^{6,0})=\boldsymbol{Z}/4\cdot\beta_6(\chi^2\mathring{\nu}_{-8n})$,

v) $\pi_S^{8n+3,-8n-4}(S_+^{6,0})=\boldsymbol{Z}/8\cdot[\eta\omega_1^{8n+3}]_6$,

vi) $\pi_S^{8n+4,-8n-5}(S_+^{6,0})=\boldsymbol{Z}/2\cdot\beta_6(\mathring{\eta}_{-8n-4})\oplus\boldsymbol{Z}/2\cdot\rho\beta_6(\mathring{\eta}_{-8n-4})\oplus\boldsymbol{Z}/4\cdot\hat{\eta}\xi^*_{6,4}\omega_4^{n+1}\overline{\omega}_4^n$,

vii) $\pi_S^{8n+5,-8n-6}(S_+^{6,0})=\boldsymbol{Z}/2\cdot\beta_6(\chi\eta\mathring{\eta}_{-8n-4})$,

viii) $\pi_S^{8n+6,-8n-7}(S_+^{6,0})=\boldsymbol{Z}/2\cdot\mathring{\nu}\xi^*_{6,4}\omega_4^{n+1}\overline{\omega}_4^n\oplus\boldsymbol{Z}/2\cdot\xi^*_{6,1}(\nu^2\omega_1^{8n+1})$

for any integer n.

Computation of $\pi_S^{7-p,-q-1}(S_+^{7,0})$ for $p+q=6$. $\delta_{6,1}(\eta\omega_1^{4n})=0$, $\delta_{6,1}(\eta\omega_1^{4n+1})=(1+\rho)\beta_6(\mathring{\eta}_{-4n})$ of order 2, $\delta_{6,1}(\eta\omega_1^{4n+2})=\beta_6(\chi\eta\mathring{\eta}_{-4n})$ of order 2, $\delta_{6,1}(\eta\omega_1^{8n+3})=\delta_{6,1}\eta^*_{1,4}(\hat{\eta}\omega_4^{n+1}\overline{\omega}_4^n)=\beta_6(\chi^3\hat{\eta}\mathring{\nu}_{-8n})=2\beta_6(\chi^2\mathring{\nu}_{-8n})$ of order 2 and $\delta_{6,1}(\eta\omega_1^{8n+7})=0$. Thus we have

(13.8)
$$\operatorname{Ker}[\delta_{6,1}\colon \bigoplus_{p+q=7}\pi_S^{7-p,-q-1}(S_+^{1,0})\to\bigoplus_{p+q=7}\pi_S^{6-p,-q}(S_+^{6,0})]$$
$$=\boldsymbol{Z}[\omega_1^8,\omega_1^{-8}]\otimes\{\boldsymbol{Z}/2\cdot\eta\oplus\boldsymbol{Z}/2\cdot\eta\omega_1^4\oplus\boldsymbol{Z}/2\cdot\eta\omega_1^{-1}\}\,.$$

The exact sequence (1.9) for $\imath=6$, (13.2) and (10.8) give the short exact sequence

$$0\to\bigoplus_{p+q=6}\pi_S^{6-p,-q-1}(S_+^{6,0})/\operatorname{Im}\delta_{6,1}\xrightarrow{\xi^*_{7,6}}\bigoplus_{p+q=6}\pi_S^{7-p,-q-1}(S_+^{7,0})$$
$$\xrightarrow{\eta^*_{1,7}}\boldsymbol{Z}[\omega_1^8,\omega_1^{-8}]\otimes\{\boldsymbol{Z}\cdot 1\oplus\sum_{i=1}^{3}\boldsymbol{Z}\cdot 2\omega_1^{2i}\}\to 0\,.$$

Thus we obtain

Proposition 13.9. $(\pi_S^{7-p,-q-1}(S_+^{7,0}),\ p+q=6)$

i) $\pi_S^{8n,-8n}(S_+^{7,0})=\boldsymbol{Z}\cdot\omega_7^n\oplus\boldsymbol{Z}/8\cdot(1+\rho)\omega_7^n\oplus\boldsymbol{Z}/2\cdot\xi^*_{7,1}(\nu^2\omega_1^{8n-6})$,

ii) $\pi_S^{8n+1,-8n-1}(S_+^{7,0})=\boldsymbol{Z}/2\cdot\beta_7(\chi\mathring{\eta}_{-8n})\oplus\boldsymbol{Z}/2\cdot\chi^3\mathring{\nu}\omega_7^n\oplus\boldsymbol{Z}/2\cdot\xi^*_{7,1}(\nu^2\omega_1^{8n-5})$,

iii) $\pi_S^{8n+2,-8n-2}(S_+^{7,0})=\boldsymbol{Z}\cdot\delta_{7,1}\omega_1^{8n+3}\oplus\boldsymbol{Z}/2\cdot\xi^*_{7,1}(\nu^2\omega_1^{8n-4})$,

iv) $\pi_S^{8n+3,-8n-3}(S_+^{7,0})=\boldsymbol{Z}/2\cdot\beta_7(\chi^3\mathring{\nu}_{-8n})$,

v) $\pi_S^{8n+4,-8n-4}(S_+^{7,0})=\boldsymbol{Z}\cdot\delta_{7,1}\omega_1^{8n+5}\oplus\boldsymbol{Z}/8\cdot\xi^*_{7,6}[\eta\omega_1^{8n+3}]_6$,

vi) $\pi_S^{8n+5,-8n-5}(S_+^{7,0})=\boldsymbol{Z}/2\cdot\beta_7(\chi\mathring{\eta}_{-8n-4})\oplus\boldsymbol{Z}/4\cdot\hat{\nu}\xi^*_{7,4}\omega_4^{n+1}\overline{\omega}_4^n$,

vii) $\pi_S^{8n+6,-8n-6}(S_+^{7,0})=\boldsymbol{Z}\cdot\delta_{7,1}\omega_1^{8n+7}$,

viii) $\pi_S^{8n+7,-8n-7}(S_+^{7,0})=\boldsymbol{Z}/2\cdot\mathring{\nu}\xi^*_{7,4}\omega_4^{n+1}\overline{\omega}_4^n\oplus\boldsymbol{Z}/2\cdot\xi^*_{7,1}(\nu^2\omega_1^{8n+1})$

for any integer n.

Computation of $\pi_S^{8-p,-q-1}(S_+^{8,0})$ for $p+q=6$. $\delta_{7,1}\omega_1^{8n}=0$, $\delta_{7,1}\omega_1^{8n+4}=\beta_7(\chi^3\mathring{\nu}_{-8n})$ of order 2, $\delta_{7,1}\omega_1^{4n+2}=\beta_7(\chi\mathring{\eta}_{-4n})$ of order 2, $\delta_{7,1}\omega_1^{8n+1}=2\omega_7^n-(1+\rho)\omega_7^n$ of order ∞, $\delta_{8,1}\omega_1^{8n+3}$, $\delta_{7,1}\omega_1^{8n+5}$ and $\delta_{7,1}\omega_1^{8n+7}$ are of order ∞. Thus we have

(13.10)
$$\operatorname{Ker}[\delta_{7,1}\colon\bigoplus_{p+q=7}\pi_S^{8-p,-p-1}(S_+^{1,0})\to\bigoplus_{p+q=7}\pi_S^{7-p,-q}(S_+^{7,0})]$$
$$=\boldsymbol{Z}[\omega_1^8,\omega_1^{-8}]\otimes\{\boldsymbol{Z}\cdot 1\oplus\sum_{i=1}^{3}\boldsymbol{Z}\cdot 2\omega_1^{2i}\}\,.$$

The exact sequence (1.9) for $r=7$ and Proposition 1.10 give the isomorphism

$$\xi^*_{8,7}\colon \bigoplus_{p+q=6} \pi_S^{7-p,-q-1}(S_+^{7,0})/\mathrm{Im}\ \delta_{7,1} \approx \bigoplus_{p+q=6} \pi_S^{8-p,-q-1}(S_+^{8,0})\,.$$

Thus we obtain

Proposition 13.11. $(\pi_S^{8-p,-q-1}(S_+^{8,0}),\ p+q=6)$

i) $\pi_S^{8n+1,-8n}(S_+^{8,0})=\boldsymbol{Z}/\mathbf{16}\cdot\chi\omega_8^n\oplus\boldsymbol{Z}/\mathbf{2}\cdot\xi^*_{8,1}(\nu^2\omega_1^{8n-6})$,

ii) $\pi_S^{8n+2,-8n-1}(S_+^{8,0})=\boldsymbol{Z}/\mathbf{2}\cdot\chi^2\eta\omega_8^n\oplus\boldsymbol{Z}/\mathbf{2}\cdot\xi^*_{8,1}(\nu^2\omega_1^{8n-5})$,

iii) $\pi_S^{8n+3,-8n-2}(S_+^{8,0})=\boldsymbol{Z}/\mathbf{2}\cdot\xi^*_{8,1}(\nu^2\omega_1^{8n-4})$,

iv) $\pi_S^{8n+4,-8n-3}(S_+^{8,0})=0$,

v) $\pi_S^{8n+5,-8n-4}(S_+^{8,0})=\boldsymbol{Z}/\mathbf{8}\cdot\xi^*_{8,6}[\eta\omega_1^{8n+3}]_6$,

vi) $\pi_S^{8n+6,-8n-5}(S_+^{8,0})=\boldsymbol{Z}/\mathbf{4}\cdot\hat{\nu}\xi^*_{8,4}\omega_4^{n+1}\overline{\omega}_4^n$,

vii) $\pi_S^{8n+7,-8n-6}(S_+^{8,0})=0$,

viii) $\pi_S^{8n+8,-8n-7}(S_+^{8,0})=\boldsymbol{Z}/\mathbf{2}\cdot\mathring{\nu}\xi^*_{8,4}\omega_4^{n+1}\overline{\omega}_4^n\oplus\boldsymbol{Z}/\mathbf{2}\cdot\xi^*_{8,1}(\nu^2\omega_1^{8n+1})$

for any integer n.

Proposition 13.11 describes $\lambda^S_{p,q}$ for $p+q=6$. By Proposition 4.8 the groups $\pi^S_{p,q}$ for $p+q=6$ are determined except $\pi^S_{3,3}$, $\pi^S_{4,2}$, $\pi^S_{5,1}$, $\pi^S_{6,0}$ and $\pi^S_{7,-1}$. By Corollary 7.2 we know that $Sqv\in\pi^S_{3,3}$ and is of order 24. Hence by the fixed-point exact sequence (1.12) for $r=8$ we see that $\pi^S_{3,3}=\boldsymbol{Z}/\mathbf{8}\cdot\tilde{\nu}_3\oplus\boldsymbol{Z}/\mathbf{24}\cdot Sqv$. By the forgetful exact sequence (1.5) we get that $\pi^S_{4,2}=\boldsymbol{Z}/\mathbf{2}\cdot\hat{\nu}^2$, $\pi^S_{5,1}=0$, $\pi^S_{6,0}=\boldsymbol{Z}\cdot\chi y_7\oplus\boldsymbol{Z}/\mathbf{2}\cdot\nu^2\delta_1\omega_1^{-5}$ and $\pi^S_{7,-1}=\boldsymbol{Z}/\mathbf{2}\cdot\nu^2\delta_1\omega_1^{-6}$. Thus we obtain

Theorem 13.12. $(\pi^S_{p,q},\ p+q=6)$

i) $\pi^S_{4,2}=\boldsymbol{Z}/\mathbf{2}\cdot\hat{\nu}^2$.

ii) $\pi^S_{5,1}=0$.

iii) $\pi^S_{6,0}=\boldsymbol{Z}\cdot\chi y_7\oplus\boldsymbol{Z}/\mathbf{2}\cdot\nu^2\delta_1\omega_1^{-5}$.

iv) $\pi^S_{7,-1}=\boldsymbol{Z}/\mathbf{2}\cdot\nu^2\delta_1\omega_1^{-6}$.

v) $\pi^S_{8n,-8n+6}\approx\boldsymbol{Z}/\mathbf{2}\cdot\mathring{\nu}\mathring{\nu}_{8n}\oplus\boldsymbol{Z}/\mathbf{2}\cdot\rho\mathring{\nu}\mathring{\nu}_{8n}\oplus\pi^S_{n+6}$ *for any integer n.*

vi) $\pi^S_{8n+1,-8n+5}\approx\pi^S_{-8n+5}$ *for any integer n.*

vii) $\pi^S_{8n+2,-8n+4}\approx\boldsymbol{Z}/\mathbf{4}\cdot\hat{\nu}\mathring{\nu}_{8n}\oplus\pi^S_{-8n+4}$ *for any integer n.*

viii) $\pi^S_{8n+3,-8n+3}\approx\boldsymbol{Z}/\mathbf{8}\cdot\tilde{\nu}_{8n+3}\oplus\pi^S_{-8n+3}$ *for any integer n.*

ix) $\pi^S_{8n+4,-8n+2}\approx\pi^S_{-8n+2}$ *for* $n\neq 0$.

x) $\pi^S_{8n+5,-8n+1}\approx\boldsymbol{Z}/\mathbf{2}\cdot\nu^2\delta_1\omega_1^{-8n-4}\oplus\pi^S_{-8n+1}$ *for* $n\neq 0$.

xi) $\pi^S_{8n+6,-8n}\approx\boldsymbol{Z}/\mathbf{2}\cdot\chi^2\eta\delta_8\omega_8^{-n}\oplus\boldsymbol{Z}/\mathbf{2}\cdot\nu^2\delta_1\omega_1^{-8n-5}\oplus\pi^S_{-8n}$ *for* $n\neq 0$.

xii) $\pi^S_{8n+7,-8n-1}\approx\boldsymbol{Z}/\mathbf{16}\cdot\chi\delta_8\omega_8^{-n}\oplus\boldsymbol{Z}/\mathbf{2}\cdot\nu^2\delta_1\omega_1^{-8n-6}\oplus\pi^S_{-8n-1}$ *for* $n\neq 0$.

Since $\delta_{4,1}(\nu\omega_1^{8n+3})=2\hat{\nu}\omega_4^{n+1}\overline{\omega}_4^n+\xi^*_{4,1}(\nu^2\omega_1^{8n-1})$, we get

Proposition 13.13. $2\hat{\nu}\mathring{\nu}_{8n}=\nu^2\delta_1\omega_1^{-8n-1}$.

14. 7 stem

Proposition 14.1. *There is an element* $\hat{\sigma}\in\pi_{4,3}^S$ *such that* $\psi(\hat{\sigma})=\sigma$ *and* $\phi(\hat{\sigma})=\nu$.

Proof. Let $\mathfrak{C}$ be the Cayley numbers and $e_0=1, e_1,\cdots,e_7$ the canonical basis of $\mathfrak{C}$ over $\boldsymbol{R}$. Put $\varepsilon=e_1e_2e_3$, then $\varepsilon e_i\varepsilon=e_i$ for $0\leqslant i\leqslant 3$ and $\varepsilon e_i\varepsilon=-e_i$ for $4\leqslant i\leqslant 7$. Identify $\mathfrak{C}=\boldsymbol{R}^{4,4}=\boldsymbol{R}\{e_4, e_5, e_6, e_7\}\oplus\boldsymbol{R}\{1, e_1, e_2, e_3\}$ by the involution $\tau(x)=\varepsilon x\varepsilon$ for $x\in\mathfrak{C}$. By parallel arguments to the cases of $\hat{\eta}$ and $\hat{\nu}$, the product of the Cayley numbers gives an element $\hat{\sigma}\in\pi_{4,3}^S$ such that $\psi(\hat{\sigma})=\pm\sigma$ and $\phi(\hat{\sigma})=\pm\nu$. Replace it with $\pm\hat{\sigma}$ or $\pm\rho\hat{\sigma}$ if necessary, then we get a desired element. □

Computation of $\pi_S^{2-p,-q-1}(S_+^{2,0})$ for $p+q=7$. Observe the short exact sequence (7.4) for $r=7$. $\eta_{1,2}^*(\nu^2\omega_2^n)=\nu^2\omega_1^{2n}$, $\nu^2\omega_2^n$ is of order 2, $\eta_{1,2}^*(\tilde{\nu}_3\omega_2^n)=\nu^2\omega_1^{2n-3}$ by Proposition 12.5 and $2\tilde{\nu}_3\omega_2^n=\chi\hat{\eta}\tilde{\nu}_3\omega_2^n=\xi_{2,1}^*(\beta_1(\hat{\eta}\tilde{\nu}_3)\omega_1^{2n})=\xi_{2,1}^*(\eta\nu^2\omega_1^{2n-4})=0$, *i.e.* $\tilde{\nu}_3\omega_2^n$ is of order 2. Thus we obtain

Proposition 14.2. $(\pi_S^{2-p,-q-1}(S_+^{2,0}), p+q=7)$

i) $\pi_S^{2n,-2n-6}(S_+^{2,0})=\boldsymbol{Z}/2\cdot\nu^2\omega_2^n\oplus\boldsymbol{Z}/240\cdot\xi_{2,1}^*(\sigma\omega_1^{2n-1})$,

ii) $\pi_S^{2n-3,-2n-3}(S_+^{2,0})=\boldsymbol{Z}/2\cdot\tilde{\nu}_3\omega_2^n\oplus\boldsymbol{Z}/2\cdot\xi_{2,1}^*(\sigma\omega_1^{2n-4})$

for any integer n.

Computation of $\pi_S^{4-p,-q-1}(S_+^{4,0})$ for $p+q=7$. $\delta_{2,1}(\nu^2\omega_1^{4n})=0$, $\delta_{2,1}(\nu^2\omega_1^{4n+1})=\delta_{2,1}\eta_{1,3}^*(\tilde{\nu}_3\omega_3^{n+1})=0$, $\delta_{2,1}(\nu^2\omega_1^{4n+2})=\delta_{2,1}\eta_{1,3}^*(\hat{\nu}\nu\omega_3^{n+1})=0$ and $\delta_{2,1}(\nu^2\omega_1^{4n+3})=2\nu^2\omega_2^{2n+1}=0$ by Proposition 8.4. Thus we have

$$\text{(14.3)}\qquad \operatorname{Ker}\Big[\delta_{2,1}\colon \bigoplus_{p+q=8}\pi_S^{3-p,-q-1}(S_+^{1,0})\to\bigoplus_{p+q=8}\pi_S^{2-p,-q}(S_+^{2,0})\Big] = \boldsymbol{Z}[\omega_1,\omega_1^{-1}]\otimes\boldsymbol{Z}/2\cdot\nu^2.$$

The exact sequences (1.9) for $r=2$ and $r=3$ give isomorphisms

$$\xi_{3,2}^*\colon \bigoplus_{p+q=7}\pi_S^{2-p,-q-1}(S_+^{2,0})\approx\bigoplus_{p+q=7}\pi_S^{3-p,-q-1}(S_+^{3,0}),$$

$$\xi_{4,3}^*\colon \bigoplus_{p+q=7}\pi_S^{3-p,-q-1}(S_+^{3,0})\approx\bigoplus_{p+q=7}\pi_S^{4-p,-q-1}(S_+^{4,0}).$$

Since $\chi^2\tilde{\nu}_{-8n+3}=\hat{\eta}\mathring{\nu}_{-8n}$ by Proposition 12.5, we obtain

Proposition 14.4. $(\pi_S^{4-p,-q-1}(S_+^{4,0}), p+q=7)$

i) $\pi_S^{8n-4,-8n}(S_+^{4,0})=\boldsymbol{Z}/2\cdot\chi^2\mathring{\nu}\mathring{\nu}\omega_4^n\bar{\omega}_4^{n-1}\oplus\boldsymbol{Z}/240\cdot\xi_{4,1}^*(\sigma\omega_1^{8n-7})$,

ii) $\pi_S^{4n-3,-4n-1}(S_+^{4,0})=\boldsymbol{Z}/2\cdot\tilde{\nu}_3\xi_{4,2}^*\omega_2^{2n-1}\oplus\boldsymbol{Z}/2\cdot\xi_{4,1}^*(\sigma\omega_1^{4n-6})$,

iii) $\pi_S^{8n-2,-8n-2}(S_+^{4,0})=\boldsymbol{Z}/2\cdot\chi^2\nu^2\omega_4^n\bar{\omega}_4^{n-1}\oplus\boldsymbol{Z}/240\cdot\xi_{4,1}^*(\sigma\omega_1^{8n-5})$,

iv) $\pi_S^{8n-1,-8n-3}(S_+^{4,0})=\boldsymbol{Z}/2\cdot\beta_4(\mathring{\eta}\mathring{\nu}_{-8n})\oplus\boldsymbol{Z}/2\cdot\xi_{4,1}^*(\sigma\omega_1^{8n-4})$,

v) $\pi_S^{8n,-8n-4}(S_+^{4,0})=\boldsymbol{Z}/2\cdot\beta_4(\chi^2\mathring{\nu}\mathring{\nu}_{-8n})\oplus\boldsymbol{Z}/240\cdot\xi_{4,1}^*(\sigma\omega_1^{8n-3})$,

vi) $\pi_S^{8n+2,-8n-6}(S_+^{4,0})=\boldsymbol{Z}/2\cdot\beta_4(\chi^2\nu\mathring{\nu}_{-8n})\oplus\boldsymbol{Z}/240\cdot\xi_{4,1}^*(\sigma\omega_1^{8n-1})$,

vii) $\pi_S^{8n+3,-8n-7}(S_+^{4,0})=\boldsymbol{Z}/2\cdot\mathring{\eta}\mathring{\nu}\omega_4^{n+1}\bar{\omega}_4^n\oplus\boldsymbol{Z}/2\cdot\xi_{4,1}^*(\sigma\omega_1^{8n})$

for any integer n.

Computation of $\pi_S^{5-p,-q-1}(S_+^{5,0})$ for $p+q=7$. The exact sequence (1.9) for $r=4$ and (13.4) give the short exact sequence

$$0\to\bigoplus_{p+q=7}\pi_S^{4-p,-q-1}(S_+^{4,0})\xrightarrow{\xi_{5,4}^*}\bigoplus_{p+q=7}\pi_S^{5-p,-q-1}(S_+^{5,0})\xrightarrow{\eta_{1,5}^*}$$
$$\boldsymbol{Z}[\omega_1^8,\omega_1^{-8}]\otimes\{\boldsymbol{Z}/24\cdot\nu\omega_1^{-2}\oplus\boldsymbol{Z}/24\cdot\nu\oplus\boldsymbol{Z}/12\cdot2\nu\omega_1^2\oplus\boldsymbol{Z}/12\cdot2\nu\omega_1^4\}$$
$$\oplus\boldsymbol{Z}[\omega_1^4,\omega_1^{-4}]\otimes\{\boldsymbol{Z}/2\cdot\eta^3\omega_1^{-1}\oplus\boldsymbol{Z}/2\cdot\eta^3\omega_1\}\to 0\,.$$

By [10], Theorem 2.2, we see that $\phi(\pi_{7,0}^S)=2^4\pi_0^S$ and $\phi(\pi_{4,0}^S)=2^5\pi_0^S$. By the forgetful exact sequence (1.5) with this fact we know that $\psi(y_7)\neq0$ and $\psi(2y_7)=0$. Hence

$$\psi(y_7)=120\sigma,\ i.e.\ \beta_1(y_7)=120\sigma\omega_1^{-7}\,. \tag{14.5}$$

$\eta_{1,5}^*(\mathring{\eta}^3\omega_5^n)=\eta^3\omega_1^{8n-3}$, $2\mathring{\eta}^3\omega_5^n=\xi_{5,4}^*(\mathring{\eta}^4\omega_4^n\bar{\omega}_4^n)=\xi_{5,4}^*(\chi^3y_7\omega_4^n\bar{\omega}_4^n)=\xi_{5,1}^*(\beta_1(y_7)\omega_1^{8n})=120\xi_{5,1}^*(\sigma\omega_1^{8n-7})$, $\eta_{1,5}^*(\mathring{\nu}\omega_5^n)=\nu\omega_1^{8n-2}$, $\mathring{\nu}\omega_5^n$ is of order 24, $\eta_{1,5}^*\beta_5(\mathring{\eta}\eta\mathring{\eta}_{-4n})=\eta^3\omega_1^{4n-1}$, $\beta_5(\mathring{\eta}\eta\mathring{\eta}_{-4n})$ is of order 2, $\eta_{1,5}^*\beta_5(\mathring{\nu}_{-8n})=\nu\omega_1^{8n}$, $\beta_5(\mathring{\nu}_{-8n})$ is of order 24, $\eta_{1,5}^*\beta_5(\nu\delta_1\omega_1^{8n+3})=2\nu\omega_1^{8n+2}$, $\beta_5(\nu\delta_1\omega_1^{8n+3})$ is of order 12 by Theorem 10.11, vi), $\eta_{1,5}^*\beta_5(\nu\delta_1\omega_1^{8n+5})=2\nu\omega_1^{8n+4}$, $\beta_5(\nu\delta_1\omega_1^{8n+5})=(\nu\delta_1\omega_1^{-3})\omega_5^{n+1}$ is of order 12 by Theorem 10.11, iii), and the periodicity, $\eta_{1,5}^*(\mathring{\eta}^2\eta_{5,6}^*[\eta\omega_1^{8n+3}]_6)=\eta^3\omega_1^{8n+1}$ and $2\mathring{\eta}^2\eta_{5,6}^*[\eta\omega_1^{8n+3}]_6=\xi_{5,4}^*(\mathring{\eta}^4\omega_4^{n+1}\bar{\omega}_4^n)=\xi_{5,4}^*(\chi^3y_7\omega_4^{n+1}\bar{\omega}_4^n)=\xi_{5,1}^*(\beta_1(y_7)\omega_1^{8n+4})=120\xi_{5,1}^*(\sigma\omega_1^{8n-3})$. Thus we obtain

Proposition 14.6. ($\pi_S^{5-p,-q-1}(S_+^{5,0})$, $p+q=7$)

i) $\pi_S^{8n-3,-8n}(S_+^{5,0})=\boldsymbol{Z}/2\cdot\{\mathring{\eta}^3\omega_5^n-60\xi_{5,1}^*(\sigma\omega_1^{8n-7})\}\oplus\boldsymbol{Z}/2\cdot\chi^2\mathring{\nu}\mathring{\nu}\xi_{5,4}^*\omega_4^n\bar{\omega}_4^{n-1}$
$\oplus\boldsymbol{Z}/240\cdot\xi_{5,1}^*(\sigma\omega_1^{8n-7})$,

ii) $\pi_S^{8n-2,-8n-1}(S_+^{5,0})=\boldsymbol{Z}/24\cdot\mathring{\nu}\omega_5^n\oplus\boldsymbol{Z}/2\cdot\tilde{\nu}_3\xi_{5,2}^*\omega_2^{4n-1}\oplus\boldsymbol{Z}/2\cdot\xi_{5,1}^*(\sigma\omega_1^{8n-6})$,

iii) $\pi_S^{8n-1,-8n-2}(S_+^{5,0})=\boldsymbol{Z}/2\cdot\beta_5(\mathring{\eta}\eta\mathring{\eta}_{-8n})\oplus\boldsymbol{Z}/2\cdot\chi^2\nu^2\xi_{5,4}^*\omega_4^n\bar{\omega}_4^{n-1}$
$\oplus\boldsymbol{Z}/240\cdot\xi_{5,1}^*(\sigma\omega_1^{8n-5})$,

iv) $\pi_S^{8n,-8n-3}(S_+^{5,0})=\boldsymbol{Z}/24\cdot\beta_5(\mathring{\nu}_{-8n})\oplus\boldsymbol{Z}/2\cdot(1+\rho)\beta_5(\mathring{\nu}_{-8n})\oplus\boldsymbol{Z}/2\cdot\xi_{5,1}^*(\sigma\omega_1^{8n-4})$,

v) $\pi_S^{8n+1,-8n-4}(S_+^{5,0})=\boldsymbol{Z}/2\cdot\{\mathring{\eta}^2\eta_{5,6}^*[\eta\omega_1^{8n+3}]_6-60\xi_{5,1}^*(\sigma\omega_1^{8n-3})\}\oplus\boldsymbol{Z}/2\cdot\beta_5(\chi^3\mathring{\nu}\mathring{\nu}_{-8n})$
$\oplus\boldsymbol{Z}/240\cdot\xi_{5,1}^*(\sigma\omega_1^{8n-3})$,

vi) $\pi_S^{8n+2,-8n-5}(S_+^{5,0})=\boldsymbol{Z}/12\cdot\beta_5(\nu\delta_1\omega_1^{8n+3})\oplus\boldsymbol{Z}/2\cdot\tilde{\nu}_3\xi_{5,2}^*\omega_2^{4n+1}\oplus\boldsymbol{Z}/2\cdot\xi_{5,1}^*(\sigma\omega_1^{8n-2})$,

vii) $\pi_S^{8n+3,-8n-6}(S_+^{5,0})=\boldsymbol{Z}/2\cdot\beta_5(\mathring{\eta}\eta\mathring{\eta}_{-8n-4})\oplus\boldsymbol{Z}/2\cdot\beta_5(\chi^3\nu\mathring{\nu}_{-8n})\oplus\boldsymbol{Z}/240\cdot\xi_{5,1}^*(\sigma\omega_1^{8n-1})$,

viii) $\pi_S^{8n+4,-8n-7}(S_+^{5,0})=\boldsymbol{Z}/12\cdot\beta_5(\nu\delta_1\omega_1^{8n+5})\oplus\boldsymbol{Z}/2\cdot\mathring{\eta}\mathring{\nu}\xi_{5,4}^*\omega_4^{n+1}\bar{\omega}_4^n\oplus\boldsymbol{Z}/2\cdot\xi_{5,1}^*(\sigma\omega_1^{8n})$

for any integer n.

Computation of $\pi_S^{6-p,-q-1}(S_+^{6,0})$ for $p+q=7$. $\delta_{5,1}(\nu\omega_1^{8n})=0$, $\delta_{5,1}(\nu\omega_1^{8n+1})=2\beta_5(\mathring{\nu}_{-8n})-(1+\rho)\beta_5(\mathring{\nu}_{-8n})$ of order 12, $\delta_{5,1}(\nu\omega_1^{8n+2})=\delta_{5,1}(\hat{\nu}\omega_1^{8n+4})=\beta_5(\chi^3\hat{\nu}\mathring{\nu}_{-8n})$ of order 2, $\delta_{5,1}(\nu\omega_1^{8n+3})=\beta_5(\nu\delta_1\omega_1^{8n+3})$ of order 12, $\delta_{5,1}(\nu\omega_1^{8n+4})=\beta_5(\chi^3\nu\mathring{\nu}_{-8n})$ of order 2, $\delta_{5,1}(\nu\omega_1^{8n+5})=\beta_5(\nu\delta_1\omega_1^{8n+5})$ of order 12, $\delta_{5,1}(\nu\omega_1^{8n+6})=\delta_{5,1}(\hat{\nu}\omega_1^{8n+8})=0$ and $\delta_{5,1}(\nu\omega_1^{8n+7})=\delta_{5,1}(\hat{\nu}\omega_1^{8n+9})=2\hat{\nu}\omega_5^{n+1}$ of order 12. Thus we have

$$(14.7)\qquad \mathrm{Ker}\,[\delta_{5,1}\colon \bigoplus_{p+q=8}\pi_S^{6-p,-q-1}(S_+^{1,0})\to\bigoplus_{p+q=8}\pi_S^{5-p,-q}(S_+^{5,0})]$$
$$=\boldsymbol{Z}[\omega_1^8,\omega_1^{-8}]\otimes\{\boldsymbol{Z}/\boldsymbol{24}\cdot\nu\omega_1^{-2}\oplus\boldsymbol{Z}/\boldsymbol{24}\cdot\nu\oplus\boldsymbol{Z}/\boldsymbol{12}\cdot 2\nu\omega_1^2\oplus\boldsymbol{Z}/\boldsymbol{12}\cdot 2\nu\omega_1^4\}$$
$$\oplus\ \boldsymbol{Z}[\omega_1^4,\omega_1^{-4}]\otimes\{\boldsymbol{Z}/2\cdot\eta^3\omega_1^{-1}\oplus\boldsymbol{Z}/2\cdot\eta^3\omega_1\}\ .$$

The exact sequence (1.9) for $r=5$ and (13.6) give the short exact sequence

$$0\to\bigoplus_{p+q=7}\pi_S^{5-p,-q-1}(S_+^{5,0})/\mathrm{Im}\,\delta_{5,1}\xrightarrow{\xi_{6,5}^*}\bigoplus_{p+q=7}\pi_S^{6-p,-q-1}(S_+^{6,0})$$
$$\xrightarrow{\eta_{1,6}^*}\boldsymbol{Z}[\omega_1^4,\omega_1^{-4}]\otimes\{\boldsymbol{Z}/2\cdot\eta^2\oplus\boldsymbol{Z}/2\cdot\eta^2\omega_1^{-1}\oplus\boldsymbol{Z}/2\cdot\eta^2\omega_1^{-2}\}\to 0\,.$$

$\eta_{1,6}^*(\hat{\eta}^2\omega_6^n)=\eta^2\omega_1^{8n-2}$, $2\hat{\eta}^2\omega_6^n=\xi_{6,5}^*(\hat{\eta}^3\omega_5^n)$, $\eta_{1,6}^*\beta_6(\hat{\eta}\mathring{\eta}_{-4n})=\eta^2\omega_1^{4n-1}$, $\eta_{1,6}^*\beta_6(\eta\mathring{\eta}_{-4n})=\eta^2\omega_1^{4n}$, $\beta_6(\hat{\eta}\mathring{\eta}_{-4n})$ and $\beta_6(\eta\mathring{\eta}_{-4n})$ are of order 2, $\eta_{1,6}^*(\hat{\eta}[\eta\omega_1^{8n+3}]_6)=\eta^2\omega_1^{8n+2}$, $2\hat{\eta}[\eta\omega_1^{8n+3}]_6=\xi_{6,5}^*(\hat{\eta}^2\eta_{5,6}^*[\eta\omega_1^{8n+3}]_6)$ and $\mathrm{Im}\,\delta_{5,1}$ has been computed above (14.7). Thus we obtain

Proposition 14.8. $(\pi_S^{6-p,-q-1}(S_+^{6,0}),\,p+q=7)$

i) $\pi_S^{8n-2,-8n}(S_+^{6,0})=\boldsymbol{Z}/\boldsymbol{4}\cdot\{\hat{\eta}^2\omega_6^n-30\xi_{6,1}^*(\sigma\omega_1^{8n-7})\}\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\chi^2\hat{\nu}\mathring{\nu}\xi_{6,4}^*\omega_4^n\overline{\omega}_4^{n-1}$ $\oplus\boldsymbol{Z}/\boldsymbol{240}\cdot\xi_{6,1}^*(\sigma\omega_1^{8n-7})$,

ii) $\pi_S^{8n-1,-8n-1}(S_+^{6,0})=\boldsymbol{Z}/\boldsymbol{2}\cdot\beta_6(\hat{\eta}\mathring{\eta}_{-8n})\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\chi\hat{\nu}\omega_6^n\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\tilde{\nu}_3\xi_{6,2}^*\omega_2^{4n-1}$ $\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\xi_{6,1}^*(\sigma\omega_1^{8n-6})$,

iii) $\pi_S^{8n,-8n-2}(S_+^{6,0})=\boldsymbol{Z}/\boldsymbol{2}\cdot\beta_6(\eta\mathring{\eta}_{-8n})\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\rho\beta_6(\eta\mathring{\eta}_{-8n})\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\chi^2\nu^2\xi_{6,4}^*\omega_4^n\overline{\omega}_4^{n-1}$ $\oplus\boldsymbol{Z}/\boldsymbol{240}\cdot\xi_{6,1}^*(\sigma\omega_1^{8n-5})$,

iv) $\pi_S^{8n+1,-8n-3}(S_+^{6,0})=\boldsymbol{Z}/\boldsymbol{4}\cdot\beta_6(\chi\mathring{\nu}_{-8n})\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\xi_{6,1}^*(\sigma\omega_1^{8n-4})$,

v) $\pi_S^{8n+2,-8n-4}(S_+^{6,0})=\boldsymbol{Z}/\boldsymbol{4}\cdot\{\hat{\eta}[\eta\omega_1^{8n+3}]_6-30\xi_{6,1}^*(\sigma\omega_1^{8n-3})\}\oplus\boldsymbol{Z}/\boldsymbol{240}\cdot\xi_{6,1}^*(\sigma\omega_1^{8n-3})$,

vi) $\pi_S^{8n+3,-8n-5}(S_+^{6,0})=\boldsymbol{Z}/\boldsymbol{2}\cdot\beta_6(\hat{\eta}\mathring{\eta}_{-8n-4})\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\tilde{\nu}_3\xi_{6,2}^*\omega_2^{4n+1}\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\xi_{6,1}^*(\sigma\omega_1^{8n-2})$,

vii) $\pi_S^{8n+4,-8n-6}(S_+^{6,0})=\boldsymbol{Z}/\boldsymbol{2}\cdot\beta_6(\eta\mathring{\eta}_{-8n-4})\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\rho\beta_6(\eta\mathring{\eta}_{-8n-4})\oplus\boldsymbol{Z}/\boldsymbol{240}\cdot\xi_{6,1}^*(\sigma\omega_1^{8n-1})$,

viii) $\pi_S^{8n+5,-8n-7}(S_+^{6,0})=\boldsymbol{Z}/\boldsymbol{2}\cdot\hat{\eta}\mathring{\nu}\xi_{6,4}^*\omega_4^{n+1}\overline{\omega}_4^n\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\xi_{6,1}^*(\sigma\omega_1^{8n})$

for any integer n.

Computation of $\pi_S^{7-p,-q-1}(S_+^{7,0})$ for $p+q=7$. $\delta_{6,1}(\eta^2\omega_1^{4n})=\delta_{6,1}(\eta^2\omega_1^{4n-1})=\delta_{6,1}(\eta^2\omega_1^{8n-2})=0$, $\delta_{6,1}(\eta^2\omega_1^{4n+1})=(1+\rho)\beta_6(\eta\mathring{\eta}_{-4n})$ of order 2 and $\delta_{6,1}(\eta^2\omega_1^{8n+2})=\delta_{6,1}(\hat{\eta}^2\omega_1^{8n+4})=\beta_6(\chi^3\hat{\eta}^2\mathring{\nu}_{-8n})=4\beta_6(\chi\mathring{\nu}_{-8n})=0$. Thus we have

$$(14.9)\qquad \mathrm{Ker}\,[\delta_{6,1}\colon\bigoplus_{p+q=8}\pi_S^{7-p,-q-1}(S_+^{1,0})\to\bigoplus_{p+q=8}\pi_S^{6-p,-q}(S_+^{6,0})]$$
$$=\boldsymbol{Z}[\omega_1^4,\omega_1^{-4}]\otimes\{\boldsymbol{Z}/2\cdot\eta^2\oplus\boldsymbol{Z}/2\cdot\eta^2\omega_1^{-1}\oplus\boldsymbol{Z}/2\cdot\eta^2\omega_1^{-2}\}\ .$$

The exact sequence (1.9) for $r=6$ and (13.8) give the short exact sequence

$$0 \to \bigoplus_{p+q=7} \pi_S^{6-p,-q-1}(S_+^{6,0})/\mathrm{Im}\ \delta_{6,1} \xrightarrow{\xi_{7,6}^*} \bigoplus_{p+q=7} \pi_S^{7-p,-q-1}(S_+^{7,0})$$
$$\xrightarrow{\eta_{1,7}^*} \boldsymbol{Z}[\omega_1^8, \omega_1^{-8}]\otimes\{\boldsymbol{Z}/2\cdot\eta\omega_1^{-1}\oplus\boldsymbol{Z}/2\cdot\eta\oplus\boldsymbol{Z}/2\cdot\eta\omega_1^4\} \to 0\,.$$

Since $\eta_{1,7}^*(\hat{\eta}\omega_7^n)=\eta\omega_1^{8n-1}$, $2\hat{\eta}\omega_7^n=\xi_{7,6}^*(\hat{\eta}^2\omega_6^n)$, $\eta_{1,7}^*\beta_7(\mathring{\eta}_{-4n})=\eta\omega_1^{4n}$, $\beta_7(\mathring{\eta}_{-4n})$ is of order 2 and Im $\delta_{6,1}$ is computed above (14.9), we obtain

Proposition 14.10. $(\pi_S^{7-p,-q-1}(S_+^{7,0}),\ p+q=7)$

i) $\pi_S^{8n-1,-8n}(S_+^{7,0})=\boldsymbol{Z}/8\cdot\{\hat{\eta}\omega_7^n-15\xi_{7,1}^*(\sigma\omega_1^{8n-7})\}\oplus\boldsymbol{Z}/2\cdot\chi^2\hat{\nu}\mathring{\nu}\xi_{7,4}^*\omega_4^n\bar{\omega}_4^{n-1}$
$\oplus\boldsymbol{Z}/240\cdot\xi_{7,1}^*(\sigma\omega_1^{8n-7})$,

ii) $\pi_S^{8n,-8n-1}(S_+^{7,0})=\boldsymbol{Z}/2\cdot\beta_7(\mathring{\eta}_{-8n})\oplus\boldsymbol{Z}/2\cdot\rho\beta_7(\mathring{\eta}_{-8n})\oplus\boldsymbol{Z}/2\cdot\chi^2\hat{\nu}\omega_7^n$
$\oplus\boldsymbol{Z}/2\cdot\tilde{\nu}_3\xi_{7,2}^*(\omega_2^{4n-1})\oplus\boldsymbol{Z}/2\cdot\xi_{7,1}^*(\sigma\omega_1^{8n-6})$,

iii) $\pi_S^{8n+1,-8n-2}(S_+^{7,0})=\boldsymbol{Z}/2\cdot\beta_7(\chi\eta\mathring{\eta}_{-8n})\oplus\boldsymbol{Z}/2\cdot\chi^2\nu^2\xi_{7,4}^*\omega_4^n\bar{\omega}_4^{n-1}$
$\oplus\boldsymbol{Z}/240\cdot\xi_{7,1}^*(\sigma\omega_1^{8n-5})$,

iv) $\pi_S^{8n+2,-8n-3}(S_+^{7,0})=\boldsymbol{Z}/4\cdot\beta_7(\chi^2\mathring{\nu}_{-8n})\oplus\boldsymbol{Z}/2\cdot\xi_{7,1}^*(\sigma\omega_1^{8n-4})$,

v) $\pi_S^{8n+3,-8n-4}(S_+^{7,0})=\boldsymbol{Z}/4\cdot\{\hat{\eta}\xi_{7,6}^*[\eta\omega_1^{8n+3}]_6-30\xi_{7,1}^*(\sigma\omega_1^{8n-3})\}$
$\oplus\boldsymbol{Z}/240\cdot\xi_{7,1}^*(\sigma\omega_1^{8n-3})$,

vi) $\pi_S^{8n+4,-8n-5}(S_+^{7,0})=\boldsymbol{Z}/2\cdot\beta_7(\mathring{\eta}_{-8n-4})\oplus\boldsymbol{Z}/2\cdot\rho\beta_7(\mathring{\eta}_{-8n-4})\oplus\boldsymbol{Z}/2\cdot\tilde{\nu}_3\xi_{7,2}^*\omega_2^{4n+1}$
$\oplus\boldsymbol{Z}/2\cdot\xi_{7,1}^*(\sigma\omega_1^{8n-2})$,

vii) $\pi_S^{8n+5,-8n-6}(S_+^{7,0})=\boldsymbol{Z}/2\cdot\beta_7(\chi\eta\mathring{\eta}_{-8n-4})\oplus\boldsymbol{Z}/240\cdot\xi_{7,1}^*(\sigma\omega_1^{8n-1})$,

viii) $\pi_S^{8n+6,-8n-7}(S_+^{7,0})=\boldsymbol{Z}/2\cdot\hat{\eta}\mathring{\nu}\xi_{7,4}^*\omega_4^{n+1}\bar{\omega}_4^n\oplus\boldsymbol{Z}/2\cdot\xi_{7,1}^*(\sigma\omega_1^{8n})$

for any integer n.

Computation of $\pi_S^{8-p,-p-1}(S_+^{8,0})$ for $p+q=7$. $\delta_{7,1}(\eta\omega_1^{4n})=\delta_{7,1}(\eta\omega_1^{8n-1})=0$, $\delta_{7,1}(\eta\omega_1^{4n+1})=(1+\rho)\beta_7(\mathring{\eta}_{-4n})$ of order 2, $\delta_{7,1}(\eta\omega_1^{4n+2})=\beta_7(\chi\eta\mathring{\eta}_{-4n})$ of order 2 and $\delta_{7,1}(\eta\omega_1^{8n+3})=\delta_{7,1}(\hat{\eta}\omega_1^{8n+4})=2\beta_7(\chi^2\mathring{\nu}_{-8n})$ of order 2. Thus we have

$$\text{(14.11)} \qquad \mathrm{Ker}[\delta_{7,1}\colon \bigoplus_{p+q=8} \pi_S^{8-p,-q-1}(S_+^{1,0}) \to \bigoplus_{p+q=8} \pi_S^{7-p,-q}(S_+^{7,0})]$$
$$= \boldsymbol{Z}[\omega_1^8, \omega_1^{-8}]\otimes\{\boldsymbol{Z}/2\cdot\eta\omega_1^{-1}\oplus\boldsymbol{Z}/2\cdot\eta\oplus\boldsymbol{Z}/2\cdot\eta\omega_1^4\}\,.$$

The exact sequence (1.9) for $r=7$ and (13.10) give the short exact sequence

$$0 \to \bigoplus_{p+q=7} \pi_S^{7-p,-q-1}(S_+^{7,0})/\mathrm{Im}\ \delta_{7,1} \xrightarrow{\xi_{8,7}^*} \bigoplus_{p+q=7} \pi_S^{8-p,-q-1}(S_+^{8,0})$$
$$\xrightarrow{\eta_{1,8}^*} \boldsymbol{Z}[\omega_1^8, \omega_1^{-8}]\otimes\{\boldsymbol{Z}\cdot 1\oplus\sum_{i=1}^{3}\boldsymbol{Z}\cdot 2\omega_1^{2i}\} \to 0\,.$$

Thus we obtain

Proposition 14.12. $(\pi_S^{8-p,-q-1}(S_+^{8,0}),\ p+q=7)$

i) $\pi_S^{8n,-8n}(S_+^{8,0})=\boldsymbol{Z}\cdot\omega_8^n\oplus\boldsymbol{Z}/\mathbf{16}\cdot(1+\rho)\omega_8^n\oplus\boldsymbol{Z}/\mathbf{2}\cdot\chi^2\hat{\nu}\mathring{\nu}\xi_{8,4}^*\omega_4^n\bar{\omega}_4^{n-1}$
$\oplus\boldsymbol{Z}/\mathbf{120}\cdot\{(1+\rho)\omega_8^n-\xi_{8,1}^*(\sigma\omega_1^{8n-7})\}$,

ii) $\pi_S^{8n+1,-8n-1}(S_+^{8,0})=\boldsymbol{Z}/\mathbf{2}\cdot\beta_8(\chi\mathring{\eta}_{-8n})\oplus\boldsymbol{Z}/\mathbf{2}\cdot\chi^3\hat{\nu}\omega_8^n\oplus\boldsymbol{Z}/\mathbf{2}\cdot\bar{\nu}_3\xi_{8,2}^*\omega_2^{4n-1}$
$\oplus\boldsymbol{Z}/\mathbf{2}\cdot\xi_{8,1}^*(\sigma\omega_1^{8n-6})$,

iii) $\pi_S^{8n+2,-8n-2}(S_+^{8,0})=\boldsymbol{Z}\cdot\delta_{8,1}\omega_1^{8n+3}\oplus\boldsymbol{Z}/\mathbf{2}\cdot\chi^2\nu^2\xi_{8,4}^*\omega_4^n\bar{\omega}_4^{n-1}\oplus\boldsymbol{Z}/\mathbf{240}\cdot\xi_{8,1}^*(\sigma\omega_1^{8n-5})$,

iv) $\pi_S^{8n+3,-8n-3}(S_+^{8,0})=\boldsymbol{Z}/\mathbf{2}\cdot\beta_8(\chi^3\mathring{\nu}_{-8n})\oplus\boldsymbol{Z}/\mathbf{2}\cdot\xi_{8,1}^*(\sigma\omega_1^{8n-4})$,

v) $\pi_S^{8n+4,-8n-4}(S_+^{8,0})=\boldsymbol{Z}\cdot\delta_{8,1}\omega_1^{8n+5}\oplus\boldsymbol{Z}/\mathbf{4}\cdot\{\hat{\eta}\xi_{8,6}^*[\eta\omega_1^{8n+3}]_6-30\xi_{8,1}^*(\sigma\omega_1^{8n-3})\}$
$\oplus\boldsymbol{Z}/\mathbf{240}\cdot\xi_{8,1}^*(\sigma\omega_1^{8n-3})$,

vi) $\pi_S^{8n+5,-8n-5}(S_+^{8,0})=\boldsymbol{Z}/\mathbf{2}\cdot\beta_8(\chi\mathring{\eta}_{-8n-4})\oplus\boldsymbol{Z}/\mathbf{2}\cdot\bar{\nu}_3\xi_{8,2}^*\omega_2^{4n+1}\oplus\boldsymbol{Z}/\mathbf{2}\cdot\xi_{8,1}^*(\sigma\omega_1^{8n-2})$,

vii) $\pi_S^{8n+6,-8n-6}(S_+^{8,0})=\boldsymbol{Z}\cdot\delta_{8,1}\omega_1^{8n+7}\oplus\boldsymbol{Z}/\mathbf{240}\cdot\xi_{8,1}^*(\sigma\omega_1^{8n-1})$,

viii) $\pi_S^{8n+7,-8n-7}(S_+^{8,0})=\boldsymbol{Z}/\mathbf{2}\cdot\hat{\eta}\mathring{\nu}\xi_{8,4}^*\omega_4^{n+1}\bar{\omega}_4^n\oplus\boldsymbol{Z}/\mathbf{2}\cdot\xi_{8,1}^*(\sigma\omega_1^{8n})$

for any integer n.

Computation of $\pi_S^{9-p,-q-1}(S_+^{9,0})$ for $p+q=7$. By a parallel argument to Proposition 10.7 we have

Proposition 14.13. i) $\delta_{8,1}\omega_1^{16n}=0$, ii) $\delta_{8,1}\omega_1^{16n+8}=\xi_{8,1}^*(\sigma\omega_1^{16n})$, iii) $\delta_{8,1}\omega_1^{8n+4}=\beta_8(\chi^3\mathring{\nu}_{-8n})$, iv) $\delta_{8,1}\omega_1^{4n+2}=\beta_8(\chi\mathring{\eta}_{-4n})$, v) $\delta_{8,1}\omega_1^{16n+1}=2\omega_8^n\bar{\omega}_8^n-(1+\rho)\omega_8^n\bar{\omega}_8^n$ *and* vi) $\delta_{8,1}\omega_1^{16n+9}=2\omega_8^{n+1}\bar{\omega}_8^n-\{(1+\rho)\omega_8^{n+1}\bar{\omega}_8^n-\xi_{8,1}^*(\sigma\omega_1^{16n+1})\}$ *for any integer n.*

This Proposition implies that

(14.14)
$$\begin{aligned}&\operatorname{Ker}[\delta_{8,1}\colon \bigoplus_{p+q=8}\pi_S^{9-p,-q-1}(S_+^{1,0})\to\bigoplus_{p+q=8}\pi_S^{8-p,-q}(S_+^{8,0})]\\&=\boldsymbol{Z}[\omega_1^{16},\omega_1^{-16}]\otimes\{\boldsymbol{Z}\cdot 1\oplus\sum_{i=1}^{7}\boldsymbol{Z}\cdot 2\omega_1^{2i}\},\end{aligned}$$

and we have the isomorphism

$$\xi_{9,8}^*\colon\bigoplus_{p+q=7}\pi_S^{8-p,-q-1}(S_+^{8,0})/\operatorname{Im}\delta_{8,1}\approx\bigoplus_{p+q=7}\pi_S^{9-p,-q-1}(S_+^{9,0})$$

by Proposition 1.10. Thus we obtain

Proposition 14.15. ($\pi_S^{9-p,-q-1}(S_+^{9,0})$, $p+q=7$)

i) $\pi_S^{16n+1,-16n}(S_+^{9,0})=\boldsymbol{Z}/\mathbf{32}\cdot\chi\omega_9^n\oplus\boldsymbol{Z}/\mathbf{120}\cdot\{2\chi\omega_9^n-\xi_{9,1}^*(\sigma\omega_1^{16n-7})\}$
$\oplus\boldsymbol{Z}/\mathbf{2}\cdot\chi^2\hat{\nu}\mathring{\nu}\xi_{9,4}^*\omega_4^{2n}\bar{\omega}_4^{2n-1}$,

ii) $\pi_S^{16n+9,-16n-8}(S_+^{9,0})=\boldsymbol{Z}/\mathbf{240}\cdot\xi_{9,8}^*\omega_8^{n+1}\bar{\omega}_8^n\oplus\boldsymbol{Z}/\mathbf{16}\cdot(1+\rho)\xi_{9,8}^*\omega_8^{n+1}\bar{\omega}_8^n$
$\oplus\boldsymbol{Z}/\mathbf{2}\cdot\chi^2\hat{\nu}\mathring{\nu}\xi_{9,4}^*\omega_4^{2n+1}\bar{\omega}_4^{2n}$,

iii) $\pi_S^{8n+2,-8n-1}(S_+^{9,0})=\boldsymbol{Z}/\mathbf{2}\cdot\chi^3\hat{\nu}\xi_{9,8}^*\omega_8^n\oplus\boldsymbol{Z}/\mathbf{2}\cdot\bar{\nu}_3\xi_{9,2}^*\omega_2^{4n-1}\oplus\boldsymbol{Z}/\mathbf{2}\cdot\xi_{9,1}^*(\sigma\omega_1^{8n-6})$,

iv) $\pi_S^{8n+3,-8n-2}(S_+^{9,0})=\boldsymbol{Z}/\mathbf{2}\cdot\chi^2\nu^2\xi_{9,4}^*\omega_4^n\bar{\omega}_4^{n-1}\oplus\boldsymbol{Z}/\mathbf{240}\cdot\xi_{9,1}^*(\sigma\omega_1^{8n-5})$,

v) $\pi_S^{8n+4,-8n-3}(S_+^{9,0})=\boldsymbol{Z}/\mathbf{2}\cdot\xi_{9,1}^*(\sigma\omega_1^{8n-4})$,

vi) $\pi_S^{8n+5,-8n-4}(S_+^{9,0})=\boldsymbol{Z}/\mathbf{4}\cdot\{\hat{\eta}\xi_{9,6}^*[\eta\omega_1^{8n+3}]_6-30\xi_{9,1}^*(\sigma\omega_1^{8n-3})\}$

$\oplus \boldsymbol{Z}/\mathbf{240}\cdot\xi^*_{9,1}(\sigma\omega_1^{8n-3})$,

vii) $\pi_S^{8n+6,-8n-5}(S_+^{9,0})=\boldsymbol{Z}/\mathbf{2}\cdot\tilde{\nu}_3\xi^*_{9,2}\omega_2^{4n+1}\oplus\boldsymbol{Z}/\mathbf{2}\cdot\xi^*_{9,1}(\sigma\omega_1^{8n-2})$,

viii) $\pi_S^{8n+7,-8n-6}(S_+^{9,0})=\boldsymbol{Z}/\mathbf{240}\cdot\xi^*_{9,1}(\sigma\omega_1^{8n-1})$,

ix) $\pi_S^{16n+8,-16n-7}(S_+^{9,0})=\boldsymbol{Z}/\mathbf{2}\cdot\hat{\eta}\mathring{\nu}\xi^*_{9,4}\omega_4^{2n+1}\overline{\omega}_4^{2n}$,

x) $\pi_S^{16n+16,-16n-15}(S_+^{9,0})=\boldsymbol{Z}/\mathbf{2}\cdot\hat{\eta}\mathring{\nu}\xi^*_{9,4}\omega_4^{2n+2}\overline{\omega}_4^{2n+1}\oplus\boldsymbol{Z}/\mathbf{2}\cdot\xi^*_{9,1}(\sigma\omega_1^{16n+8})$

for any integer n.

Proposition 14.15 describes $\lambda^S_{p,q}$ for $p+q=7$. By Proposition 4.8 the groups $\pi^S_{p,q}$ for $p+q=7$ are determined except $\pi^S_{4,3}$, $\pi^S_{5,2}$, $\pi^S_{6,1}$, $\pi^S_{7,0}$ and $\pi^S_{8,-1}$.

Observe the following forgetful exact sequence (1.5):

$$\pi^S_{3,4}\xrightarrow{\beta_1}\pi_S^{-3,-4}(S_+^{1,0})\xrightarrow{\delta_1}\pi^S_{4,3}\xrightarrow{\chi}\pi^S_{3,3}\xrightarrow{\beta_1}\pi_S^{-3,-3}(S_+^{1,0})\,.$$

Since $\pi^S_{3,4}=\boldsymbol{Z}/\mathbf{2}\cdot\tilde{\nu}_3\mathring{\eta}\oplus\boldsymbol{Z}/\mathbf{2}\cdot\sigma\delta_1\omega_1^{-2}$, β_1: $\pi^S_{3,4}\to\pi_S^{-3,-4}(S_+^{1,0})$ is a zero-map. Since $\phi(\hat{\sigma})=\nu$, we see that $\pi^S_{3,3}=\boldsymbol{Z}/\mathbf{24}\cdot\chi\hat{\sigma}\oplus\boldsymbol{Z}/\mathbf{8}\cdot\tilde{\nu}_3$ by Theorem 13.12. Thus we get the short exact sequence

$$0\to\boldsymbol{Z}/\mathbf{240}\cdot(1-\rho)\hat{\sigma}\to\pi^S_{4,3}\xrightarrow{\chi}\boldsymbol{Z}/\mathbf{24}\cdot\chi\hat{\sigma}\oplus\boldsymbol{Z}/\mathbf{4}\cdot 2\tilde{\nu}_3\to 0\,,$$

as $\delta_1(\sigma\omega_1^{-3})=\delta_1(\omega_1\hat{\sigma})=(1-\rho)\hat{\sigma}$. Since $\chi(24\hat{\sigma})=0$ and $\rho(1-\rho)\hat{\sigma}=-(1-\rho)\hat{\sigma}$, we have $24(1+\rho)\hat{\sigma}=24\hat{\sigma}+\rho 24\hat{\sigma}=24\hat{\sigma}-24\hat{\sigma}=0$. On the other hand $24(1-\rho)\hat{\sigma}$ is of order 10, hence $48\hat{\sigma}=48\hat{\sigma}-24(1+\rho)\hat{\sigma}=24(1-\rho)\hat{\sigma}$ is of order 10. Thus $\hat{\sigma}$ is of order 480 and

$$12(1+\rho)\hat{\sigma}=0 \text{ or } 120(1-\rho)\hat{\sigma}\,.$$

This problem will be solved by discussions of 8 stem and, in fact, we get

$$12(1+\rho)\hat{\sigma}=120(1-\rho)\hat{\sigma}=240\hat{\sigma}\,, \tag{14.16}$$

of which the proof is postponed until the next section below Proposition 15.25. (Here we remark that the relation (14.16) is also proved by H. Minami [12] by certain arguments of equivariant J-images.) Thus we obtain

$$\pi^S_{4,3}=\boldsymbol{Z}/\mathbf{480}\cdot\hat{\sigma}\oplus\boldsymbol{Z}/\mathbf{12}\cdot\{(1+\rho)\hat{\sigma}-20\hat{\sigma}\}\oplus\boldsymbol{Z}/\mathbf{4}\cdot\{\hat{\eta}\tilde{\nu}_3-30(1-\rho)\hat{\sigma}\}\,.$$

By the forgetful exact sequence (1.5) it is easy to see that $\pi^S_{5,2}=0$, $\pi^S_{6,1}=\boldsymbol{Z}/\mathbf{240}\cdot\sigma\delta_1\omega_1^{-5}$, $\pi^S_{7,0}=\boldsymbol{Z}\cdot y_7\oplus\boldsymbol{Z}/\mathbf{2}\cdot\tilde{\nu}_3\mathring{\eta}_4\oplus\boldsymbol{Z}/\mathbf{2}\cdot\sigma\delta_1\omega_1^{-6}$ and $\pi^S_{8,-1}=\boldsymbol{Z}/\mathbf{2}\cdot\chi^2\hat{\nu}\mathring{\nu}\mathring{\nu}_8\oplus\boldsymbol{Z}/\mathbf{120}\cdot\sigma\delta_1\omega_1^{-7}$.

Put

$$\mathring{\sigma}_{16n}=\delta_8\omega_8^{1-n}\overline{\omega}_8^{-n} \text{ and } \mathring{\sigma}=\mathring{\sigma}_0=\delta_8\omega_8\,. \tag{14.17}$$

Then $\beta_1(\mathring{\sigma}_{16n})=\sigma\omega_1^{-16n}$ and $\beta_1(\mathring{\sigma})=\beta_1(\sigma)=\sigma$ by Theorem 3.5.

Summarizing the aboves we get

Theorem 14.18. $(\pi^S_{p,q},\ p+q=7)$

i) $\pi^S_{4,3}=\boldsymbol{Z}/\mathbf{480}\cdot\hat{\sigma}\oplus\boldsymbol{Z}/\mathbf{12}\cdot\{(1+\rho)\hat{\sigma}-20\hat{\sigma}\}\oplus\boldsymbol{Z}/\mathbf{4}\cdot\{\hat{\eta}\tilde{\nu}_3-30(1-\rho)\hat{\sigma}\}$.

ii) $\pi^S_{5,2}=0$.

iii) $\pi^S_{6,1}=\boldsymbol{Z}/\mathbf{240}\cdot\sigma\delta_1\omega_1^{-5}$.

iv) $\pi^S_{7,0}=\boldsymbol{Z}\cdot y_7\oplus\boldsymbol{Z}/\mathbf{2}\cdot\tilde{\nu}_3\mathring{\eta}_4\oplus\boldsymbol{Z}/\mathbf{2}\cdot\sigma\delta_1\omega_1^{-6}$.

v) $\pi^S_{8,-1}=\boldsymbol{Z}/\mathbf{2}\cdot\chi^2\hat{\nu}\mathring{\nu}\mathring{\nu}_8\oplus\boldsymbol{Z}/\mathbf{120}\cdot\sigma\delta_1\omega_1^{-7}$.

vi) $\pi^S_{16n,-16n+7}\approx\boldsymbol{Z}/\mathbf{240}\cdot\mathring{\sigma}_{16n}\oplus\boldsymbol{Z}/\mathbf{16}\cdot(1+\rho)\mathring{\sigma}_{16n}\oplus\boldsymbol{Z}/\mathbf{2}\cdot\chi^2\hat{\nu}\mathring{\nu}\mathring{\nu}_{16n}\oplus\pi^S_{-16n+7}$

for any integer n.

vii) $\pi^S_{16n+8,-16n-1}\approx\boldsymbol{Z}/\mathbf{32}\cdot\chi\delta_9\omega_9^{-n}\oplus\boldsymbol{Z}/\mathbf{120}\cdot\{2\chi\omega_9^{-n}-\sigma\delta_1\omega_1^{-16n-7}\}\oplus\boldsymbol{Z}/\mathbf{2}\cdot\chi^2\hat{\nu}\mathring{\nu}\mathring{\nu}_{16n+8}$
$\oplus\pi^S_{-16n-1}$ *for* $n\neq 0$.

viii) $\pi^S_{16n+1,-16n+6}\approx\boldsymbol{Z}/\mathbf{2}\cdot\hat{\eta}\mathring{\nu}\mathring{\nu}_{16n}\oplus\pi^S_{-16n+6}$ *for any integer n.*

ix) $\pi^S_{16n+9,-16n-2}\approx\boldsymbol{Z}/\mathbf{2}\cdot\hat{\eta}\mathring{\nu}\mathring{\nu}_{16n+8}\oplus\boldsymbol{Z}/\mathbf{2}\cdot\sigma\delta_1\omega_1^{-16n-8}\oplus\pi^S_{-16n-2}$ *for any integer n.*

x) $\pi^S_{8n+2,-8n+5}\approx\boldsymbol{Z}/\mathbf{240}\cdot\sigma\delta_1\omega_1^{-8n-1}\oplus\pi^S_{-8n+5}$ *for any integer n.*

xi) $\pi^S_{8n+3,-8n+4}\approx\boldsymbol{Z}/\mathbf{2}\cdot\tilde{\nu}_3\mathring{\eta}_{8n}\oplus\boldsymbol{Z}/\mathbf{2}\cdot\sigma\delta_1\omega_1^{-8n-2}\oplus\pi^S_{-8n+4}$ *for any integer n.*

xii) $\pi^S_{8n+4,-8n+3}\approx\boldsymbol{Z}/\mathbf{4}\cdot\{\hat{\eta}\tilde{\nu}_{8n+3}-30\sigma\delta_1\omega_1^{-8n-3}\}\oplus\boldsymbol{Z}/\mathbf{240}\cdot\sigma\delta_1\omega_1^{-8n-3}\oplus\pi^S_{-8n+3}$

for $n\neq 0$.

xiii) $\pi^S_{8n+5,-8n+2}\approx\boldsymbol{Z}/\mathbf{2}\cdot\sigma\delta_1\omega_1^{-8n-4}\oplus\pi^S_{-8n+2}$ *for* $n\neq 0$.

xiv) $\pi^S_{8n+6,-8n+1}\approx\boldsymbol{Z}/\mathbf{2}\cdot\chi^2\nu^2\mathring{\nu}_{8n+8}\oplus\boldsymbol{Z}/\mathbf{240}\cdot\sigma\delta_1\omega_1^{-8n-5}\oplus\pi^S_{-8n+1}$ *for* $n\neq 0$.

xv) $\pi^S_{8n+7,-8n}\approx\boldsymbol{Z}/\mathbf{2}\cdot\chi^3\hat{\nu}\delta_8\omega_8^{-n}\oplus\boldsymbol{Z}/\mathbf{2}\cdot\tilde{\nu}_3\mathring{\eta}_{8n+4}\oplus\boldsymbol{Z}/\mathbf{2}\cdot\sigma\delta_1\omega_1^{-8n-6}\oplus\pi^S_{-8n}$ *for* $n\neq 0$.

Since $(1-\rho)y_7=\delta_1(\omega_1\beta_1(y_7))=\delta_1(120\sigma\omega_1^{-6})=\delta_1(120\eta^*_{1,2}(\hat{\sigma}\omega_2^{-1}))=120\chi\hat{\sigma}\mathring{\eta}_4=0$, we get

Proposition 14.19. $\rho y_7=y_7$ *in* $\pi^S_{7,0}$.

15. 8 stem

Computation of $\pi_S^{2-p,-q-1}(S_+^{2,0})$ for $p+q=8$. Observe the short exact sequence (7.4) for $r=8$. $\eta^*_{1,2}(\mathring{\sigma}\omega_2^n)=\sigma\omega_1^{2n}$, $\mathring{\sigma}\omega_2^n$ is of order 240, $\eta^*_{1,2}(y_7\omega_2^n)=120\sigma\omega_1^{2n-7}$ by (14.5) and $2y_7\omega_2^n=\chi\hat{\eta}y_7\omega_2^n=\xi^*_{2,1}(\hat{\eta}y_7\omega_1^{2n})=120\xi^*_{2,1}(\eta\sigma\omega_1^{2n-6})=0$, *i.e.* $y_7\omega_2^n$ is of order 2. Thus we obtain

Proposition 15.1 $(\pi_S^{2-p,-q-1}(S_+^{2,0}),\ p+q=8)$

i) $\pi_S^{2n,-2n-7}(S_+^{2,0})=\boldsymbol{Z}/\mathbf{240}\cdot\mathring{\sigma}\omega_2^n\oplus\boldsymbol{Z}/\mathbf{2}\cdot\xi^*_{2,1}(\eta\sigma\omega_1^{2n-1})\oplus\boldsymbol{Z}/\mathbf{2}\cdot\xi^*_{2,1}(\varepsilon\omega_1^{2n-1})$,

ii) $\pi_S^{8n-4,-8n-3}(S_+^{2,0})=\boldsymbol{Z}/\mathbf{240}\cdot\hat{\sigma}\omega_2^{4n}\oplus\boldsymbol{Z}/\mathbf{2}\cdot\xi^*_{2,1}(\eta\sigma\omega_1^{8n-5})\oplus\boldsymbol{Z}/\mathbf{2}\cdot\xi^*_{2,1}(\varepsilon\omega_1^{8n-5})$,

iii) $\pi_S^{2n-7,-2n}(S_+^{2,0})=\boldsymbol{Z}/\mathbf{2}\cdot y_7\omega_2^n\oplus\boldsymbol{Z}/\mathbf{2}\cdot\xi^*_{2,1}(\eta\sigma\omega_1^{2n-8})\oplus\boldsymbol{Z}/\mathbf{2}\cdot\xi^*_{2,1}(\varepsilon\omega_1^{2n-8})$

for any integer n.

Computation of $\pi_S^{3-p,-q-1}(S_+^{3,0})$ for $p+q=8$.

Since $2\nu\mathring{\nu}=0$ and $2\xi^*_{3,2}\omega_2^{2n+1}=0$, the stable equivariant Toda bracket

$$\langle \nu\mathring{\nu},\ 2,\ \xi^*_{3,2}\omega_2^{2n+1}\rangle^\tau$$

is well-defined. By Proposition 6.8, i), we see that $\eta^*_{1,3}\langle \nu\mathring{\nu},\ 2,\ \xi^*_{3,2}\omega_2^{2n+1}\rangle^\tau=\nu\mathring{\nu}\circ\delta_{2,1}^{-1}(2\omega_2^{2n+1})=\{\nu^2\omega_1^{4n+3}\}$. By Propositions 6.7 and 9.3, $2\langle \nu\mathring{\nu},\ 2,\ \xi^*_{3,2}\omega_2^{2n+1}\rangle^\tau\subset\langle \nu\mathring{\nu},\ 2,\ 0\rangle^\tau=\nu\mathring{\nu}\circ\pi_S^{4n+3,-4n-3}(S_+^{3,0})=0$, *i.e.* all elements of $\langle \nu\mathring{\nu},\ 2,\ \xi^*_{3,2}\omega_2^{2n+1}\rangle^\tau$ are of order 2. Next $\psi(\delta_3\langle \nu\mathring{\nu},\ 2,\ \xi^*_{3,2}\omega_2^{2n+1}\rangle^\tau)\subset\psi(\langle \nu\mathring{\nu},\ 2,\ \mathring{\eta}_{-4n}\rangle^\tau)\subset\langle \nu^2,\ 2,\ \eta\rangle=\{\varepsilon,\ \bar{\nu}\}$ by [14], (6.1). Moreover $\sigma\xi^*_{3,2}\omega_2^{2n+1}$ is an element of the indeterminacy of $\langle \nu\mathring{\nu},\ 2,\ \xi^*_{3,2}\omega_2^{2n+1}\rangle^\tau$ and $\psi(\delta_3(\sigma\xi^*_{3,2}\omega_2^{2n+1}))=\eta\sigma$. Thus there exists an element

$$[\nu^2\omega_1^{4n+3}]_3\in\langle \nu\mathring{\nu},\ 2,\ \xi^*_{3,2}\omega_2^{4n+1}\rangle^\tau\subset\pi_S^{4n+3,-4n-9}(S_+^{3,0}) \tag{15.2}$$

such that

$$\psi(\delta_3[\nu^2\omega_1^{4n+3}]_3)=\varepsilon\,. \tag{15.2'}$$

Put

$$\mathring{\varepsilon}_{4n}=\delta_3[\nu^2\omega_1^{-4n+3}]_3\in\langle \nu\mathring{\nu},\ 2,\ \mathring{\eta}_{4n}\rangle^\tau\subset\pi^S_{4n,-4n+8}\,. \tag{15.2''}$$

By Proposition 6.8, i), we see that $\eta^*_{1,3}[\nu^2\omega_1^{4n+3}]_3=\nu^2\omega_1^{4n+3}$ and $\chi^2\mathring{\varepsilon}_{4n}=\chi^2\delta_3[\nu^2\omega_1^{-4n+3}]_3=\delta_1\eta^*_{1,3}[\nu^2\omega_1^{-4n+3}]_3=\delta_1(\nu^2\omega_1^{-4n+3})=\nu^2\delta_1\omega_1^{-4n+3}$. Hence we have

Proposition 15.3. i) $\eta^*_{1,3}[\nu^2\omega_1^{4n+3}]_3=\nu^2\omega_1^{4n+3}$, $[\nu^2\omega_1^{4n+3}]_3$ *is of order* 2, ii) $\beta_1(\mathring{\varepsilon}_{4n})=\varepsilon\omega_1^{-4n}$ *and* $\chi^2\mathring{\varepsilon}_{4n}=\nu^2\delta_1\omega_1^{-4n+3}$.

Since $\mathring{\eta}\mathring{\nu}=0$ and $\mathring{\nu}\xi^*_{7,1}(\nu\omega_1^{8n-1})=\xi^*_{7,1}(\nu^2\omega_1^{8n-3})=\xi^*_{7,1}\beta_1(\bar{\nu}_{-8n+3})=\beta_7\chi^6\delta_6[\eta\omega_1^{8n+3}]_6=0$, the stable equivariant Toda bracket

$$\langle \mathring{\eta},\ \mathring{\nu},\ \xi^*_{7,1}(\nu\omega_1^{8n-1})\rangle^\tau$$

is well-defined. By Proposition 6.8, i), we see that $\eta^*_{6,7}\langle \mathring{\eta},\ \mathring{\nu},\ \xi^*_{7,1}(\nu\omega_1^{8n-1})\rangle^\tau=\mathring{\eta}\circ\delta_{1,6}^{-1}(\mathring{\nu}\nu\omega_1^{8n-1})=\mathring{\eta}\circ\delta_{1,6}^{-1}(\nu^2\omega_1^{8n-3})\ni\mathring{\eta}[\eta\omega_1^{8n+3}]_6$. Let

$$[\eta^2\omega_1^{8n+2}]_7\in\langle \mathring{\eta},\ \mathring{\nu},\ \xi^*_{7,1}(\nu\omega_1^{8n-1})\rangle^\tau\subset\pi_S^{8n+2,-8n-4}(S_+^{7,0}) \tag{15.4}$$

by the element such that $\eta^*_{6,7}[\eta^2\omega_1^{8n+2}]_7=\mathring{\eta}[\eta\omega_1^{8n+3}]_6$. Put

$$\mathring{\varepsilon}_{8n+5}=\delta_7[\eta^2\omega_1^{-8n+2}]_7\in\langle \mathring{\eta},\ \mathring{\nu},\ \nu\delta_1\omega_1^{-8n-1}\rangle^\tau\subset\pi^S_{8n+5,-8n+3}\,, \tag{15.4'}$$

then $\psi(\mathring{\varepsilon}_{8n+5})\in\psi(\langle \mathring{\eta},\ \mathring{\nu},\ \nu\delta_1\omega_1^{-8n-1}\rangle^\tau)\subset\langle \eta,\ \nu,\ 2\nu\rangle=\{\varepsilon,\ \bar{\nu}\}$.

Proposition 15.5. $\pi_S^{8n+2,-8n-4}(S_+^{7,0})=\boldsymbol{Z}/8\cdot[\eta^2\omega_1^{8n+2}]_7$, $\eta^*_{1,7}[\eta^2\omega_1^{8n+2}]_7=\eta^2\omega_1^{8n+2}$ *and* $\rho[\eta^2\omega_1^{8n+2}]_7=[\eta^2\omega_1^{8n+2}]_7$.

Proof. Observe the following exact sequence:

$$\pi_S^{8n-4,-8n-4}(S_+^{1,0})\xrightarrow{\xi^*_{7,1}}\pi_S^{8n+2,-8n-4}(S_+^{7,0})\xrightarrow{\eta^*_{6,7}}$$

$$\pi_S^{8n+2,-8n-4}(S_+^{6,0}) \xrightarrow{\delta_{1,6}} \pi_S^{8n-4,-8n-3}(S_+^{1,0}) .$$

Since $\pi_S^{8n-4,-8n-4}(S_+^{1,0})=Z/2\cdot\varepsilon\omega_1^{8n-4}\oplus Z/2\cdot\eta\sigma\omega_1^{8n-4}$, $\xi_{7,1}^*(\varepsilon\omega_1^{8n-4})=\xi_{7,1}^*\beta_1(\mathring{\varepsilon}_{-8n+4})=\beta_7\chi^6\delta_3[\nu^2\omega_1^{-8n-1}]_3=0$, $\xi_{7,1}^*(\eta\sigma\omega_1^{8n-4})=\xi_{7,1}^*\beta_1(\mathring{\eta}_{-8n+4}\sigma)=\beta_7\chi^6\delta_2(\sigma\omega_2^{4n-1})=0$, $\pi_S^{8n+2,-8n-4}(S_+^{6,0})=Z/4\cdot\{\hat{\eta}[\eta\omega_1^{8n+3}]_6-30\xi_{6,1}^*(\sigma\omega_1^{8n-3})\}\oplus Z/240\cdot\xi_{6,1}^*(\sigma\omega_1^{8n-3})$, $\delta_{1,6}(\hat{\eta}[\eta\omega_1^{8n+3}]_6)=\beta_1(\hat{\eta}\bar{\nu}_{-8n+3})=\eta\nu^2\omega_1^{8n-3}=0$ and $\delta_{1,6}\xi_{6,1}^*(\sigma\omega_1^{8n-3})=-2\sigma\omega_1^{8n-4}$ of order 120, from the above exact sequence we see that

$$\eta_{6,7}^*\colon \pi_S^{8n+2,-8n-4}(S_+^{7,0}) \to Z/8\cdot\hat{\eta}[\eta\omega_1^{8n+3}]_6$$

is isomorphic. □

The exact sequence (1.9) for $r=2$ and (14.3) give the short exact sequence

$$0 \to \bigoplus_{p+q=8} \pi_S^{2-p,-q-1}(S_+^{2,0})/\mathrm{Im}\,\delta_{2,1} \xrightarrow{\xi_{3,2}^*} \bigoplus_{p+q=8} \pi_S^{3-p,-q-1}(S_+^{3,0})$$
$$\xrightarrow{\eta_{1,3}^*} Z[\omega_1,\omega_1^{-1}]\otimes Z/2\cdot\nu^2 \to 0 .$$

$\delta_{2,1}(\sigma\omega_1^{4n})=0$, $\delta_{2,1}(\sigma\omega_1^{4n+1})=2\mathring{\sigma}\omega_2^{2n}-(1+\rho)\mathring{\sigma}\omega_2^{2n}=2\mathring{\sigma}\omega_2^{2n}-\chi\hat{\eta}\mathring{\sigma}\omega_2^{2n}=2\mathring{\sigma}\omega_2^{2n}-\xi_{2,1}^*(\eta\sigma\omega_1^{4n-1})$ of order 120, $\delta_{2,1}(\sigma\omega_1^{8n-3})=2\hat{\sigma}\omega_2^{4n}-\xi_{2,1}^*(\eta\sigma\omega_1^{8n-5})$ of order 120, $\delta_{2,1}(\sigma\omega_1^{4n+2})=\beta_2(\chi\mathring{\eta}_{-4n}\sigma)=\xi_{2,1}^*(\eta\sigma\omega_1^{4n})$ of order 2 and $\delta_{2,1}(\sigma\omega_1^{4n+3})=2\sigma\omega_2^{2n+1}$ of order 120. Thus we have

(15.6) $$\mathrm{Ker}[\delta_{2,1}\colon \bigoplus_{p+q=9}\pi_S^{3-p,-q-1}(S_+^{1,0}) \to \bigoplus_{p+q=9}\pi_S^{2-p,-q}(S_+^{2,0})]$$
$$= Z[\omega_1^4,\omega_1^{-4}]\otimes\{Z/240\cdot\sigma\oplus Z/120\cdot 2\sigma\omega_1^2\oplus Z/2\cdot 120\sigma\omega_1\oplus Z/2\cdot 120\sigma\omega_1^{-1}\}$$

$\eta_{1,3}^*(\bar{\nu}_3\omega_3^n)=\nu^2\omega_1^{4n-3}$, $2\bar{\nu}_3\omega_3^n=\chi\hat{\eta}\bar{\nu}_3\omega_3^n=\chi^2\mathring{\varepsilon}_5\omega_3^n=\xi_{3,1}^*(\beta_1(\mathring{\varepsilon}_5)\omega_1^{4n})=\xi_{3,1}^*(\varepsilon\omega_1^{4n-5})$ or $\xi_{3,1}^*(\bar{\nu}\omega_1^{4n-5})$, $\eta_{1,3}^*(\hat{\nu}\mathring{\nu}\omega_3^n)=\nu^2\omega_1^{4n-2}$, $2\hat{\nu}\mathring{\nu}\omega_3^n=\beta_3(\nu^2\delta_1\omega_1^{4n-1})=\beta_3(\chi^2\mathring{\varepsilon}_{-4n+4})=\xi_{3,1}^*(\varepsilon\omega_1^{4n-4})$ by Propositions 13.13 and 15.3, $\eta_{1,3}^*[\nu^2\omega_1^{4n-1}]_3=\nu^2\omega_1^{4n-1}$, $[\nu^2\omega_1^{4n-1}]_3$ is of order 2 by Proposition 15.3, $\eta_{1,3}^*(\nu\mathring{\nu}\omega_3^n)=\nu^2\omega_1^{4n}$, $\nu\mathring{\nu}\omega_3^n$ is of order 2 and Im $\delta_{2,1}$ is computed above (15.6). Thus we obtain

Proposition 15.7. $(\pi_S^{3-p,-q-1}(S_+^{3,0}), p+q=8)$

i) $\pi_S^{8n-3,-8n-3}(S_+^{3,0})=Z/4\cdot\chi\hat{\sigma}\omega_3^{2n}\oplus Z/4\cdot\beta_3(\bar{\nu}_{-8n+3})$,

ii) $\pi_S^{4n-2,-4n-4}(S_+^{3,0})=Z/4\cdot\hat{\nu}\mathring{\nu}\omega_3^n\oplus Z/2\cdot\chi y_7\omega_3^{n+1}$,

iii) $\pi_S^{4n-1,-4n-5}(S_+^{3,0})=Z/2\cdot[\nu^2\omega_1^{4n-1}]_3\oplus Z/2\cdot\xi_{3,2}^*\omega_2^{2n-1}\oplus Z/2\cdot\xi_{3,1}^*(\eta\sigma\omega_1^{4n-3})$
$\oplus Z/2\cdot\xi_{3,1}^*(\varepsilon\omega_1^{4n-3})$,

iv) $\pi_S^{4n,-4n-6}(S_+^{3,0})=Z/2\cdot\nu\mathring{\nu}\omega_3^n\oplus Z/2\cdot y_7\xi_{3,2}^*\omega_2^{2n+3}\oplus Z/2\cdot\xi_{3,1}^*(\eta\sigma\omega_1^{4n-2})$
$\oplus Z/2\cdot\xi_{3,1}^*(\varepsilon\omega_1^{4n-2})$,

v) $\pi_S^{8n+1,-8n-7}(S_+^{3,0})=Z/4\cdot\chi\mathring{\sigma}\omega_3^{2n}\oplus Z/4\cdot\bar{\nu}_3\omega_3^{2n+1}$

for any integer n.

Computation of $\pi_S^{5-p,-q-1}(S_+^{5,0})$ for $p+q=8$. $\delta_{3,1}(\nu^2\omega_1^{4n})=0$, $\delta_{3,1}(\nu^2\omega_1^{4n+1})=$

$\delta_{3,1}(\tilde{\nu}_3\omega_1^{4n+4})=0$, $\delta_{3,1}(\nu^2\omega_1^{4n+2})=\delta_{3,1}(\hat{\nu}\nu\omega_1^{4n+4})=0$ and $\delta_{3,1}(\nu^2\omega_1^{4n+3})=\delta_{3,1}(\omega_1^{4n+5}\hat{\nu}\mathring{\nu})=2\hat{\nu}\mathring{\nu}\omega_3^{n+1}$ of order 2. Hence

$$\text{(15.8)}\qquad \begin{aligned}&\mathrm{Ker}[\delta_{3,1}\colon \bigoplus_{p+q=9}\pi_S^{4-p,-q-1}(S_+^{1,0})\to\bigoplus_{p+q=9}\pi_S^{3-p,-q}(S_+^{3,0})]\\ &=\boldsymbol{Z}[\omega_1^4,\omega_1^{-4}]\otimes\{\boldsymbol{Z}/2\cdot\nu^2\oplus\boldsymbol{Z}/2\cdot\nu^2\omega_1\oplus\boldsymbol{Z}/2\cdot\nu^2\omega_1^2\}\,.\end{aligned}$$

The exact sequence (1.9) for $r=3$ and 4 give isomorphisms

$$\xi_{4,3}^*\colon\bigoplus_{p+q=8}\pi_S^{3-p,-q-1}(S_+^{3,0})/\mathrm{Im}\,\delta_{3,1}\approx\bigoplus_{p+q=8}\pi_S^{4-p,-q-1}(S_+^{4,0})\,,$$
$$\xi_{5,4}^*\colon\bigoplus_{p+q=8}\pi_S^{4-p,-p-1}(S_+^{4,0})\approx\bigoplus_{p+q=8}\pi_S^{5-p,-q-1}(S_+^{5,0})\,.$$

Thus we obtain

Proposition 15.9. $(\pi_S^{5-p,-q-1}(S_+^{5,0}),\ p+q=8)$

i) $\pi_S^{8n-4,-8n}(S_+^{5,0})=\boldsymbol{Z}/2\cdot\hat{\eta}^4\omega_5^n\oplus\boldsymbol{Z}/2\cdot\chi\hat{\nu}\mathring{\nu}\xi_{5,4}^*\omega_4^n\bar{\omega}_4^{n-1}$,

ii) $\pi_S^{4n-3,-4n-1}(S_+^{5,0})=\boldsymbol{Z}/2\cdot\mathring{\sigma}\xi_{5,2}^*\omega_2^{2n-3}\oplus\boldsymbol{Z}/2\cdot\xi_{5,3}^*[\nu^2\omega_1^{4n-5}]_3\oplus\boldsymbol{Z}/2\cdot\xi_{5,1}^*(\varepsilon\omega_1^{4n-7})$
$\oplus\boldsymbol{Z}/2\cdot\xi_{5,1}^*(\eta\sigma\omega_1^{4n-7})$,

iii) $\pi_S^{8n-2,-8n-2}(S_+^{5,0})=\boldsymbol{Z}/2\cdot\chi\nu\mathring{\nu}\xi_{5,4}^*\omega_4^n\bar{\omega}_4^{n-1}\oplus\boldsymbol{Z}/2\cdot y_7\xi_{5,2}^*\omega_2^{4n+1}\oplus\boldsymbol{Z}/2\cdot\xi_{5,1}^*(\varepsilon\omega_1^{8n-6})$
$\oplus\boldsymbol{Z}/2\cdot\xi_{5,1}^*(\eta\sigma\omega_1^{8n-6})$,

iv) $\pi_S^{8n-1,-8n-3}(S_+^{5,0})=\boldsymbol{Z}/4\cdot\chi^3\hat{\sigma}\omega_5^n\oplus\boldsymbol{Z}/4\cdot\beta_5(\hat{\eta}\mathring{\nu}_{-8n})$,

v) $\pi_S^{8n,-8n-4}(S_+^{5,0})=\boldsymbol{Z}/2\cdot\beta_5(\chi^2\hat{\nu}\mathring{\nu}_{-8n})\oplus\boldsymbol{Z}/2\cdot\chi^2y_7\xi_{5,4}^*\omega_4^{n+1}\bar{\omega}_4^n$,

vi) $\pi_S^{8n+2,-8n-6}(S_+^{5,0})=\boldsymbol{Z}/2\cdot\beta_5(\chi^2\nu\mathring{\nu}_{-8n})\oplus\boldsymbol{Z}/2\cdot y_7\xi_{5,2}^*\omega_2^{4n+3}\oplus\boldsymbol{Z}/2\cdot\xi_{5,1}^*(\varepsilon\omega_1^{8n-2})$
$\oplus\boldsymbol{Z}/2\cdot\xi_{5,1}^*(\eta\sigma\omega_1^{8n-2})$,

vii) $\pi_S^{8n+3,-8n-7}(S_+^{5,0})=\boldsymbol{Z}/4\cdot\chi^3\mathring{\sigma}\omega_5^n\oplus\boldsymbol{Z}/4\cdot\chi\tilde{\nu}_3\xi_{5,4}^*\omega_4^{n+1}\bar{\omega}_4^n$

for any integer n.

Computation of $\pi_S^{6-p,-q-1}(S_+^{6,0})$ for $p+q=8$. The exact sequence (1.9) for $r=5$ and (14.7) give the short exact sequence

$$\begin{aligned}0\to\bigoplus_{p+q=8}\pi_S^{5-p,-q-1}(S_+^{5,0})&\xrightarrow{\xi_{6,5}^*}\bigoplus_{p+q=8}\pi_S^{6-p,-q-1}(S_+^{6,0})\xrightarrow{\eta_{1,6}^*}\\ &\boldsymbol{Z}[\omega_1^8,\omega_1^{-8}]\otimes\{\boldsymbol{Z}/24\cdot\nu\omega_1^{-2}\oplus\boldsymbol{Z}/24\cdot\nu\oplus\boldsymbol{Z}/12\cdot2\nu\omega_1^2\oplus\boldsymbol{Z}/12\cdot2\nu\omega_1^4\}\\ &\oplus\boldsymbol{Z}[\omega_1^4,\omega_1^{-4}]\otimes\{\boldsymbol{Z}/2\cdot\eta^3\omega_1^{-1}\oplus\boldsymbol{Z}/2\cdot\eta^3\omega_1\}\to0\,.\end{aligned}$$

$\eta_{1,6}^*(\hat{\eta}^3\omega_6^n)=\eta^3\omega_1^{8n-3}$, $2\hat{\eta}^3\omega_6^n=\xi_{6,5}^*(\hat{\eta}^4\omega_5^n)$, $\eta_{1,6}^*(\hat{\nu}\omega_6^n)=\nu\omega_1^{8n-2}$, $\hat{\nu}\omega_6^n$ is of order 24, $\eta_{1,6}^*\beta_6(\mathring{\nu}_{-8n})=\nu\omega_1^{8n}$, $\beta_6(\mathring{\nu}_{-8n})$ is of order 24, $\eta_{1,6}^*\beta_6(\hat{\eta}\eta\mathring{\eta}_{-4n})=\eta^3\omega_1^{4n-1}$, $\beta_6(\hat{\eta}\eta\mathring{\eta}_{-4n})$ is of order 2, $\eta_{1,6}^*(\hat{\eta}^2[\eta\omega_1^{8n+3}]_6)=\eta^3\omega_1^{8n+1}$, $2\hat{\eta}^2[\eta\omega_1^{8n+3}]_6=\xi_{6,5}^*(\chi^2y_7\xi_{5,4}^*\omega_4^n\bar{\omega}_4^{n-1})$, $\eta_{1,6}^*\beta_6(\nu\delta_1\omega_1^{8n+3})=2\nu\omega_1^{8n+2}$, $\beta_6(\nu\delta_1\omega_1^{8n+3})$ is of order 12 by Theorem 10.11, vi), $\eta_{1,6}^*\beta_6(\nu\delta_1\omega_1^{8n+5})=2\nu\omega_1^{8n+4}$ and $\beta_6(\nu\delta_1\omega_1^{8n+5})$ is of order 12 by Theorem 11.11, iii), and the periodicity. Thus we obtain

Proposition 15.10. $(\pi_S^{6-p,-q-1}(S_+^{6,0}),\ p+q=8)$

i) $\pi_S^{8n-3,-8n}(S_+^{6,0})=\boldsymbol{Z}/4\cdot\hat{\eta}^3\omega_6^n\oplus\boldsymbol{Z}/2\cdot\chi\hat{\nu}\mathring{\nu}\xi_{6,4}^*\omega_4^n\bar{\omega}_4^{n-1}$,

ii) $\pi_S^{8n-2,-8n-1}(S_+^{6,0})=\boldsymbol{Z}/24\cdot\hat{\nu}\omega_6^n\oplus\boldsymbol{Z}/2\cdot\mathring{\sigma}\xi_{6,2}^*\omega_2^{4n-3}\oplus\boldsymbol{Z}/2\cdot\xi_{6,3}^*[\nu^2\omega_1^{8n-5}]_3$
$\oplus\boldsymbol{Z}/2\cdot\xi_{6,1}^*(\varepsilon\omega_1^{8n-7})\oplus\boldsymbol{Z}/2\cdot\xi_{6,1}^*(\eta\sigma\omega_1^{8n-7})$,

iii) $\pi_S^{8n-1,-8n-2}(S_+^{6,0})=\boldsymbol{Z}/2\cdot\beta_6(\hat{\eta}\eta\mathring{\eta}_{-8n})\oplus\boldsymbol{Z}/2\cdot\chi\nu\mathring{\nu}\xi_{6,4}^*\omega_4^n\bar{\omega}_4^{n-1}\oplus\boldsymbol{Z}/2\cdot y_7\xi_{6,2}^*\omega_2^{4n+1}$
$\oplus\boldsymbol{Z}/2\cdot\xi_{6,1}^*(\varepsilon\omega_1^{8n-6})\oplus\boldsymbol{Z}/2\cdot\xi_{6,1}^*(\eta\sigma\omega_1^{8n-6})$,

iv) $\pi_S^{8n,-8n-3}(S_+^{6,0})=\boldsymbol{Z}/24\cdot\beta_6(\mathring{\nu}_{-8n})\oplus\boldsymbol{Z}/4\cdot\chi^4\hat{\sigma}\omega_6^n\oplus\boldsymbol{Z}/4\cdot(1+\rho)\beta_6(\mathring{\nu}_{-8n})$,

v) $\pi_S^{8n+1,-8n-4}(S_+^{6,0})=\boldsymbol{Z}/4\cdot\hat{\eta}^2[\eta\omega_1^{8n+3}]_6\oplus\boldsymbol{Z}/2\cdot\beta_6(\chi^3\hat{\nu}\mathring{\nu}_{-8n})$,

vi) $\pi_S^{8n+2,-8n-5}(S_+^{6,0})=\boldsymbol{Z}/12\cdot\beta_6(\nu\delta_1\omega_1^{8n+3})\oplus\boldsymbol{Z}/2\cdot\mathring{\sigma}\xi_{6,2}^*\omega_2^{4n-1}\oplus\boldsymbol{Z}/2\cdot\xi_{6,3}^*[\nu^2\omega_1^{8n-1}]_3$
$\oplus\boldsymbol{Z}/2\cdot\xi_{6,1}^*(\varepsilon\omega_1^{8n-3})\oplus\boldsymbol{Z}/2\cdot\xi_{6,1}^*(\eta\sigma\omega_1^{8n-3})$,

vii) $\pi_S^{8n+3,-8n-6}(S_+^{6,0})=\boldsymbol{Z}/2\cdot\beta_6(\hat{\eta}\eta\mathring{\eta}_{-8n-4})\oplus\boldsymbol{Z}/2\cdot\beta_6(\chi^3\nu\mathring{\nu}_{-8n})\oplus\boldsymbol{Z}/2\cdot y_7\xi_{6,2}^*\omega_2^{4n+3}$
$\oplus\boldsymbol{Z}/2\cdot\xi_{6,1}^*(\varepsilon\omega_1^{8n-2})\oplus\boldsymbol{Z}/2\cdot\xi_{6,1}^*(\eta\sigma\omega_1^{8n-2})$,

viii) $\pi_S^{8n+4,-8n-7}(S_+^{6,0})=\boldsymbol{Z}/12\cdot\beta_6(\nu\delta_1\omega_1^{8n+5})\oplus\boldsymbol{Z}/4\cdot\chi^4\mathring{\sigma}\omega_6^n\oplus\boldsymbol{Z}/4\cdot\chi\tilde{\nu}_3\xi_{6,4}^*\omega_4^{n+1}\bar{\omega}_4^n$

for any integer n.

Computation of $\pi_S^{7-p,-q-1}(S_+^{7,0})$ for $p+q=8$. $\delta_{6,1}(\nu\omega_1^{8n})=0$, $\delta_{6,1}(\nu\omega_1^{8n+1})=2\beta_6(\mathring{\nu}_{-8n})-(1+\rho)\beta_6(\mathring{\nu}_{-8n})$ of order 12, $\delta_{6,1}(\nu\omega_1^{8n+2})=\delta_{6,1}(\hat{\nu}\omega_1^{8n+4})=\beta_6(\chi^3\hat{\nu}\mathring{\nu}_{-8n})$ of order 2, $\delta_{6,1}(\nu\omega_1^{8n+3})=\beta_6(\nu\delta_1\omega_1^{8n+3})$ of order 12, $\delta_{6,1}(\nu\omega_1^{8n+4})=\beta_6(\chi^3\nu\mathring{\nu}_{-8n})$ of order 2, $\delta_{6,1}(\nu\omega_1^{8n+5})=\beta_6(\nu\delta_1\omega_1^{8n+5})$ of order 12, $\delta_{6,1}(\nu\omega_1^{8n+6})=\delta_{6,1}(\hat{\nu}\omega_1^{8n+8})=0$ and $\delta_{6,1}(\nu\omega_1^{8n+7})=\delta_{6,1}(\hat{\nu}\omega_1^{8n+9})=2\hat{\nu}\omega_6^{n+1}$ of order 12. Hence

$$(15.11)\qquad \mathrm{Ker}\Big[\delta_{6,1}\colon \bigoplus_{p+q=9}\pi_S^{7-p,-q-1}(S_+^{1,0})\to\bigoplus_{p+q=9}\pi_S^{6-p,-q}(S_+^{6,0})\Big]$$
$$=\boldsymbol{Z}[\omega_1^8,\omega_1^{-8}]\otimes\{\boldsymbol{Z}/24\cdot\nu\omega_1^{-2}\oplus\boldsymbol{Z}/24\cdot\nu\oplus\boldsymbol{Z}/12\cdot 2\nu\omega_1^2\oplus\boldsymbol{Z}/12\cdot 2\nu\omega_1^4\}$$
$$\oplus\,\boldsymbol{Z}[\omega_1^4,\omega_1^{-4}]\otimes\{\boldsymbol{Z}/2\cdot\eta^3\omega_1^{-1}\oplus\boldsymbol{Z}/2\cdot\eta^3\omega_1\}\,.$$

The exact sequence (1.9) for $r=6$ and (14.9) give the short exact sequence

$$0\to\bigoplus_{p+q=8}\pi_S^{6-p,-q-1}(S_+^{6,0})/\mathrm{Im}\,\delta_{6,1}\xrightarrow{\xi_{7,6}^*}\bigoplus_{p+q=8}\pi_S^{7-p,-q-1}(S_+^{7,0})$$
$$\xrightarrow{\eta_{1,7}^*}\boldsymbol{Z}[\omega_1^4,\omega_1^{-4}]\otimes\{\boldsymbol{Z}/2\cdot\eta^2\oplus\boldsymbol{Z}/2\cdot\eta^2\omega_1^{-1}\oplus\boldsymbol{Z}/2\cdot\eta^2\omega_1^{-2}\}\to 0\,.$$

$\eta_{1,7}^*(\hat{\eta}^2\omega_7^n)=\eta^2\omega_1^{8n-2}$, $2\hat{\eta}^2\omega_7^n=\xi_{7,6}^*(\hat{\eta}^3\omega_6^n)$, $\eta_{1,7}^*\beta_7(\hat{\eta}\mathring{\eta}_{-4n})=\eta^2\omega_1^{4n-1}$, $\beta_7(\hat{\eta}\mathring{\eta}_{-4n})$ is of order 2, $\eta_{1,7}^*\beta_7(\eta\mathring{\eta}_{-4n})=\eta^2\omega_1^{4n}$, $\beta_7(\eta\mathring{\eta}_{-4n})$ is of order 2, $\eta_{1,7}^*[\eta^2\omega_1^{8n+2}]_7=\eta^2\omega_1^{8n+2}$, $[\eta^2\omega_1^{8n+2}]_7$ is of order 8 and $\mathrm{Im}\,\delta_{6,1}$ is computed above (15.11). Thus we obtain

Proposition 15.12. $(\pi_S^{7-p,-q-1}(S_+^{7,0}),\ p+q=8)$

i) $\pi_S^{8n-2,-8n}(S_+^{7,0})=\boldsymbol{Z}/8\cdot\hat{\eta}^2\omega_7^n\oplus\boldsymbol{Z}/2\cdot\chi\hat{\nu}\mathring{\nu}\xi_{7,4}^*\omega_4^n\bar{\omega}_4^{n-1}$,

ii) $\pi_S^{8n-1,-8n-1}(S_+^{7,0})=\boldsymbol{Z}/2\cdot\beta_7(\hat{\eta}\mathring{\eta}_{-8n})\oplus\boldsymbol{Z}/2\cdot\chi\hat{\nu}\omega_7^n\oplus\boldsymbol{Z}/2\cdot\mathring{\sigma}\xi_{7,2}^*\omega_2^{4n-3}$
$\oplus\boldsymbol{Z}/2\cdot\xi_{7,3}^*[\nu^2\omega_1^{8n-5}]_3\oplus\boldsymbol{Z}/2\cdot\xi_{7,1}^*(\varepsilon\omega_1^{8n-7})\oplus\boldsymbol{Z}/2\cdot\xi_{7,1}^*(\eta\sigma\omega_1^{8n-7})$,

iii) $\pi_S^{8n,-8n-2}(S_+^{7,0})=\boldsymbol{Z}/2\cdot\beta_7(\eta\mathring{\eta}_{-8n})\oplus\boldsymbol{Z}/2\cdot\rho\beta_7(\eta\mathring{\eta}_{-8n})\oplus\boldsymbol{Z}/2\cdot\chi\nu\mathring{\nu}\xi_{7,4}^*\omega_4^n\bar{\omega}_4^{n-1}$
$\oplus\boldsymbol{Z}/2\cdot y_7\xi_{7,2}^*\omega_2^{4n+1}\oplus\boldsymbol{Z}/2\cdot\xi_{7,1}^*(\varepsilon\omega_1^{8n-6})\oplus\boldsymbol{Z}/2\cdot\xi_{7,1}^*(\eta\sigma\omega_1^{8n-6})$,

iv) $\pi_S^{8n+1,-8n-3}(S_+^{7,0})=\boldsymbol{Z}/\boldsymbol{8}\cdot\beta_7(\chi\mathring{\nu}_{-8n})\oplus\boldsymbol{Z}/\boldsymbol{4}\cdot\chi^5\mathring{\sigma}\omega_7^n$,

v) $\pi_S^{8n+2,-8n-4}(S_+^{7,0})=\boldsymbol{Z}/\boldsymbol{8}\cdot[\eta^2\omega_1^{8n+2}]_7$,

vi) $\pi_S^{8n+3,-8n-5}(S_+^{7,0})=\boldsymbol{Z}/\boldsymbol{2}\cdot\beta_7(\mathring{\eta}\mathring{\eta}_{-8n-4})\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\mathring{\sigma}\xi_{7,2}^*\omega_2^{4n-1}\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\xi_{7,3}^*[\nu^2\omega_1^{8n-1}]_3$
$\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\xi_{7,1}^*(\varepsilon\omega_1^{8n-3})\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\xi_{7,1}^*(\eta\sigma\omega_1^{8n-3})$,

vii) $\pi_S^{8n+4,-8n-6}(S_+^{7,0})=\boldsymbol{Z}/\boldsymbol{2}\cdot\beta_7(\eta\mathring{\eta}_{-8n-4})\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\rho\beta_7(\eta\mathring{\eta}_{-8n-4})\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot y_7\xi_{7,2}^*\omega_2^{4n+3}$
$\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\xi_{7,1}^*(\varepsilon\omega_1^{8n-2})\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\xi_{7,1}^*(\eta\sigma\omega_1^{8n-2})$,

viii) $\pi_S^{8n+5,-8n-7}(S_+^{7,0})=\boldsymbol{Z}/\boldsymbol{4}\cdot\chi^5\mathring{\sigma}\omega_7^n\oplus\boldsymbol{Z}/\boldsymbol{4}\cdot\chi\tilde{\nu}_3\xi_{7,4}^*\omega_4^{n+1}\bar{\omega}_4^n$

for any integer n.

Computation of $\pi_S^{8-p,-q-1}(S_+^{8,0})$ for $p+q=8$. $\delta_{7,1}(\eta^2\omega_1^{4n})=\delta_{7,1}(\eta^2\omega_1^{4n+3})=\delta_{7,1}(\eta^2\omega_1^{8n-2})=0$, $\delta_{7,1}(\eta^3\omega_1^{4n+1})=(1+\rho)\beta_7(\eta\mathring{\eta}_{-4n})$ of order 2 and $\delta_{7,1}(\eta^2\omega_1^{8n+2})=\delta_{7,1}(\mathring{\eta}^2\omega_1^{8n+4})=4\beta_7(\chi\mathring{\nu}_{-8n})$ of order 2. Hence

$$\text{(15.13)}\quad \operatorname{Ker}[\delta_{7,1}\colon \bigoplus_{p+q=9}\pi_S^{8-p,-q-1}(S_+^{1,0})\to\bigoplus_{p+q=9}\pi_S^{7-p,-q}(S_+^{7,0})]$$
$$=\boldsymbol{Z}[\omega_1^4,\omega_1^{-4}]\otimes\{\boldsymbol{Z}/\boldsymbol{2}\cdot\eta^2\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\eta^2\omega_1^{-1}\}\oplus\boldsymbol{Z}[\omega_1^8,\omega_1^{-8}]\otimes\{\boldsymbol{Z}/\boldsymbol{2}\cdot\eta^2\omega_1^{-2}\}\,.$$

The exact sequence (1.9) for $r=7$ and (14.11) give the short exact sequence

$$0\to\bigoplus_{p+q=8}\pi_S^{7-p,-q-1}(S_+^{7,0})/\operatorname{Im}\delta_{7,1}\xrightarrow{\xi_{8,7}^*}\bigoplus_{p+q=8}\pi_S^{8-p,-q-1}(S_+^{8,0})$$
$$\xrightarrow{\eta_{1,8}^*}\boldsymbol{Z}[\omega_1^8,\omega_1^{-8}]\otimes\{\boldsymbol{Z}/\boldsymbol{2}\cdot\eta\omega_1^{-1}\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\eta\omega\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\eta\omega_1^4\}\to 0\,.$$

Since $\eta_{1,8}^*(\mathring{\eta}\omega_8^n)=\eta\omega_1^{8n-1}$, $2\mathring{\eta}\omega_8^n=\xi_{8,7}^*(\mathring{\eta}^2\omega_7^n)$, $\eta_{1,8}^*\beta_8(\mathring{\eta}_{-4n})=\eta\omega_1^{4n}$, $\beta_8(\mathring{\eta}_{-4n})$ is of order 2 and Im $\delta_{7,1}$ is computed above (15.13), we obtain

Proposition 15.14. $(\pi_S^{8-p,-q-1}(S_+^{8,0}),\ p+q=8)$

i) $\pi_S^{8n-1,-8n}(S_+^{8,0})=\boldsymbol{Z}/\boldsymbol{16}\cdot\mathring{\eta}\omega_8^n\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\chi\mathring{\nu}\mathring{\nu}\xi_{8,4}^*\omega_4^n\bar{\omega}_4^{n-1}$,

ii) $\pi_S^{8n,-8n-1}(S_+^{8,0})=\boldsymbol{Z}/\boldsymbol{2}\cdot\beta_8(\mathring{\eta}_{-8n})\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\rho\beta_8(\mathring{\eta}_{-8n})\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\chi^2\mathring{\nu}\omega_8^n\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\mathring{\sigma}\xi_{8,2}^*\omega_2^{4n-3}$
$\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\xi_{8,3}^*[\nu^2\omega_1^{8n-5}]_3\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\xi_{8,1}^*(\varepsilon\omega_1^{8n-7})\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\xi_{8,1}^*(\eta\sigma\omega_1^{8n-7})$,

iii) $\pi_S^{8n+1,-8n-2}(S_+^{8,0})=\boldsymbol{Z}/\boldsymbol{2}\cdot\beta_7(\chi\eta\mathring{\eta}_{-8n})\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\chi\nu\mathring{\nu}\xi_{8,4}^*\omega_4^n\bar{\omega}_4^{n-1}\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot y_7\xi_{8,2}^*\omega_2^{4n+1}$
$\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\xi_{8,1}^*(\varepsilon\omega_1^{8n-6})\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\xi_{8,1}^*(\eta\sigma\omega_1^{8n-6})$,

iv) $\pi_S^{8n+2,-8n-3}(S_+^{8,0})=\boldsymbol{Z}/\boldsymbol{4}\cdot\beta_8(\chi^2\mathring{\nu}_{-8n})\oplus\boldsymbol{Z}/\boldsymbol{4}\cdot\chi^6\mathring{\sigma}\omega_8^n$,

v) $\pi_S^{8n+3,-8n-4}(S_+^{8,0})=\boldsymbol{Z}/\boldsymbol{8}\cdot\xi_{8,7}^*[\eta^2\omega_1^{8n+2}]_7$,

vi) $\pi_S^{8n+4,-8n-5}(S_+^{8,0})=\boldsymbol{Z}/\boldsymbol{2}\cdot\beta_8(\mathring{\eta}_{-8n-4})\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\rho\beta_8(\mathring{\eta}_{-8n-4})\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\sigma\xi_{8,2}^*\omega_2^{4n-1}$
$\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\xi_{8,3}^*[\nu^2\omega_1^{8n-1}]_3\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\xi_{8,1}^*(\varepsilon\omega_1^{8n-3})\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\xi_{8,1}^*(\eta\sigma\omega_1^{8n-3})$,

vii) $\pi_S^{8n+5,-8n-6}(S_+^{8,0})=\boldsymbol{Z}/\boldsymbol{2}\cdot\beta_8(\chi\eta\mathring{\eta}_{-8n-4})\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot y_7\xi_{8,2}^*\omega_2^{4n+3}\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\xi_{8,1}^*(\varepsilon\omega_1^{8n-2})$
$\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\xi_{8,1}^*(\eta\sigma\omega_1^{8n-2})$,

viii) $\pi_S^{8n+6,-8n-7}(S_+^{8,0})=\boldsymbol{Z}/\boldsymbol{4}\cdot\chi^6\mathring{\sigma}\omega_8^n\oplus\boldsymbol{Z}/\boldsymbol{4}\cdot\chi\tilde{\nu}_3\xi_{8,4}^*\omega_4^{n+1}\bar{\omega}_4^n$

for any integer n.

Computation of $\pi_S^{9-p,-q-1}(S_+^{9,0})$ for $p+q=8$. $\delta_{8,1}(\eta\omega_1^{4n})=0$, $\delta_{8,1}(\eta\omega_1^{4n+1})=(1+\rho)\beta_8(\mathring{\eta}_{-4n})$ of order 2, $\delta_{8,1}(\eta\omega_1^{4n+2})=\beta_8(\chi\eta\mathring{\eta}_{-4n})$ of order 2, $\delta_{8,1}(\eta\omega_1^{8n+3})=\delta_{8,1}(\hat{\eta}\omega_1^{8n+4})=2\beta_8(\chi^2\mathring{\nu}_{-8n})$ of order 2, $\delta_{8,1}(\eta\omega_1^{16n+7})=\delta_{8,1}(\hat{\eta}\omega_1^{16n+8})=2\beta_8(\chi^6\mathring{\sigma}_{-16n})$ of order 2 and $\delta_{8,1}(\eta\omega_1^{16n-1})=0$. Hence

$$
\begin{aligned}
(15.15)\qquad & \operatorname{Ker}[\delta_{8,1}\colon \bigoplus_{p+q=9}\pi_S^{9-p,-q-1}(S_+^{1,0}) \to \bigoplus_{p+q=9}\pi_S^{8-p,-q}(S_+^{8,0})] \\
&= \boldsymbol{Z}[\omega_1^{16}, \omega_1^{-16}]\otimes\boldsymbol{Z}/2\cdot\eta\omega_1^{-1}\oplus\boldsymbol{Z}[\omega_1^{4}, \omega_1^{-4}]\otimes\boldsymbol{Z}/2\cdot\eta\,.
\end{aligned}
$$

The exact sequence (1.9) for $r=8$ and (14.14) give the short exact sequence

$$
\begin{aligned}
0 \to \bigoplus_{p+q=8}\pi_S^{8-p,-q-1}(S_+^{8,0})/\operatorname{Im}\delta_{8,1} &\xrightarrow{\xi_{9,8}^*} \bigoplus_{p+q=8}\pi_S^{9-p,-q-1}(S_+^{9,0}) \\
&\xrightarrow{\eta_{1,9}^*} \boldsymbol{Z}[\omega_1^{16}, \omega_1^{-16}]\otimes\{\boldsymbol{Z}\cdot\oplus\sum_{i=1}^{7}\boldsymbol{Z}\cdot 2\omega_1^{2i}\} \to 0\,.
\end{aligned}
$$

Thus we obtain

Proposition 15.16. ($\pi_S^{9-p,-q-1}(S_+^{9,0})$, $p+q=8$)

i) $\pi_S^{16n,-16n}(S_+^{9,0})=\boldsymbol{Z}\cdot\omega_9^n\oplus\boldsymbol{Z}/\boldsymbol{16}\cdot(1+\rho)\omega_9^n\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\chi\hat{\nu}\mathring{\nu}\xi_{9,4}^*\omega_4^{2n}\overline{\omega}_4^{2n-1}$,

ii) $\pi_S^{16n+8,-16n-8}(S_+^{9,0})=\boldsymbol{Z}\cdot\delta_{9,1}\omega_1^{16n+9}\oplus\boldsymbol{Z}/\boldsymbol{16}\cdot\hat{\eta}\xi_{9,8}^*\omega_8^{n+1}\overline{\omega}_8^{n}\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\chi\hat{\nu}\mathring{\nu}\xi_{9,4}^*\omega_4^{2n+1}\overline{\omega}_4^{2n}$,

iii) $\pi_S^{8n+1,-8n-1}(S_+^{9,0})=\boldsymbol{Z}/\boldsymbol{2}\cdot\beta_9(\chi\mathring{\eta}_{-8n})\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\mathring{\sigma}\xi_{9,2}^*\omega_2^{4n-3}\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\chi^2\hat{\nu}\xi_{9,8}^*\omega_8^n$
$\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\xi_{9,3}^*[\nu^2\omega_1^{8n-5}]_3\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\xi_{9,1}^*(\varepsilon\omega_1^{8n-7})\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\xi_{9,1}^*(\eta\sigma\omega_1^{8n-7})$,

iv) $\pi_S^{8n+2,-8n-2}(S_+^{9,0})=\boldsymbol{Z}\cdot\delta_{9,1}\omega_1^{8n+3}\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\chi\nu\mathring{\nu}\xi_{9,4}^*\omega_4^{n}\overline{\omega}_4^{n-1}\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot y_7\xi_{9,2}^*\omega_2^{4n+1}$
$\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\xi_{9,1}^*(\varepsilon\omega_5^{8n-6})\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\xi_{9,1}^*(\eta\sigma\omega_1^{8n-6})$,

v) $\pi_S^{8n+3,-8n-3}(S_+^{9,0})=\boldsymbol{Z}/\boldsymbol{2}\cdot\beta_9(\chi^3\mathring{\nu}_{-8n})\oplus\boldsymbol{Z}/\boldsymbol{4}\cdot\chi^6\hat{\sigma}\xi_{9,8}^*\omega_8^n$,

vi) $\pi_S^{8n+4,-8n-4}(S_+^{9,0})=\boldsymbol{Z}\cdot\delta_{9,1}\omega_1^{8n+5}\oplus\boldsymbol{Z}/\boldsymbol{8}\cdot\xi_{9,7}^*[\eta^2\omega_1^{8n+2}]_7$,

vii) $\pi_S^{8n+5,-8n-5}(S_+^{9,0})=\boldsymbol{Z}/\boldsymbol{2}\cdot\beta_9(\chi\mathring{\eta}_{-8n-4})\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\mathring{\sigma}\xi_{9,2}^*\omega_2^{4n-1}\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\xi_{9,3}^*[\nu^2\omega_1^{8n-1}]_3$
$\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\xi_{9,1}^*(\varepsilon\omega_1^{8n-3})\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\xi_{9,1}^*(\eta\sigma\omega_1^{8n-3})$,

viii) $\pi_S^{8n+6,-8n-6}(S_+^{9,0})=\boldsymbol{Z}\cdot\delta_{9,1}\omega_1^{8n+7}\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot y_7\xi_{9,2}^*\omega_2^{4n+3}\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\xi_{9,1}^*(\varepsilon\omega_1^{8n-2})$
$\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\xi_{9,1}^*(\eta\sigma\omega_1^{8n-2})$,

ix) $\pi_S^{16n+7,-16n-7}(S_+^{9,0})=\boldsymbol{Z}/\boldsymbol{2}\cdot\beta_9(\chi^7\mathring{\sigma}_{-16n})\oplus\boldsymbol{Z}/\boldsymbol{4}\cdot\chi\tilde{\nu}_3\xi_{9,4}^*\omega_4^{2n+1}\overline{\omega}_4^{2n}$,

x) $\pi_S^{16n+15,-16n-15}(S_+^{9,0})=\boldsymbol{Z}/\boldsymbol{4}\cdot\chi^6\mathring{\sigma}\xi_{9,8}^*\omega_8^{n+1}\overline{\omega}_8^{n}\oplus\boldsymbol{Z}/\boldsymbol{4}\cdot\chi\tilde{\nu}_3\xi_{9,4}^*\omega_4^{2n+2}\overline{\omega}_4^{2n+1}$

for any integer n.

Computation of $\pi_S^{10-p,-q-1}(S_+^{10,0})$ for $p+q=8$. By a parallel argument to Proposition 10.7 we have

Proposition 15. 17. i) $\delta_{9,1}\omega_1^{32n}=0$, ii) $\delta_{9,1}\omega_1^{32n+16}=2\chi^6\mathring{\sigma}\xi_{9,8}^*\omega_8^{2n+1}\overline{\omega}_8^{2n}$, iii) $\delta_{9,1}\omega_1^{16n+8}=\beta_9(\chi^7\mathring{\sigma}_{-16n})$, iv) $\delta_{8,1}\omega_1^{8n+4}=\beta_9(\chi^3\mathring{\nu}_{-8n})$, v) $\delta_{9,1}\omega_1^{4n+2}=\beta_9(\chi\mathring{\eta}_{-4n})$, vi) $\delta_{9,1}\omega_1^{32n+1}=2\omega_9^{2n}-(1+\rho)\omega_9^{2n}$ *and* vii) $\delta_{9,1}\omega_1^{32n+17}=2\omega_9^{2n+1}-(1+\rho)\omega_9^{2n+1}$ *for any integer n.*

This Proposition implies that

$$\begin{aligned}(15.18)\qquad &\operatorname{Ker}\Big[\delta_{9,1}\colon \bigoplus_{p+q=9} \pi_S^{10-p,-q-1}(S_+^{1,0}) \to \bigoplus_{p+q=9} \pi_S^{9-p,-q}(S_+^{9,0})\Big]\\ &= \boldsymbol{Z}[\omega_1^{32}, \omega_1^{-32}] \otimes \Big\{\boldsymbol{Z}\cdot 1 \oplus \sum_{i=1}^{15} \boldsymbol{Z}\cdot 2\omega_1^{2i}\Big\}\,,\end{aligned}$$

and we have the isomorphism

$$\xi_{10,9}^{*}\colon \bigoplus_{p+q=8} \pi_S^{9-p,-q-1}(S_+^{9,0})/\operatorname{Im}\delta_{9,1} \approx \bigoplus_{p+q=8} \pi_S^{10-p,-q-1}(S_+^{10,0})$$

by Proposition 1.10. Thus we obtain

Proposition 15.19. $(\pi_S^{10-p,-q-1}(S_+^{10,0}),\ p+q=8)$

i) $\pi_S^{32n+1,-32n}(S_+^{10,0}) = \boldsymbol{Z}/32\cdot\chi\omega_{10}^{n} \oplus \boldsymbol{Z}/2\cdot\chi\hat{\nu}\mathring{\nu}\xi_{10,4}^{*}\omega_4^{4n}\bar{\omega}_4^{4n-1}$,

ii) $\pi_S^{32n+17,-32n-16}(S_+^{10,0}) = \boldsymbol{Z}/32\cdot\xi_{10,9}^{*}\omega_9^{2n+1} \oplus \boldsymbol{Z}/2\cdot\chi\hat{\nu}\mathring{\nu}\xi_{10,4}^{*}\omega_4^{4n+2}\bar{\omega}_4^{4n+1}$,

iii) $\pi_S^{16n+9,-16n-8}(S_+^{10,0}) = \boldsymbol{Z}/16\cdot\hat{\eta}\xi_{10,8}^{*}\omega_8^{n+1}\bar{\omega}_8^{n} \oplus \boldsymbol{Z}/2\cdot\chi\hat{\nu}\mathring{\nu}\xi_{10,4}^{*}\omega_4^{2n+1}\bar{\omega}_4^{2n}$,

iv) $\pi_S^{8n+2,-8n-1}(S_+^{10,0}) = \boldsymbol{Z}/2\cdot\chi^2\hat{\nu}\xi_{10,8}^{*}\omega_8^{n} \oplus \boldsymbol{Z}/2\cdot\xi_{10,3}^{*}[\nu^2\omega_1^{8n-5}]_3 \oplus \boldsymbol{Z}/2\cdot\mathring{\sigma}\xi_{10,2}^{*}\omega_2^{4n-3}$
$\oplus \boldsymbol{Z}/2\cdot\xi_{10,1}^{*}(\varepsilon\omega_1^{8n-7}) \oplus \boldsymbol{Z}/2\cdot\xi_{10,1}^{*}(\eta\sigma\omega_1^{8n-7})$,

v) $\pi_S^{8n+3,-8n-2}(S_+^{10,0}) = \boldsymbol{Z}/2\cdot\chi\nu\mathring{\nu}\xi_{10,4}^{*}\omega_4^{n}\bar{\omega}_4^{n-1} \oplus \boldsymbol{Z}/2\cdot y_7\xi_{10,2}^{*}\omega_2^{4n+1}$
$\oplus \boldsymbol{Z}/2\cdot\xi_{10,1}^{*}(\varepsilon\omega_1^{8n-6}) \oplus \boldsymbol{Z}/2\cdot\xi_{10,1}^{*}(\eta\sigma\omega_1^{8n-6})$,

vi) $\pi_S^{8n+4,-8n-3}(S_+^{10,0}) = \boldsymbol{Z}/4\cdot\chi^6\hat{\sigma}\xi_{10,8}^{*}\omega_8^{n}$,

vii) $\pi_S^{8n+5,-8n-4}(S_+^{10,0}) = \boldsymbol{Z}/8\cdot\xi_{10,7}^{*}[\eta^2\omega_1^{8n+2}]_7$,

viii) $\pi_S^{8n+6,-8n-5}(S_+^{10,0}) = \boldsymbol{Z}/2\cdot\xi_{10,3}^{*}[\nu^2\omega_1^{8n-1}]_3 \oplus \boldsymbol{Z}/2\cdot\mathring{\sigma}\xi_{10,2}^{*}\omega_2^{4n-1} \oplus \boldsymbol{Z}/2\cdot\xi_{10,1}^{*}(\varepsilon\omega_1^{8n-3})$
$\oplus \boldsymbol{Z}/2\cdot\xi_{10,1}^{*}(\eta\sigma\omega_1^{8n-3})$,

ix) $\pi_S^{8n+7,-8n-6}(S_+^{10,0}) = \boldsymbol{Z}/2\cdot y_7\xi_{10,2}^{*}\omega_2^{4n+3} \oplus \boldsymbol{Z}/2\cdot\xi_{10,1}^{*}(\varepsilon\omega_1^{8n-2}) \oplus \boldsymbol{Z}/2\cdot\xi_{10,1}^{*}(\eta\sigma\omega_1^{8n-2})$,

x) $\pi_S^{16n+8,-16n-7}(S_+^{10,0}) = \boldsymbol{Z}/4\cdot\chi\tilde{\nu}_3\xi_{10,4}^{*}\omega_4^{2n+1}\bar{\omega}_4^{2n}$,

xi) $\pi_S^{32n+16,-32n-15}(S_+^{10,0}) = \boldsymbol{Z}/2\cdot\chi^6\mathring{\sigma}\xi_{10,8}^{*}\omega_8^{2n+1}\bar{\omega}_8^{2n} \oplus \boldsymbol{Z}/4\cdot\chi\tilde{\nu}_3\xi_{10,4}^{*}\omega_4^{4n+2}\bar{\omega}_4^{4n+1}$,

xii) $\pi_S^{32n+32,-32n-31}(S_+^{10,0}) = \boldsymbol{Z}/4\cdot\chi^6\mathring{\sigma}\xi_{10,8}^{*}\omega_8^{2n+2}\bar{\omega}_8^{2n+1} \oplus \boldsymbol{Z}/4\cdot\chi\tilde{\nu}_3\xi_{10,4}^{*}\omega_4^{4n+4}\bar{\omega}_4^{4n+3}$

for any integer n.

Proposition 15.20. $\rho\xi_{10,9}^{*}\omega_9^{2n+1} = \xi_{10,9}^{*}\omega_9^{2n+1}$.

Proposition 15.19 describes $\lambda_{p,q}^{S}$ for $p+q=8$. By Proposition 4.8 the groups $\pi_{p,q}^{S}$ for $p+q=8$ are determined except $\pi_{4,4}^{S}$, $\pi_{5,3}^{S}$, $\pi_{6,2}^{S}$, $\pi_{7,1}^{S}$, $\pi_{8,0}^{S}$ and $\pi_{9,-1}^{S}$.

The stable equivariant Toda bracket

$$\langle\hat{\nu},\ \hat{\eta},\ \hat{\nu}\rangle^{\tau} \subset \pi_{5,3}^{S}$$

is well-defined and $\phi(\langle\hat{\nu}, \hat{\eta}, \hat{\nu}\rangle^{\tau}) = \langle\eta, 2, \eta\rangle = \{6\nu, -6\nu\}$ by [14], (5.4). Let

$$(15.21)\qquad \hat{\hat{\nu}} \in \langle\hat{\nu},\ \hat{\eta},\ \hat{\nu}\rangle^{\tau} \subset \pi_{5,3}^{S}$$

be the element such that $\phi(\hat{\bar{\nu}})=6\nu$. Then $\hat{\bar{\nu}}$ is of order $\geqq 4$. Since $\psi(\langle\hat{\nu}, \hat{\eta}, \hat{\nu}\rangle^{\tau}) \subset\langle\nu, \eta, \nu\rangle=\{\bar{\nu}\}$ by [14], Lemma 6.2, we see that $\psi(\hat{\bar{\nu}})=\bar{\nu}$. Next, $2\langle\hat{\nu}, \hat{\eta}, \hat{\nu}\rangle^{\tau} \subset\langle\hat{\nu}, \hat{\eta}, 2\hat{\nu}\rangle^{\tau}=\langle\hat{\nu}, \hat{\eta}, \delta_1(\nu\omega_1^{-1})\rangle^{\tau}=\boldsymbol{Z}/\boldsymbol{2}\cdot\chi\hat{\nu}^3$ because: $\langle\hat{\nu}, \hat{\eta}, \nu\omega_1^{-1}\rangle^{\tau}\subset\pi_S^{4,-4}(S_+^{1,0})$, $\delta_1(\varepsilon\omega_1^{-4})=\delta_1\beta_1(\mathring{\varepsilon}_4)=0$, $\delta_1(\eta\sigma\omega_1^{-4})=\delta_1\beta_1(\mathring{\eta}_4\sigma)=0$, and hence $\langle\hat{\nu}, \hat{\eta}, \delta_1(\nu\omega_1^{-1})\rangle^{\tau}\supset \delta_1(\langle\hat{\nu}, \hat{\eta}, \nu\omega_1^{-1}\rangle^{\tau})=0$, i.e. $\langle\hat{\nu}, \hat{\eta}, \delta_1(\nu\omega_1^{-1})\rangle^{\tau}=$ its indeterminacy$=\boldsymbol{Z}/\boldsymbol{2}\cdot\chi\hat{\nu}^3$. Thus $\hat{\bar{\nu}}$ is of order 4.

Observe the fixed-point exact sequence (1.12) for $r=10$;

$$\pi_{-5,4}^S \xrightarrow{\beta_{10}} \pi_S^{5,-4}(S_+^{10,0}) \xrightarrow{\delta_{10}} \pi_{5,3}^S \xrightarrow{\chi^{10}} \pi_{-5,3}^S \xrightarrow{\beta_{10}} \pi_S^{5,-3}(S_+^{10,0})\,.$$

Since $\pi_{-5,4}^S\approx\pi_4^S=0$, $\pi_S^{5,-4}(S_+^{10,0})=\boldsymbol{Z}/\boldsymbol{8}\cdot\xi_{10,7}^*[\eta^2\omega_1^2]_7$, $\pi_{-5,3}^S=\boldsymbol{Z}/\boldsymbol{24}\cdot\chi^5\nu$ and $\beta_{10}(\chi^5\nu)=\beta_{10}(\chi^9\hat{\sigma})=\xi_{10,1}^*(\sigma\omega_1^{-4})$ is of order 2 by Proposition 14.15, the above exact sequence yields the short exact sequence

$$0\to\boldsymbol{Z}/\boldsymbol{8}\cdot\mathring{\varepsilon}_5\to\pi_{5,3}^S\xrightarrow{\phi}2\cdot\pi_3^S\to 0\,. \tag{15.22}$$

Since $\phi(3\hat{\eta}\hat{\sigma}-\hat{\bar{\nu}})=0$, there exists $a\in\boldsymbol{Z}/\boldsymbol{8}$ satisfying

$$3\hat{\eta}\hat{\sigma}-\hat{\bar{\nu}}=a\cdot\mathring{\varepsilon}_5\,. \tag{15.23}$$

Apply ψ to both sides of this equality. Then

$$\varepsilon=\eta\sigma+\bar{\nu}=3\eta\sigma+\bar{\nu}=\psi(3\hat{\eta}\hat{\sigma}-\hat{\bar{\nu}})=a\cdot\psi(\mathring{\varepsilon}_5)\,,$$

hence a is of order 8 and $\psi(\mathring{\varepsilon}_5)=\varepsilon$. Multiply both sides of (15.23) by four. Since $\hat{\bar{\nu}}$ is of order 4, we see that $12\hat{\eta}\hat{\sigma}=4\mathring{\varepsilon}_5$. Thus we obtain

Proposition 15.24. i) $\pi_{5,3}^S=\boldsymbol{Z}/\boldsymbol{24}\cdot\hat{\eta}\hat{\sigma}\oplus\boldsymbol{Z}/\boldsymbol{4}\cdot\hat{\bar{\nu}}$.
ii) $\mathring{\varepsilon}_5$ *is of order* 8, $\psi(\mathring{\varepsilon}_5)=\varepsilon$ *and* $4\mathring{\varepsilon}_5=12\hat{\eta}\hat{\sigma}$.
iii) $\hat{\bar{\nu}}$ *is the element of* $\langle\hat{\nu}, \hat{\eta}, \hat{\nu}\rangle^{\tau}$ *such that* $\psi(\hat{\bar{\nu}})=\bar{\nu}$ *and* $\phi(\hat{\bar{\nu}})=6\nu$.

Proposition 15.25. i) $\mathring{\varepsilon}_{8n+5}\in\pi_{8n+5,-8n+3}^S$ *is of order* 8 *and* $\rho\mathring{\varepsilon}_{8n+5}=\mathring{\varepsilon}_{8n+5}$.
ii) $\beta_1(\mathring{\varepsilon}_{8n+5})=\varepsilon\omega_1^{-8n-5}$ *and* $\chi\mathring{\varepsilon}_{8n+5}=\hat{\eta}\bar{\nu}_{8n+3}$.

Proof. i) follows from Proposition 15.5.

ii). Since $\eta_{1,7}^*([\eta^2\omega_1^2]_7\omega_7^n)=\eta^2\omega_1^{8n+2}$, there exists $a\in\boldsymbol{Z}/\boldsymbol{4}$ satisfying

$$[\eta^2\omega_1^{8n+2}]_7=[\eta^2\omega_1^2]_7\omega_7^n+a\cdot\xi_{7,6}^*(\hat{\eta}^2[\eta\omega_1^{8n+3}]_6)\,.$$

Apply $\delta_{1,7}$ to both sides of this equality. Then we see that $\beta_1(\mathring{\varepsilon}_{-8n+5})=\delta_{1,7}[\eta^2\omega_1^{8n+2}]_7=\delta_{1,7}([\eta^2\omega_1^2]_7\omega_7^n)+a\cdot\delta_{1,7}\xi_{7,6}^*(\hat{\eta}^2[\eta\omega_1^{8n+3}]_6)=\beta_1(\mathring{\varepsilon}_5)\omega_1^{8n}+a\cdot\beta_1(\hat{\eta}^2\bar{\nu}_{-8n+3})=\varepsilon\omega_1^{8n-5}$ by Proposition 15.24.

$\chi\mathring{\varepsilon}_{8n+5}=\chi\delta_7[\eta^2\omega_1^{-8n+2}]_7=\delta_6\eta_{6,7}^*[\eta^2\omega_1^{-8n+2}]_7=\delta_6(\hat{\eta}[\eta\omega_1^{-8n+3}]_6)=\hat{\eta}\bar{\nu}_{8n+3}$. $\square$

Here we prove (14.16).

Proof of (14.16). Propositions 15.24 and 15.25 imply that $12(1+\rho)\hat{\sigma}=$

$12\chi\hat{\eta}\hat{\sigma}=4\chi\mathring{\varepsilon}_5=4\hat{\eta}\tilde{\nu}_3=120(1-\rho)\hat{\sigma}=240\hat{\sigma}$. $\square$

It is now easy to compute the groups $\pi^S_{4,4}$, $\pi^S_{6,2}$, $\pi^S_{7,1}$, $\pi^S_{8,0}$ and $\pi^S_{9,-1}$ and we obtain

Theorem 15.26. ($\pi^S_{p,q}$, $p+q=8$)

i) $\pi^S_{5,3}=\boldsymbol{Z}/24\cdot\hat{\eta}\hat{\sigma}\oplus\boldsymbol{Z}/4\cdot\hat{\hat{\nu}}$.

ii) $\pi^S_{6,2}=0$.

iii) $\pi^S_{7,1}=\boldsymbol{Z}/2\cdot y_7\mathring{\eta}\oplus\boldsymbol{Z}/2\cdot\eta\sigma\delta_1\omega_1^{-6}\oplus\boldsymbol{Z}/2\cdot\varepsilon\delta_1\omega_1^{-6}$.

iv) $\pi^S_{8,0}=\boldsymbol{Z}\cdot\hat{\eta}y_7\oplus\boldsymbol{Z}/2\cdot\mathring{\eta}_8\sigma\oplus\boldsymbol{Z}/2\cdot\rho\mathring{\eta}_8\sigma\oplus\boldsymbol{Z}/2\cdot\mathring{\varepsilon}_8\oplus\boldsymbol{Z}/2\cdot\rho\mathring{\varepsilon}_8$.

v) $\pi^S_{9,-1}=\boldsymbol{Z}/2\cdot\chi\hat{\nu}\mathring{\nu}\mathring{\nu}_8$.

vi) $\pi^S_{8n,-8n+8}\approx\boldsymbol{Z}/2\cdot\chi^2\hat{\nu}\delta_8\omega_8^{1-n}\oplus\boldsymbol{Z}/2\cdot\mathring{\eta}_{8n}\mathring{\sigma}\oplus\boldsymbol{Z}/2\cdot\rho\mathring{\eta}_{8n}\mathring{\sigma}\oplus\boldsymbol{Z}/2\cdot\mathring{\varepsilon}_{8n}\oplus\boldsymbol{Z}/2\cdot\rho\mathring{\varepsilon}_{8n}$ $\oplus\pi^S_{-8n+8}$ *for* $n\neq1$.

vii) $\pi^S_{16n+1,-16n+7}\approx\boldsymbol{Z}/16\cdot\hat{\eta}\mathring{\sigma}_{16n}\oplus\boldsymbol{Z}/2\cdot\chi\hat{\nu}\mathring{\nu}\mathring{\nu}_{16n}\oplus\pi^S_{-16n+7}$ *for any integer* n.

viii) $\pi^S_{32n+9,-32n-1}\approx\boldsymbol{Z}/32\cdot\delta_{10}\omega_{10}^{-n}\oplus\boldsymbol{Z}/2\cdot\chi\hat{\nu}\mathring{\nu}\mathring{\nu}_{32n+8}\oplus\pi^S_{-32n-1}$ *for* $n\neq0$.

ix) $\pi^S_{32n+25,-32n-17}\approx\boldsymbol{Z}/32\cdot\delta_9\omega_9^{-2n-1}\oplus\boldsymbol{Z}/2\cdot\chi\hat{\nu}\mathring{\nu}\mathring{\nu}_{32n+24}\oplus\pi^S_{-32n-17}$ *for any integer* n.

x) $\pi^S_{16n+2,-16n+6}\approx\boldsymbol{Z}/4\cdot\chi\tilde{\nu}_3\mathring{\nu}_{16n}\oplus\pi^S_{-16n+6}$ *for any integer* n.

xi) $\pi^S_{32n+10,-32n-2}\approx\boldsymbol{Z}/4\cdot\chi\tilde{\nu}_3\mathring{\nu}_{32n+8}\oplus\boldsymbol{Z}/4\cdot\chi^6\mathring{\sigma}\mathring{\sigma}_{32n+16}\oplus\pi^S_{-32n-2}$ *for any integer* n.

xii) $\pi^S_{32n+26,-32n-18}\approx\boldsymbol{Z}/4\cdot\chi\tilde{\nu}_3\mathring{\nu}_{32n+24}\oplus\boldsymbol{Z}/2\cdot\chi^6\mathring{\sigma}\mathring{\sigma}_{32n+32}\oplus\pi^S_{-32n-18}$ *for any integer* n.

xiii) $\pi^S_{8n+3,-8n+5}\approx\boldsymbol{Z}/2\cdot y_7\mathring{\eta}_{8n-4}\oplus\boldsymbol{Z}/2\cdot\eta\sigma\delta_1\omega_1^{-8n-2}\oplus\boldsymbol{Z}/2\cdot\varepsilon\delta_1\omega_1^{-8n-2}\oplus\pi^S_{-8n+5}$ *for any integer* n.

xiv) $\pi^S_{8n+4,-8n+4}\approx\boldsymbol{Z}/2\cdot\mathring{\eta}_{8n+4}\mathring{\sigma}\oplus\boldsymbol{Z}/2\cdot\rho\mathring{\eta}_{8n+4}\mathring{\sigma}\oplus\boldsymbol{Z}/2\cdot\mathring{\varepsilon}_{8n+4}\oplus\boldsymbol{Z}/2\cdot\rho\mathring{\varepsilon}_{8n+4}\oplus\pi^S_{-8n+4}$ *for any integer* n.

xv) $\pi^S_{8n+5,-8n+3}\approx\boldsymbol{Z}/8\cdot\mathring{\varepsilon}_{8n+5}\oplus\pi^S_{-8n+3}$ *for* $n\neq0$.

xvi) $\pi^S_{8n+6,-8n+2}\approx\boldsymbol{Z}/4\cdot\chi^6\hat{\sigma}\delta_8\omega_8^{-n}\oplus\pi^S_{-8n+2}$ *for* $n\neq0$.

xvii) $\pi^S_{8n+7,-8n+1}\approx\boldsymbol{Z}/2\cdot\chi\nu\mathring{\nu}\mathring{\nu}_{8n+8}\oplus\boldsymbol{Z}/2\cdot\eta\sigma\delta_1\omega_1^{-8n-6}\oplus\boldsymbol{Z}/2\cdot\varepsilon\delta_1\omega_1^{-8n-6}\oplus\pi^S_{-8n+1}$ *for* $n\neq0$.

Proposition 15.27. *There holds the relation* $\rho\delta_9\omega_9^{-2n+1}=\delta_9\omega_9^{-2n+1}$ *for any integer* n.

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