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HOMOLOGOUS FIBRES AND TOTAL SPACES

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0. Introduction

In this note we consider fibrations of the form $F \to E \to B$ where all spaces involved have the homotopy type of pointed connected CW-complexes. Well-known work on the plus-construction (for algebraic K-theory, *et al*) reveals the following situation concerning when $E \to B$ induces an isomorphism of homology groups with trivial integer coefficients.

Theorem 0.1 [4]. *The following are equivalent.*
(i) *F is acyclic;*
(ii) *$H_*(E) \to H_*(B)$ is an isomorphism, and $\pi_1(B)$ acts trivially on $H_*(F)$.*

We focus here on a dual problem of when $H_*(F) \to H_*(E)$ can be an isomorphism. In general, mere acyclicity of B does not suffice, as evidenced by the following.

Example 0.2. Let $Re \rightarrowtail Fr \twoheadrightarrow G$ be a free presentation of a finitely generated acyclic group G, with Fr of finite rank. By passing to classifying spaces we obtain a fibration as in the first sentence above. If G is non-trivial, then it is well-known that the rank of Re and $H_1(Re)$ exceeds that of Fr and $H_1(Fr)$ [13 I §3].

Here is the counterpart to Theorem 0.1.

Theorem 0.3 [8]. *The following are equivalent.*
(i) *B is acyclic, and $\pi_1(B)$ acts trivially on $H_*(F)$;*
(ii) *$H_*(F) \to H_*(E)$ is an isomorphism.*

However we shall now show how it is possible to remove the hypothesis of trivial fundamental group action (orientability) in favourable circumstances. We thereby derive assumptions under which acyclicity of B implies that $H_*(F) \to H_*(E)$ is an isomorphism for **all** fibrations involving the given F, E and B. The price is a further condition, either on B or on F. For the former approach we use our previous work [9]. While the arguments there are adequate to apply to regular coverings, for more general fibrations we introduce the concept of a *fundamentally torsion-generated* (ftg) space B.

1. Conditions on the base: fundamentally torsion-generated spaces

We define a space B to be *fundamentally torsion-generated* if it admits a map

$$\bigvee_{\substack{\pi \\ \text{finite}}} K(\pi, 1) \to B$$

inducing on fundamental groups an epimorphism

$$*\pi \twoheadrightarrow \pi_1(B)$$

whose domain is a free product $*\pi$ of finite groups π. In particular, $\pi_1(B)$ is a torsion-generated group. Indeed, a group G is torsion-generated if and only if its classifying space $K(G, 1)$ is ftg. Evidently the class of ftg spaces also includes all simply-connected spaces. It is easily seen to be closed under finite unions whose intersections are connected (by the Seifert-van Kampen Theorem), direct limits and finite products.

In this work we are interested in acyclic ftg spaces. We now indicate some classes of examples of these.

Example 1.1. Let A be an abelian group which is an R-module for some torsion ring R. (For example, A could be any bounded abelian group [12 (120.8)].) Let Λ be a dense ordering with first and last elements. Then the group $M(\Lambda, (R, A))$, with centre A, constructed in [6] is both acyclic and torsion-generated, so that its classifying space is acyclic and ftg.

Example 1.2. Let G be a group which is the quotient of some acyclic torsion-generated group N. Then the map $K(N, 1) \to K(G, 1)$ lifts to Dror's acyclic space $\mathcal{A}K(G, 1)$ [4 ch. 7], which we claim is also ftg. This is because $U = \pi_1(\mathcal{A}K(G, 1))$ is the universal central extension [4 ch. 8] of G. So any epimorphism $N \to U \to G$ has $U = \operatorname{Im} N . \mathcal{Z}(U)$. Since $U = [U, U]$, this makes $N \to U$ surjective as required.

Note from Example 1.1 that any bounded abelian group is (naturally) the centre of the fundamental group of an acyclic ftg space. If we abandon functoriality then this result generalizes to any abelian group, as follows.

Proposition 1.3. *Let A be any abelian group. Then there exists an acyclic ftg space X such that $A \cong \mathcal{Z}(\pi_1(X))$.*

Proof. Using [3 Lemma 7], we start with a group G_1 whose abelianization is A and whose higher homology vanishes. As in [2 p. 17], G_1 may be embedded in an algebraically closed group L, say. Let G_2 be the free product of two copies of L with the subgroup G_1 amalgamated. Since by [2] L is acyclic, G_2 has a single nonzero homology group, namely A in dimension two. In particular, G_2 is perfect. Also, L is simple by [13 Theorem 8.2] (since for nontrivial groups the notions of algebraically closed and existentially closed coincide). From the algebraic closure property, L must contain elements of (any) finite order. So, from simplicity, L is torsion-generated. Hence G_2 is torsion-generated too.

Now apply Quillen's plus-construction to $K(G_2, 1)$ to form the simply-connected space $K(G_2, 1)^+$ [4 ch.5]. Again, it has no homology in dimensions higher than 2, making it homotopy equivalent to its 3-skeleton. Then for any finite cyclic group π, any homomorphism $\pi \to G_2$ induces

$$K(\pi, 1) \to K(G_2, 1) \to K(G_2, 1)^+$$

which must be nullhomotopic, by [15 Thm A]. So $K(\pi, 1) \to K(G_2, 1)$ lifts to the acyclic fibre $X = \mathcal{A}K(G_2, 1)$ of $K(G_2, 1) \to K(G_2, 1)^+$. Since G_2 is torsion-generated, we obtain $\vee K(\pi, 1) \to X$ which on fundamental groups induces a surjection $*\pi \to \pi_1(X) \to G_2$. The argument given in Example 1.2 shows that $*\pi \to \pi_1(X)$ is already surjective, as required. □

Next we show how to associate an acyclic ftg space to any space.

Proposition 1.4. *Given any space X there exists an (aspherical) acyclic ftg space Y and $f\colon X \to Y$ such that f induces an injection of fundamental groups. Moreover, there exists a universal such Y having this property with respect to all finite CW-complexes.*

Proof. Given an acyclic torsion-generated group L in which $\pi_1(X)$ embeds, one may simply take the obvious composition $f\colon X \to K(\pi_1(X), 1) \to K(L, 1) = Y$. In general, one can choose L to be an algebraically closed group, as above. However, when X is finite, $\pi_1 X$ is finitely presented, in which case L may be taken to be the universal finitely presented acyclic torsion generated group constructed in [10 Theorem 6]. □

Recall from [4 Chapter 7] that any space X has a canonical cover by a space X_2 whose fundamental group is isomorphic to the maximum perfect subgroup $\mathcal{P}\pi_1(X)$ of $\pi_1(X)$, and further, the fibre X_3 of the canonical map $X_2 \to K(H_2(X_2), 2)$ has as fundamental group the (superperfect) universal central extension of $\mathcal{P}\pi_1(X)$. Thus the next result associates a ftg space to any given space in an especially nice way.

Proposition 1.5. *Let X be a space with superperfect fundamental group.*

Then there exists a ftg space Z and $g: X \to Z$ such that g induces an injection of fundamental groups and an isomorphism of all homology groups.

Proof. Again consider Dror's acyclic space $\mathcal{A}X$ and the map $d\colon \mathcal{A}X \to X$ which induces an isomorphism on fundamental groups. Now apply (1.4) with respect to $\mathcal{A}X$ to obtain an acyclic ftg space Y and $f\colon \mathcal{A}X \to Y$ with $\pi_1(f)$ injective. Define Z and $g\colon X \to Z$ from the following pushout diagram.

$$\begin{tikzcd} \mathcal{A}X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\ulcorner\cdot"] & Y \arrow[d] \\ X \arrow[r, "g"] & Z \end{tikzcd}$$

Then the claimed properties of g follow. $\square$

2. Conditions on the base: the main result

To state our result we recall that a space F is of *finite type* if each of its homology groups is finitely generated. (If we use a universal coefficients argument in the proof below then we require only the weaker property whereby each $H_i(F)$ is of finite rank at all primes p–including 0–, or equally $H_i(F; k)$ is finitely generated for each i and each prime field k.)

Theorem 2.1. *Let B be a fundamentally torsion-generated space. The following are equivalent.*
(i) B is acyclic;
(ii) for any fibration $F \to E \to B$ with F of finite type, $H_*(F) \to H_*(E)$ is an isomorphism.

Proof. Since by Theorem 0.3 above (i) is a consequence of (ii), we are left with the derivation of (ii) from (i). It suffices, by Theorem 0.3 again, to establish orientability. By hypothesis, each of the automorphism groups $\mathrm{Aut}(H_i(F))$ is residually finite [13 IV (4.8)]. Therefore any nontrivial element of the image M of $\pi_1(B)$ in $\mathrm{Aut}(H_i(F))$ maps nontrivially to some finite quotient $\mathrm{Aut}(H_i(F))/N$ and so maps nontrivially to the finite quotient $Q = M/(M \cap N)$ ($\cong MN/N$) of $\pi_1(B)$. Hence it suffices to establish the triviality of any finite quotient Q of $\pi_1(B)$ (thus, with Q embeddable in some $GL_n(\boldsymbol{C})$). Using the canonical map $B \to K(\pi_1(B), 1)$ and the hypothesis on B, we are therefore led to consider

$$K(\pi, 1) \to B \to K(\pi_1(B), 1) \to K(Q, 1) \to BGL(\boldsymbol{C})$$

with π finite, Q linear and B acyclic, and $K(Q, 1) \to BGL(\boldsymbol{C})$ induced from the given embedding. Because $BGL(\boldsymbol{C}) \simeq BU$ is simply-connected, any map from B to $BGL(\boldsymbol{C})$ factors through the contractible plus-construction B^+ of B [4 ch. 5], making $\pi \to Q \to GL(\boldsymbol{C})$ nullhomotopic on classifying spaces. It follows from [1 (6.11), (7.2)] that this homomorphism is trivial when π has prime power order. By hypothesis $\pi_1(B)$ is generated by the image of such π, so we deduce that the image Q of $\pi_1(B)$ in $GL_n(\boldsymbol{C})$ is trivial as required. $\square$

A somewhat analogous argument occurs in [7] where a different class of acyclic groups (binate groups) is considered.

The discussion above suggests the following problem.

QUESTION. *Can one give a group-theoretic characterization of the fundamental groups of acyclic ftg spaces?*

The arguments given above reveal the following conditions on a group G to be necessary.

(i) G is torsion-generated.

(ii) G is superperfect (that is, $H_1(G)=H_2(G)=0$).

(iii) G admits no non-trivial representation $G \to GL(\boldsymbol{C})$.

Note that (i) is necessary and sufficient to make G the fundamental group of a ftg space (namely $K(G, 1)$), while (ii) is equivalent to G being the fundamental group of an acyclic space (namely $\mathcal{A}K(G, 1)$ [4 p. 65]). The binary icosahedral group of order 120 is the smallest example of a group G satisfying (i) and (ii) but not (iii).

3. Conditions on the fibre

It is also possible to dispense with orientability in Theorem 0.3 by placing stronger conditions on the fibre F instead of the base. Let $\mathcal{E}_0(F)$ (resp. $\mathcal{E}(F)$) denote the group of free (resp. based) homotopy classes of self-homotopy equivalences of F, and again let F^+ denote the plus-construction on F (so that $\pi_1(F^+)$ has no non-trivial perfect subgroups).

Theorem 3.1. *The following are equivalent.*

(i) *B is acyclic*;

(ii) *$H_*(F) \to H_*(E)$ is an isomorphism whenever $\mathcal{E}_0(F)$ or $\mathcal{E}_0(F^+)$ has no non-trivial perfect subgroup.*

Proof. In view of Theorem 0.3 we have to establish orientability when (i) holds and $\mathcal{E}_0(F)$ or $\mathcal{E}_0(F^+)$ is hypoabelian (that is, has no non-trivial perfect subgroup). First observe that by applying the fibre-wise plus-construction of [5] we may as well assume that $\mathcal{E}_0(F)$ is hypoabelian. Now the fibration is induced from the universal fibration $F \to B\,\mathrm{aut}^\circ F \to B\,\mathrm{aut}\,F$ where $\pi_1(B\,\mathrm{aut}\,F) \cong \mathcal{E}_0(F)$ [12]. So the classifying map $B \to B\,\mathrm{aut}\,F$ gives rise to a commuting square

$$\begin{array}{ccl} B & \to & B\,\mathrm{aut}\,F \\ \downarrow & & \downarrow \\ B^+ & \to & (B\,\mathrm{aut}\,F)^+ = B\,\mathrm{aut}\,F\,. \end{array}$$

Here $(B\,\mathrm{aut}\,F)^+ = B\,\mathrm{aut}\,F$ precisely because $\mathcal{E}_0(F)$ is hypoabelian. Because B^+ is contractible the fibration is trivial and the result follows. $\square$

EXAMPLE 3.2. For an interesting class of spaces X with $\mathcal{E}_0(X)$ admitting no non-trivial perfect subgroups (indeed, solvable), consider a finite group G acting freely on an odd-dimensional sphere with orbit space X. From [16] there is a commuting diagram of group extensions

$$\begin{tikzcd}
\mathrm{Inn}\,G \arrow[r, tail] \arrow[d, equal] & \mathcal{E}(X) \arrow[r, two heads] \arrow[d, tail] & \mathcal{E}_0(X) \arrow[d, tail] \\
\mathrm{Inn}\,G \arrow[r, tail] & \mathrm{Aut}\,G \arrow[r, two heads] \arrow[d, two heads] & \mathrm{Out}\,G \arrow[d, two heads] \\
& \mathrm{Aut}\,G/\mathcal{E}(X) \arrow[r, equal] & \mathrm{Aut}\,G/\mathcal{E}(X)
\end{tikzcd}$$

where the quotient group $\mathrm{Aut}\,G/\mathcal{E}(X)$ is abelian by [17] (1.7). Hence the finite group $\mathcal{E}_0(X)$ is solvable precisely when $\mathrm{Out}\,G$ is. Although I have not verified all cases, it is clear from the classification of periodic groups in, for example, [14], that for such G, $\mathrm{Out}\,G$ very often is solvable (and in fact I am not yet aware of any counterexamples).

Results of [12] and [18] provide classes of spaces F as in Theorem 3.1, as follows.

Corollary 3.3. *The following are equivalent.*
(i) *B is acyclic;*
(ii) *$H_*(F) \to H_*(E)$ is an isomorphism whenever F or F^+ is nilpotent and $H_j(F) \neq 0$ for only finitely many j;*
(iii) *$H_*(F) \to H_*(E)$ is an isomorphism whenever F or F^+ is homotopy equivalent to a connected CW-complex whose Postnikov system is finite and whose homotopy groups all have solvable automorphism groups.*

Proof. As previously, we may simplify notation by assuming that the conditions apply to F. For (ii), observe from [12] Theorem D that in this case $\mathcal{E}_0(F)$ is a nilpotent group, so that Theorem 3.1 applies. For (iii), note that $\mathcal{E}_0(F)$ is a homomorphic image of the corresponding group $\mathcal{E}(F)$ of based homotopy classes. Now $\mathcal{E}(F)$ maps into the solvable group $\prod \mathrm{Aut}\,\pi_i(F)$, with kernel denoted $G_\sharp(F)$ in [18]. If $\{F_n\}$ is a Postnikov system for F, then [18] shows that for each n the natural homomorphism $G_\sharp(F_n) \to G_\sharp(F_{n-1})$ has abelian kernel. So, by induction on n, $G_\sharp(F)$ is solvable. Hence in turn $\mathcal{E}(F)$ and $\mathcal{E}_0(F)$ are also solvable. Hence we are again in the situation of Theorem 3.1. $\square$

REMARK 3.4. Under the circumstances of (3.3) (ii) above, it follows from [5] that the fibration $F \to E \to B$ is plus-constructive; that is, induces a fibration

$F^+ \to E^+ \to B^+$. However the space B^+ is here contractible, making $F^+ \to E^+$ in fact a homotopy equivalence.

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