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A UNIQUE CONTINUATION THEOREM FOR AN ELLIPTIC OPERATOR OF TWO INDEPENDENT VARIABLES WITH NON-SMOOTH DOUBLE CHARACTERISTICS

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1. Introduction. Let $P = P(x; \partial/\partial x)$ be an elliptic homogeneous differential operator of order $m (\geq 2)$ with complex valued C^∞ coefficients defined near the origin in the 2-dimensional real Euclidean space R^2 . We say that P is A-elliptic at x_0 if x_0 has a neighbourhood U such that for any open and connected neighbourhood $V (\subset U)$ of x_0 , there is no non-trivial solution $u \in C^m(V)$ of the differential inequality in V

$$(1) \quad |P(x; \partial/\partial x)u| \leq C \sum_{|\alpha| < m} |(\partial/\partial x)^\alpha u|$$

such that $u=0$ in some open subset (of V) whose closure contains the point x_0 . It is well known that P is A-elliptic at each point where P has simple characteristics, or P has double characteristics and has Lipschitz continuous characteristic roots (see Hörmander [1], Pederson [2]).

In the present paper we shall give a sufficient condition for the operator P to be A-elliptic when P has double characteristics and its symbol $P(x; \xi)$ has a factorization of the form in a neighbourhood of the origin

$$(2) \quad p(x; \xi) = a(x) \prod_{j=1}^N (\xi_1^2 + 2a_j(x)\xi_1\xi_2 + b_j(x)\xi_2^2)$$

or

$$(3) \quad p(x; \xi) = a(x) \prod_{j=1}^N (\xi_1^2 + 2a_j(x)\xi_1\xi_2 + b_j(x)\xi_2^2) \prod_{j=N+1}^{N+s} (\xi_1 + a_j(x)\xi_2).$$

Here a , a_j and b_k are $C^\infty(\omega)$ functions such that $a(0) \neq 0$, $a_j(0)^2 = b_j(0)$ ($j=1, \dots, N$) and $a_j(0) \neq a_k(0)$ ($1 \leq j \neq k \leq N+s$).

Set $c_j(x) = b_j(x) - a_j(x)^2$ and let R_j be the set of points $y \in \omega$ which has a neighbourhood where $c_j(x) = k(x)^2$ for some $C^{1+1/2}$ function $k(x)$. Then we have our main result as follows.

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Theorem. *If there is an open neighbourhood $\omega_1 (\subset \omega)$ of the origin such that the following conditions (4) and (5) hold for each $j=1, \dots, N$, then $P(x; \partial/\partial x)$ is A-elliptic at the origin.*

- (4) $\text{grad}_x c_j(x)=0$ when $c_j(x) = 0, x \in \omega_1$.
- (5) *The order of zero for c_j is finite at each point of $\omega_1 \setminus R_j$.*

In the next section we shall give a proposition on differentiable square roots in order to prove Theorem.

2. Differentiable square roots and the proof of Theorem. Let $\omega_2 (\subset \omega_1)$ be an open and connected neighbourhood of the origin such that $\min \{ |a_j(x) - a_k(x)|; 1 \leq j \neq k \leq N + s \} > \max \{ 2|b_j(x) - a_j(x)^2|^{1/2}; 1 \leq j \leq N \}$. This means that P has at most double characteristics at each point of ω_1 . Then, applying the results by Hörmander [1] or Pederson [2], we have P is A-elliptic at each point of $\omega_2 \cap R$ where $R = \bigcap_{j=1}^N R_j$. So it is sufficient to prove that $\omega_2 \cap R$ is a dense subdomain of ω_2 . From now on, we shall use new notation. Let Ω be domain in $R^n (n \geq 2)$, and let f_j and $g_j (1 \leq j < \infty)$ real valued C^∞ functions defined in Ω . We also denote by R_j the set defined §1 for $f_j + \sqrt{-1}g_j$ and Ω instead of c_j and ω . Then we have

Proposition. *$R = \bigcap_{j=1}^N R_j$ is a dense subdomain of Ω if the following conditions (6) and (7) are satisfied for each $j=1, \dots, N$.*

- (6) $\text{grad}_x f_j(x) = \text{grad}_x g_j(x) = 0$ when $f_j(x) = g_j(x) = 0, x \in \Omega$.
- (7) *At least one of orders of zeros for f_j and g_j is finite at each point of $\Omega \setminus R_j$.*

REMARKS OF THEOREM. 1) Since $p(x; \xi)$ has factorization (2) or (3), we that the condition:

- (4)' $\text{grad}_x p(x; \xi) = 0$ when $p(x; \xi) = \text{grad}_\xi p(x, \xi) = 0, (x, \xi) \in \omega_1 \times C^2 \setminus 0$ is equivalent to the condition (4).

2) We denote by $D(x, \xi_2)$ the discriminant of polynomial $p(x; \xi)$ in ξ_1 . Since $p(x; \xi)$ is homogeneous in $\xi = (\xi_1, \xi_2)$ of order m , we have $D(x, \xi_2) = \delta(x) \xi_2^{m(m-1)}$ and $\delta(x) = \delta_0(x) \prod_{j=1}^N (b_j(x) - a_j(x)^2)$. Here $\delta_0 \in C^\infty$ and $\delta_0(0) \neq 0$. So that we have the condition:

- (5)' the order of zero for δ is finite at the origin implies the condition (5) if we take sufficiently small ω_1 .

REMARKS OF PROPOSITION. 1) In order that x_0 belongs to R_j , it is necessary that the following inequality holds in some neighbourhood of x_0 ,

$$(8) \quad |\text{grad}_x f_j(x)|^2 + |\text{grad}_x g_j(x)|^2 \leq C \{|f_j(x)| + |g_j(x)|\}$$

for some constant C . Moreover the condition:

$$(9) \quad \text{for each } j=1, \dots, N, \text{ setting } \text{Re } c_j=f_j \text{ and } \text{Im } c_j=g_j, \text{ the inequality (8) holds in a neighbourhood of the origin}$$

is sufficient in order that P is A -elliptic at the origin.

2) There is a pair f and g of C^∞ functions satisfying the condition (6), but not satisfying both (7) and (8) such that R is not connected. For example, near $t=0$, $f(t, x_2)=\exp(-1/t) \sin(1/t)$, $g(t, x_2)=(\log 1/t)^{-1/t}$ if $t < 0$, $f=g=0$ if $t > 0$.

Proof of Proposition. It is easy to see that R_j and R are open and dense in Ω . So that we have only to show that R_j and R are connected. Now we first prove the following two Lemmas under the conditions (6) and (7)

Lemma 1. For any $(n-1)$ -dimensional C^∞ manifold Γ in Ω , $\Gamma \cap R_j$ is dense in Γ .

Lemma 2. For each point $x \in S_j = \Omega \setminus R_j$, there is a fundamental system $\{U(x)\}$ of open neighbourhoods of x such that $U(x) \setminus S_j$ is connected.

Proof of Lemma 1. Without loss of generality, we may assume that Γ is defined by the equation $x_1 = \phi(x')$ near $x_0 \in S_j$ and that f_j and g_j vanish at any point x on Γ if x is near x_0 . Here $\phi \in C^\infty$ and we use notation $x = (x_1, x')$. In addition, we may assume that the order of zero for f_j is finite at x_0 by the assumption. So, near x_0 , f_j and g_j have a factorization of the form

$$f_j(x) = f'_j(x)(x_1 - \phi(x'))^\lambda.$$

If, for some positive integer k , $\partial^k g_j / \partial x_1^k$ does not vanish identically in any neighbourhood of x_0 in Γ , we have

$$g_j(x) = g'_j(x)(x_1 - \phi(x'))^\nu$$

and if other case occurs, for any positive integer i we have

$$g_j(x) = g_{j,i}(x)(x_1 - \phi(x'))^i.$$

Here λ and ν are positive integers which are independent on x and f'_j, g'_j and $g_{j,i}$ are C^∞ functions such that f'_j and g'_j do not vanish identically in any neighbourhood of x_0 in Γ . By the assumption (6), we have $\lambda, \nu \geq 2$. The above factorizations imply that when for g_j the first case occurs and $\lambda \leq \nu$, or the second case occurs, any point such that f'_j does not vanish is in R_j and when other case occurs, any point such that g'_j does not vanish is in R_j . This means that $\Gamma \cap R_j$ is dense in Γ .

Proof of Lemma 2. Take any point x_0 in S_j . Without loss of generality,

we may assume that $\alpha(x)$, the order of zero for f_j at x , is finite at x_0 . Since $\alpha(x)$ is upper semi-continuous, we can choose an open neighbourhood W of x_0 such that $W \cap S_j = \{x \in W \cap S_j; \alpha(x) \leq \alpha(x_0)\}$. Setting $T^{(k)} = \{x \in W \cap S_j; \alpha(x) = k\}$, we have $W \cap S_j = \bigcup_{k=2}^{\alpha(x_0)} T^{(k)}$, so that, using the induction on k such that $x \in T^{(k)}$, we prove this Lemma for any $x \in W \cap S_j$. When $x \in T^{(2)}$, $T^{(2)} = W \cap S_j$ and $T^{(2)}$ is contained in a $(n-1)$ -dimensional C^∞ manifold near x . Hence, using Lemma 1, the result is clear. When $x \in T^{(k+1)}$, by similar reason, we can take a fundamental system $\{U(x)\}$ of open neighbourhoods of x such that $U(x) \cap S_j = U(x) \cap (\bigcup_{i=2}^{k+1} T^{(i)})$ and that $U(x) \setminus T^{(k+1)}$ is connected. We show that this system $\{U(x)\}$ has that required property. If $U(x) \setminus S_j$ is not connected, we can take two disjoint components C_0 and C_1 and a continuous curve $x(t)$ ($0 \leq t \leq 1$) in $U(x) \setminus T^{(k+1)}$ such that $x(0) \in C_0$ and $x(1) \in C_1$. Take a small positive number ε such that $B(x(t), \varepsilon) \subset U(x) \setminus T^{(k+1)}$ for any t , $B(x(0), \varepsilon) \subset C_0$ and $B(x(1), \varepsilon) \subset C_1$ where $B(y, \varepsilon)$ is the closed ball with the center y and radius ε . Set $H = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n; x_1 = 0\}$, $H(t) = B(x(t), \varepsilon) \cap [x(t) + H]$ and $t_{\max} = \sup \{t \in [0, 1]; H(t) \cap C_0 \neq \phi\}$. Since the set $\{t; H(t) \cap C_0 \neq \phi\}$ is open in $[0, 1]$ we have $0 < t_{\max} < 1$ and $H(t_{\max}) \cap C_0 = \phi$. On the other hand, by the definition of t_{\max} , there are two convergent sequences $\{t_v\}$ and $\{x_v\}$ such that $x_v \in H(t_v) \cap C_0$ with the limit point t_{\max} and y , respectively. This limit point y is in $H(t_{\max}) \cap S_j$ since $H(t_{\max}) \cap C_0 = \phi$. So that $y \in \bigcup_{i=2}^k T^{(i)}$. By the induction hypothesis, there is a neighbourhood $U(y)$ of y such that $U(y) \subset U(x)$ and that $U(y) \setminus S_j$ is connected. Hence, using Lemma 1, we have $U(y) \setminus S_j \subset C_0$ and $H(t_{\max}) \cap C_0 \neq \phi$. This gives a contradiction and then completes the proof of Lemma 2.

Then we can prove easily that R_j and R are connected if we shall use the similar method by the reduction to absurdity as that in the proof of Lemma 2.

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