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Author(s)	Oja, Kirsti
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Osaka University

ON THE SET OF REGULAR BOUNDARY POINTS

KIRSTI OJA

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Introduction

Let X be a \mathcal{P} -harmonic space with a countable base in the sense of the axiomatics of Constantinescu and Cornea [3], U an open set of X and U_{reg} the set of regular boundary points of U . If X is a connected Brelot space, it is known that U_{reg} is dense on $\partial\bar{U}$ (see e.g. Hervé [4], Ikegami [6]). This is not valid for more general harmonic spaces. We prove two results related to this question. Assuming that the space has a base of regular sets, we obtain a necessary condition (by means of absorbent sets) for the case that U_{reg} is not dense on $\partial\bar{U}$.

1. Preliminaries

Let X be a \mathcal{P} -harmonic space with a countable base in the sense of Constantinescu and Cornea [3] and U an open set of X . We denote the set of regular (resp. irregular) points of ∂U by U_{reg} (resp. U_{ir}). If U is relatively compact and $M \subset \partial U$ with $\mu_x^U(M) = 0$ for all $x \in U$, M is called *negligible*. Since X has a countable base, if M is negligible, $\bar{H}_{x,M}^U(x) = \mu_x^U(M) = 0$ for all $x \in U$ (cf. [2, Satz 4.1.7]).

REMARK 1.1. Let $y \in \partial U$. A strictly positive hyperharmonic function u defined on the intersection of U and an open neighbourhood V of y is called a barrier at y if

$$\lim_{U \cap V \ni z \rightarrow y} u(z) = 0.$$

Then $y \in U_{reg}$ if and only if there exists a barrier at y . This follows from [3, Proposition 2.4.7], [3, Theorem 6.3.3] and [3, Proposition 7.2.2]. Thus $y \in U_{reg}$ implies that for every open subset U' of U with $y \in \partial U'$, we have $y \in U'_{reg}$.

A relatively compact open set U is called a *Keldyš set*, if U_{ir} is negligible [8, Proposition 2].

The following result was proved by Lukeš and Netuka [9, Theorem 3]: Let U be an open set of X . If K is an arbitrary compact set of U , there is a Keldyš set V with $K \subset V \subset \bar{V} \subset U$.

Lemma 1.2. *Let U be an open set of X and $M \subset \partial U$ with $\bar{H}_{x_M}^U = 0$. Let U' be an open subset of U . Then $\bar{H}_{x_M \cap \partial U'}^{U'} = 0$.*

Proof. Cf. [3, Proposition 2.4.4].

In the sequel we shall need the following two well-known minimum principles.

Theorem 1.3. *Let U be relatively compact. Let $M \subset \partial U$ be a negligible set. For every lower bounded hyperharmonic function u on U , if*

$$\liminf_{x \rightarrow z} u(x) \geq 0$$

for all $z \in \partial U \setminus M$, then $u \geq 0$.

Proof. This has been proved in [2, Satz 4.4.6]. The same proof carries over into the present situation.

Let U be relatively compact and \mathcal{F}_U the set of finite, continuous functions on \bar{U} whose restrictions to U are hyperharmonic. A point $x \in \bar{U}$ is called *extremal* if ε_x is the only measure μ on \bar{U} such that

$$\int u d\mu \leq u(x)$$

for all $u \in \mathcal{F}_U$. Then any extremal point is a regular point of ∂U (cf. [2, Satz 4.4.1], [3, Exercise 2.4.7]).

Theorem 1.4. *Let U be relatively compact. Any $u \in \mathcal{F}_U$ is positive if it is positive at any extremal point.*

Proof. The proof is a modification of [1, Satz 33]. We have to use [3, Lemma 2, p. 26].

In the following lemma we denote by $S(p)$ the smallest closed set outside which a potential p is harmonic. Let G be a relatively compact open set. The set of potentials p on X , for which $\emptyset \neq S(p) \subset \bar{G}$, is denoted by \mathcal{P}_G ; $\mathcal{P}_G \neq \emptyset$ by [3, Proposition 2.3.1].

Lemma 1.5. *Let W and G be open relatively compact sets of X with $G \subset \bar{G} \subset W$. For every potential $p \in \mathcal{P}_G$ we denote*

$$A_p = \{z \in W \mid \hat{R}_p^{X \setminus W}(z) = p(z)\}.$$

Then there exists a $p \in \mathcal{P}_G$ such that $G \subset W \setminus A_p$.

Proof. Let p_0 be a finite strict potential on X . Then $W \subset \{z \in X \mid \hat{R}_{p_0}^{X \setminus W}(z) < p_0(z)\}$ by [3, Proposition 7.2.2]. Let $p = \hat{R}_{p_0}^{\bar{G}}$; p is a potential and $p \in \mathcal{P}_G$. Since $\hat{R}_p^{X \setminus W} \leq \hat{R}_{p_0}^{X \setminus W}$, for every $x \in G$

$$\hat{R}_p^{X \setminus W}(x) \leq \hat{R}_{p_0}^{X \setminus W}(x) < p_0(x) = p(x),$$

and $x \in W \setminus A_p$.

2. On the set of regular points

Let U be an open set of X . We shall investigate the conditions under which the set $\partial \bar{U} \setminus \overline{U_{reg}}$ may be nonempty.

Theorem 2.1. *Let U be a Keldyš set. Every $x \in \partial \bar{U} \setminus \overline{U_{reg}}$ has an open neighbourhood V with $\partial U \cap \bar{V} \subset \partial U \setminus \overline{U_{reg}}$ such that $\bar{U} \cap V$ is a nontrivial absorbent set of V . Moreover, $\bar{U} \setminus \overline{U_{reg}}$ is an absorbent set of $X \setminus \overline{U_{reg}}$.*

Proof. Let V be a Keldyš set, $V \ni x$ such that $\partial U \cap \bar{V} \subset \partial U \setminus \overline{U_{reg}}$. Obviously we can assume that V is connected (Lemma 1.2).

We have $V \setminus \bar{U} \neq \emptyset$ by the assumption $x \in \partial \bar{U}$. Let G be an open set with $G \subset \bar{G} \subset V \setminus \bar{U}$. We consider the set of potentials \mathcal{P}_G (see p. 276).

First, let there exist a $G, \bar{G} \subset V \setminus \bar{U}$, and a $p \in \mathcal{P}_G$ with

$$(2.1) \quad (p - \hat{R}_p^{X \setminus V})|_{\bar{U} \cap V} \neq 0.$$

The function $u := p - \hat{R}_p^{X \setminus V}$ is positive and harmonic on $U \cap V$, continuous on $\partial U \cap V$ and bounded on $\overline{U \cap V}$. Also, u does not vanish identically on $U \cap V$ and has the limit zero at every regular boundary point of V . Further,

$$H_{x_{ir} \cap \partial(U \cap V)}^{u \cap V} = 0, \quad H_{x_{r'} \cap \partial(U \cap V)}^{u \cap V} = 0,$$

by Lemma 1.2. Thus the set $U_{ir} \cup V_{ir}$ is negligible on $\partial(U \cap V)$. Since $\partial U \cap \bar{V} \subset \partial U \setminus \overline{U_{reg}}$, everywhere else on $\partial(U \cap V)$, u has the limit zero. Then Theorem 1.3 gives $u=0$ on $U \cap V$, a contradiction.

Thus, for every G such that $\bar{G} \subset V \setminus \bar{U}$, and every $p \in \mathcal{P}_G$, the function $p - \hat{R}_p^{X \setminus V}$ equals zero on $\bar{U} \cap V$.

Let $y \in V \setminus \bar{U}$ be arbitrary and G an open set with $y \in G \subset \bar{G} \subset V \setminus \bar{U}$. Then by Lemma 1.5 there is a potential p_y , such that $G \subset V \setminus A_{p_y} = \{z \in V \mid \hat{R}_{p_y}^{X \setminus V} < p_y(z)\}$. Thus

$$\bigcap_{y \in V \setminus \bar{U}} A_{p_y} = \bar{U} \cap V$$

is an absorbent set of V .

Hence for every $x \in \partial \bar{U} \setminus \overline{U_{reg}}$ there is an open neighbourhood $V \subset X \setminus \overline{U_{reg}}$ such that $\bar{U} \cap V$ is an absorbent set of V . By the sheaf property of hyperharmonic functions, the function v which is 0 on $\bar{U} \setminus \overline{U_{reg}}$ and ∞ on $(X \setminus \overline{U_{reg}}) \setminus \bar{U}$ is hyperharmonic on $X \setminus \overline{U_{reg}}$. Thus $\bar{U} \setminus \overline{U_{reg}}$ is an absorbent set of $X \setminus \overline{U_{reg}}$. This still holds if $\partial \bar{U} \setminus \overline{U_{reg}} = \emptyset$.

REMARK 2.2. If $\partial\bar{U}\setminus\overline{U_{reg}}=\emptyset$, then $\bar{U}\setminus\overline{U_{reg}}$ is a union of some components of $X\setminus\overline{U_{reg}}$.

Theorem 2.3. *Let X have a base of regular sets and U an open set of X . Then all the assertions of Theorem 2.1 are valid.*

Proof. Let $x\in\partial\bar{U}\setminus\overline{U_{reg}}$ be arbitrary and the connected set V in the proof of Theorem 2.1 be regular [2, Satz 4.3.5].

We assume that there exist the set G and the potential p such that (2.1) holds. Then, the function u has the same properties as previously. Moreover, u is continuous on $\bar{U}\cap\bar{V}$ and equals 0 at every point of ∂V . Since $\partial U\cap V\subset U_{ir}$, by the barrier criterion also $\partial U\cap V\subset(U\cap V)_{ir}$. Thus the set of regular, and hence of extremal boundary points is contained in ∂V . From Theorem 1.4 we obtain $u=0$ on $U\cap V$, a contradiction.

Everything else needed for the conclusion may be proved exactly as for Theorem 2.1.

The following result was obtained for Brelot spaces (cf. [4, Théorème 8.2], [6, Theorem 7]).

Corollary 2.4. *Let X be elliptic and U an open set of X . Then $\partial\bar{U}\setminus\overline{U_{reg}}=\emptyset$.*

Proof. X has a base of regular sets.

EXAMPLE 2.5. It is known that for the heat equation $\partial\bar{U}\setminus\overline{U_{reg}}$ may be nonempty. Let $X=\mathbf{R}^2$ and

$$U = (0, 1) \times (0, 1).$$

Then $U_{reg} = ([0, 1] \times \{0\}) \cup (\{0\} \times [0, 1]) \cup (\{1\} \times [0, 1])$, and $\bar{U}\setminus\overline{U_{reg}}$ is absorbent on $X\setminus\overline{U_{reg}}$, which may be seen directly. The same observation follows immediately by Theorem 2.3, and since U is a Keldyš set [7, p. 1501], also by Theorem 2.1.

EXAMPLE 2.6. Let X be the space of [3, Example 3.2.13] and

$$U = \{(x, y, 0) \in X \mid 0 < x^2 + y^2 < 1\}.$$

Then $X\setminus U$ is thin at $(0, 0, 0)$, and $\{(0, 0, 0)\} = \partial\bar{U} = U_{ir}$. Now $\bar{U} = \bar{U}\setminus\overline{U_{reg}}$ is an absorbent set of $X\setminus\overline{U_{reg}}$, which can be seen directly and by Theorem 2.3.

REMARK 2.7. If U is a Keldyš set, then for every $x\in U$, $\text{supp}(\mu_x^U) \subset \overline{U_{reg}}$. Denoting

$$T := \bigcup_{x\in U} \overline{\text{supp}(\mu_x^U)},$$

$T \subset \overline{U_{reg}}$. As $\overline{U_{reg}} \subset T$ always, $T = \overline{U_{reg}}$. It was proved in [5, Lemma 1.4] that $\overline{U} \setminus T$ is an absorbent set of $X \setminus T$. Writing $T = \overline{U_{reg}}$, this gives the assertion of Theorem 2.1. However, Theorem 2.3 cannot be obtained in this way, since $T = \overline{U_{reg}}$ does not always hold.

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Helsinki University of Technology
 Institute of Mathematics
 SF-02150 Espoo 15
 Finland

