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ON THE SET OF REGULAR BOUNDARY POINTS

KIRSTI OJA

(Received February 3, 1983)

Introduction

Let \( X \) be a \( \mathcal{P} \)-harmonic space with a countable base in the sense of the axiomatics of Constantinescu and Cornea [3], \( U \) an open set of \( X \) and \( U_{\text{reg}} \) the set of regular boundary points of \( U \). If \( X \) is a connected Brelot space, it is known that \( U_{\text{reg}} \) is dense on \( \partial U \) (see e.g. Hervé [4], Ikegami [6]). This is not valid for more general harmonic spaces. We prove two results related to this question. Assuming that the space has a base of regular sets, we obtain a necessary condition (by means of absorbent sets) for the case that \( U_{\text{reg}} \) is not dense on \( \partial U \).

1. Preliminaries

Let \( X \) be a \( \mathcal{P} \)-harmonic space with a countable base in the sense of Constantinescu and Cornea [3] and \( U \) an open set of \( X \). We denote the set of regular (resp. irregular) points of \( \partial U \) by \( U_{\text{reg}} \) (resp. \( U_{\text{ir}} \)). If \( U \) is relatively compact and \( M \subset \partial U \) with \( \mu^U(M)=0 \) for all \( x \in U, M \) is called negligible. Since \( X \) has a countable base, if \( M \) is negligible, \( H^U_{\infty}(x)=\mu^U_x(M)=0 \) for all \( x \in U \) (cf. [2, Satz 4.1.7]).

Remark 1.1. Let \( y \in \partial U \). A strictly positive hyperharmonic function \( u \) defined on the intersection of \( U \) and an open neighbourhood \( V \) of \( y \) is called a barrier at \( y \) if

\[
\lim_{V \cap V \ni z \to y} u(z) = 0.
\]

Then \( y \in U_{\text{reg}} \) if and only if there exists a barrier at \( y \). This follows from [3, Proposition 2.4.7], [3, Theorem 6.3.3] and [3, Proposition 7.2.2]. Thus \( y \in U_{\text{reg}} \) implies that for every open subset \( U' \) of \( U \) with \( y \in \partial U' \), we have \( y \in U'_{\text{reg}} \).

A relatively compact open set \( U \) is called a Keldyš set, if \( U_{\text{ir}} \) is negligible [8, Proposition 2].

The following result was proved by Lukeš and Netuka [9, Theorem 3]: Let \( U \) be an open set of \( X \). If \( K \) is an arbitrary compact set of \( U \), there is a Keldyš set \( V \) with \( K \subset V \subset V' \subset U \).
Lemma 1.2. Let $U$ be an open set of $X$ and $M \subset \partial U$ with $\mathcal{H}^{U}_{\partial U} = 0$. Let $U'$ be an open subset of $U$. Then $\mathcal{H}^{U'}_{\partial U' \cap \partial U} = 0$.

Proof. Cf. [3, Proposition 2.4.4].

In the sequel we shall need the following two well-known minimum principles.

Theorem 1.3. Let $U$ be relatively compact. Let $M \subset \partial U$ be a negligible set. For every lower bounded hyperharmonic function $u$ on $U$, if

$$\liminf_{z \to x} u(z) \geq 0$$

for all $z \in \partial U \setminus M$, then $u \geq 0$.

Proof. This has been proved in [2, Satz 4.4.6]. The same proof carries over into the present situation.

Let $U$ be relatively compact and $\mathcal{H}_U$ the set of finite, continuous functions on $\bar{U}$ whose restrictions to $U$ are hyperharmonic. A point $x \in \bar{U}$ is called extremal if $\mathcal{E}_x$ is the only measure $\mu$ on $\bar{U}$ such that

$$\int u d\mu \leq u(x)$$

for all $u \in \mathcal{H}_U$. Then any extremal point is a regular point of $\partial U$ (cf. [2, Satz 4.4.1], [3, Exercise 2.4.7]).

Theorem 1.4. Let $U$ be relatively compact. Any $u \in \mathcal{H}_U$ is positive if it is positive at any extremal point.

Proof. The proof is a modification of [1, Satz 33]. We have to use [3, Lemma 2, p. 26].

In the following lemma we denote by $S(p)$ the smallest closed set outside which a potential $p$ is harmonic. Let $G$ be a relatively compact open set. The set of potentials $p$ on $X$, for which $\emptyset \neq S(p) \subset G$, is denoted by $\mathcal{P}_G$; $\mathcal{P}_G \neq \emptyset$ by [3, Proposition 2.3.1].

Lemma 1.5. Let $W$ and $G$ be open relatively compact sets of $X$ with $G \subset \bar{G} \subset W$. For every potential $p \in \mathcal{P}_G$ we denote

$$A_p = \{ z \in W \mid \hat{R}^{X\setminus W}_{\partial p}(z) = p(z) \} .$$

Then there exists a $p \in \mathcal{P}_G$ such that $G \subset W \setminus A_p$.

Proof. Let $p_0$ be a finite strict potential on $X$. Then $W \subset \{ z \in X \mid \hat{R}^{X\setminus W}_{\partial p_0}(z) < p_0(z) \}$ by [3, Proposition 7.2.2]. Let $p = \hat{R}^G_{p_0}$; $p$ is a potential and $p \in \mathcal{P}_G$. Since $\hat{R}^{X\setminus W}_p \leq \hat{R}^{X\setminus W}_{p_0}$, for every $x \in G$
\[ \hat{R}_p^X(x) \leq \hat{R}_p^\overline{U}(x) < p_0(x) = p(x), \]
and \( x \in W \setminus A_p. \)

2. **On the set of regular points**

Let \( U \) be an open set of \( X \). We shall investigate the conditions under which the set \( \partial \overline{U} \setminus \overline{U}_{\text{reg}} \) may be nonempty.

**Theorem 2.1.** Let \( U \) be a Keldyš set. Every \( x \in \partial \overline{U} \setminus \overline{U}_{\text{reg}} \) has an open neighbourhood \( V \) with \( \partial U \cap V \subset \partial \overline{U} \setminus \overline{U}_{\text{reg}} \) such that \( \overline{U} \cap V \) is a nontrivial absorbent set of \( V \). Moreover, \( \overline{U} \setminus \overline{U}_{\text{reg}} \) is an absorbent set of \( X \setminus \overline{U}_{\text{reg}} \).

**Proof.** Let \( V \) be a Keldyš set, \( V \subseteq x \) such that \( \partial U \cap V \subset \partial \overline{U} \setminus \overline{U}_{\text{reg}} \). Obviously we can assume that \( V \) is connected (Lemma 1.2).

We have \( V \setminus \overline{U} \neq \emptyset \) by the assumption \( x \in \partial \overline{U} \). Let \( G \) be an open set with \( G \subset G \subset V \setminus \overline{U} \). We consider the set of potentials \( \mathcal{P}_G \) (see p. 276).

First, let there exist a \( G, G \subset V \setminus \overline{U} \), and \( p \in \mathcal{P}_G \) with

\[ (2.1) \quad (p - \hat{R}_p^\overline{V})|_{U \cap V} \neq 0. \]

The function \( u := p - \hat{R}_p^\overline{V} \) is positive and harmonic on \( U \cap V \), continuous on \( \partial U \cap V \) and bounded on \( U \setminus V \). Also, \( u \) does not vanish identically on \( U \cap V \) and has the limit zero at every regular boundary point of \( V \). Further,

\[ \overline{R}_u^\overline{V} \in \bar{U} \cap V = 0, \quad \overline{R}_u^\overline{V} \in \bar{U} \cap V = 0, \]
by Lemma 1.2. Thus the set \( U_{ir} \cup V_{ir} \) is negligible on \( \partial(U \cap V) \). Since \( \partial U \cap V \subset \partial \overline{U} \setminus \overline{U}_{\text{reg}} \), everywhere else on \( \partial(U \cap V) \), \( u \) has the limit zero. Then Theorem 1.3 gives \( u = 0 \) on \( U \cap V \), a contradiction.

Thus, for every \( G \) such that \( G \subset V \setminus \overline{U} \), and every \( p \in \mathcal{P}_G \), the function \( p - \hat{R}_p^\overline{V} \) equals zero on \( \overline{U} \cap V \).

Let \( y \in V \setminus \overline{U} \) be arbitrary and \( G \) an open set with \( y \in G \subset G \subset V \setminus \overline{U} \). Then by Lemma 1.5 there is a potential \( p_y \) such that \( G \subset V \setminus A_{p_y} = \{ z \in V \mid \hat{R}_p^\overline{V} < p_y(z) \} \). Thus

\[ \bigcap_{y \in V \setminus \overline{U}} A_{p_y} = \overline{U} \cap V \]
is an absorbent set of \( V \).

Hence for every \( x \in \partial \overline{U} \setminus \overline{U}_{\text{reg}} \) there is an open neighbourhood \( V \subset X \setminus \overline{U}_{\text{reg}} \) such that \( \overline{U} \cap V \) is an absorbent set of \( V \). By the sheaf property of hyperharmonic functions, the function \( v \) which is 0 on \( \overline{U} \setminus \overline{U}_{\text{reg}} \) and \( \infty \) on \( (X \setminus \overline{U}_{\text{reg}}) \setminus \overline{U} \) is hyperharmonic on \( X \setminus \overline{U}_{\text{reg}} \). Thus \( \overline{U} \setminus \overline{U}_{\text{reg}} \) is an absorbent set of \( X \setminus \overline{U}_{\text{reg}} \). This still holds if \( \partial \overline{U} \setminus \overline{U}_{\text{reg}} = \emptyset \).
REMARK 2.2. If $\partial \overline{U \setminus U_{\text{reg}}} = \emptyset$, then $\overline{U \setminus U_{\text{reg}}}$ is a union of some components of $X \setminus U_{\text{reg}}$.

Theorem 2.3. Let $X$ have a base of regular sets and $U$ an open set of $X$. Then all the assertions of Theorem 2.1 are valid.

Proof. Let $x \in \partial \overline{U \setminus U_{\text{reg}}}$ be arbitrary and the connected set $V$ in the proof of Theorem 2.1 be regular [2, Satz 4.3.5].

We assume that there exist the set $G$ and the potential $p$ such that (2.1) holds. Then, the function $u$ has the same properties as previously. Moreover, $u$ is continuous on $\overline{U \cap V}$ and equals 0 at every point of $\partial V$. Since $\partial U \cap V \subset U_{ir}$, by the barrier criterion also $\partial U \cap V \subset (U \cap V)_{ir}$. Thus the set of regular, and hence of extremal boundary points is contained in $\partial V$. From Theorem 1.4 we obtain $u = 0$ on $U \cap V$, a contradiction.

Everything else needed for the conclusion may be proved exactly as for Theorem 2.1.

The following result was obtained for Brelot spaces (cf. [4, Théorème 8.2], [6, Theorem 7]).

Corollary 2.4. Let $X$ be elliptic and $U$ an open set of $X$. Then $\partial \overline{U \setminus U_{\text{reg}}} = \emptyset$.

Proof. $X$ has a base of regular sets.

Example 2.5. It is known that for the heat equation $\partial \overline{U \setminus U_{\text{reg}}}$ may be nonempty. Let $X = \mathbb{R}^2$ and

$$U = (0, 1) \times (0, 1).$$

Then $U_{\text{reg}} = ([0, 1] \times \{0\}) \cup (\{0\} \times [0, 1]) \cup (\{1\} \times [0, 1])$, and $\overline{U \setminus U_{\text{reg}}}$ is absorbent on $X \setminus U_{\text{reg}}$, which may be seen directly. The same observation follows immediately by Theorem 2.3, and since $U$ is a Keldyš set [7, p. 1501], also by Theorem 2.1.

Example 2.6. Let $X$ be the space of [3, Example 3.2.13] and

$$U = \{(x, y, 0) \in X \mid 0 < x^2 + y^2 < 1\}.$$

Then $X \setminus U$ is thin at $(0, 0, 0)$, and $\{(0, 0, 0)\} = \partial \overline{U} = U_{ir}$. Now $\overline{U \setminus U_{\text{reg}}}$ is an absorbent set of $X = X \setminus U_{\text{reg}}$, which can be seen directly and by Theorem 2.3.

Remark 2.7. If $U$ is a Keldyš set, then for every $x \in U$, $\text{supp}(\mu_x^U) \subset U_{\text{reg}}$. Denoting

$$T := \bigcup_{x \in U} \text{supp}(\mu_x^U),$$
T ⊆ \overline{U_{\text{reg}}}. As \overline{U_{\text{reg}}} \subseteq T always, T = \overline{U_{\text{reg}}}. It was proved in [5, Lemma 1.4] that \overline{U \setminus T} is an absorbent set of \overline{X \setminus T}. Writing T = \overline{U_{\text{reg}}}, this gives the assertion of Theorem 2.1. However, Theorem 2.3 cannot be obtained in this way, since T = \overline{U_{\text{reg}}} does not always hold.

References


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