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RANDERS METRICS WITH SPECIAL CURVATURE PROPERTIES

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1. Introduction

A Finsler metric on a manifold is a family of Minkowski norms on tangent spaces. There are several notions of curvatures in Finsler geometry. The flag curvature **K** is an analogue of the sectional curvature in Riemannian geometry. The distortion τ is a basic invariant which characterizes Riemannian metrics among Finsler metrics, namely, $\tau = 0$ if and only if the Finsler metric is Riemannian. The vertical derivative of τ on tangent spaces gives rise to the mean Cartan torsion **I**. The horizontal derivative of τ along geodesics is the so-called S-curvature **S**. The vertical Hessian of $(1/2)\mathbf{S}$ on tangent spaces is called the E-curvature. Thus if the S-curvature is isotropic, so is the E-curvature. The horizontal derivative of **I** along geodesics is called the mean Landsberg curvature **J**. Thus $\mathbf{J}/\mathbf{I}$ is regarded as the relative growth rate of the mean Cartan torsion along geodesics. We see how these quantities are generated from the distortion. Except for the flag curvature **K**, the above quantities are all non-Riemannian, namely, they vanish when F is Riemannian. See Section 2 for a brief discussion and [8] for a detailed discussion. In this paper, we will study a special class of Finsler metrics — Randers metrics with special curvature properties.

Randers metrics are among the simplest Finsler metrics, which arise from many areas in mathematics, physics and biology [1]. They are expressed in the form $F = \alpha+\beta$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form with $\|\beta\|_p := \sup_{\mathbf{y}\in T_pM} \beta(\mathbf{y})/\alpha(\mathbf{y}) < 1$ for any point p. Randers metrics were first studied by physicist, G. Randers, in 1941 [7] from the standard point of general relativity [1]. Since then, many Finslerian geometers have made efforts in investigation on the geometric properties of Randers metrics.

The shortest time problem on a Riemannian manifold also gives rise to a Randers metric. Given an object which can freely move over a Riemannian manifold (M, α), the object is pushed by a constant internal force $\mathbf{u}$ with $\alpha(\mathbf{u}) = 1$. We may assume that the object moves at a constant speed, due to friction. In this case, any path of shortest time is a shortest path of α. If there is an external force field $\mathbf{x}$ acting on the object with $\alpha(\mathbf{x}) < 1$, then any path of shortest time is a shortest path of the following

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Randers metric

$$F := \frac{\sqrt{\langle \mathbf{x}, \mathbf{y}\rangle_\alpha^2 + \alpha(\mathbf{y})^2(1-\alpha(\mathbf{x})^2)}}{1-\alpha(\mathbf{x})^2} - \frac{\langle \mathbf{x}, \mathbf{y}\rangle_\alpha}{1-\alpha(\mathbf{x})^2}, \qquad \mathbf{y} \in T_pM. \tag{1}$$

where $\langle\ ,\ \rangle_\alpha$ denotes the inner product on T_pM associated with α such that $\alpha(\mathbf{y}) = \sqrt{\langle \mathbf{y}, \mathbf{y}\rangle_\alpha}$. An interesting fact is that the Busemann-Hausdorff volume form of F is still equal to that of α. By choosing appropriate α and $\mathbf{x}$, one obtains a Randers metric defined in (1) with many special curvature properties. See [10] [11] [3] for more details.

On the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$, taking $\mathbf{x}_p = \mp(x^i)$ at $p = (x^i) \in \mathbb{B}^n$ in (1), we obtain the well-known Funk metrics on $\mathbb{B}^n$,

$$F = \frac{\sqrt{|\mathbf{y}|^2 - (|\mathbf{x}|^2|\mathbf{y}|^2 - \langle \mathbf{x}, \mathbf{y}\rangle^2)}}{1-|\mathbf{x}|^2} \pm \frac{\langle \mathbf{x}, \mathbf{y}\rangle}{1-|\mathbf{x}|^2}, \qquad \mathbf{y} \in T_{\mathbf{x}}\mathrm{R}^n, \tag{2}$$

The Funk metrics have many special curvature properties: (i) $\mathbf{S} = \pm(1/2)(n+1)F$, (ii) $\mathbf{E} = \pm(1/4)(n+1)F^{-1}h$ and (iii) $\mathbf{J} \pm (1/2)F\ \mathbf{I} = 0$ and (iv) $\mathbf{K} = -1/4$. Note that (ii) follows from (i). The geometric meaning of $\mathbf{S} = \pm(1/2)(n+1)F$ is that the rate of change of the distortion τ along geodesics is a constant and the geometric meaning of $\mathbf{J} \pm (1/2)F\ \mathbf{I} = 0$ is that the relative rate of change of the mean Cartan torsion $\mathbf{I}$ along geodesics is a constant.

Motivated by the properties of Funk metrics, we first study Randers metrics satisfying (i), (ii) or (iii). We prove the following

Theorem 1.1. *Let $F = \alpha + \beta$ be a Randers metric on an n-dimensional manifold M.*

(i) *For a scalar function $c = c(x)$ on M, the following are equivalent,*
  (ia) $\mathbf{S} = (n+1)cF$;
  (ib) $\mathbf{E} = (1/2)(n+1)cF^{-1}h$;

(ii) *For a scalar function $c = c(x)$ on M, the following are equivalent,*
  (iia) $\mathbf{J} + cF\ \mathbf{I} = 0$;
  (iib) $\mathbf{S} = (n+1)cF$ *and β is closed.*

Assume that a Randers metric $F = \alpha + \beta$ satisfies that $\mathbf{J} = 0$. By Theorem 1.1 (ii), $\mathbf{S} = 0$ and β is closed. In this case, it is easy to verify that β is a Killing form. Then we conclude that β is parallel with respect to α. Recently, D. Bao has shown to the author that for a Randers metric, $\mathbf{J} = 0$ if and only if it is of Landsberg type, since Randers metrics are C-reducible in the sense of Matsumoto [4]. A long time ago, M. Matsumoto [5] proved that if a Randers metric $F = \alpha + \beta$ is of Landsberg type, then β is parallel with respect to α. Thus Theorem 1.1(ii) is a generalization of Matsumoto's result.

There are lots of Randers metrics with constant flag curvature satisfying that $\mathbf{S} =$

0, but β is not closed [10] [11]. There are also lots of Randers metrics with $\mathbf{J}+cF\,\mathbf{I}=0$ for a function $c=c(x)\neq constant$. See Example 4.2 below.

Recently, the second author has classified all locally projectively flat Randers metrics with constant flag curvature [9]. He proves that a locally projectively flat Randers metric with constant flag curvature $\mathbf{K}=\lambda$ is either locally Minkowskian or after a scaling, isometric to the a Finsler metric on the unit ball $\mathbb{B}^n$ in the following form

$$(3)\qquad F_{\mathbf{a}}=\frac{\sqrt{|\mathbf{y}|^2-(|\mathbf{x}|^2|\mathbf{y}|^2-\langle\mathbf{x},\mathbf{y}\rangle^2)}}{1-|\mathbf{x}|^2}\pm\frac{\langle\mathbf{x},\mathbf{y}\rangle}{1-|\mathbf{x}|^2}\pm\frac{\langle\mathbf{a},\mathbf{y}\rangle}{1+\langle\mathbf{a},\mathbf{x}\rangle},\qquad \mathbf{y}\in T_{\mathbf{x}}\mathrm{R}^n,$$

where $\mathbf{a}\in\mathrm{R}^n$ is a constant vector with $|\mathbf{a}|<1$. One can directly verify that $F_{\mathbf{a}}$, $\mathbf{a}\neq 0$, are locally projectively flat Finsler metrics with negative constant flag curvature. Moreover, they have the above mentioned properties of the Funk metrics.

Based on Theorem 1.1 and the main result in [9] we classify Randers metrics with constant flag curvature $\mathbf{K}=\lambda$ and $\mathbf{J}/\mathbf{I}=-c(x)F$.

Theorem 1.2. *Let $F=\alpha+\beta$ be a Randers metric on an n-dimensional manifold M satisfying $\mathbf{J}+cF\,\mathbf{I}=0$ for some scalar function $c=c(x)$ on M. Suppose that F has constant flag curvature $\mathbf{K}=\lambda$. Then $\lambda=-c^2\leq 0$. F is either locally Minkowskian ($\lambda=-c^2=0$) or in the form* (3) ($\lambda=-c^2=-1/4$) *after a scaling.*

In fact, we prove that for a Randers metric of constant flag curvature, the mean Landsberg curvature is proportional to the mean Cartan torsion if and only if it is locally projectively flat. Then Theorem 1.2 follows from the main result in [9].

Several people have made effects to find simple equivalent and sufficient conditions for a Randers metric to be of constant flag curvature [3] [6] [13] [14], etc. The classification of Randers metrics with constant flag curvature has just been completed recently by D. Bao, C. Robles and Z. Shen.

2. Preliminaries

Let F be a Finsler metric on a manifold M. In a standard local coordinate system (x^i,y^i) in TM, $F=F(x,y)$ is a function of (x^i,y^i). Let

$$g_{ij}(x,y):=\frac{1}{2}[F^2]_{y^iy^j}(x,y)$$

and $(g^{ij}):=(g_{ij})^{-1}$. For a non-zero vector $\mathbf{y}=y^i(\partial/\partial x^i)|_p\in T_pM$, F induces an inner product on T_pM,

$$g_{\mathbf{y}}(\mathbf{u},\mathbf{v})=g_{ij}(x,y)u^iv^j,$$

where $\mathbf{u} = u^i(\partial/\partial x^i)|_p, \mathbf{v} = v^i(\partial/\partial x^i)|_p \in T_pM$. $g = \{g_{\mathbf{y}}\}$ is called the fundamental metric of F. For a non-zero vector $\mathbf{y} \in T_pM$, define

$$h_{\mathbf{y}}(\mathbf{u},\mathbf{v}) := g_{\mathbf{y}}(\mathbf{u},\mathbf{v}) - F^{-2}(\mathbf{y})g_{\mathbf{y}}(\mathbf{y},\mathbf{u})g_{\mathbf{y}}(\mathbf{y},\mathbf{v}), \qquad \mathbf{u},\ \mathbf{v} \in T_pM.$$

$h = \{h_{\mathbf{y}}\}$ is called the angular metric of F.

The geodesics of F are characterized locally by

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0,$$

where

$$G^i = \frac{1}{4}g^{ik}\left\{2\frac{\partial g_{pk}}{\partial x^q} - \frac{\partial g_{pq}}{\partial x^k}\right\}y^py^q.$$

The Riemann curvature is a family of endomorphisms $\mathbf{R}_{\mathbf{y}} = R^i{}_k\, dx^k \otimes (\partial/\partial x^i)$: $T_pM \to T_pM$, defined by

$$R^i{}_k := 2\frac{\partial G^i}{\partial x^k} - y^j\frac{\partial^2 G^i}{\partial x^j\partial y^k} + 2G^j\frac{\partial G^i}{\partial y^j\partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{4}$$

F is said to be of constant flag curvature $\mathbf{K} = \lambda$, if

$$g_{\mathbf{y}}\Big(\mathbf{R}_{\mathbf{y}}(\mathbf{u}), \mathbf{v}\Big) = \lambda F(\mathbf{y})^2 h_{\mathbf{y}}(\mathbf{u},\mathbf{v}),$$

or equivalently,

$$R^i{}_k = \lambda\Big\{F^2\delta^i_k - FF_{y^k}y^i\Big\}.$$

There are many interesting non-Riemannian quantities in Finsler geometry [8]. Let $dV_F = \sigma(x)\,dx^1\cdots dx^n$ denote the volume form of F, where

$$\sigma(x) := \frac{\mathrm{Vol}(\mathbb{B}^n(1))}{\mathrm{Vol}\{(y^i) \in \mathrm{R}^n \mid F(y^i(\partial/\partial x^i)|_x) < 1\}}.$$

For a non-zero vector $\mathbf{y} \in T_pM$, the distortion $\tau(\mathbf{y})$ is defined by

$$\tau(\mathbf{y}) := \ln\left[\frac{\sqrt{\det(g_{ij}(x,y))}}{\sigma(x)}\right].$$

The distortion characterizes Riemannian metrics among Finsler metrics, namely, F is Riemannian if and only if $\tau = 0$.

The mean Cartan torsion $\mathbf{I}_{\mathbf{y}} = I_i(x,y)\,dx^i : T_pM \to \mathrm{R}$ is defined as the vertical derivative of τ on T_pM,

$$I_i := \frac{\partial\tau}{\partial y^i} = \frac{1}{4}g^{jk}[F^2]_{y^iy^jy^k}. \tag{5}$$

The S-curvature $\mathbf{S}$ is defined as the horizontal derivative of τ along geodesics,

$$\mathbf{S}(\mathbf{y}) := \frac{d}{dt}\Big[\tau\Big(\dot{c}(t)\Big)\Big]\Big|_{t=0},$$

where $c(t)$ is the geodesic with $c(0) = p$ and $\dot{c}(0) = \mathbf{y} \in T_pM$. A direction computation yields

(6) $$\mathbf{S}(\mathbf{y}) = \frac{\partial G^i}{\partial y^i}(x, y) - \frac{y^i}{\sigma(x)}\frac{\partial \sigma}{\partial x^i}(x).$$

For a non-zero vector $\mathbf{y} \in T_pM$, the E-curvature $\mathbf{E_y} = E_{ij}(x, y)\, dx^i \otimes dx^j : T_pM \times T_pM \to \mathrm{R}$ is defined as the vertical Hessian of $(1/2)\mathbf{S}$ on T_pM,

(7) $$E_{ij} := \frac{1}{2}\mathbf{S}_{y^i y^j} = \frac{1}{2}\frac{\partial^3 G^m}{\partial y^m \partial y^i \partial y^j}(x, y).$$

F is said to be *weakly Berwaldian* if $\mathbf{E} = 0$.

For a non-zero vector $\mathbf{y} \in T_pM$, the mean Landsberg curvature $\mathbf{J_y} = J_i(x, y)\, dx^i : T_pM \to \mathrm{R}$ is defined as the horizontal derivative of $\mathbf{I}$ along geodesics,

(8) $$J_i := y^j \frac{\partial I_i}{\partial x^j} - I_j \frac{\partial G^j}{\partial y^i} - 2G^j \frac{\partial I_i}{\partial y^j} = -\frac{1}{2} F F_{y^l} g^{jk} \frac{\partial^3 G^l}{\partial y^i \partial y^j \partial y^k}.$$

F is said to be *weakly Landsbergian* if $\mathbf{J} = 0$.

The above mentioned geometric quantities are computable for Randers metrics. Let $F = \alpha + \beta$ be a Randers metric on a manifold M, where

$$\alpha(y) = \sqrt{a_{ij}(x) y^i y^j}, \qquad \beta(y) = b_i(x) y^i$$

with $\|\beta\|_x := \sup_{y \in T_xM} \beta(y)/\alpha(y) < 1$. An easy computation yields

(9) $$g_{ij} = \frac{F}{\alpha}\Big(a_{ij} - \frac{y_i}{\alpha}\frac{y_j}{\alpha}\Big) + \Big(\frac{y_i}{\alpha} + b_i\Big)\Big(\frac{y_j}{\alpha} + b_j\Big),$$

where $y_i := a_{ij} y^j$. By an elementary argument in linear algebra, we obtain

(10) $$\det(g_{jk}) = \Big(\frac{F}{\alpha}\Big)^{n+1} \det(a_{ij}).$$

Define $b_{i|j}$ by

$$b_{i|j}\theta^j := db_i - b_j \theta_i^{\ j},$$

where $\theta^i := dx^i$ and $\theta_i^{\ j} := \tilde{\Gamma}^j_{ik}\, dx^k$ denote the Levi-Civita connection forms of α. Let

$$r_{ij} := \frac{1}{2}\Big(b_{i|j} + b_{j|i}\Big), \qquad s_{ij} := \frac{1}{2}\Big(b_{i|j} - b_{j|i}\Big),$$

$$s^i{}_j := a^{ih}s_{hj}, \qquad s_j := b_i s^i{}_j, \qquad e_{ij} := r_{ij} + b_i s_j + b_j s_i.$$

Then the geodesic coefficients G^i are given by

$$G^i = \bar{G}^i + \frac{e_{00}}{2F}y^i - s_0 y^i + \alpha s^i{}_0, \tag{11}$$

where $\bar{G}^i$ denote the geodesic coefficients of α, $e_{00} := e_{ij}y^i y^j$, $s_0 := s_i y^i$ and $s^i{}_0 := s^i{}_j y^j$. See [1].

Lemma 2.1. *For a Randers metric $F = \alpha + \beta$, the mean Cartan torsion $\mathbf{I} = I_i\, dx^i$ and the mean Landsberg curvature $\mathbf{J} = J_i\, dx^i$ are given by*

$$I_i = \frac{1}{2}(n+1)F^{-1}\alpha^{-2}\Big\{\alpha^2 b_i - \beta y_i\Big\} \tag{12}$$

$$\begin{aligned} J_i = {} & \frac{1}{4}(n+1)F^{-2}\alpha^{-2}\Big\{2\alpha\Big[(e_{i0}\alpha^2 - y_i e_{00}) - 2\beta(s_i\alpha^2 - y_i s_0) + s_{i0}(\alpha^2+\beta^2)\Big] \\ & + \alpha^2(e_{i0}\beta - b_i e_{00}) + \beta(e_{i0}\alpha^2 - y_i e_{00}) \\ & - 2(s_i\alpha^2 - y_i s_0)(\alpha^2+\beta^2) + 4s_{i0}\alpha^2\beta\Big\}. \end{aligned} \tag{13}$$

Proof. First plugging (10) into (5) yield (12). Now we are going to compute J_i. Let

$$H^i := \frac{e_{00}}{2F}y^i - s_0 y^i + \alpha s^i{}_0.$$

We can rewrite (8) as follows

$$J_i = y^j I_{i|j} - I_j H^j{}_{\cdot i} - 2H^j I_{i\cdot j}, \tag{14}$$

where $H^j{}_{\cdot i} := \partial H^j/\partial y^i$, $I_{i\cdot j} = \partial I_i/\partial y^j$ and $I_{i|j}$ are defined by

$$dI_i - I_j \frac{\partial \bar{G}^j}{\partial y^i \partial y^k}\, dx^k = I_{i|j}\, dx^j + I_{i\cdot j}\Big(dy^j + \frac{\partial \bar{G}^j}{\partial y^k}\, dx^k\Big).$$

By a direct computation, we obtain

$$\begin{aligned} I_{i\cdot j} = {} & -\frac{n+1}{2}F^{-2}\alpha^{-2}\Big(y_j\alpha^{-1} + b_j\Big)\Big(\alpha^2 b_i - \beta y_i\Big) \\ & -(n+1)F^{-1}\alpha^{-4}y_j\Big(\alpha^2 b_i - \beta y_i\Big) \\ & + \frac{n+1}{2}F^{-1}\alpha^{-2}\Big(2y_j b_i - b_j y_i - \beta a_{ij}\Big) \\ H^j{}_{\cdot i} = {} & \frac{e_{00}}{2F}\delta^j_i + \frac{e_{i0}}{F}y^j - \frac{e_{00}}{2F^2}\Big(y_i\alpha^{-1} + b_i\Big)y^j \\ & - s_0\delta^j_i - s_i y^j + y_i\alpha^{-1}s^j{}_0 + \alpha s^j{}_i. \end{aligned}$$

where $b_{i|0} = b_{i|j} y^j$ and $b_{0|0} = b_{i|j} y^i y^j$. Observe that

$$b_{i|j} = r_{ij} + s_{ij} = e_{ij} - b_i s_j - b_j s_i + s_{ij}.$$

We have

$$b_{i|0} = e_{i0} - b_i s_0 - s_i \beta + s_{i0}, \qquad b_{0|0} = e_{00} - 2 s_0 \beta.$$

By these identities, we obtain

$$\begin{aligned} y^j I_{i|j} &= -\frac{n+1}{2} F^{-2} \alpha^{-2} b_{0|0} \Big(\alpha^2 b_i - \beta y_i \Big) + \frac{n+1}{2} F^{-1} \alpha^{-2} \Big(\alpha^2 b_{i|0} - b_{0|0} y_i \Big) \\ &= -\frac{n+1}{2} F^{-2} \alpha^{-2} \Big(e_{00} - 2 s_0 \beta \Big) \Big(\alpha^2 b_i - \beta y_i \Big) \\ &\quad + \frac{n+1}{2} F^{-1} \alpha^{-2} \Big((\alpha^2 e_{i0} - e_{00} y_i) - \alpha^2 (b_i s_0 + s_i \beta) + 2 s_0 \beta y_i + \alpha^2 s_{i0} \Big). \end{aligned}$$

Plugging them into (14) yields (13). □

3. Randers metrics with $\mathbf{S} = (n+1)cF$

In this section, we are going to find a sufficient and necessary condition on α and β for $\mathbf{S} = (n+1)c\, F$.

Consider a Randers metric $F = \alpha + \beta$ on a manifold M, where $\alpha = \sqrt{a_{ij} y^i y^j}$ and $\beta = b_i y^i$. Define r_{ij}, s_{ij}, $s^i{}_j$, s_j and e_{ij} as above in Section 2. The geodesic coefficients G^i of F are related to the geodesic coefficients $\bar{G}^i$ of α by (11). Let

$$\rho := \ln \sqrt{1 - \|\beta\|_\alpha^2}$$

and $d\rho = \rho_i\, dx^i$. According to [8], the S-curvature of $F = \alpha + \beta$ is given by

$$\mathbf{S} = (n+1) \Big\{ \frac{e_{00}}{2F} - (s_0 + \rho_0) \Big\}, \tag{15}$$

where e_{00} and s_0 are defined in Section 2 and $\rho_0 := \rho_p y^p$.

We have the following

Lemma 3.1. *Let $F = \alpha + \beta$ be a Randers metric on an n-dimensional manifold M, where $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ and $\beta = b_i(x) y^i$. For a scalar function $c = c(x)$ on M, the following are equivalent*

(a) $\mathbf{S} = (n+1)c\, F$;

(b) $e_{00} = 2c(\alpha^2 - \beta^2)$.

Proof. From (15), we see that $\mathbf{S} = (n+1)cF$ if and only if

$$e_{ij} = (s_i + \rho_i) b_j + (s_j + \rho_j) b_i + 2c(a_{ij} + b_i b_j) \tag{16}$$

$$s_i + \rho_i + 2cb_i = 0. \tag{17}$$

On the other hand, $e_{00} = 2c(\alpha^2 - \beta^2)$ is equivalent to the following identity,

$$e_{ij} = 2c(a_{ij} - b_i b_j). \tag{18}$$

First suppose that $\mathbf{S} = (n+1)cF$. Then (16) and (17) hold. Plugging (17) into (16) gives (18).

Conversely, suppose that (18) holds. Contracting (18) with b^j yields

$$r_{ij}b^j + \|\beta\|^2 s_i = 2c(1 - \|\beta|^2)b_i, \tag{19}$$

where we have used the fact $s_j b^j = 0$. Note that

$$-b^j b_{j|i} = (1 - \|\beta\|^2)\rho_i. \tag{20}$$

Adding (20) to (19) gives

$$-(1 - \|\beta\|^2)s_i = 2c(1 - \|\beta\|^2)b_i + (1 - \|\beta\|^2)\rho_i. \tag{21}$$

This is equivalent to (17) since $1 - \|\beta\|^2 \neq 0$. From (18) and (17), one immediately obtains (16). This proves the lemma. □

Lemma 3.2. *Let $F = \alpha + \beta$ be a Randers metric on an n-dimensional manifold M. For a scalar function $c = c(x)$ on M, the following are equivalent*
(a) $\mathbf{E} = (1/2)(n+1)cF^{-1}h$;
(b) $e_{00} = 2c(\alpha^2 - \beta^2)$.

Proof. It follows from (7) and (15) that

$$E_{ij} = \frac{1}{4}(n+1)\left[\frac{e_{00}}{F}\right]_{y^i y^j}. \tag{22}$$

Suppose that $e_{00} = 2c(\alpha^2 - \beta^2)$. Then

$$\frac{e_{00}}{F} = 2c(\alpha - \beta).$$

Plugging it into (22) we obtain

$$E_{ij} = \frac{1}{2}(n+1)c\ \alpha_{y^i y^j} = \frac{1}{2}(n+1)c\ F_{y^i y^j}. \tag{23}$$

That is, $\mathbf{E} = (1/2)(n+1)c\ F^{-1}h$.

Conversely, suppose that (23) holds. It follows from (22) and (23) that

$$\left[\frac{e_{00}}{F}\right]_{y^i y^j} = 2c\ F_{y^i y^j}.$$

Thus at each point $p \in M$, the following holds on $T_pM \setminus \{0\}$,

$$\frac{e_{00}}{F} = 2cF + \eta + \tau,$$

where $\eta \in T_p^*M$ and τ is a constant. By the homogeneity, we conclude that $\tau = 0$. Then

$$e_{00} = 2cF^2 + \eta F. \tag{24}$$

Equation (24) is equivalent to the following equations,

$$e_{00} = 2c(\alpha^2 + \beta^2) + \eta\beta \tag{25}$$

$$0 = 4c\beta + \eta. \tag{26}$$

By (26), we obtain $\eta = -4c\beta$. Plugging it into (25), we obtain

$$e_{00} = 2c(\alpha^2 - \beta^2).$$

This completes the proof. □

4. Randers metrics with $\mathbf{J} + cF\,\mathbf{I} = 0$

As we know, the mean Landsberg curvature $\mathbf{J}$ can be expressed in terms of α and β. But the formula (13) is very complicated. So the equation $\mathbf{J} + cF\ \mathbf{I} = 0$ is complicated too. In this section, we are going to find a simpler necessary and sufficient condition for $\mathbf{J} + cF\mathbf{I} = 0$.

Lemma 4.1. *Let $F = \alpha + \beta$ be a Randers metric on a manifold M. For a scalar function $c = c(x)$ on M, the following are equivalent*
(a) $\mathbf{J} + cF\ \mathbf{I} = 0$;
(b) $e_{00} = 2c(\alpha^2 - \beta^2)$ *and β is closed.*

Proof. Let

$$f_{ij} := e_{ij} - 2c(a_{ij} - b_i b_j)$$

and $f_{i0} := f_{ij}y^j$, $f_{00} := f_{ij}y^i y^j$. We have

$$2\alpha(e_{i0}\alpha^2 - y_i e_{00}) + \alpha^2(e_{i0}\beta - b_i e_{00}) + \beta(e_{i0}\alpha^2 - y_i e_{00}) =$$

$$2\alpha(f_{i0}\alpha^2 - y_i f_{00}) + \alpha^2(f_{i0}\beta - b_i f_{00}) + \beta(f_{i0}\alpha^2 - y_i e_{00}) - 2c(b_i\alpha^2 - y_i\beta)F^2.$$

Plugging it into (13), we see that $\mathbf{J} + cF\,\mathbf{I} = 0$ if and only if

$$(f_{i0}\beta - b_i f_{00})\alpha^2 + (f_{i0}\alpha^2 - y_i f_{00})\beta + 4s_{i0}\alpha^2\beta - 2(s_i\alpha^2 - y_i s_0)(\alpha^2 + \beta^2) = 0, \tag{27}$$

$$(f_{i0}\alpha^2 - y_i f_{00}) + s_{i0}(\alpha^2 + \beta^2) - 2(s_i\alpha^2 - y_i s_0)\beta = 0. \tag{28}$$

Differentiating (28) with respect to y^j, y^k and y^l, we obtain

$$\begin{aligned} 0 = {} & f_{ij}a_{kl} + f_{ik}a_{jl} + f_{il}a_{jk} - a_{ij}f_{kl} - a_{ik}f_{jl} - a_{il}f_{jk} \\ & + s_{ij}(a_{kl} + b_k b_l) + s_{ik}(a_{jl} + b_j b_l) + s_{il}(a_{jk} + b_j b_k) \\ & - (2a_{kl}s_i - a_{ik}s_l - a_{il}s_k)b_j \\ & - (2a_{jl}s_i - a_{ij}s_l - a_{il}s_j)b_k \\ & - (2a_{jk}s_i - a_{ij}s_k - a_{ik}s_j)b_l. \end{aligned} \tag{29}$$

Contracting (29) with a^{kl} yields

$$nf_{ij} - \lambda a_{ij} + s_{ij}(n + 2 + \|\beta\|^2) - 2(n+1)s_i b_j + 2(b_i s_j - b_j s_i) = 0, \tag{30}$$

where $\lambda := a^{kl}f_{kl}$. Here we have made the use of the identity $b_k a^{kl} s_{il} = -s_i$. It follows from (30) that

$$f_{ij} = \frac{\lambda}{n}\, a_{ij} + \frac{n+1}{n}(s_i b_j + s_j b_i), \tag{31}$$

$$s_{ij}(n + 2 + \|\beta\|^2) = (n-1)(s_i b_j - s_j b_i). \tag{32}$$

Contracting (32) with $b^i := b_r a^{ri}$ yields

$$s_j = 0.$$

Plugging it into (32) we obtain that

$$s_{ij} = 0$$

and

$$f_{ij} = \frac{\lambda}{n} a_{ij}. \tag{33}$$

Now equation (27) simplifies to

$$\lambda(b_i\alpha^2 - y_i\beta) = 0. \tag{34}$$

Taking $y_i = b_i$ in (34) we obtain

$$\lambda(\|\beta\|^2 - 1)b_i = 0. \tag{35}$$

Assume that $\beta \not\equiv 0$. It follows from (35) that $\lambda = 0$. From (33), we conclude that $f_{ij} = 0$.

Conversely, we suppose that $e_{00} = 2c(\alpha^2 - \beta^2)$. Then

$$e_{i0} = 2c(y_i - b_i\beta), \qquad e_{00} = 2c(\alpha^2 - \beta^2).$$

We obtain

$$e_{i0}\alpha^2 - y_i e_{00} = -2c(b_i\alpha^2 - y_i\beta)\beta, \tag{36}$$

$$e_{i0}\beta - b_i e_{00} = -2c(b_i\alpha^2 - y_i\beta). \tag{37}$$

Plugging (36) and (37) into (13) yields

$$J_i = \frac{1}{2}(n+1)\alpha^{-2}\Big\{ -c\Big[(b_i\alpha^2 - y_i\beta) + (s_i\alpha^2 - y_i s_0)\Big] + s_{i0}\alpha\Big\}. \tag{38}$$

Further, suppose that β is closed, hence $s_{ij} = 0$. From (12) and (38), we obtain

$$J_i = -\frac{1}{2}(n+1)c\alpha^{-2}\Big\{b_i\alpha^2 - y_i\beta\Big\} = -cF\, I_i.$$

This proves the lemma. □

There are lots of Randers metrics satisfying

$$\mathbf{E} = \frac{1}{2}(n+1)cF^{-1}h, \qquad \mathbf{J} + cF\,\mathbf{I} = 0.$$

Besides the Randers metrics in (3) with $c = \pm(1/2)$, we have the following example with $c = c(x) \neq constant$.

EXAMPLE 4.2. For an arbitrary number ϵ with $0 < \epsilon \leq 1$, define

$$\alpha := \frac{\sqrt{(1-\epsilon^2)(xu+yv)^2 + \epsilon(u^2+v^2)(1+\epsilon(x^2+y^2))}}{1+\epsilon(x^2+y^2)}$$

$$\beta := \frac{\sqrt{1-\epsilon^2}\,(xu+yv)}{1+\epsilon(x^2+y^2)}.$$

We have

$$\|\beta\|_\alpha = \sqrt{1-\epsilon^2}\sqrt{\frac{x^2+y^2}{\epsilon+x^2+y^2}} < 1.$$

Thus $F := \alpha + \beta$ is a Randers metric on R^2. By a direct computation, we obtain

$$\mathbf{J} + cF\,\mathbf{I} = 0,$$

where

$$c = \frac{\sqrt{1-\epsilon^2}}{2(\epsilon + x^2 + y^2)}.$$

Moreover, the Gauss curvature of F is given by

$$\mathbf{K} = -\frac{3\sqrt{1-\epsilon^2}\,(xu + yv)}{(\epsilon + x^2 + y^2)^2 F} + \frac{7(1-\epsilon^2)}{4(\epsilon + x^2 + y^2)^2} + \frac{2\epsilon}{\epsilon + x^2 + y^2}.$$

Thus F does not have constant Gauss curvature.

5. Proof of Theorem 1.2

By assumption that $\mathbf{J} + cF\mathbf{I} = 0$ and Lemma 4.1, we know that

$$e_{ij} = 2c(a_{ij} - b_i b_j), \qquad s_{ij} = 0.$$

Plugging them into (11) yields

$$G^i = \bar{G}^i + c(\alpha - \beta)y^i. \tag{39}$$

Thus $F = \alpha + \beta$ is pointwise projectively equivalent to α and the Douglas curvature vanishes [2] (see also Example 13.2.1 in [8]). By assumption $\mathbf{K} = \lambda$, the Weyl curvature (the Berwald-Weyl curvature in dimension two) vanishes [12] (see also Chapter 13 in [8]). Therefore, F is locally projectively flat. Then Theorem 1.2 follows from Theorem 1.1 in [9].

For reader's convenience, we sketch a direct proof below under the assumption that

$$e_{ij} = 2c(a_{ij} - b_i b_j), \qquad s_{ij} = 0, \qquad R^i{}_k = \lambda F^2\Big\{\delta^i_k - \frac{F_{y^k}}{F}y^i\Big\}.$$

First, plugging (39) into (4) and using

$$b_{i|j}y^i y^j = e_{00} = 2c(\alpha^2 - \beta^2), \tag{40}$$

we obtain

$$R^i{}_k = \bar{R}^i{}_k + \Xi\,\delta^i_k + \tau_k y^i, \tag{41}$$

where

$$\Xi = 3c^2\alpha^2 - 2c^2\alpha\beta - c^2\beta^2 - c_{|0}(\alpha - \beta).$$

Then

$$\bar{R}^i{}_k = R^i{}_k - \Xi\delta^i_k - \tau_k y^i = \Big[(\lambda - 3c^2)\alpha^2 + 2(\lambda + c^2)\alpha\beta + (\lambda + c^2)\beta^2 + c_{|0}(\alpha - \beta)\Big]\delta^i_k + \tilde{\tau}_k y^i.$$

This implies that

$$\frac{(\lambda - 3c^2)\alpha^2 + 2(\lambda + c^2)\alpha\beta + (\lambda + c^2)\beta^2 + c_{|0}(\alpha - \beta)}{\alpha^2} = \mu,$$

where $\mu = \mu(x)$ is a scalar function on M and it must be a constant when $n = \dim M > 2$. The above equation is equivalent to the following two equations

$$2(\lambda + c^2)\beta + c_{|0} = 0 \tag{42}$$

$$(\lambda - 3c^2 - \mu)\alpha^2 + (\lambda + c^2)\beta^2 - c_{|0}\beta = 0. \tag{43}$$

From (42) we obtain that $c_{|0} = -2(\lambda + c^2)\beta$. Plugging it into (43) yields

$$(\lambda - 3c^2 - \mu)\alpha^2 + 3(\lambda + c^2)\beta^2 = 0.$$

It follows that

$$\lambda - 3c^2 - \mu = 0, \qquad \lambda + c^2 = 0.$$

Thus c is a constant, $\lambda = -c^2$ and $\mu = -4c^2$, i.e.,

$$\bar{R}^i{}_k = -4c^2\alpha^2\Big\{\delta^i_k - \frac{\alpha_{y^k}}{\alpha}y^i\Big\}.$$

We will follow [9] closely. First, we suppose that $c = 0$. It follows from (39) that $G^i = \bar{G}^i(x, y)$ are quadratic in $y \in \mathrm{R}^n$ for any x. Hence F is a Berwald metric. Moreover,

$$R^i{}_k = \bar{R}^i{}_k.$$

On the other hand, $\mu = -4c^2 = 0$ implies that α is flat, $\bar{R}^i{}_k = 0$. Thus $F = \alpha + \beta$ is flat. We conclude that F is locally Minkowskian.

Now suppose that $c \neq 0$. After an appropriate scaling, we may assume that $c = \pm 1/2$. We can express α in the following Klein form

$$\alpha = \frac{\sqrt{|\mathbf{y}|^2 - (|\mathbf{x}|^2|\mathbf{y}|^2 - \langle \mathbf{x}, \mathbf{y}\rangle^2)}}{1 - |\mathbf{x}|^2}.$$

Since β is closed, we can express it in the following form

$$\beta = \pm\frac{\langle \mathbf{x}, \mathbf{y}\rangle}{1 - |\mathbf{x}|^2} \pm d\varphi(\mathbf{y}), \qquad \mathbf{y} = (y^i) \in T_{\mathbf{x}}\mathbb{B}^n.$$

It follows from (40) that

$$b_{i|j} = \pm(a_{ij} - b_i b_j). \tag{44}$$

The Christoffel symbols of α are given by

$$\bar{\Gamma}^i_{jk} = \frac{x^k \delta^i_j + x^j \delta^i_k}{1 - |\mathbf{x}|^2}.$$

The covariant derivatives of β with respect to α are given by

$$b_{i|j} = \pm\Big\{ \frac{\partial^2 \varphi}{\partial x^i \partial x^j} + \frac{1}{1 - |\mathbf{x}|^2}\Big(\delta_{ij} - x^i \frac{\partial \varphi}{\partial x^j} - x^j \frac{\partial \varphi}{\partial x^i}\Big)\Big\},$$

and

$$a_{ij} - b_i b_j = \frac{1}{1 - |\mathbf{x}|^2}\Big(\delta_{ij} - x^i \frac{\partial \varphi}{\partial x^j} - x^j \frac{\partial \varphi}{\partial x^i}\Big) - \frac{\partial \varphi}{\partial x^i}\frac{\partial \varphi}{\partial x^j}.$$

Plugging them into (44) yields

$$\frac{\partial^2 \varphi}{\partial x^i \partial x^j} + \frac{\partial \varphi}{\partial x^i}\frac{\partial \varphi}{\partial x^j} = 0. \tag{45}$$

Let $f = \exp(\varphi)$. Then (45) simplifies to

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = 0. \tag{46}$$

Thus f is a linear function

$$f = k(1 + \langle \mathbf{a}, \mathbf{x} \rangle), \qquad k > 0.$$

We obtain that

$$\varphi = \ln k + \ln(1 + \langle \mathbf{a}, \mathbf{x} \rangle).$$

Finally, we find the most general solution for β,

$$\beta = \pm \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{1 - |\mathbf{x}|^2} \pm \frac{\langle \mathbf{a}, \mathbf{y} \rangle}{1 + \langle \mathbf{a}, \mathbf{x} \rangle}, \qquad \mathbf{y} \in T_{\mathbf{x}}\mathbb{B}^n. \tag{47}$$

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