Title	Non-contractible acyclic normal spines
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Citation	Osaka Journal of Mathematics. 10(3) P.511-P.520
Issue Date	1973
Text Version	publisher
URL	https://doi.org/10.18910/5989
DOI	10.18910/5989
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Ikeda, H.
Osaka J. Math.
10 (1973), 511–520

NON-CONTRACTIBLE ACYCLIC NORMAL SPINES

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(Received July 24, 1972)

1. Introduction

In [3], we have defined fake surfaces to study 3-manifolds with boundary from their spines. We use the notations in [3] and [4], for example, $\mathcal{F}(s, t)$ denotes the set of all the acyclic closed fake surfaces P with $\#\mathfrak{S}_2(P)=s$ and $\#\mathfrak{S}_3(P)=t$, where $\mathfrak{S}_i(P)$ means the i-th singularity of P and $\#$ denotes the number of the connected components. And, $\mathcal{E}(s, t)$ is the subset of $\mathcal{F}(s, t)$ each of whose elements is a noraml spine, that is, for any element P of $\mathcal{E}(s, t)$, there exists a 3-manifold in which P can be embedded as a spine. The following theorems are proved in [3] and [4].

Theorem. $\mathcal{F}(s, t)=\phi$, *if and only if* $t=0$.

Theorem. $\mathcal{E}(s, t)=\phi$, *if and only if* $s\geqq 2t$.

Then, when $t\geqq 1$, it is known that the difference $\mathcal{F}(s, t)-\mathcal{E}(s, t)$ is non-empty.

Let $\mathcal{C}(s, t)$ denote the subset of $\mathcal{E}(s, t)$ each of whose elements is contractible and $\mathcal{B}(s, t)$ the subset of $\mathcal{C}(s, t)$ each of whose elements is a normal spine of a 3-ball. Define the two difference sets $\mathcal{D}(s, t)$ and $\mathcal{A}(s, t)$ by

$$\mathcal{D}(s, t) = \mathcal{E}(s, t)-\mathcal{C}(s, t)\,,$$
$$\mathcal{A}(s, t) = \mathcal{C}(s, t)-\mathcal{B}(s, t)\,.$$

Then, Poincaré conjecture asks "Is the set $\bigcup_{s,t} \mathcal{A}(s, t)$ empty?". On the other hand, the following theorem is well-known.

Theorem. $\bigcup_{s,t} \mathcal{D}(s, t)\neq\phi$.

And, in [3] and [4], we proved the following.

Theorem. $\mathcal{D}(s, t)=\phi=\mathcal{A}(s, t)$ *for the cases* $s=2t-1$ *and* $s=2t-2$, *and* $\mathcal{D}(1, 2)=\phi=\mathcal{A}(1, 2)$.

In this paper, we show the following.

Theorem 1. *For the case* $1 \leqq s \leqq 2t-11$ *and* $t \geqq 6$, *the set* $\mathcal{D}(s, t)$ *is non-empty.*

In § 2, we construct a non-contractible acyclic mormal spine P_k with $\#\mathfrak{S}_2(P_k)=1$ and $\#\mathfrak{S}_3(P_k)=8k-1$ for any integer $k \geqq 1$. And, in § 3, we can prove that a 3-manifold W_1 has a normal spine P' with $\#\mathfrak{S}_2(P')=1$ and $\#\mathfrak{S}_3(P')=6$, where W_k is the 3-manifold containing P_k as its normal spine. And, the proof of Theorem 1 is obtained. It is known, by the uniqueness theorem of [1], that W_k is uniquely determined. In § 4, we define the *Dehn space of type k* and show, in Theorem 2, that W_k is the Dehn space of type k.

The author thanks Mr. Y. Tsukui for pointing out the existence of P' and to all the membres of All Japan Combinatorial Topology Study Group for many useful discussions.

2. The construction of non-contractible acyclic normal spines P_k

It has been proved in Theorem 4 [3] that $\mathcal{E}(1, 1)$ contains a unique element $F^1_{1,1}$, called an abalone. Let the set $\{M_1, M_2, f\}$ be the polygonal representation of the abalone, that is, M_i is a 2-ball for $i=1, 2$, and f means the identification map from $M_1 \cup M_2$ to $F^1_{1,1}$ (for M_1, M_2 and the identification by f, see Theorem 2 [3]).

Through out this paper, the subpolyhedron $f(M_2)$ of the abalone is denoted by F, which is written in Fig. 1. Then, F is a closed fake surface with $\#\mathfrak{S}_2(F)=1$ and $\#\mathfrak{S}_3(F)=0$, more precisely, $U(F)=S \times_\sigma T$. And, by a little geometrical consideration, it is seen that F is a normal spine of the exterior of the clover-leaf knot in 3-sphere. The fundamental group of F is as follows.

$$\pi_1(F) = (S_1, S_2; S_1 S_2^{-1} S_1 S_2^2 = 1)$$

(for the generators S_1 and S_2, see Fig. 1).

Lemma 1. *For any integer* $k \geqq 0$, *there exists an embedding* h_k *from* 1*-sphere* S *into* F *which represents the homotopy class* $S_2^{3k} S_1^{6k-1}$ *and the intersection* $h_k(S) \cap S_2$ *consists of* $|8k-1|$ *points.*

Proof. When $k=0$, we can take h_0 to be the homeomorphism from S onto S_1 which reverses the orientation. Then, clearly, h_0 represents the homotopy class S_1^{-1} and we have $\#(h_0(S) \cap S_2)=1$. Let us construct the required embedding h_k for the cases $k \geqq 1$.

Step 1. Suppose $k=1$. For the point a, a', b, b', c, c', d and x_i, $i=1, 2, 3$, see Fig. 2. Now, starting from the point a, go to b along the orientation of S_2. From b', go to c along S_2. Intersecting with S_2 at the point x_1, go to c' as shown in Fig.2. From c', go to d along S_2. And, intersecting with S_2 at x_2, go to x_3.

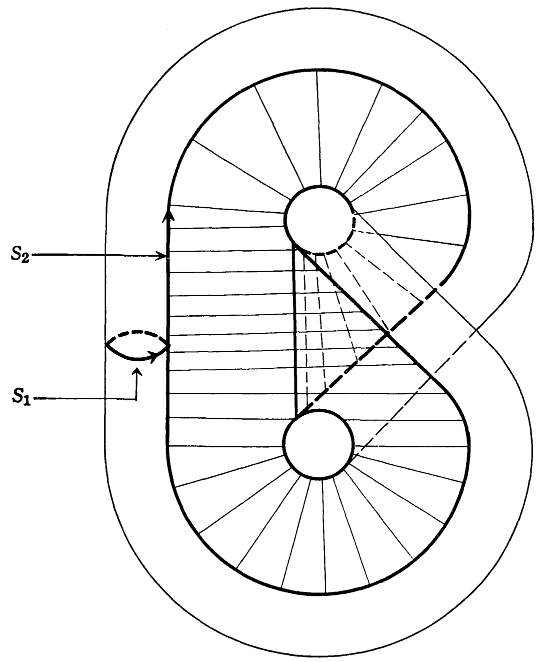

Fig. 1

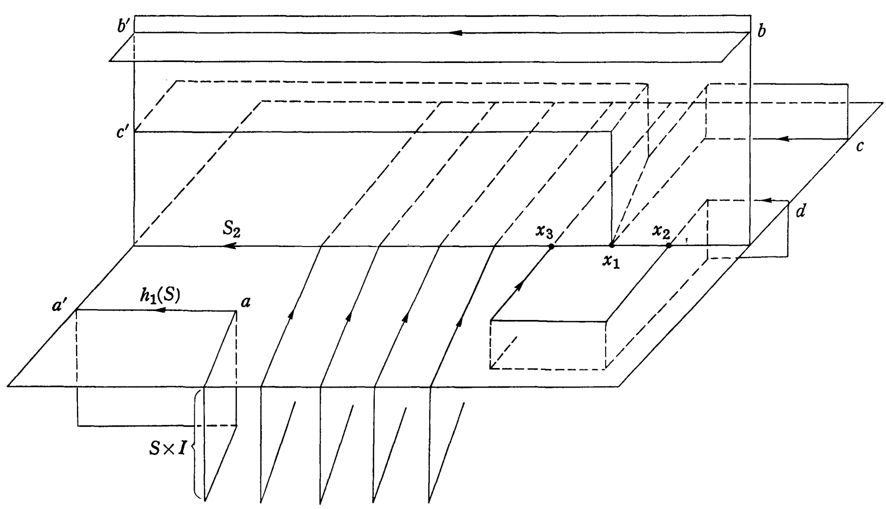

Fig.2

Finally, going along S_1 five times from x_3, we come back to the starting point a. Thus, we obtain an embedding h_1 representing the homotopy class $S_2^3 S_1^5$ and $\#(h_1(S) \cap S_2) = 7$.

Step 2. Because S_2^3 lies in the center of $\pi_1(F)$, we obtain the following.

$$S_2^{3k}S_1^{6k-1}=(S_2^3S_1^5)\prod_{p=2}^{k}(S_2^3S_1^6)_p,\ k\geqq 2\,.$$

So, we try to construct the required embedding h_k to represent the homotopy class $(S_2^3S_1^5)\prod_{p=2}^{k}(S_2^3S_1^6)_p$, as follows. Let $a_2, \cdots, a_k$ be the points between a and S_1 as shown in Fig. 3. And formally, set $a_1=a$. Then, by the same way as in Step 1, we obtain an embedding h_p' from S into F which represents the homotopy class $(S_2^3S_1^6)_p$ and whose initial point and end point is a_p. And set $h_1'=h_1$. More strictly, we can choose h_p' to satisfy the following conditions.

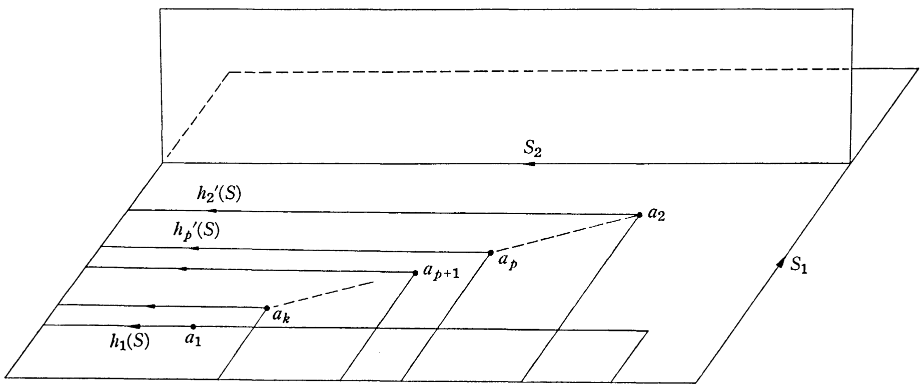

Fig. 3

(1) $h_p'(S)\cap h_q'(S)=\phi$, if $p\neq q$ and $p,\ q\geqq 2$.
(2) $h_p'(S)\cap h_1(S)$ is one point in the small neighborhood of a_1 (see Fig. 3).
(3) $\sharp(h_p'(S)\cap S_2)=8$.

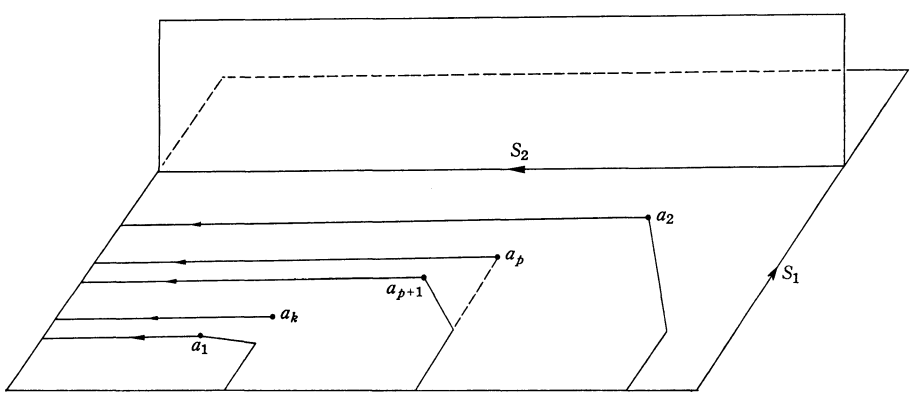

Fig. 4

Now, changing the end point a_p of h_p' to the initial point a_{p+1} of h'_{p+1}, for $1 \leqq p \leqq k-1$, and the end point a_k to a_1 (see Fig. 4), we obtain the required embedding h_k from S into F.

DEFINITION 1. Let P_k be the closed fake surface obtained from F by attaching a 2-ball B by the homeomorphism h_k from B to F.

REMARK. From the construction, it is clear that P_0 is homeomorphic to an abalone $F^1_{1,1}$.

Lemma 2. *If $k \geqq 1$, then P_k is a non-contractible element of $\mathcal{E}(1, 8k-1)$.*

Proof. We can prove that P_k is acyclic, because

$$\begin{aligned} H_1(P_k) &= (S_1, S_2; 2S_1+S_2 = 0 \quad (6k-1)S_1+3kS_2 = 0) \\ &= 0 \end{aligned}$$

and $H_2(P_k)$ is trivially trivial. And the fact that $\pi_1(P_k)$ is non-trivial follows from the calculation in [2]. Hence, P_k is a non-contractible acyclic closed fake surface. It follows from the construction of P_k that $U(P_k)$ can be embedded in the euclidean 3–space R^3. Then, by Lemma 2 [4], P_k is a normal spine. And, again from the construction, we see $\#\mathfrak{S}_2(P_k)=1$ and $\#\mathfrak{S}_3(P_k)=8k-1$, more precisely, $\mathfrak{S}_2(P_k)=S_2 \cup h_k(S)$ and $\mathfrak{S}_3(P_k)=S_2 \cap h_k(S)$ is the union $\bigcup_{p=1}^{k} (S_2 \cap h_p'(S))$, and we obtain $\#\mathfrak{S}_3(P_k)=8k-1$.

3. The element P' of $\mathcal{D}(1, 6)$ and the proof of Theorem 1

Let W_k denote the 3-manifold containing P_k as its normal spine, $k=1, 2, \dots$. In this section, we consider P_1 in W_1 and consturct another normal spine P' of W_1 from P_1 in $\mathcal{D}(1, 6)$. For the polygonal representation of P_1, see Fig. 5.

Proposition 1. *W_1 has a normal spine P' in $\mathcal{D}(1, 6)$.*

Proof. Let us consider M_1 of the polygonal representation of P_1, and let N be the regular neighborhood of M_1 mod $\dot{M}_1$ in W_1 chosen to satisfy

$$N \cap (P_1 - \mathring{M}_1) = \dot{N} \cap P_1 = \dot{M}_1 \times I \,,$$

as shown in Fig. 6, where I is the closed unit interval [0, 1] and $M_1 = M_1 \times 1/2$. Put $A = \dot{N} \cap P_1$. Then, $A = (A \cap \mathfrak{S}_2(P_1))$ has three connected components each of whose closures is a 2-ball. Take such a 2-ball B. Regarding B as a free face of $P_1 \cup N$, we can collapse $P_1 \cup N$ to $(P_1 - (N \cap P_1)) \cup (\dot{N} - \mathring{B})$ (see Fig. 7). Put $P' = (P_1 - (N \cap P_1)) \cup (\dot{N} - \mathring{B})$. Then, it is clear that P' is a closed fake surface embedded in the 3-manifold W_1. Since P_1 expands to $P_1 \cup N$ and $P_1 \cup N$ col-

Polygonal representation of P_1

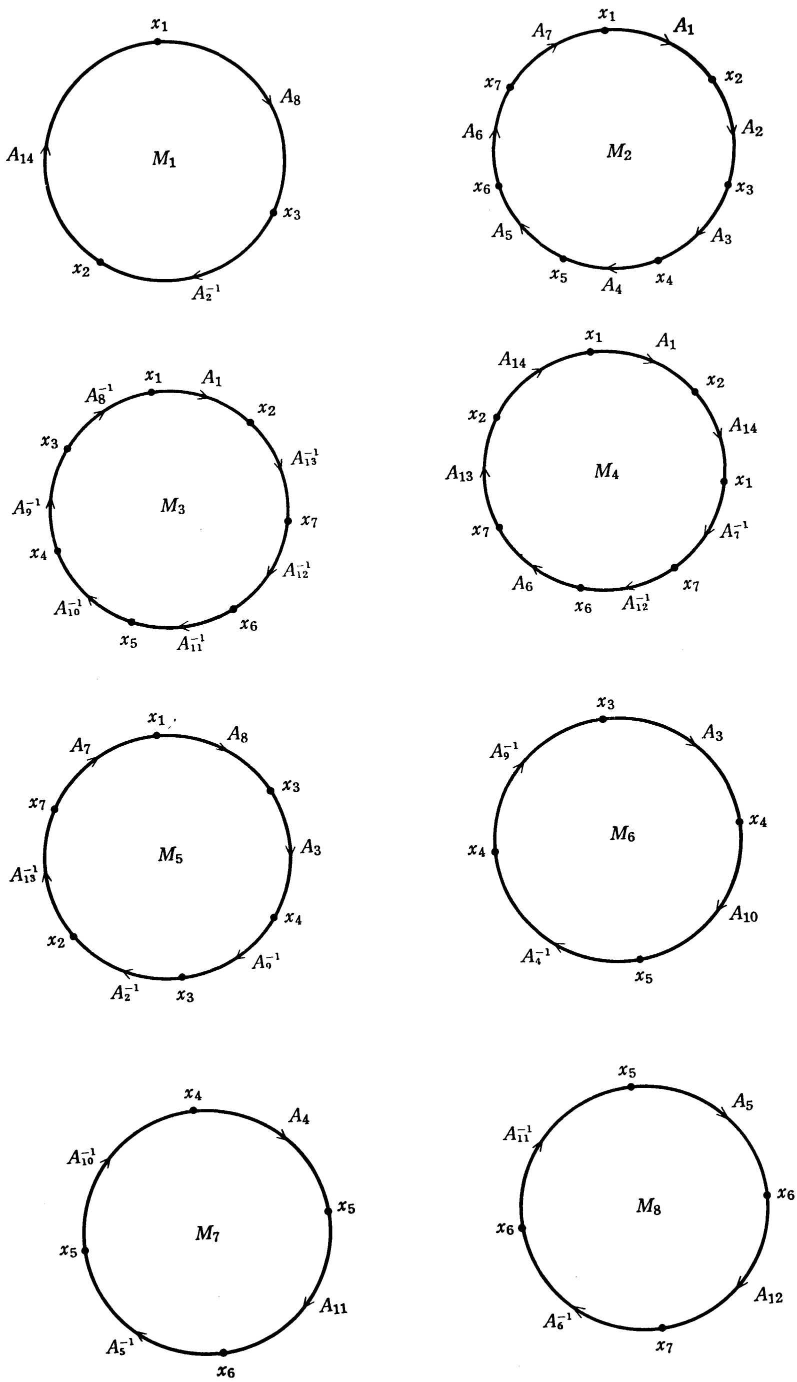

Fig. 5

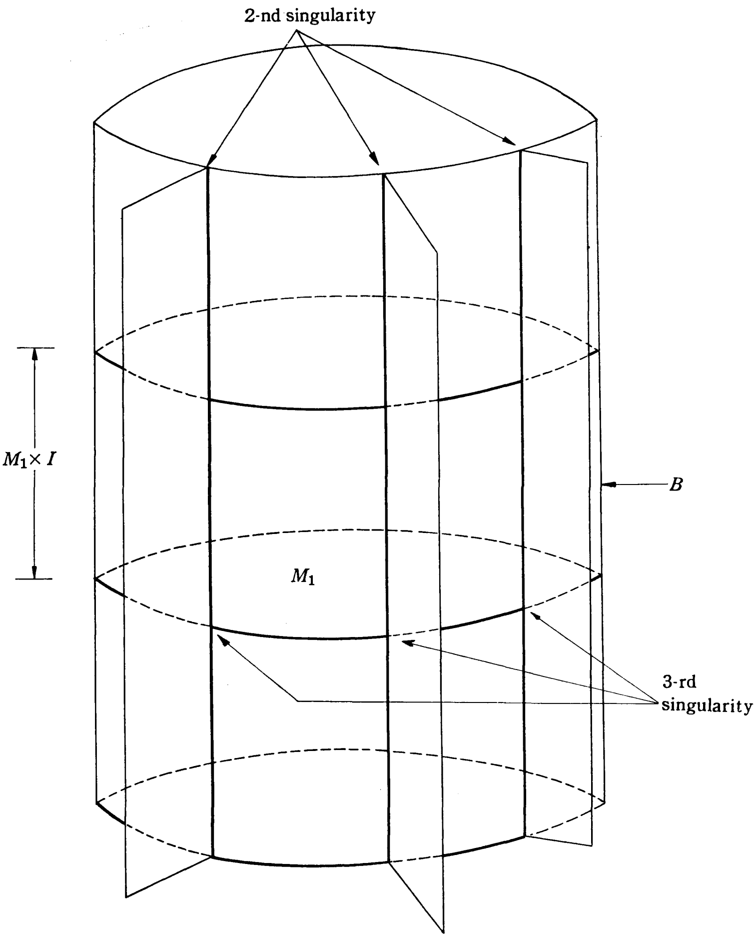

Fig. 6

lapses to P' in W_1, P_1 and P' belong to the same simple homotopy type in W_1, that is P' is also a spine of W_1. By the above construction, the conditions $\#\mathfrak{S}_2(P')=1$ and $\#\mathfrak{S}_3(P')=6$ are easily seen. Thus, W_1 has a normal spine P' in $\mathcal{D}(1.6)$.

REMARK. The polygonal representation of P' is shown in Fig. 8.

Now, we can prove Theorem 1.

Theorem 1. *For the case* $1 \leqq s \leqq 2t-11$ *and* $t \geqq 6$. *the set* $\mathcal{D}(s, t)$ *is non-empty.*

Proof. First, it is shown that $\mathcal{D}(1, t)$ is non-empty for $t \geqq 6$ by the same argument as that of the proof of Lemma 12 [4], because P' and P_1 belong to

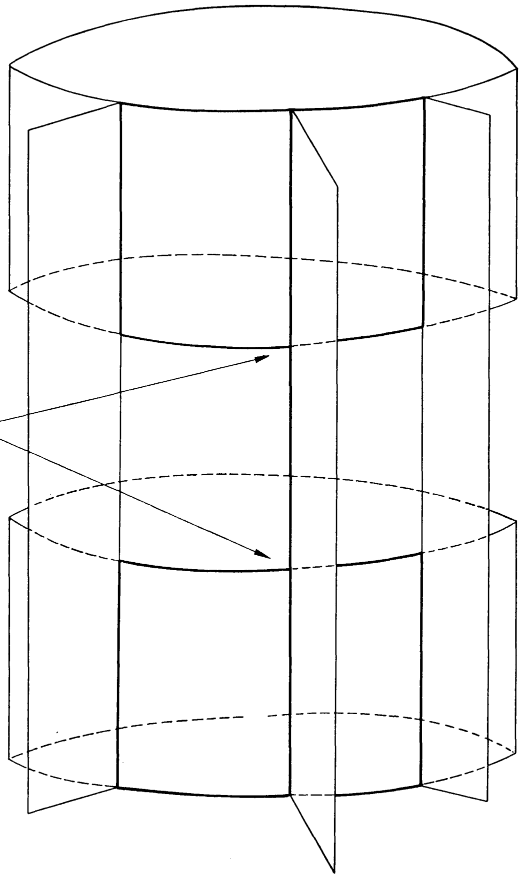

Fig. 7

$\mathcal{D}(1, 6)$ and $\mathcal{D}1, 7)$, respectively. And, we obtain an element of $\mathcal{D}(s, t)$ with $1 \leqq s \leqq 2t-11$ as in the proof of Theorem 6 [4].

4. The Dehn spaces

Let E denote the exterior of a clover knot k in a 3-sphere Σ, that is, $E=\Sigma-\mathring{N}(k, \Sigma)$ where $\mathring{N}(k, \Sigma)$ means the interior of a regular neighborhood $N(k, \Sigma)$ of k in Σ. Then, there exists a subpolyhedron F_0 in E which is homeomorphic to F. Of course, F_0 is a spine of E. Regarding the generators S_1 and S_2 of $\pi_1(F)$ as those of $\pi_1(F_0)$, we can write

$$\pi_1(E) = (S_1, S_2\colon S_1 S_2^{-1} S_1 S_2^2 = 1)\,.$$

Take S_1 and $S_1^{-1} S_2$ as the generators of $\pi_1(E)$, and let i_* denote the homomorphism from $\pi_1(\dot{E})$ to $\pi_1(E)$ induced by the inclusion map. Since E is an exterior

Polygonal representation of P'

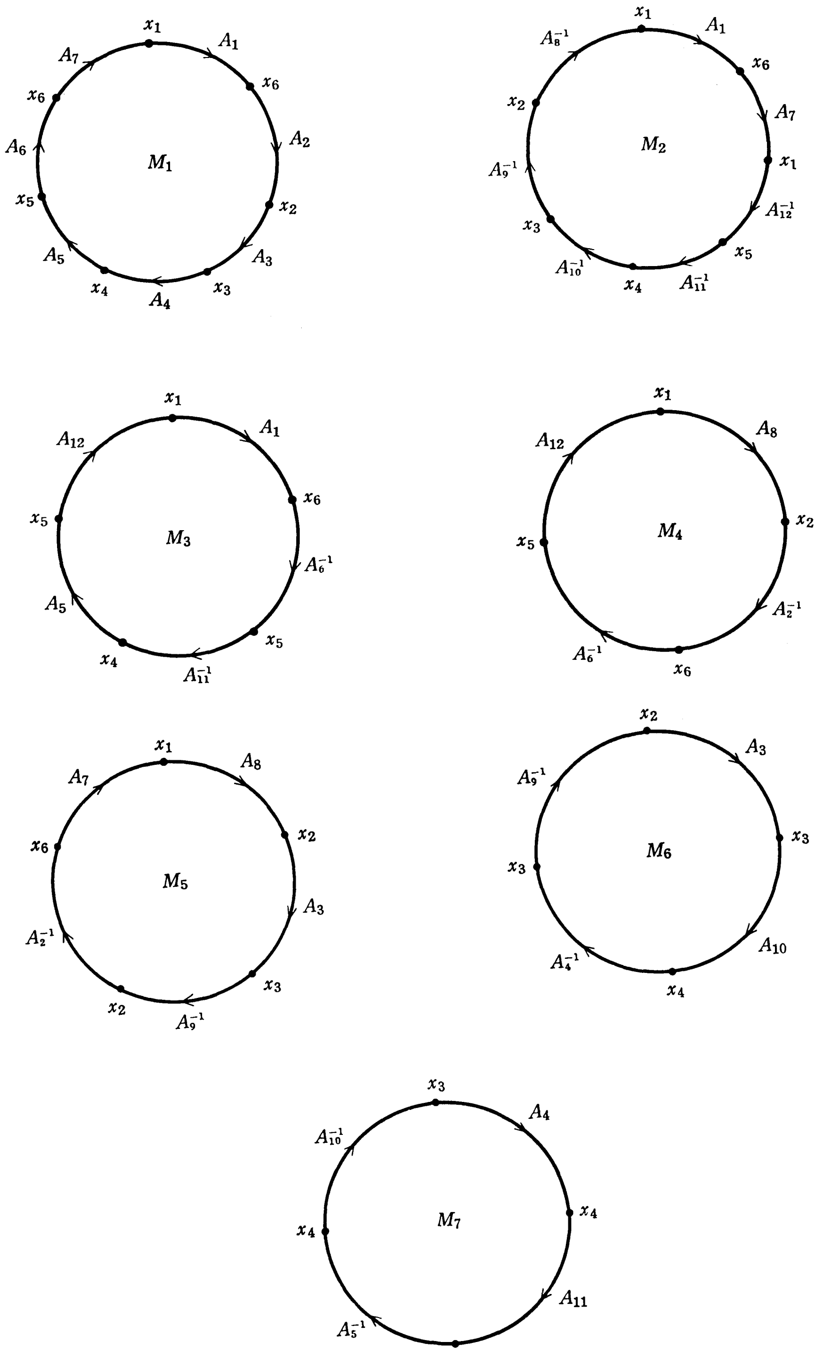

Fig. 8

of a knot, i_* is a monomorphism and we have $i_*^{-1}(S_2^{3k}S_1^{6k-1})=(S_1^{-1}S_2)^{2k}S_1^{6k-1}$. Let C_k denote the 1-sphere in $\dot{E}$ representing the homotopy class $(S_1^{-1}S_2)^{2k}S_1^{2k-1}$. Note that C_k exists because $2k$ and $6k-1$ are relatively prime.

DEFINITION 2. Define the *Dehn space V_k of type k* to be the 3-manifold obtained from E by attaching a 2-handle along C_k. (Cf. [2])

Theorem 2. *Let W_k be the 3-manifold containing P_k as its spine. Then, W_k is the Dehn space of type k.*

Proof. By the uniqueness theorem of [1], it is sufficient to prove that the Dehn space V_k contains P_k as its spine, because P_k clearly satisfies the conditions of *standard spine* of [1]. Let N_0 be the 3-rd derived neighborhood of $U(F_0)$ in E mod $\dot{U}(F_0)$. We can embed a cylinder $S\times I$ in N_0 in order to satisfy $(S\times I)\cap U(F_0)=S\times 0=h_k(S)$ and $(S\times I)\cap \dot{N}_0=S\times 1$ as shown in Fig. 2. Now, let $F_1=F\cup(S\times I)$ and N_1 the regular neighborhood of F_1 in E mod $\dot{F}_1=S\times 1$. Then, N_1 is homeomorphic to E keeping F_0 fixed, because F_1 collapses to F_0 by collapsing $S\times I$ to $S\times 0$ from $S\times 1$. And hence $S\times 1$ represents the homotopy class $(S_1^{-1}S_2)^{2k}S_2^{6k-1}$ in $\pi_1(N_1)$. Thus, V_k may be regarded as the 3-manifold obtained from N_1 by attaching a 2-handle along $S\times 1$. Then, the 2-handle $B^2\times I$ collapses to $(\dot{B}^2\times I)\cup(B^2\times 1/2)$, where B^2 is a 2-ball and $\dot{B}^2\times 1/2=S\times 1$. Thus, V_k collapses to $N_1\cup(B^2\times 1/2)$. Since N_1 is a regular neighborhood of F_1, $N_1\cup(B^2\times 1/2)$ collapses to $F_1\cup(B^2\times 1/2)$ which is clearly homeomorphic to P_k. Thus, V_k has a spine homeomorphic to P_k. This completes the proof of Theorem 2.

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