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FPF RINGS CHARACTERIZED BY TWO-GENERATED FAITHFUL MODULES

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0. Introduction

Azumaya [2], Osofsky [20] and Utumi [26] considered various properties of QF rings, e.g., the completely faithfulness (i.e., generator) of modules, the injective cogenerator and Morita duality. They introduced and studied one new class of, so-called, *right PF (pseudo-Frobenius) rings*, i.e., rings R whose every faithful right R-module is a generator for Mod-R, the category of all right R-modules. Afterward, Endo [6] and Tachikawa [25] naturally studied (commutative noetherian or perfect) rings R satisfying the condition that every *finitely generated* faithful right R-module is a generator for Mod-R. Rings satisfying this condition are called *right FPF (finitely pseudo-Frobenius) rings* whose general studies were made, at first over semiperfect rings, by Faith [7], [8]. The study of commutative or semiperfect FPF rings was improved in more detail ([9], [10] [11], [13], [14], [21]). Most of the basic results on FPF rings may be found in Faith and Page [12].

We now consider, for each positive integer n, the condition "right n-PF" on a ring R that every n-generated (i.e., generated by at most n elements) faithful right R-module is a generator for Mod-R ([27]). Thus the rings that are right n-PF for all positive integers n are just the right FPF rings, and there exists a chain of conditions:

$$\text{FPF} \Rightarrow \cdots \Rightarrow (n+1)\text{-PF} \Rightarrow n\text{-PF} \Rightarrow \cdots \Rightarrow 1\text{-PF}.$$

Concerning this, it was shown in [4], [17] that a right self-injective ring is right FPF if and only if it is right 1-PF, while a commutative semiprime or von Neumann regular ring is right FPF if and only if it is right 2-PF. We then ask generally whether the chain of conditions from FPF to n-PF for some positive integer n collapses to a single condition, i.e., n-PF $\Rightarrow$ FPF. Obviously, 1-PF does not imply FPF in general (for every commutative ring is 1-PF). Thus it is natural to ask whether 2-PF implies FPF. We do not know whether this is true in general case.

In this paper, we shall study semiperfect or commutative FPF rings in connection with the noted above. In Section 1, we present a characterization of semiperfect FPF rings, which shows that 2-PF $\Rightarrow$ FPF for semiperfect rings. Section 2 is concerned with commutative FPF rings. In the section, we characterize these rings R by overmodules of R in its injective hull and the stalks of R, from which 2-PF $\Rightarrow$ FPF for commutative rings. In the last section, by using the theorem of Section 2 we present additional results on commutative FPF rings, e.g., the "invertibility" of finitely generated overmodules in the injective hull, the integrally closedness and flat epimorphisms of overrings in the maximal quotient rings.

NOTATION and TERMINOLOGY. Throughout this paper, all rings considered are associative rings with identity and all modules are unitary.

Let R be a ring. For an R-module M and a positive integer n, we denote by $E(M)$, $J(M)$ and $Z(M)$ the injective hull, the Jacobson radical and the singular submodule of M, respectively, and by $M^{(n)}$ the direct sum of n copies of M. The notations $N \lesssim M$ and $N \lesssim_{\oplus} M$ mean that N is an R-module isomorphic to a submodule and a direct summand of M, respectively. For subsets A, B of M, we set $(A : B) = \{r \in R \mid Br \subset A\}$. For finitely-many $x_1, \ldots, x_n \in M$, we abbreviate $(A : \{x_1, \ldots, x_n\})$ to $(A : x_1, \ldots, x_n)$.

Recall that a ring R is *right FPF* if every finitely generated faithful right R-module is a generator for Mod-R.

1. Semiperfect FPF rings

In this section, we shall prove the following.

Theorem 1.1. *Let R be a semiperfect ring with basic idempotent e. Then the following conditions are equivalent*:

(1) *R is right FPF*;

(2) *Every faithful factor module of $(eR)^{(2)}$ is a generator for* Mod-R;

(3) (i) *If I is a submodule of eR such that eR/I is faithful, then $I = 0$*;

(ii) *For every $x \in E(eR)$, $eR + xeR$ is a generator for* Mod-R;

(4) *eR is the unique faithful factor module of eR and the unique finitely generated faithful submodule of $E(eR)$, to within isomorphism.*

The theorem above immediately implies the following result.

Corollary 1.2. *A semiperfect ring R is right FPF if and only if every 2-generated faithful right R-module is a generator for* Mod-R.

Concerning the corollary, we note that in [25, (the proof of) Proposition 2.4 and Theorem 2.5], Tachikawa proves the following strong theorem for perfect rings.

Theorem ([25]). *A left perfect ring R is right PF if and only if every 2-generated faithful right R-module is a generator for* Mod-R.

To prove Theorem 1.1, we use the following lemmas. The first one is well-known (e.g. [3, Lemma I, 3.5]).

Lemma 1.3. (1) *Let M and $M_1, \ldots, M_n$ be right R-modules such that* $\mathrm{End}_R(M)$ *is a local ring. If*

$$M \lesssim_{\oplus} M_1 \oplus \cdots \oplus M_n,$$

then there exists an $i \in \{1, \ldots, n\}$ such that $M \lesssim_{\oplus} M_i$.

(2) *Let R be a semiperfect ring with basic idempotent e. If G is a generator for* Mod-R, *then $eR \lesssim_{\oplus} G$.*

For an R-module M, we denote by $T(M)$ (= the *top* of M) the factor module of M modulo its Jacobson radical.

Lemma 1.4. *Let P be a finitely generated projective right R-module whose top is semisimple. If X is a submodule of P such that $P \lesssim_{\oplus} P/X$, then $X = 0$.*

Proof. Note by the semisimpleness of $T(P)$ that the top of every factor module of P is semisimple. Now, assume that

$$P/X \cong P \oplus Y$$

for some right R-module Y, and set $J(P/X) = X'/X$. Then,

$$P/X' \cong T(P/X) \cong T(P \oplus Y) \cong (P/J(P)) \oplus (Y/J(Y)).$$

Comparing the (composition) length of the semisimple R-modules above, we obtain $X' = J(P)$ and $Y = J(Y)$, i.e., $Y = 0$. Thus, $P/X \cong P$, whence by the projectivity of P, X is a direct summand of P. But then, $X \subset J(P)$, from which we conclude that $X = 0$. $\square$

Recall that a right R-module is *co-faithful* if $R_R \lesssim M^{(n)}$ for some positive integer n, or equivalently there exist finitely-many $x_1, \ldots, x_n \in M$ such that $(0 : x_1, \ldots, x_n) = 0$. Note also an easy fact that for right R-modules M and N with $N \lesssim M$, there exists a homomorphic image of M in $E(N)$ containing N.

Lemma 1.5. *For a ring R, the following conditions are equivalent:*

(1) *R is right FPF;*

(2) (i) *Every cyclic faithful right R-module is co-faithful;*

(ii) *Every finitely generated submodule of $E(R_R)$ containing R is a generator for* Mod-R.

Proof. (1) $\Rightarrow$ (2). Obvious.

(2) $\Rightarrow$ (1). Let $M = \sum_{i=1}^{n} x_i R$ be a finitely generated faithful right R-module. Set $I = (0 : x_1, \dots, x_n)$. Then, R/I is faithful and $R/I \lesssim M^{(n)}$, whence by (2)(i), $R_R \lesssim M^{(mn)}$ for some integer m. Thus, $M^{(mn)}$ has a homomorphic image F in $E(R)$ containing R. It then follows from (2)(ii) that F, and hence M, is a generator for Mod-R. Therefore, R is right FPF. □

In general, infinite direct products of FPF rings are not FPF. For example, the direct product of simple artinian rings R_n ($n = 1, 2, \dots$) of length n is not FPF. However, as a consequence of the lemmas above, we have the following results for semiperfect rings (c.f. [9, Corollary 18]).

Proposition 1.6. *Let R_λ be a semiperfect ring with basic idempotent e_λ for $\lambda \in \Lambda$. Then, the ring $\prod_{\lambda\in\Lambda} R_\lambda$ is right FPF if and only if* (i) *each R_λ is right FPF, and* (ii) *there exists a positive integer n such that $R_\lambda \lesssim_\oplus (e_\lambda R_\lambda)^{(n)}$ for all $\lambda \in \Lambda$.*

Proof. Set $R = \prod R_\lambda$, $E_\lambda = E(R_{\lambda_{R_\lambda}})$ and $E = \prod E_\lambda$, the injective hull of R_R.

"If part". Assume (i) and (ii). To prove that R is right FPF, we show Lemma 1.5(2). First, let I be a right ideal of R such that R/I is faithful. For each $\lambda \in \Lambda$, let $p_\lambda : R \to R_\lambda$ be the λ-th projection, and set $I_\lambda = p_\lambda(I)$. Then each R_λ-module R_λ/I_λ is faithful, whence by the assumption and Lemma 1.3, $R_\lambda \lesssim (R_\lambda/I_\lambda)^{(n)}$. Thus, $R \lesssim (R/\prod I_\lambda)^{(n)}$, and hence $R \lesssim (R/I)^{(n)}$, i.e., R/I is co-faithful. Next, let M be a finitely generated submodule of E containing R. Then, $M = \prod M_\lambda$, where each M_λ is a finitely generated submodule of E_λ containing R_λ. By the assumption and Lemma 1.3, $R_\lambda \lesssim_\oplus M_\lambda^{(n)}$ so that $R \lesssim_\oplus M^{(n)}$, i.e., M is a generator for Mod-R. Therefore, R is right FPF.

"Only if part". This part immediately follows from the fact that any direct summand of an FPF ring is FPF, and the one that the condition (ii) is equivalent to the condition that the cyclic faithful R-module $(e_\lambda)R$ is a generator for Mod-R. □

Lemma 1.7 (c.f. [7, Theorem 1], [21, Theorem 2.1]). *Let R be a semiperfect ring with basic idempotent e such that for every $x \in E(eR)$, $eR+xeR$ is a generator for* Mod-*R. Then,*

(1) *R has finite uniform dimension as a right R-module.*

(2) *Every finitely generated submodule of $E(eR)$ containing eR is isomorphic to eR.*

Proof. Set $E = E(eR)$, and let the basic idempotent e be expressed as

$$e = e_1 + \cdots + e_k,$$

where $e_1, \dots, e_k$ is a basic set of primitive idempotents of R.

(1) It suffices to show that the right R-module e_iR is uniform for each $i = 1, \dots, k$. So, let I, J be submodules of e_iR with $I \cap J = 0$, and set

$$X = (eR/I) \oplus (eR/J).$$

Then the R-homomorphism

$$\varphi : eR \to X : a \mapsto (a + I, a + J)$$

is monic and

$$X = \varphi(eR) + (e + I, 0)R.$$

Extending the inclusion map $eR \to E$ to an R-homomorphism $\psi : X \to E$ through φ, we have

$$\operatorname{Im} \psi = eR + \psi(e + I, 0)eR.$$

By hypothesis, $\operatorname{Im} \psi$, and hence X, is a generator, whence by Lemma 1.3 we obtain either $e_iR \lesssim_\oplus eR/I$ or $e_iR \lesssim_\oplus eR/J$. We may assume the first case so that

$$e_iR \lesssim_\oplus eR/I \cong (e_iR/I) \oplus (e_1R \oplus \cdots \oplus e_{i-1}R \oplus e_{i+1}R \oplus \cdots \oplus e_kR),$$

from which $e_iR \lesssim_\oplus e_iR/I$, because e_iR is not isomorphic to e_jR for $i \neq j$. It then follows from Lemma 1.4 that $I = 0$. Thus, e_iR is uniform, as desired.

(2) Let M be a finitely generated submodule of E containing eR. Since eR is a generator for Mod-R, there exist $x_1, \dots, x_n \in M$ such that

$$M = x_1eR + \cdots + x_neR, \text{where } x_1 = e.$$

If $n = 1$, then the result is obvious. Assume that $n > 1$ and that there exists an R-isomorphism

$$\varphi : x_1eR + \cdots + x_{n-1}eR \to eR.$$

Then we may extend φ to an R-monomorphism $M \to E$ so that

$$\varphi(M) = eR + \varphi(x_ne)eR.$$

Since by hypothesis, $\varphi(M) \cong M$ is a generator, it follows from Lemma 1.3 that $eR \lesssim_\oplus M$. But then, by (1), eR and M have the same finite uniform dimension, from which we obtain $eR \cong M$. $\square$

Now, we prove Theorem 1.1.

Proof of Theorem 1.1. Set $E = E(eR)$.

(1) $\Rightarrow$ (2). Obvious.

(2) $\Rightarrow$ (3). Assuming (2), we see that (3)(i) immediately follows from Lemma 1.3 and 1.4, while (3)(ii) is obvious.

(3) $\Rightarrow$ (4). Assume (3). Then one only needs to prove that eR is the unique finitely generated faithful submodule of $E(eR)$, to within isomorphism. So, let M be a finitely generated faithful submodule of E. Then there exist $x_1, \ldots, x_n \in M$ such that

$$M = x_1 eR + \cdots + x_n eR.$$

We set

$$I = eR \cap (0 : x_1, \ldots, x_n).$$

Since eR/I is faithful, (3)(i) implies that $I = 0$, from which the R-homomorphism

$$eR \to M^{(n)} : a \mapsto (x_1 a, \ldots, x_n a)$$

is monic. Thus, $M^{(n)}$ has a homomorphic image N in E containing eR. By (3)(ii) and Lemma 1.7(2), N, and hence M, is a generator, i.e., $eR \lesssim_{\oplus} M$, while by Lemma 1.7(1), E has finite uniform dimension. This shows that $eR \cong M$.

(4) $\Rightarrow$ (1). To this end, we use Lemma 1.5. The condition (2)(i) of the lemma follows from the proof of (3) $\Rightarrow$ (4) combined with Lemma 1.4, while (2)(ii) does from noting that every R-module containing R has a homomorphic image in E containing eR. $\square$

REMARK. Over semiperfect rings, 1-PF condition (i.e., every cyclic faithful module is a generator) does not imply FPF one in general. For example, any non-semihereditary local commutative domain is 1-PF, but not FPF (see [10, Corollary Part II, 1.9]).

2. Commutative FPF rings

Faith [10] has already given the following decisive characterization of commutative FPF rings.

Theorem ([10, Theorem Part II, 5.1]). *For a commutative ring R, the following conditions are equivalent:*

(1) *R is FPF;*

(2) *The classical quotient ring of R is self-injective, and every finitely generated faithful ideal of R is a generator for* Mod-R;

(3) *Every finitely generated submodule of $E(R)$ containing R and every finitely generated faithful ideal of R is a generator for* Mod-R;
(4) *Every finitely generated submodule of $E(R)$ containing R and every finitely generated faithful ideal of R is projective.*

In this section, we present a characterization of commutative FPF rings, which sharpens a part of the theorem above and is concerned with the noted in § 0.

For a ring R, we denote by $B(R)$ the set of all central idempotents in R, which forms a *Boolean algebra* with the join $e \vee f = e+f-ef$ and the meet $e \wedge f = ef$, and by $\mathfrak{X}(R)$ the set of all maximal ideals of the Boolean algebra $B(R)$. Let X be an R-module and x an element of X. For each $m \in \mathfrak{X}(R)$, we denote by X_m the (Pierce) *stalk* of X at m, i.e., the factor R-module X/Xm of X, and by x_m the image of x in X/Xm. If Y is a submodule of X, then we may naturally identify Y_m with the submodule $(Y+Xm)/Xm$ of X_m. Elementary results on $B(R)$, $\mathfrak{X}(R)$ and the stalks may be found in [22, Part I, § 1 $\sim$ 4].

Our aim in this section is to prove the following theorem.

Theorem 2.1. *Let R be a commutative ring with $E = E(R)$. Then the following conditions are equivalent:*
(1) *R is FPF;*
(2) *For every $x \in E$, $R + xR$ is a generator for* Mod-R;
(3) *For every $x \in E$, $R + xR$ is projective;*
(4) *For every $x \in E$, there exist $a \in (R : x)$ and $b \in R$ such that $a + xb = 1$;*
(5) *For every $m \in \mathfrak{X}(R)$ and $x_m \in E_m$, there exist $a_m \in (R_m : x_m)$ and $b_m \in R_m$ such that $a_m + x_m b_m = 1_m$.*

The theorem immediately implies the following result, which is shown in [17, Corollary 2] for commutative semiprime rings.

Corollary 2.2. *A commutative ring R is FPF if and only if every 2-generated faithful R-module is a generator for* Mod-R.

To prove the theorem, we provide several lemmas.

The following facts are elementary and well-known (e.g. [10, Part II, Chapt. 2, 3], [24, Chapt. XIV]).

Let R be a ring with Q the maximal right quotient ring, and set $E = E(R_R)$, $S = \mathrm{End}_R(E)$ and $J = J(S)$. Then:
(1) $J = \{\varphi \in S \mid \mathrm{Ker}\,\varphi$ is essential in $E_R\}$, and S/J is a (von Neumann) regular and right self-injective ring.
(2) $J1_R = JE = Z(E)$.
(3) $Q = \{x \in E \mid$ for every $a \in R, (R : xa)$ has zero left annihilator in $R\}$.

(4) Q is right self-injective if and only if $Q = E$ if and only if for every $x \in E$, $(R : x)$ has zero left annihilator in R.

(5) If R is a commutative ring and M is a finitely generated R-module that is generated by n elements $x_1, \dots, x_n$, then M is a generator for Mod-R if and only if there exist $\varphi_1, \dots, \varphi_n \in \mathrm{Hom}_R(M, R)$ such that $\sum_{i=1}^n \varphi_i(x_i) = 1$.

Lemma 2.3. *Let R be a commutative ring such that $Z(R) = Z(E(R))$. If M is a submodule of $E(R)$ containing R, then*

$$M(R : M) = \mathrm{Tr}_R(M),$$

where $\mathrm{Tr}_R(M)$ is the trace ideal of M.

Proof. Set $E = E(R)$, $S = \mathrm{End}_R(E)$ and $J = J(S)$. It is then immediate that $M(R : M) \subset \mathrm{Tr}_R(M)$. To the converse, let $\varphi \in \mathrm{Hom}_R(M, R)$, and set

$$a = \varphi(1) \text{ and } \psi = \varphi - a^*,$$

where φ is extended to an R-endomorphism of E, and $a^* : E \to E$ is the multiplication map of a. Since $R \subset \mathrm{Ker}\,\psi$, it follows that $\psi \in J$, from which

$$\psi(M) \subset JE = Z(E) = Z(R),$$
$$Ma \subset \varphi(M) - \psi(M) \subset R, \text{ i.e., } a \in (R : M).$$

Thus we have

$$\varphi(M) \subset M(R : M) + Z(R).$$

On the other hand,

$$MZ(R) \subset Z(E) = Z(R), \text{ i.e., } Z(R) \subset (R : M).$$

Therefore we obtain $\varphi(M) \subset M(R : M)$, which shows that $\mathrm{Tr}_R(M) \subset M(R : M)$, as desired. □

Lemma 2.4. *Let E be a (right) R-module extension of a ring R such that for every $x \in E$, there exist $a \in (R : x)$ and $b \in R$ for which $a + xb = 1$. Let F be a submodule of E such that $E(R \cap F) \subset F$. Set $\overline{R} = (R + F)/F$ and $\overline{E} = E/F$, and denote by $\overline{x}$ the image of each element x of E in $\overline{E}$. Then,*

(1) *For every $\overline{x} \in \overline{E}$, there exist $\overline{a} \in (\overline{R} : \overline{x})$ and $\overline{b} \in \overline{R}$ such that $\overline{a} + \overline{x}\overline{b} = \overline{1}$.*

(2) *$\overline{R}$ is essential in $\overline{E}$ as an $\overline{R}$-module.*

(3) *If E is a ring extension of R and F is an ideal of E, then all idempotents of the ring $\overline{E}$ are contained in $\overline{R}$.*

Proof. (1) Immediate.

(2) and (3) Let $\overline{x}$ be a nonzero element of $\overline{E}$. Then, by (1) there exist $\overline{a} \in (\overline{R} : \overline{x})$ and $\overline{b} \in \overline{R}$ such that $\overline{a} + \overline{x}\overline{b} = \overline{1}$. If $\overline{x}\overline{b} = \overline{0}$, then $\overline{0} \neq \overline{x} = \overline{x}\overline{a} \in \overline{R}$. If otherwise, then $\overline{0} \neq \overline{x}\overline{b} = \overline{1} - \overline{a} \in \overline{R}$. Thus, $\overline{R}$ is essential in $\overline{E}_{\overline{R}}$.

If $\overline{x}$ is an idempotent of $\overline{E}$, then $\overline{x} = \overline{x}\overline{a} + \overline{x}\overline{b} \in \overline{R}$. □

For a ring R and $e \in B(R)$, we set

$$\mathcal{N}(e) = \{\boldsymbol{m} \in \mathfrak{X}(R) \mid e \notin \boldsymbol{m}\}.$$

Then it is well-known that $\mathfrak{X}(R)$ is a *Boolean space* in which

$$\{\mathcal{N}(e) \mid e \in B(R)\}$$

is the set of all clopen (= closed and open) sets, and that $\mathfrak{X}(R)$ has the following, so-called (see [22, p.12–13]), *partition property*.

Partition Property. *For every open covering $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ of $\mathfrak{X}(R)$, there exist finitely-many clopen sets $\mathcal{N}_1, \ldots, \mathcal{N}_k$ of $\mathfrak{X}(R)$ such that:*
(i) *for each $i = 1, \ldots, k$, there exists a $\lambda_i \in \Lambda$ for which $\mathcal{N}_i \subset \mathcal{O}_{\lambda_i}$;*
(ii) *$\mathfrak{X}(R) = \mathcal{N}_1 \cup \cdots \cup \mathcal{N}_k$;*
(iii) *$\mathcal{N}_i \cap \mathcal{N}_j = \emptyset$ for $i \neq j$.*

We use the following "sheaf theoretical" result as in [22, Proposition 3.4]. We briefly give its direct proof.

Lemma 2.5. *Let $M_1, \ldots, M_t$ be submodules of a (right) R-module M, and let $x_1, \ldots, x_s \in M$. Let $f_1, \ldots, f_n$ be integral polynomials in noncommuting variables $X_1, \ldots, X_s, Y_1, \ldots, Y_t$ such that for each $\boldsymbol{m} \in \mathfrak{X}(R)$, there exists a*

$$(y'_1, \ldots, y'_t) \in M_1 \times \cdots \times M_t$$

for which

$$f_j(x_{1\boldsymbol{m}}, \ldots, x_{s\boldsymbol{m}}, y'_{1\boldsymbol{m}}, \ldots, y'_{t\boldsymbol{m}}) = 0_{\boldsymbol{m}} \text{ (in } M_{\boldsymbol{m}} \text{ for } j = 1, \ldots, n).$$

Then there exists a

$$(y_1, \ldots, y_t) \in M_1 \times \cdots \times M_t$$

for which

$$f_j(x_1, \ldots, x_s, y_1, \ldots, y_t) = 0 \text{ (in } M \text{ for } j = 1, \ldots, n).$$

Proof. For each $m \in \mathfrak{X}(R)$, there exist a neighborhood $\mathcal{U}(m)$ of m and

$$(y'_1, \dots, y'_t) \in M_1 \times \cdots \times M_t$$

such that for each j and $m' \in \mathcal{U}(m)$,

$$f_j(x_{1_{m'}}, \dots, x_{s_{m'}}, y'_{1_{m'}}, \dots, y'_{t_{m'}}) = 0_{m'}.$$

Indeed, by hypothesis there exists a

$$(y'_1, \dots, y'_t) \in M_1 \times \cdots \times M_t$$

such that for each j,

$$f_j(x_{1_m}, \dots, x_{s_m}, y'_{1_m}, \dots, y'_{t_m}) = 0_m,$$

which means that there exist $z_j \in M$ and $e_j \in m$ such that

$$f_j(x_1, \dots, x_s, y'_1, \dots, y'_t) = z_j e_j.$$

We then only set $\mathcal{U}(m) = \cap_{j=1}^n \mathcal{N}(1 - e_j)$.

By Partition Property of $\mathfrak{X}(R)$, there exist finitely-many

$$e_1, \dots, e_k \in B(R),$$
$$(y_{11}, \dots, y_{1t}), \dots, (y_{k1}, \dots, y_{kt}) \in M_1 \times \cdots \times M_t$$

such that:

(i) for each $i = 1, \dots, k$, $f_j(x_{1_m}, \dots, x_{s_m}, y_{i1_m}, \dots, y_{it_m}) = 0_m$ for each j and $m \in \mathcal{N}(e_i)$;

(ii) $\mathfrak{X}(R) = \mathcal{N}(e_1) \cup \cdots \cup \mathcal{N}(e_k)$;

(iii) $\mathcal{N}(e_i) \cap \mathcal{N}(e_{i'}) = \emptyset$ for $i \neq i'$.

Note by (ii) and (iii) that $e_1 \vee \cdots \vee e_k = 1$ and $e_i \wedge e_{i'} = 0$ for $i \neq i'$. Now, for each $h = 1, \dots, t$, we set

$$y_h = y_{1h} e_1 + \cdots + y_{kh} e_k.$$

Then it is easy to see that for each j and $m \in \mathfrak{X}(R)$,

$$f_j(x_{1_m}, \dots, x_{s_m}, y_{1_m}, \dots, y_{t_m}) = 0_m.$$

Therefore we conclude that for each j,

$$f_j(x_1, \dots, x_s, y_1, \dots, y_t) = 0,$$

as desired. □

Lemma 2.6. *Let $R \subset Q$ be a ring extension such that $B(R) \subset B(Q)$. If for each $m \in \mathfrak{X}(R)$, the ring Q_m is the classical (right) quotient ring of R_m, then Q is the classical (right) quotient ring of R.*

Proof. First, let c be a regular element of R. Then it is immediate that for each $m \in \mathfrak{X}(R)$, the element c_m is regular in R_m and hence invertible in Q_m. Now, applying Lemma 2.5 (in which $M_1 = M = Q$) to the element c and the polynomials

$$f_1(X_1, Y_1) = X_1Y_1 - 1,\ f_2(X_1, Y_1) = Y_1X_1 - 1,$$

we see that c is invertible in Q.

Next, to see that every element of Q is of the form ab^{-1} for some a, $b \in R$, one only needs to apply Lemma 2.5 (in which $M_1 = M_2 = R$ and $M_3 = M = Q$) to any element $x \in Q$ and the following three polynomials (in variables X_1, Y_1, Y_2, Y_3):

$$X_1Y_1 - Y_2,\ Y_1Y_3 - 1,\ Y_3Y_1 - 1.$$ □

We denote by $Q_{cl}(R)$ the classical quotient ring of R.

Lemma 2.7. *Let $R \subset Q$ be a commutative ring extension such that for every $x \in Q$, there exist $a \in (R : x)$ and $b \in R$ for which $a + xb = 1$. Then,*

(1) *If I is an ideal of Q such that Q/I is a regular ring, then $Q/I = Q_{cl}((R + I)/I)$.*

(2) *If $Q_{cl}(R) \subset Q$ and Q is a regular ring modulo its Jacobson radical, then $Q = Q_{cl}(R)$.*

Proof. (1) According to Lemma 2.4 and 2.6, it suffices by passing through I to show that in case Q is a regular ring, $Q_m = Q_{cl}(R_m)$ for every $m \in \mathfrak{X}(R)$. By the regularity of Q and Lemma 2.4, the ring Q_m is a field and R_m is essential in Q_m as an R_m-module. Therefore, Q_m is the quotient field of R_m.

(2) Set $J = J(Q)$, $\overline{R} = (R + J)/J$ and $\overline{Q} = Q/J$, and denote by $\overline{x}$ the image of each element x of Q in $\overline{Q}$. If $x \in J$, then by hypothesis there exist $a \in (R : x)$ and $b \in R$ such that $a + xb = 1$. Since $a = 1 - xb$ is invertible in Q, the element a ($\in (R : x)$) is regular in R, which shows that $x \in Q_{cl}(R)$. Thus we obtain $J \subset Q_{cl}(R)$.

Now, let $y \in Q$ be arbitrary. By (1), $\overline{Q} = Q_{cl}(\overline{R})$, whence there exist c, $d \in R$ and $q \in Q$ such that $yc - d$, $cq - 1 \in J$. Since cq is invertible in Q, c is regular in R. Thus, $y \in dc^{-1} + Jc^{-1} \subset Q_{cl}(R)$. Therefore we conclude that $Q = Q_{cl}(R)$. □

The following lemma is well-known (see [2, Theorem 8] and [1, Proposition 17.9]).

Lemma 2.8. *Every finitely generated faithful projective module over a commutative ring R is a generator for* Mod-R.

An ideal of a ring R is called a *regular ideal* if it contains a regular element of R.

Griffin [15] characterized commutative rings in which every finitely generated regular ideal is invertible, i.e., projective (see [24, Proposition II, 4.3]). The following lemma may be obtained by the proof of (10) $\Rightarrow$ (1) of his theorem.

Lemma 2.9 ([15, Theorem 13]). *For a commutative ring R, the following conditions are equivalent:*
(1) *Every finitely generated regular ideal of R is projective;*
(2) *For every a, $b \in R$ such that a is regular in R, the ideal $aR + bR$ is projective.*

Now, we are in position to prove Theorem 2.1.

Proof of Theorem 2.1. Throughout the proof, let Q denote the maximal quotient ring of R.

(1) $\Rightarrow$ (2). Obvious.

(2) $\Rightarrow$ (3) and (4). First we show the following.

CLAIM. Q is a self-injective ring, i.e., $Q = E$.

Proof of Claim. First note that $E = E(Q_Q)$, and by (2) that for every $x \in E$, the Q-module $Q + xQ$ is a generator for Mod-Q, because every R-homomorphism $R + xR \to R$ may be extended to a Q-homomorphism $Q + xQ \to Q$. Thus, to prove the claim, we may assume that $R = Q$. This then implies that

$$Z(R) = Z(E).$$

Indeed, obviously, $Z(R) \subset Z(E)$. To the converse, let $x \in Z(E)$. Then there exists a $\theta \in J(\mathrm{End}_R(E))$ such that $x = \theta(1)$, while by hypothesis, the R-module $R + xR$ is a generator; hence there exist φ, $\psi \in \mathrm{Hom}_R(R + xR, R)$ such that

$$\varphi(1) + \psi(x) = 1.$$

Extending φ and ψ to R-endomorphisms of E, we see that $\varphi(1) = (1 - \psi\theta)(1)$ and $1 - \psi\theta$ is invertible in $\mathrm{End}_R(E)$. Consequently, $\mathrm{Ker}\,\varphi \cap R = 0$, i.e., φ is monic. Since $\varphi(x\varphi(1) - \varphi(x)) = 0$ and hence $x\varphi(1) = \varphi(x) \in R$, it follows that $(R : x)$ contains the regular element $\varphi(1)$. In particular, $(R : x)$ has zero annihilator in R, i.e, $x \in R$ (because $R = Q$). Thus, we conclude that $Z(R) = Z(E)$, as desired.

Now, let $y \in E$ be arbitrary. Then, the R-module $R + yR$ is a generator, whence by Lemma 2.3 we obtain $(R+yR)(R:y)=R$. Therefore, $(R:y)$ has zero annihilator in R, i.e, $y \in R$, which completes the proof of Claim. $\square$

To show (4), let $x \in E$. Then by (2) there exist φ_1, $\varphi_2 \in \mathrm{Hom}_R(R+xR,R)$ such that

$$\varphi_1(1)+\varphi_2(x)=1.$$

Since by Claim each φ_i may be extended to a Q-endomorphism of Q, it follows that $x\varphi_i(1)=\varphi_i(x)\in R$, i.e.,

$$\varphi_i(1)\in(R:x) \quad \text{and} \quad \varphi_1(1)+x\varphi_2(1)=1,$$

which means (4).

To show (3), one only needs to use Dual Basis Lemma for projective modules.

$(3)\Rightarrow(2)$. This follows from Lemma 2.8.

$(4)\Rightarrow(1)$. It follows immediately from (4) that for every $x \in E$, the annihilator of $(R:x)$ is zero, i.e., $E=Q$, a self-injective ring, and that R satisfies (2) and hence (3). Thus by Lemma 2.7 we have $Q=Q_{cl}(R)$, while we see by Lemma 2.9 that every finitely generated regular ideal of R is projective.

Now, to show that R is FPF, it suffices by Lemma 1.5 to show that every finitely generated submodule of $E=Q_{cl}(R)$ containing R is a generator. So, let M be such a submodule. Then there exists a regular element c of R such that $Mc \subset R$; hence Mc is a finitely generated regular ideal of R. As mentioned above, Mc is projective, whence by Lemma 2.8, $M \cong Mc$ is indeed a generator. Therefore, R is an FPF ring.

$(4)\Rightarrow(5)$. This follows from Lemma 2.4.

$(5)\Rightarrow(4)$. One only needs to apply Lemma 2.5 (in which $M_1=M_2=M_3=R$ and $M=E$) to any element $x\in E$ and the polynomials

$$f_1(X_1,Y_1,Y_2,Y_3)=Y_1+X_1Y_2-1,\ f_2(X_1,Y_1,Y_2,Y_3)=X_1Y_1-Y_3. \qquad \square$$

Recall that a domain R is *Prüfer* if every finitely generated ideal of R is projective. Note also that every commutative semiprime ring R has the regular and self-injective maximal quotient ring that is the injective hull of R.

As consequences of Theorem 2.1, we obtain the following corollaries.

Corollary 2.10. *Let R be a commutative ring with Q the maximal quotient ring. Then the following conditions are equivalent:*

(1) *R is a semiprime FPF ring;*

(2) *For every $m \in \mathfrak{X}(R)$, R_m is a Prüfer domain with Q_m the quotient field.*

Proof. Note by Lemma 2.8 and Dual Basis Lemma that a domain is FPF if and only if it is Prüfer.

(1) $\Rightarrow$ (2) follows from Theorem 2.1 and the proof of Lemma 2.7(1).

(2) $\Rightarrow$ (1) follows from Theorem 2.1 and an easy fact that if R_m is a domain for every $m \in \mathfrak{X}(R)$, then R is a semiprime ring. $\square$

REMARK 1. In the corollary above, we may not drop the condition "*with Q_m the quotient field*" of (2). For example, let R be a non-self-injective commutative (von Neumann) regular ring. Then, R_m is a field for every $m \in \mathfrak{X}(R)$, but R is not FPF.

REMARK 2. Let R be a commutative ring with $E = E(R)$ and consider the following two conditions for R:

(1) *R is FPF*;

(2) *For every $m \in \mathfrak{X}(R)$, R_m is an FPF ring with E_m the R_m-injective hull.*

Then, Theorem 2.1 shows that (2) $\Rightarrow$ (1), while we do not know whether the converse holds in general. However, we see by Theorem 2.1 and [24, Proposition XI, 3.11] that this is equivalent to the following:

($*$) *Every stalk Q_m of an arbitrary self-injective commutative ring Q is also a self-injective ring.*

The condition (4) of Theorem 2.1 immediately implies the following.

Corollary 2.11 ([9, Corollary 18], [10, Proposition Part II, 2.9]).

(1) *Let R_λ be a commutative ring for $\lambda \in \Lambda$. Then, the ring $\prod_{\lambda \in \Lambda} R_\lambda$ is FPF if and only if each R_λ is FPF.*

(2) *Let R be a commutative FPF ring with Q the maximal quotient ring. Then every subring of Q containing R is FPF.*

3. Additional results on commutative FPF rings

In this section, by using Theorem 2.1 we present another characterization of commutative FPF rings.

Recall that a ring homomorphism $\varphi : R \to S$ is a *(right) flat epimorphism* if φ is an epimorphism in the category of rings and S is flat as a (left) R-module.

The following is well-known (c.f. [24, Theorem XI, 2.1]).

Lemma 3.1 ([23, Théorème 2.7]). *For a ring extension $R \subset S$, the following conditions are equivalent*:

(1) *The inclusion map $R \to S$ is a right flat epimorphism*;

(2) *For every $x \in S$, $(R : x)S = S$.*

We obtain the following theorem, which somewhat generalizes [11, § 8, Proposition, (the first) Corollary].

Theorem 3.2. *For a commutative ring R, the following conditions are equivalent*:

(1) *R is FPF*;

(2) *Every finitely generated submodule M of $E(R)$ containing R is invertible, i.e., $M(R:M) = R$*;

(3) *The maximal quotient ring Q of R is self-injective, and for every subring S of Q containing R, the inclusion map $R \to S$ is a flat epimorphism*;

(4) *The maximal quotient ring Q of R is self-injective, and every subring of Q containing R is integrally closed in Q.*

Proof. (1) $\Rightarrow$ (2). Let M be a finitely generated submodule of $E(R)$ ($= Q$, the maximal quotient ring of R by (1)) containing R. Since M is a generator, there exist $x_1, \ldots, x_n \in M$, $\varphi_1, \ldots, \varphi_n \in \mathrm{Hom}_R(M, R)$ such that

$$\varphi_1(x_1) + \cdots + \varphi_n(x_n) = 1.$$

Extending each φ_i to a Q-endomorphism of Q, we obtain

$$1 = x_1\varphi_1(1) + \cdots + x_n\varphi_n(1) \in M(R:M).$$

Thus, $M(R:M) = R$.

(2) $\Rightarrow$ (3). Given any $x \in E$, we see by (2) that $(R + xR)(R:x) = R$. This combined with Lemma 3.1 implies (3).

(3) $\Rightarrow$ (4). Let S be a subring of Q containing R and $\overline{S}$ its integral closure in Q. Let $x \in \overline{S}$ be arbitrary. Applying Lemma 3.1 to the inclusion map $R \to \overline{S}$, we have $(R:x)\overline{S} = \overline{S}$, from which

$$(S:x)_S\overline{S} = \overline{S}, \text{ where } (S:x)_S = \{s \in S \mid xs \in S\}.$$

It then follows from the Lying Over Theorem (e.g. [16, Theorem 44]) that $(S:x)_S = S$, i.e., $x \in S$. Therefore, $S = \overline{S}$ is integrally closed in Q.

(4) $\Rightarrow$ (1). This follows from [5, Theorem 2] and Theorem 2.1. □

REMARK. Let R be a commutative ring with Q the maximal quotient ring. For each prime ideal P of R, we set

$$R_{[P]} = \{x \in Q \mid (R:x) \cap (R - P) \neq \emptyset\},$$
$$[P] = \{y \in Q \mid (P:y) \cap (R - P) \neq \emptyset\}.$$

It then follows from Theorem 3.2 and [5, Theorem 2] that the following conditions for R are equivalent:

(1) *R is FPF*;

(2) *Q is a self-injective ring on which each $(R_{[P]}, [P])$ is a valuation pair in the sense of* Manis [18], [19].

The theorem above implies the following corollary in which (1) $\Leftrightarrow$ (3) is [11, § 8, (the second) Corollary].

Corollary 3.3. *Let R be a commutative semiprime ring with Q the maximal quotient ring. Then the following conditions are equivalent*:

(1) *R is FPF*;

(2) *For every subring S of Q containing R, the inclusion map $R \to S$ is a flat epimorphism*;

(3) *Every subring of Q containing R is integrally closed in Q.*

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