

Title	Immersion and embeddings of orbit manifolds $D_p(l, m)$ of $S^{2l+1} \times S^m$ by dihedral group D_p
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Citation	Osaka Journal of Mathematics. 12(2) P.299-P.304
Issue Date	1975
Text Version	publisher
URL	https://doi.org/10.18910/6010
DOI	10.18910/6010
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IMMERSIONS AND EMBEDDINGS OF ORBIT MANIFOLDS $D_p(l, m)$ OF $S^{2l+1} \times S^m$ BY DIHEDRAL GROUP D_p

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(Received January 31, 1974)

(Revised November 5, 1974)

1. Let a dihedral group D_p , p an odd prime, act on $S^{2l+1} \times S^m$ by $g^{it^j}(z, x) = (\rho^i c^j(z), (-1)^j x)$ where g and t are generators of order p and 2 respectively, $c(z)$ is the conjugate point of z and $\rho = \exp 2\sqrt{-1}/p$. We denote by $D_p(l, m)$ the orbit space [6].

Let $\phi(m)$ = the number of integers s with $0 < s \leq m$ and $s \equiv 0, 1, 2$ or $4 \pmod{8}$;

$\bar{\sigma}(l, m)$ = the largest integer s with $2^{s-1} \binom{l+m+s+1}{s} \not\equiv 0 \pmod{2^{\phi(m)}}$;

$L(l, p)$ = the largest integer s with $s \leq \left\lfloor \frac{l}{2} \right\rfloor$ and $\binom{l+s}{s} \not\equiv 0 \pmod{p^{1+\lfloor l-2s/p-1 \rfloor}}$;

$\sigma^*(l, m) = \begin{cases} \max(\bar{\sigma}(l, m), 2L(l, p)) & \text{if } m > 0, \\ 2L(l, p) & \text{if } m = 0. \end{cases}$

In this paper we obtain

Theorem 1.1.

- (i) $D_p(l, m)$ cannot be immersed in $R^{2l+m+\sigma^*(l, m)}$,
- (ii) $D_p(l, m)$ cannot be embedded in $R^{2l+m+\sigma^*(l, m)+1}$.

In §2, we discuss about $\widetilde{KO}(D_p(l, m))$, $l \not\equiv 0 \pmod{4}$. In §3, we study the tangent bundle of $D_p(l, m)$. In §4, the Grothendieck operators γ^i in $KO(D_p(l, m))$ are computed and Theorem 1.1 is proved.

The author wishes to express his thanks to Professors N. Ishikawa and M. Kamata for valuable suggestions and discussions.

2. Let G be a finite group. The symbol Λ will denote either O or U . Let $K\Lambda_G(X)$ be equivariant $K\Lambda$ -group of a G -space X . It is well-known that if the action of G is free then $K\Lambda_G(X) \cong K\Lambda(X/G)$. There is canonical homomorphism from the representation ring $RA(G)$ to $K\Lambda_G(X)$ which maps a representation space M over real field or complex field to an equivariant G -bundle $X \times M$. If X is a free G -space then there is a homomorphism

$$\pi_\Delta: R\Lambda(G) \rightarrow K\Lambda_G(X) \cong K\Lambda(X/G).$$

Let $a: H \rightarrow G$ be a homomorphism and $f: Y \rightarrow X$ be an equivariant map from an H -space Y to a G -space X , $f(hy) = \alpha(h)f(y)$. The equivariant map f induces the homomorphism

$$f_\Delta^!: K\Lambda_G(X) \rightarrow K\Lambda_H(Y) \quad [8].$$

We take a Z_2 -action on S^m , a Z_p -action on S^{2l+1} and a D_p -action on $S^{2l+1} \times S^m$ defined by $t x = -x$, $g z = \rho z$ and $g t(x, z) = (\rho z, -x)$ respectively [4]. These orbit space are an m -dimensional real projective space RP^m , a $(2l+1)$ -dimensional lens space $L^l(p)$ and $D_p(l, m)$. There exist equivariant maps

$$\begin{aligned} i: S^{2l+1} &\rightarrow S^{2l+1} \times S^m, & i(z) &= (z, (1, 0, \dots, 0)), \\ j: S^m &\rightarrow S^{2l+1} \times S^m, & j(x) &= ((1, 0, \dots, 0), x) \end{aligned}$$

and

$$p: S^{2l+1} \times S^m \rightarrow S^m, \quad p(z, x) = x$$

compatible with injections $\tilde{i}: Z_p \rightarrow D_p$, $\tilde{j}: Z_2 \rightarrow D_p$ and a projection $p: D_p \rightarrow Z_2$ respectively. It follows immediately that $j_\Delta^! p_\Delta^! = 1$.

Let H be a normal subgroup of a finite group G and A be a representation of H . Denote by A^G the induced representation (§2 in [4]). Throughout this section, we suppose that $l \equiv 0 \pmod 4$. Consider the following commutative diagram

$$\begin{array}{ccc} \widetilde{RU}(Z_p) & \xrightarrow{\pi_U} & \widetilde{RU}(L^l(p)) \\ \downarrow r_1 & & \downarrow r \\ \widetilde{RO}(Z_p) & \xrightarrow{\pi_\Delta} & \widetilde{KO}_{Z_p}(S^{2l+1}) \cong \widetilde{KO}(L^l(p)) \end{array}$$

where r_1, r are real restrictions. It follows from T. Kambe [5] and N. Mohammed [7], that r and π_U are surjective. Therefore we have that π_Δ is surjective. We define the homomorphism

$$i_*^{\Delta}: \widetilde{K}\Lambda_{Z_p}(S^{2l+1}) \rightarrow \widetilde{K}\Lambda_{D_p}(S^{2l+1} \times S^m)$$

by $i_*^{\Delta}(S^{2l+1} \times M) = S^{2l+1} \times S^m \times M^{D_p}$ where M is a representation space of Z_p and M^{D_p} is the induced representation space.

Lemma 2.1. *The following diagram is commutative.*

$$\begin{array}{ccc} \widetilde{KU}_{Z_p}(S^{2l+1}) & \xrightleftharpoons{i_*^U} & \widetilde{KU}_{D_p}(S^{2l+1} \times S^m) \\ \uparrow c & & \uparrow c \\ \widetilde{KO}_{Z_p}(S^{2l+1}) & \xrightleftharpoons{i_*^O} & \widetilde{KO}_{D_p}(S^{2l+1} \times S^m) \end{array}$$

where c is the complexification.

Proof. This lemma is obtained by the naturality of complexification c and $(cM)^{D_p} = c(M^{D_p})$, where M is a representation space of Z_p over real field. q.e.d.

Proposition 2.2. For $\eta \in \widetilde{KO}_{Z_p}(S^{2l+1})$,

$$i_{\partial}^1 i_{*}^0(\eta) = 2\eta .$$

Proof. Since r is surjective, there is β in $\widetilde{KU}_{Z_p}(S^{2l+1})$ with $r(\beta) = \eta$. $cr(\beta) = \beta + \bar{\beta}$ is an element of $KU_{Z_p}(S^{2l+1})^{Z_2}$ which is a subgroup of $KU_{Z_p}(S^{2l+1})$ consisting of elements fixed under the conjugation automorphism. We obtain $i_{\partial}^1 i_{*}^0(cr(\beta)) = 2cr(\beta)$ (cf [4] Proposition 2.1). By Lemma 2.1, $c i_{\partial}^1 i_{*}^0(\eta) = c(2\eta)$. Since c is injective, we have $i_{\partial}^1 i_{*}^0(\eta) = 2\eta$. q.e.d.

By the same way as Theorem 2.2 in [4], we have the following.

Theorem 2.3. The homomorphism

$$\theta: \widetilde{KO}_{Z_p}(S^{2l+1}) \oplus \widetilde{KO}_{Z_2}(S^{m-1}) \rightarrow \widetilde{KO}_{D_p}(S^{2l+1} \times S^m)$$

given by $\theta(\eta, \nu) = i_{*}^0(\eta) + p_{\partial}^1(\nu)$ is injective.

3. Consider maps $i: L^l(p) \rightarrow D_p(l, m)$, $j: RP^m \rightarrow D_p(l, m)$ and $p: D_p(l, m) \rightarrow RP^m$ which are induced by i, j and p in §2. Let $\pi: L^l(p) \rightarrow CP^l$ be a canonical projection. Denote by η and ξ the canonical line bundles over the complex projective space CP^l and the real projective space RP^m respectively.

Proposition 3.1. (cf. [3], [9]). There is a real 2-plane bundle η_1 over $D_p(l, m)$ satisfying the following conditions:

- (i) $i^1 \eta_1$ is equivalent to $r\pi^1 \eta$,
- (ii) η_1 for $l=0$ is the 2-plane bundle $1 \oplus p^1 \xi$,
- (iii) $\eta_1 \otimes p^1 \xi$ is equivalent to η_1 ,
- (iv) $j^1 \eta_1$ is equivalent to $1 \oplus \xi$,

where r is the real restriction.

Proof. Each point of $D_p(l, m)$ can be represented by $[z, x]$ under the identification $(z, x) = (\rho^k c(z), -x)$ for $z \in S^{2l+1} \subset C^{l+1}$, $x \in S^m \subset R^{m+1}$. Then the total space $E(\eta_1)$ of η_1 is defined as set of all triples $[(z, x), y]$ under the identification $((z, x), y) = ((\rho^k c(z), -x), \rho^k y)$, where $y \in C$ and z, x are as above. Let $U_{\alpha\beta}$ be the set of points $[z, x]$ of $D_p(l, m)$ such that z_{α} and X_{β} are non-zero. $\{U_{\alpha\beta}: \alpha=0, 1, \dots, l; \beta=0, 1, \dots, m\}$ is an open covering of $D_p(l, m)$. The projection of the bundle η_1 is given by $p_1([(z, x), y]) = [z, x]$

Define $\phi_{\alpha\beta}: U_{\alpha\beta} \times R^2 \rightarrow p_1^{-1}(U_{\alpha\beta})$ by

$$\phi_{\alpha\beta}([z, x], y) = \begin{cases} [(z, x), z_\alpha y] & \text{if } x_\beta > 0, \\ [(z, x), z_\alpha \bar{y}] & \text{if } x_\beta < 0. \end{cases}$$

Then $\phi_{\alpha\beta}$ is a chart of η_1 over $U_{\alpha\beta}$ and the transition functions are given as follows:

$$g_{\langle\gamma, \delta\rangle\langle\alpha, \beta\rangle}[z, x] = \begin{cases} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} & x_\beta, x_\delta > 0, \\ \begin{pmatrix} a & -b \\ -b & -a \end{pmatrix} & x_\beta > 0, x_\delta < 0, \\ \begin{pmatrix} a & -b \\ b & -a \end{pmatrix} & x_\beta < 0, x_\delta > 0, \\ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} & x_\beta, x_\delta < 0, \end{cases}$$

where $z_\alpha/z_\gamma = a + b\sqrt{-1}$, $a, b \in R$.

Let $U_\alpha = \{[z_0, \dots, z_l] : z_\alpha \neq 0\} \subset L^l(p)$, then the transition function of η , $g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow GL(1, C)$, is given by $g_{\alpha\beta}[z] = z_\beta/z_\alpha$. Take $V_\alpha = \{[x_0, \dots, x_m] : x_\alpha \neq 0\} \subset RP^m$, then the transition function of ξ , $g'_{\alpha\beta} : V_\alpha \cap V_\beta \rightarrow O(1)$, is given by the following

$$g'_{\alpha\beta}[x] = \begin{cases} 1 & x_\alpha x_\beta > 0, \\ -1 & x_\alpha x_\beta < 0. \end{cases}$$

Therefore, we can complete the proof. q.e.d.

Let $\tau(M)$ denote the tangent bundle of manifold M .

Proposition 3.2.

$$\tau(D_p(l, m)) \oplus 1 \cong p^1 \tau(RP^m) \oplus (l+1)\eta_1.$$

Proof. Let $\langle | \rangle$ and $(|)$ denote the real and complex inner products of R^{m+1} and C^{l+1} respectively. The total space of the real tangent vector bundle of $D_p(l, m)$ can be represented as the set of all pairs $[(z, x), (u, v)]$ with $z \in S^{2l+1}$, $x \in S^m$, $u \in C^{l+1}$, $v \in R^{m+1}$, $(z|u) = 0$ or $u = r\sqrt{-1} \cdot z$ for some $r \in R$, and $\langle x|v \rangle = 0$, under the identification $((z, x), (u, v)) = ((\rho^k \bar{z}, -x), (\rho^k \bar{u}, -v))$. We have the following decomposition

$$\tau(D_p(l, m)) = p^1 \tau(RP^m) \oplus \xi,$$

where the total space $E(\xi)$ of ξ is the set of all triple $[(z, x), u]$ with $(z|u) = 0$ or $u = r\sqrt{-1} \cdot z$ for some $r \in R$, under the identification $((z, x), u) = ((\rho^k \bar{z}, -x), \rho^k \bar{u})$ in $S^{2l+1} \times S^m \times C^{l+1}$. The total space $E((l+1)\eta_1)$ of the $(l+1)$ -fold bundle sum $(l+1)\eta_1$ can be represented as the set of all triple $[(z, x), u]$ with the identification $((z, x), u) = ((\rho^k \bar{z}, -x), \rho^k \bar{u})$ in $S^{2l+1} \times S^m \times C^{l+1}$. Then we have $E((l+1)\eta_1)$

$\supset E(\zeta)$. Consider the trivial line bundle θ over $D_p(l, m)$ whose total space is represented as $[(z, x), t z]$ modulo the identification $((z, x), t z) = ((\rho^k \bar{z}, -x), t \rho^k \bar{z})$, where $z \in S^{2l+1}$, $x \in S^m$ and $t \in R$. Then we have

$$\tau(D_p(l, m)) \oplus \theta \cong p^1 \tau(RP^m) \oplus (l+1)\eta_1. \quad \text{q.e.d.}$$

4. We use λ^i - and γ^i -operations in KO -theory [2] to study the immersion and embedding of the manifold $D_p(l, m)$. Put $\tau_0(D_p(l, m)) = \tau(D_p(l, m)) - (2l + m + 1)$ and $x = p^1 \xi - 1$, $z = \eta_1 - 2$ and $y = z - x (= \eta_1 - 1 - p^1 \xi)$ in $\widetilde{KO}(D_p(l, m))$. It follows from Proposition 3.2, that

$$-\tau_0(D_p(l, m)) = -(m+1)x - (l+1)z = -(l+m+2)x - (l+1)y$$

and

$$(4.1) \quad \gamma_t(-\tau_0(D_p(l, m))) = \gamma_t(x)^{-(l+m+2)} \gamma_t(y)^{-(l+1)}.$$

Then, we have the following.

- Lemma 4.1.** (i) $\gamma_t(z) = 1 + zt - yt^2$,
 (ii) $x \cdot y = 0$, $\gamma_t(x) = 1 + xt$,
 (iii) $\gamma_t(y) = 1 + yt - yt^2$.

Proof. (i) Comparing the transition functions $\lambda^2(\eta_1)$ with one of $p^1 \xi$, we have $\lambda^2(\eta_1) \cong p^1 \xi$. Therefore we obtain $\lambda_t(\eta_1) = 1 + \eta_1 t + p^1 \xi t$ and $\lambda_t(z) = \lambda_t(\eta_1 - 2) = \lambda_t(\eta_1) \lambda_t(1)^{-2} = (1 + \eta_1 t + p^1 \xi t)^2 / (1 + t)^2$. Hence, $\gamma_t(z) = \lambda_{t/(1-t)}(z) = 1 + zt - yt^2$.

(ii) We note that $p^1 \xi \otimes p^1 \xi \cong 1$. And recall $p^1 \xi \otimes \eta_1 \cong \eta_1$, from Proposition 3.1, (iii). We have

$$x \cdot y = (p^1 \xi - 1) \cdot (\eta_1 - 1 - p^1 \xi) = 0$$

in $\widetilde{KO}(D_p(l, m))$. Since $\gamma_t(\xi - 1) = 1 + (\xi - 1)t$, we have

$$\gamma_t(x) = \gamma_t(p^1 \xi - 1) = 1 + xt.$$

(iii) Making use of the relation $x \cdot y = 0$, we have

$$\gamma_t(y) = \gamma_t(z - x) = \gamma_t(z) \gamma_t(x)^{-1} = 1 + yt - yt^2. \quad \text{q.e.d.}$$

Noting that $x^2 = -2x$, we obtain the following proposition from (4.1) and Lemma 4.1.

Proposition 4.2.

$$\gamma_t(-\tau_0(D_p(l, m))) = \sum_{s=0}^{\infty} \binom{m+l+s+1}{s} 2^{s-1} x t^s.$$

$$\left(\sum_{k=0}^{\infty} (-1)^k \binom{l+k}{k} y^k (t-t^2)^k \right).$$

Proof of Theorem 1.1.

Since $pj=1$, $j^!x=\xi-1$ is the generator of $\widetilde{KO}(RP^m)$ and $j^!x$ is of order $2^{p(m)}$ by J.F. Adams [1]. On the other hand, $i^!y=r(\pi^! \eta-1_C)$ by $i^!p^! \xi-1$ and Proposition 3.1. (i). By T. Kambe [5], $i^!y, i^!y^2, \dots, i^!y^{p-1/2}$ are additive generators of p -components of $\widetilde{KO}(L^!(p))$ and $i^!y^k$ is of order $p^{1+[(l-2k)/(p-1)]}$. We investigate the power of t having non-zero coefficient $\gamma^k(-\tau_0(D_p(l, m)))$ in the expansion of $\gamma_t(-\tau_0(D_p(l, m)))$ and apply the theorem of Atiyah [2] to the non-immersion and non-embedding of $D_p(l, m)$ in $R^{2l+m+k+1}$. Then, we obtain the theorem.

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