Title	The real K-groups of SO(n) for n≡3,4 and 5 mod 8
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Citation	Osaka Journal of Mathematics. 1988, 25(1), p. 185-211
Version Type	VoR
URL	https://doi.org/10.18910/6011
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Osaka J. Math.
25 (1988), 185–211

THE REAL K-GROUPS OF $SO(n)$ FOR $n\equiv 3, 4$ AND 5 MOD 8

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(Received December 5, 1986)

In [14] we determined the algebra structure of $KO^*(SO(n))$ for $n\equiv 0, 1, 7$ mod 8 assuming information about the K- and KO-groups of Spin(n) and P^{n-1}.

In this paper we compute $KO^*(SO(n))$ for $n\equiv 3, 4, 5$ mod 8 in the same way as in [14]. However in the present case some generators appear in dgree -5. So we first study the squares of elements in $KO^{-5}(X)$ following the method of Crabb [7] for elements in $KO^{-1}(X)$. We then provide a short exact sequence in KO_{Z_2}-theory similar to those in Lemma 4.1 of [14] which is a main tool for our computation.

We write $A\cdot g$ for an A-module with a single generator g throughout this paper.

1. Preliminaries

a) Let G be the multiplicative group consisting of ± 1. Denote by $R^{p,q}$ the $\boldsymbol{R}^{p+q}$ with non trivial G-action on the first p coordinates, and denote by $B^{p,q}$, $S^{p,q}$ and $\Sigma^{p,q}$ the unit ball and unit sphere in $R^{p,q}$ and the quotient space $B^{p,q}/S^{p,q}$ with the collapsed $S^{p,q}$ as base point respectively.

Let X be a compact G-space with base point. According to [12, 5], if $p+q\equiv 0$ mod 8 and $p\equiv 0$ mod 4, there is a Thom element $\omega_{p,q}\in\widetilde{KO}_G(\Sigma^{p,q})$, so that we have an isomorphism

$$\phi_{p,q}\colon \widetilde{KO}_G^*(X)\simeq\widetilde{KO}_G^*(\Sigma^{p,q}\wedge X)$$

given by $\phi_{p,q}(x)=\omega_{p,q}\wedge x$ for $x\in\widetilde{KO}_G^*(X)$ where $\wedge$ denotes the smash product.

We now specify the elements $\omega_{8p,0}$ and $\omega_{4,4}$. Let us take $\omega_{8p,0}$ to be the element ω_p^+ given in [14; p. 793]. Then we have

$$(1.1)\qquad i^*(\omega_{8p,0}) = 2^{4p-1}(1-H) \qquad in \quad \widetilde{KO}_G(\Sigma^{0,0})=RO(G)$$

where i is the inclusion of $\Sigma^{0,0}$ into $\Sigma^{8p,0}$ and $H=R^{1,0}$.

We may assume that $\psi(\omega_{8p,0})=1$ through the periodicity isomorphism, by replacing $\omega_{8p,0}$ by $-H\omega_{8p,0}$ if necessary. Here ψ denotes the forgetful homomorphism.

Applying KO_G-functor to the cofibering $S^{4,0} \xrightarrow{i} B^{4,0} \xrightarrow{p} \Sigma^{4,0}$ where i and p are obvious maps, we have an exact sequence

$$\begin{array}{ccccccc} \to \widetilde{KO}_G^{-4}(\Sigma^{4,0}) & \xrightarrow{p^*} & KO_G^{-4}(B^{4,0}) & \xrightarrow{i^*} & KO_G^{-4}(S^{4,0}) & \to \\ \| & & \| & & \wr\| & \\ RO(G)\cdot\omega_{4,4} & & RO(G)\cdot\eta_4 & & KO^{-4}(P^3) & \end{array}$$

in which η_4 is a generator of $KO^{-4}(+)\cong Z$ ($+$=point) and P^3 is the real projective 3-space. Since $\widetilde{KO}^{-4}(P^3)=0$ [8], from this we see that $i^*((1-H)\eta_4)=0$ and hence $p^*(\omega_{4,4})=(1-H)\eta_4$ up to sign. So we suppose that $\omega_{4,4}$ is chosen so that

$$(1.2) \qquad i^*(\omega_{4,4}) = (1-H)\eta_4 \quad in \quad \widetilde{KO}_G(\Sigma^{0,4})=RO(G)\cdot\eta_4$$

where i is the inclusion of $\Sigma^{0,4}$ into $\Sigma^{4,4}$, and also $\psi(\omega_{4,4})=1$.

Similar results hold for K_G. Let $\tau_{2p,0}$ denote the element $\tau_p^+\in\tilde{K}_G(\Sigma^{2p,0})$ as in [14], i the inclusion of $\Sigma^{0,0}$ into $\Sigma^{2p,0}$ and $L=\boldsymbol{C}\otimes_{\boldsymbol{R}}H$. Then we have

$$(1.3) \qquad i^*(\tau_{2p,0}) = 2^{p-1}(1-L) \quad in \quad \tilde{K}_G(\Sigma^{0,0}) = R(G)\,.$$

By construction we obviously have

$$(1.4) \qquad c(\omega_{8p,0}) = \tau_{8p,0}$$

where c denotes the complexification.

Let $\mu\in K^{-2}(+)$ be the Bott class and denote by ψ also the forgetful homomorphism $K_G(X)\to K(X)$. Similarly we may assume that $\tau_{2p,0}$ satisfies the relation $\psi(\tau_{2p,0})=\mu^p$. Here let $K^*(X, Y)$ be regarded as a Z_8-graded cohomology theory, so that $K^*(+)=Z[\mu]/\mu^4=1$.

b) Let KH denote the quaternionic K-functor and KR the Real K-functor in the sense of [3]. We recall the following isomorphism

$$t\colon\ \widetilde{KH}(X)\cong\widetilde{KR}^{-4}(X) = \widetilde{KR}(\Sigma^{0,4}\wedge X) \quad [6, 15]$$

where X is a Real space with base point. By a quaternionic vector bundle over X we mean a complex vector bundle $E\to X$ together with a conjugate linear anti-involution J_E: $E\to E$ commuting with the Real structure on X. We assume henceforth that the quaternionic structure on $\boldsymbol{H}$ is right multiplication by j.

Define an isomorphism

$$\alpha\colon\ S^{0,4}\times\boldsymbol{H}\cong S^{0,4}\times\boldsymbol{H}$$

by $\alpha(v, w)=(v,v\bar{w})$ for $v\in S^{0,4}$ (=the unit quaternions), $w\in\boldsymbol{H}$ and denote by ε_H

the trivial line bundle $B^{0,4}\times \boldsymbol{H}\to B^{0,4}$. Then by σ we denote the element of $KH(B^{0,4}, S^{0,4})=\widetilde{KH}(\Sigma^{0,4})$ which $(\mathcal{E}_H, \mathcal{E}_H, \alpha)$ represents.

Let Y also be a Real space with base point. If E and F are quaternionic vector bundle over X and Y, the external tensor product $E\otimes_{\boldsymbol{C}} F$ of the underlying complex vector bundles E and F can be viewed as a Real vector bundle over $X\times Y$, since $J_E\otimes J_F$ becomes a conjugate linear involution commuting the Real structure on $X\times Y$. Hence the functor $(E, F)\mapsto E\otimes_{\boldsymbol{C}}F$ induces a smash product $\wedge_C\colon \widetilde{KH}(X)\otimes\widetilde{KH}(Y)\to\widetilde{KR}(X\wedge Y)$. Using this the required isomorphism t is given by

$$t(x)=\sigma\wedge_C x \qquad \text{for} \quad x\in\widetilde{KH}(X)\,.$$

Let s denote an obvious complexification $KH(X)\to K(X)$. By construction we then have

$$(1.5)\qquad s(\sigma)=\mu^2, \quad \textit{so that} \quad \sigma\wedge_C\mathcal{E}_H=\eta_4\,, \ \sigma\wedge_C\sigma=1 \quad [15]$$

where $\mathcal{E}_H$ denotes the quaternionic trivial line bundle over $\Sigma^{0,0}$.

In the above let take $X=Y$ and $E=F$. Then $E\otimes_{\boldsymbol{C}}E$ can be viewed as a Real G-vector bundle [5] with G-action switching factors. Therefore a similar functor $E\mapsto E\otimes_{\boldsymbol{C}}E$ induces a natural transformation

$$\widetilde{KH}(X)\to\widetilde{KR}_G(X\wedge X)$$

which we denote also by $\wedge_C$. (Here we consider that $X\wedge X$ is a Real G-space with G-action interchanging factors and also G has trivial involution.)

In particular, when $X=\Sigma^{0,q}\wedge Y$, $X\wedge X$ is identified with $\Sigma^{q,q}\wedge(Y\wedge Y)$ as Real spaces through a canonical homeomorphism $\Sigma^{0,q}\wedge\Sigma^{0,q}\approx\Sigma^{q,q}$. This is obtained from the homeomorphism $R^{0,q}\times R^{0,q}\approx R^{q,0}\times R^{0,q}$ given by the map $(u, v)\mapsto(u-v, u+v)$ for $u, v\in R^q$, by taking one-point compactifications of both sides.

As in the non equivariant case [3], if X is a Real G-space with trivial involution, the functor $E\mapsto\boldsymbol{C}\otimes_{\boldsymbol{R}}E$ yields an isomorphism $KO_G(X)\cong KR_G(X)$ where E is an ordinary real G-vector bundle over X and $\boldsymbol{C}$ has a standard real structure by complex conjugation. In this paper we regard as

$$KO(X)=KR(X) \quad \text{and} \quad KO_G(X)=KR_G(X)$$

via the isomorphisms as in [3] and above.

2. Squares of elements in $KO^{-5}(X)$

Let $c\colon KR(X)\to K(X)$ be the complexification. From now on we assume that η_4 is chosen so that $c(\eta_4)=2\mu^2$.

Lemma 2.1. $\sigma\wedge_C\sigma=\omega_{4,4}$

as an element of $\widetilde{KR}_G(\Sigma^{4,4})=RO(G)\cdot\omega_{4,4}$.

Proof. View $\mu^2\wedge\mu^2$ as an element of $\tilde{K}_G(\Sigma^{4,4})$ with G-action switching factors. According to (2.6) when $p=2$ in [2] we then have

$$i^*(\mu^2\wedge\mu^2)=1\otimes\psi i^*(\mu^2\wedge\mu^2)+(L-1)\otimes\lambda^2\mu^2 \quad \text{in} \quad \tilde{K}_G(\Sigma^{0,4})=R(G)\otimes K(S^4)$$

where i is the inclusion of $\Sigma^{0,4}$ into $\Sigma^{4,4}$. However because of $\psi i^*(\mu^2\wedge\mu^2)=0$ we get

$$i^*(\mu^2\wedge\mu^2)=(L-1)\lambda^2\mu^2 \quad \text{in} \quad \tilde{K}_G(\Sigma^{0,4}).$$

Let ψ^2 be the 2nd Adams operation. Then $\psi^2(\mu^2)=-2\lambda^2\mu^2$ by definition and also $\psi^2(\mu^2)=2\mu^2$ by Proposition 3.2.2 in [4]. Hence $\lambda^2\mu^2=-2\mu^2$. Thus we obtain

$$i^*(\mu^2\wedge\mu^2)=2(1-L)\mu^2 \quad \text{in} \quad \tilde{K}_G(\Sigma^{0,4}).$$

From this formula it follows that

$$ci^*(\sigma\wedge_C\sigma)=2(1-L)\mu^2 \quad \text{in} \quad \tilde{K}_G(\Sigma^{0,4}) \tag{a}$$

because $c(\sigma\wedge_C\sigma)=\mu^2\wedge\mu^2$ by (1.5).

On the other hand by (1.2) we have

$$ci^*(\omega_{4,4})=2(1-L)\mu^2 \quad \text{in} \quad \tilde{K}_G(\Sigma^{0,4}) \tag{b}$$

Furthermore by (1.5) again and the assumption in (1.2) we have

$$\psi c(\sigma\wedge_G\sigma)=\psi c(\omega_{4,4})=1 \tag{c}$$

where ψ denotes the forgetful homomorphism.

Compare $\sigma\wedge_C\sigma$ with $\omega_{4,4}$ using (a), (b) and (c), then the assertion follows immediately.

Let $\xi\colon E\to X$ be a quaternionic vector bundle over X and ξ_C its underlying complex vector bundle. Then the 2nd exterior power $\lambda^2\xi_C$ of ξ_C becomes a Real vector bundle over X with $J_E\wedge J_E$ as a Real structure. If we write $\lambda_C^2\xi$ for this vector bundle, the functor $\xi\mapsto\lambda_C^2\xi$ is extended to a natural transformation $\lambda_C^2\colon KH(X)\to KR(X)$ in an obvious way.

Proposition 2.2. *Let X be a Real space with trivial involution. If $x=t(\mathfrak{x})\in\widetilde{KR}^{-5}(X)$ for $\mathfrak{x}\in\widetilde{KH}^{-1}(X)$, then*

$$x^2=\eta_1\lambda_C^2\mathfrak{x}$$

where η_1 denotes the generator of $KO^{-1}(+)\cong Z_2$.

Proof. Under the identification $(S^5\wedge X)\wedge(S^5\wedge X)=\Sigma^{5,5}\wedge X\wedge X$ of Real G-spaces stated above we have by definition

$$(\sigma\wedge_C\tilde{x})\wedge_C(\sigma\wedge_C\tilde{x})=(\sigma\wedge_C\sigma)\wedge_C(\tilde{x}\wedge_C\tilde{x}) \quad \text{in} \quad \widetilde{KR}_G(\Sigma^{5,5}\wedge X\wedge X).$$

By Lemma 2.1 we therefore have

$$x\wedge_C x=\omega_{4,4}\wedge(\tilde{x}\wedge_C\tilde{x}) \quad \text{in} \quad \widetilde{KR}_G(\Sigma^{5,5}\wedge X\wedge X)=\widetilde{KR}_G(\Sigma^{1,1}\wedge X\wedge X)\cdot\omega_{4,4} \quad \text{(a)}.$$

Arguments parallel to (2.6) when $p=2$ in [2] yield

$$i^*(1\wedge d)^*(\tilde{x}\wedge_C\tilde{x})=(H-1)\lambda_C^2\tilde{x} \quad \text{in} \quad \widetilde{KR}_G(\Sigma^{0,1}\wedge X)=RO(G)\otimes\widetilde{KR}(S^1\wedge X) \quad \text{(b)}$$

where $d\colon X\to X\wedge X$ is the diagonal map and $i\colon \Sigma^{0,1}\wedge X\to\Sigma^{1,1}\wedge X$ is the inclusion.

To analyze i^* we consider the exact sequence for $(\Sigma^{1,1}\wedge X, \Sigma^{0,1}\wedge X)$. Because of $\Sigma^{1,1}/\Sigma^{0,1}\approx S^{1,0}_+\wedge\Sigma^{0,2}$ [12], we have the following exact sequence:

$$\begin{array}{ccccccc}
\widetilde{KR}_G^{-1}(\Sigma^{0,1}\wedge X) & \xrightarrow{\delta} & \widetilde{KR}_G(S^{1,0}_+\wedge\Sigma^{0,2}\wedge X) & \xrightarrow{j^*} & \widetilde{KR}_G(\Sigma^{1,1}\wedge X) & \xrightarrow{i^*} & \widetilde{KR}_G(\Sigma^{0,1}\wedge X) \\
\wr\Vert & & \wr\Vert & & & & \wr\Vert \\
RO(G)\otimes\widetilde{KR}(S^2\wedge X) & & \widetilde{KR}(S^2\wedge X) & & & & RO(G)\otimes\widetilde{KR}(S^1\wedge X)
\end{array}$$

Here it is easily verified that δ is induced by the map $k\colon S^{1,0}_+\wedge\Sigma^{0,1}\wedge\Sigma^{0,1}\wedge X\to \Sigma^{0,1}\wedge\Sigma^{0,1}\wedge X$ given by $k(\pm1, t, y)=(-t, y)$ for $t\in\Sigma^{0,1}$, $y\in\Sigma^{0,1}\wedge X$. Hence we see that δ agrees with $\psi\colon \widetilde{KR}_G(\Sigma^{0,2}\wedge X)\to\widetilde{KR}(S^2\wedge X)$. From this it follows that δ is surjective and so i^* is injective.

Let $\xi\colon B^{1,0}\times R^{1,0}\to B^{1,0}$ and $\varepsilon\colon B^{1,0}\times R^{0,1}\to B^{1,0}$ be the product bundles and $\alpha\colon S^{1,0}\times R^{1,0}\cong S^{1,0}\times R^{0,1}$ be an equivariant isomorphism given by $\alpha(1, v)=(1, v)$, $\alpha(-1, v)=(-1, -v)$ for $v\in\boldsymbol{R}$. Denote by $\tilde{\eta}$ the element of $\widetilde{KR}_G(\Sigma^{1,0})$ represented by $(\xi, \varepsilon, \alpha)$. Then evidently the restriction of $\tilde{\eta}$ to $\widetilde{KR}_G(\Sigma^{0,0})=RO(G)$ is $H-1$, so that

$$i^*(\tilde{\eta}\wedge\lambda_C^2\tilde{x})=(H-1)\lambda_C^2\tilde{x} \quad \text{in} \quad \widetilde{KR}_G(\Sigma^{0,1}\wedge X) \qquad \text{(c)}.$$

Because i^* is injective, we see by (b) and (c) that

$$(1\wedge d)^*(\tilde{x}\wedge_C\tilde{x})=\tilde{\eta}\wedge\lambda_C^2\tilde{x} \quad \text{in} \quad \widetilde{KR}_G(\Sigma^{1,1}\wedge X).$$

Consequently from this and (a) it follows that

$$(1\wedge d)^*(x\wedge x)=\omega_{4,4}\wedge\tilde{\eta}\wedge\lambda_C^2\tilde{x} \quad \text{in} \quad \widetilde{KR}(\Sigma^{5,5}\wedge X).$$

Applying ψ to both sides of this equality, we obtain

$$x^2=\eta_1\lambda_C^2\tilde{x} \quad \text{in} \quad \widetilde{KR}^{-2}(X)$$

since $\psi(\tilde{\eta})=\eta_1$ by definition and $\psi(\omega_{4,4})=1$. This completes the proof.

Let $f: X \rightarrow GL(n, F)$, $F=\boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{H}$, be a base point preserving map where the unit element of $GL(n, F)$ is taken as its base point. We denote by $\beta(f)$ the element of $\widetilde{KO}^{-1}(X)$, $\tilde{K}^{-1}(X)$ or $\widetilde{KH}^{-1}(X)$ which f defines in a natural manner according as $F=\boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{H}$. If $f: X \rightarrow GL(m, \boldsymbol{H})$ and $g: X \rightarrow GL(n, \boldsymbol{H})$ are base point preserving maps, the isomorphism $\alpha: H \times (\boldsymbol{H}^m \wedge_C \boldsymbol{H}^n) \cong X \times (\boldsymbol{H}^m \wedge_C \boldsymbol{H}^n)$ given by $\alpha(x, u \wedge v)=(x, f(x)u \wedge g(x)v)$ for $u \in \boldsymbol{H}^m$, $v \in \boldsymbol{H}^n$ becomes an isomorphism of Real vector bundles. So α defines an element of $\widetilde{KR}^{-1}(X)$ in a similar way to $\beta(f)$. We denote this element also by $\beta(f \wedge_C g)$ and by $\beta(\lambda_C^2 f)$ if $f=g$. Then by Proposition 2.2 we have

Corollary 2.3. *Let $f: X \rightarrow GL(n, \boldsymbol{H})$ be a base point preserving map. Then for $t(\beta(f)) \in \widetilde{KR}^{-5}(X)$*

$$(t(\beta(f)))^2 = \eta_1(\beta(\lambda_C^2 f)+n\beta(\iota_H \wedge_C f)) \qquad in \quad \widetilde{KR}^{-2}(X)$$

where ι_H is the constant map from X to $GL(1, \boldsymbol{H})$.

3. $KO^*(P^{l-1})$ and $KO^*(\mathrm{Spin}(l))$ for $l \equiv 3, 4, 5 \bmod 8$

In this section we observe the algebra structure of the KO-groups of the real projective l-space P^l and the spinor group $\mathrm{Spin}(l)$. Then for the additive structures of them we refer to [1, 8, 16] and [15].

We consider that G acts on $\mathrm{Spin}(l)$ as a subgroup of $\mathrm{Spin}(l)$, and we regards as $S^{l,0}/G=P^{l-1}$ and $\mathrm{Spin}(l)/G=SO(l)$, the rotation group of dgree l. Assume that $\bar{\iota}: S^{l,0} \rightarrow \mathrm{Spin}(l)$ is an equivariant embedding which induces a well-known embedding $\iota: P^{l-1} \rightarrow SO(l)$, and denote by π the canonical projections $S^{l,0} \rightarrow P^{l-1}$ and $\mathrm{Spin}(l) \rightarrow SO(l)$. Then clearly

$$\pi \bar{\iota} = \iota \pi . \tag{3.1}$$

Let γ_{l-1}' and ξ_l' be the real 1-dimensional vector bundles over P^{l-1} and $SO(l)$ associated with the 2-fold coverings above respectively. Moreover let $\gamma_{1-l}=\gamma_{l-1}'-1$ and $\xi_l=\xi_l'-1$ as elements of $\widetilde{KO}(P^{l-1})$ and $\widetilde{KO}(SO(l))$. Obviously we then have

$$i^*\xi_l = \gamma_{l-1}, \ \xi_l^2 = -2\xi_l \quad and \quad \gamma_{l-1}^2 = -2\gamma_{l-1} \tag{3.2}$$

and arguing as in [14] we see

(3.3) *The order of ξ_l agrees with that of γ_{l-1}.*

Note that in $KO^*(+)$ there hold the relations such that $2\eta_1=\eta_1^3=\eta_1\eta_4=0$ and $\eta_4^2=4$ and that generators of $\widetilde{KO}^0(P^l)$ and $\widetilde{KO}(P)$ are already specified in [1, 8] as described below.

For the proof of Proposition 3.4 below we use freely the following three types of exact sequences: the ones obtained by applying KO- and KO_G-functors to the cofiberings

$$P^{l-1} \xrightarrow{i} P^l \xrightarrow{p} P^l/P^{l-1} = S^l \quad \text{and} \quad S^{l,0} \xrightarrow{i} B^{l,0} \xrightarrow{p} \Sigma^{l,0}$$

where i and p are evident maps and the Atiyah's in [3], (3.4)

$$\cdots \to KO^{1-q}(X) \xrightarrow{\eta_1 \cdot} KO^{-q}(X) \xrightarrow{c} K^{-q}(X) \xrightarrow{\delta} KO^{2-q}(X) \to \cdots$$

where $\eta_1 \cdot$ is multiplication by η_1 and δ is given by $\delta(\mu x) = r(x)$ for $x \in K^{2-q}(X)$ (cf. [15], (2.4)). Here r denotes the realification $K(X) \to KO(X)$. Then we also refer to the table in [13] for the additive structure of $\widetilde{KO}^*(\Sigma^{p,q})$.

Proposition 3.4. 1) $\widetilde{KO}^0(P^{8n+2}) = Z_{2^{4n+2}} \cdot \gamma_{8n+2}$,

$$\widetilde{KO}^{-1}(P^{8n+2}) = Z_2 \cdot \eta_1 \gamma_{8n+2},$$
$$\widetilde{KO}^{-2}(P^{8n+2}) = Z_2 \cdot \eta_1^2 \gamma_{8n+2},$$
$$\widetilde{KO}^{-3}(P^{8n+2}) = 0,$$
$$\widetilde{KO}^{-4}(P^{8n+2}) = Z_{2^{4n}} \cdot \eta_4 \gamma_{8n+2},$$
$$\widetilde{KO}^{-5}(P^{8n+2}) = 0,$$
$$\widetilde{KO}^{-6}(P^{8n+2}) = Z_2 \cdot \mu_{8n+2},$$
$$\widetilde{KO}^{-7}(P^{8n+2}) = Z_2 \cdot \eta_1 \mu_{8n+2}$$

with relations

$$\gamma_{8n+2}^2 = -2\gamma_{8n+2}, \quad \mu_{8n+2}^2 = \eta_4 \mu_{8n+2} = \gamma_{8n+2} \mu_{8n+2} = 0,$$
$$\eta_1^2 \mu_{8n+2} = 2^{4n+1} \gamma_{8n+2}.$$

2) $\widetilde{KO}^0(P^{8n+3}) = Z_{2^{4n+2}} \cdot \gamma_{8n+3}$,

$$\widetilde{KO}^{-1}(P^{8n+3}) = Z \cdot \eta_4 \bar{\nu}_{8n+3} \oplus Z_2 \cdot \eta_1 \gamma_{8n+3},$$
$$\widetilde{KO}^{-2}(P^{8n+3}) = Z_2 \cdot \eta_1^2 \gamma_{8n+3},$$
$$\widetilde{KO}^{-3}(P^{8n+3}) = 0,$$
$$\widetilde{KO}^{-4}(P^{8n+3}) = Z_{2^{4n}} \cdot \eta_4 \gamma_{8n+3},$$
$$\widetilde{KO}^{-5}(P^{8n+3}) = Z \cdot \bar{\nu}_{8n+3},$$
$$\widetilde{KO}^{-6}(P^{8n+3}) = Z_2 \cdot \eta_1 \bar{\nu}_{8n+3} \oplus Z_2 \cdot \mu_{8n+3},$$
$$\widetilde{KO}^{-7}(P^{8n+3}) = Z_2 \cdot \eta_1^2 \bar{\nu}_{8n+3} \oplus Z_2 \cdot \eta_1 \mu_{8n+3}$$

with relations

$$\gamma_{8n+3}^2 = -2\gamma_{8n+3}, \quad \bar{\nu}_{8n+3}^2 = \mu_{8n+3}^2 = \eta_4 \mu_{8n+3} = \gamma_{8n+3} \bar{\nu}_{8n+3} = \bar{\nu}_{8n+3} \mu_{8n+3} = 0,$$

$$\eta_1 \bar{\nu}_{8n+3} = \gamma_{8n+3}\mu_{8n+3}, \quad \eta_1^2 \mu_{8n+3} = 2^{4n+1}\gamma_{8n+3}.$$

3) $\widetilde{KO}^0(P^{8n+4}) = Z_{2^{4n+3}} \cdot \gamma_{8n+4}$,
$\widetilde{KO}^{-1}(P^{8n+4}) = Z_2 \cdot \eta_1 \gamma_{8n+4}$,
$\widetilde{KO}^{-2}(P^{8n+4}) = Z_2 \cdot \eta_1^2 \gamma_{8n+4}$,
$\widetilde{KO}^{-3}(P^{8n+4}) = 0$,
$\widetilde{KO}^{-4}(P^{8n+4}) = Z_{2^{4n+1}} \cdot \eta_4 \gamma_{8n+4}$,
$\widetilde{KO}^{-5}(P^{8n+4}) = 0$,
$\widetilde{KO}^{-6}(P^{8n+4}) = Z_2 \cdot \mu_{8n+4}$,
$\widetilde{KO}^{-7}(P^{8n+4}) = Z_2 \cdot \eta_1 \mu_{8n+4}$,

with relations

$$\gamma_{8n+4}^2 = -2\gamma_{8n+4}, \quad \mu_{8n+4}^2 = \eta_4 \mu_{8n+4} = \gamma_{8n+4}\mu_{8n+4} = 0,$$
$$\eta_1^2 \mu_{8n+4} = 2^{4n+2}\gamma_{8n+4}.$$

Proof. See (3.2) for the first relations.

1) Because of $\widetilde{KO}^{-6}(P^{8n+1})=0$ we see that $p^*: \widetilde{KO}^{-6}(S^{8n+2}) \to \widetilde{KO}^{-6}(P^{8n+2}) \cong Z_2$ is surjective. So we define $\mu_{8n+2} \in \widetilde{KO}^{-6}(P^{8n+2}) \cong Z_2$ as

$$\mu_{8n+2} = p^*(g)$$

where g denotes a generator of $\widetilde{KO}^{-6}(S^{8n+2}) \cong Z$. Then evidently $KO^{-6}(P^{8n+2}) = Z_2 \cdot \mu_{8n+2}$ and $\mu_{8n+2}^2 = 0$.

Since $\widetilde{KO}^{-7}(P^{8n+1}) \cong Z$ and $\widetilde{KO}^{-1}(S^{8n+2}) = 0$, $p^*: \widetilde{KO}^{-7}(S^{8n+2}) \cong \widetilde{KO}^{-7}(P^{8n+2})$ and $p^*=0: \widetilde{KO}^{-2}(S^{8n+2}) \to \widetilde{KO}^{-2}(P^{8n+2})$. Therefore it follows that $\widetilde{KO}^{-7}(P^{8n+2}) = Z_2 \cdot \eta_1 \mu_{8n+2}$ and $\eta_4 \mu_{8n+2} = 0$.

Since $i^*(\gamma_{8n+2}) = \gamma_{8n+1}$ and γ_{8n+1} has order 2^{4n+1}, $i^*(2^{4n+1}\gamma_{8n+2}) = 0$, so that $p^*(\eta_1^2 g) = 2^{4n+1}\gamma_{8n+2}$ which shows $\eta_1^2 \mu_{8n+2} = 2^{4n+1}\gamma_{8n+2}$ immediately.

$\gamma_{8n+2} g$ lies in $KO^{-6}(P^{8n+2}, P^{8n+1}) \cong Z$ and has finite order because so does γ_{8n+2}. Hence $\gamma_{8n+2} g = 0$, so that $\gamma_{8n+2}\mu_{8n+2} = 0$.

From the following isomorphism with an obvious identification:

$$\begin{array}{ccc} i^*: KO_G^{-1}(B^{8n+3,0}) & \cong & KO_G^{-1}(S^{8n+3,0}) \\ \| & & \| \\ RO(G) \cdot \eta_1 & & KO^{-1}(P^{8n+2}) \end{array}$$

it follows that $i^*((H-1)\eta_1) = \eta_1 \gamma_{8n+2}$ is a generator of $KO^{-1}(P^{8n+2}) \cong Z_2$. In a similar way we get also $\widetilde{KO}^{-2}(P^{8n+2}) = Z_2 \cdot \eta_1^2 \gamma_{8n+2}$.

2) According to [11; § 13] the half spin representations Δ_{8n+4}^+ and Δ_{8n+4}^- of Spin$(8n+4)$ are quaternionic and may be viewed as homomorphisms from

Spin$(8n+4)$ to $GL(2^{4n}, \boldsymbol{H})$. We define a map

$$\delta\colon P^{8n+3} \to GL(2^{4n}, \boldsymbol{H})$$

by $\delta\pi(x)=\Delta^{+}_{8n+4}(\bar{\iota}(x))\Delta^{-}_{8n+4}(\bar{\iota}(x))^{-1}$ for $x\in S^{8n+4,0}$ and set

$$\bar{\nu}_{8n+3} = t(\beta(\delta))\in\widetilde{KO}^{-5}(P^{8n+3})\,.$$

Let $\bar{\nu}_{8n+3}\in\tilde{K}^{-1}(P^{8n+3})\cong Z$ be a generator given in [14]. Then we have

$$c(\bar{\nu}_{8n+3}) = \mu^2\nu_{8n+3} \tag{a}$$

since $s(\sigma)=\mu^2$ by (1.5) and $s(\beta(\delta))=\nu_{8n+3}$ by construction.

Because of $\widetilde{KO}^{-4}(P^{8n+3})=Z_{2^{4n}}\cdot\eta_4\gamma_{8n+3}$ and $\widetilde{KO}^{-3}(P^{8n+3})=0$ we see that $c\colon\widetilde{KO}^{-5}(P^{8n+3})\cong\tilde{K}^{-5}(P^{8n+3})$. Therefore by (a) we obtain $\widetilde{KO}^{-5}(P^{8n+3})=Z\cdot\bar{\nu}_{8n+3}$.

Since $\widetilde{KO}^{-5}(P^{8n+2})=0$ we see that $\bar{\nu}_{8n+3}$ lies in the image of $p^*\colon\widetilde{KO}^{-5}(S^{8n+3})\to\widetilde{KO}^{-5}(P^{8n+3})$, so that $\bar{\nu}^2_{8n+3}=0$.

We have $\tilde{K}^{-6}(P^{8n+3})=Z_{2^{4n+1}}\cdot\mu^3c(\gamma_{8n+3})$ by [4] and $\delta(\mu^3c(\gamma_{8n+3}))=\eta_4\gamma_{8n+3}$ using $r(\mu^2)=\eta_4$. Therefore, since $\eta_4\gamma_{8n+3}$ has order 2^{4n} there is an element $\mu_{8n+3}\in\widetilde{KO}^{-6}(P^{8n+3})\cong Z_2\oplus Z_2$ such that

$$c(\mu_{8n+3}) = 2^{4n}\mu^3c(\gamma_{8n+3}) \tag{b}.$$

Hence by observing Atiyah's exact sequence we see readily that $\eta_1\bar{\nu}_{8n+3}$ and μ_{8n+3} generate $KO^{-6}(P^{8n+3})\cong Z_2\oplus Z_2$ additively.

$\eta_1\cdot\colon\widetilde{KO}^{-6}(P^{8n+3})\to KO^{-7}(P^{8n+3})$ is injective because $c\colon\widetilde{KO}(P^{8n+3})\to\tilde{K}(P^{8n+3})$ is surjective. This leads to $\widetilde{KO}^{-7}(P^{8n+3})=Z_2\cdot\eta_1^2\bar{\nu}_{8n+3}\oplus Z_2\cdot\eta_1\mu_{8n+3}$.

Since the order of $c(\gamma_{8n+3})$ is 2^{4n+1} by [4], $c(2^{4n+1}\gamma_{8n+3})=0$ and hence $\eta_1^2\mu_{8n+3}=2^{4n+1}\gamma_{8n+3}$.

By (a) we get $c(\mu^2_{8n+3})=0$, because $2^{4n+2}c(\gamma_{8n+3})=0$. But $c\colon\widetilde{KO}^{-4}(P^{8n+3})\to\tilde{K}^{-4}(P^{8n+3})$ is injective since $\widetilde{KO}^{-3}(P^{8n+3})=0$, therefore we have $\mu^2_{8n+3}=0$.

Using $r(xc(y))=r(x)y$, $c(\eta_4)=2\mu^2$ and $r(\mu^2)=\eta_4$ we have $\delta c(2^{4n-1}\eta_4\gamma_{8n+3})=\eta_4\mu_{8n+3}$ by (b), so that $\eta_4\mu_{8n+3}=0$ by exactness.

The relation $\bar{\nu}_{8n+3}\mu_{8n+3}=0$ is clear because of $\widetilde{KO}^{-3}(P^{8n+3})=0$.

Using $r(\mu)=\eta_1^2$ it follows from (a) that $\delta(\nu_{8n+3})=\eta_1^2\bar{\nu}_{8n+3}$ which is of order 2. Also by (a) we have $c(\eta_4\bar{\nu}_{8n+3})=2\nu_{8n+3}$. These lead to $\widetilde{KO}^{-1}(P^{8n+3})=Z_2\cdot\eta_1\gamma_{8n+3}\oplus Z\cdot\eta_4\bar{\nu}_{8n+3}$.

Consider the following exact sequence:

$$\begin{array}{ccccc} KO_G^{-5}(S^{8n+4,0}) & \xrightarrow{\delta} & \widetilde{KO}_G^{-4}(\Sigma^{8n+4,0}) & \xrightarrow{j^*} & KO_G^{-4}(B^{8n+4,0}) \\ \| & & \| & & \| \\ KO^{-5}(P^{8n+4}) & & RO(G)\cdot\omega_{8n,0}\wedge\omega_{4,4} & & RO(G)\cdot\eta_4 \end{array}$$

Through the obvious identifications above we then have

$$\delta(\bar{\nu}_{8n+3}) = (1+H)\omega_{8n,0}\wedge\omega_{4,4} \qquad \text{(c)}.$$

By (1.1) and (1.2) we have $j^*(\omega_{8n,0}\wedge\omega_{4,4})=2^{4n}(1-H)\eta_4$ and so $\mathrm{Ker}\,j^*=((1+H)\omega_{8n,0}\wedge\omega_{4,4})$. Hence we can write $\delta(\bar{\nu}_{8n+3})=l(1+H)\omega_{8n,0}\wedge\omega_{4,4}$, $l\in Z$. Applying ψ_C to both sides of this equality and arguing as in Proof of Lemma 1.8 in [14] we get $l=1$ under the assumption stated in § 1, a).

Since $\delta(\gamma_{8n+3}\bar{\nu}_{8n+3})=(H-1)\delta(\bar{\nu}_{8n+3})$ we have $\delta(\gamma_{8n+3}\bar{\nu}_{8n+3})=0$ by (c) and therefore $\gamma_{8n+3}\bar{\nu}_{8n+3}=0$ because δ is injective.

Observe the following isomorphism with an evident identification:

$$\begin{array}{ccc} \delta:\ KO_G^{-6}(S^{8n+4,0}) & \cong & \widetilde{KO}_G^{-5}(\Sigma^{8n+4,0}) \\ \| & & \| \\ KO^{-6}(P^{8n+3}) & & RO(G)\cdot\eta_1\omega_{8n,0}\wedge\omega_{4,4} \end{array}$$

Since $\delta(\eta_1\bar{\nu}_{8n+3})=(1+H)\eta_1\omega_{8n,0}\wedge\omega_{4,4}$ by (c) we then have $\delta(\mu_{8n+3})=\varepsilon\eta_1\omega_{8n,0}\wedge\omega_{4,4}$ where $\varepsilon=1$ or H. Hence we have $\delta(\gamma_{8n+3}\mu_{8n+3})=\delta(\eta_1\bar{\nu}_{8n+3})$, from which it follows immediately that $\gamma_{8n+3}\mu_{8n+3}=\eta_1\bar{\nu}_{8n+3}$.

It is clear from 1) that $\widetilde{KO}^{-2}(P^{8n+3})=Z_2\cdot\eta_1^2\gamma_{8n+3}$, because $i^*\colon \widetilde{KO}^{-2}(P^{8n+3})\cong\widetilde{KO}^{-2}(P^{8n+2})$.

3) In the exact sequence

$$\widetilde{KO}^{-6}(P^{8n+4}) \xrightarrow{c} \tilde{K}^{-6}(P^{8n+4}) \xrightarrow{\delta} \widetilde{KO}^{-4}(P^{8n+4})$$

we have $\tilde{K}^{-6}(P^{8n+4})=Z_{2^{4n+2}}\cdot\mu^3c(\gamma_{8n+4})$ by [4], $\widetilde{KO}^{-4}(P^{8n+4})=Z_{2^{4n+1}}\cdot\eta_4\gamma_{8n+4}$ and $\delta(\mu^3c(\gamma_{8n+4}))=\eta_4\gamma_{8n+4}$. Hence we see $\delta(2^{4n+1}\mu^3c(\gamma_{8n+4}))=0$, which shows that there is an element $\mu_{8n+4}\in\widetilde{KO}^{-6}(P^{8n+4})\cong Z_2$ such that

$$c(\mu_{8n+4}) = 2^{4n+1}\mu^3c(\gamma_{8n+4})\,,$$

so that $\widetilde{KO}^{-6}(P^{8n+4})=Z_2\cdot\mu_{8n+4}$.

Clearly $c(\gamma_{8n+4}\mu_{8n+4})=c(\mu_{8n+4}^2)=0$, therefore we see that $\gamma_{8n+4}\mu_{8n+4}=\mu_{8n+4}^2=0$, because $c\colon \widetilde{KO}^{-i}(P^{8n+4})\to\tilde{K}^{-i}(P^{8n+4})$ is is injective for $i=4, 6$. Since $\eta_1\cdot\colon \widetilde{KO}^{-6}(P^{8n+4})\cong\widetilde{KO}^{-7}(P^{8n+4})$ we get $\widetilde{KO}^{-7}(P^{8n+4})=Z_2\cdot\eta_1\mu_{8n+4}$ immediately. Moreover, from the fact that $\eta_1\cdot\colon \widetilde{KO}^{-7}(P^{8n+4})\to\widetilde{KO}^0(P^{8n+4})$ is injective and $c(2^{4n+2}\gamma_{8n+4})=0$ it follows that $\eta_1^2\mu_{8n+4}=2^{4n+2}\gamma_{8n+4}$. Observe the equality $c(2^{4n}\eta_4\gamma_{8n+4})=2^{4n+1}\mu^2c(\mu_{8n+4})$. Then applying δ to both sides of this and using the above formula for μ_{8n+4} we have $\eta_4\mu_{8n+4}=0$.

The rest is easily checked by arguments parallel to 1). This completes the proof.

Let

$$\rho_l\colon SO(l)\subset GL(l, \boldsymbol{R})$$

be the obvious inclusion and let us denote by the same letter ρ_l the composite

$$\rho_l \pi \colon \mathrm{Spin}(l) \to SO(l) \subset GL(l, \boldsymbol{R}) .$$

As we noted before, the half spin representations of Spin$(8n+4)$ are viewed as homomorphisms

$$\Delta^+_{8n+4},\ \Delta^-_{8n+4} \colon \mathrm{Spin}(8n+4) \to GL(2^{4n}, \boldsymbol{H})$$

According to [11; § 13], similarly we may view the spin representations Δ_{8n+3} and Δ_{8n+5} of Spin$(8n+3)$ and Spin$(8n+5)$ as homomorphisms

$$\Delta_{8n+3} \colon \mathrm{Spin}(8n+3) \to GL(2^{4n}, \boldsymbol{H})$$

and

$$\Delta_{8n+5} \colon \mathrm{Spin}(8n+5) \to GL(2^{4n+1}, \boldsymbol{H}) .$$

Set

$$\tilde{\kappa}^+_{8n+4} = t(\beta(\Delta^+_{8n+4})) , \quad \tilde{\kappa}^-_{8n+4} = t(\beta(\Delta^-_{8n+4}))$$

and

$$\tilde{\kappa}_{8n+3} = t(\beta(\Delta_{8n+3})) , \quad \tilde{\kappa}_{8n+5} = t(\beta(\Delta_{8n+5})) .$$

Then we have

Proposition 3.5.

$$\begin{aligned} KO^*(\mathrm{Spin}(8n+3)) &= \Lambda_{KO^*(+)}(\beta(\lambda^1 \rho_{8n+3}), \cdots, \beta(\lambda^{4n} \rho_{8n+3}), \tilde{\kappa}_{8n+3}) , \\ KO^*(\mathrm{Spin}(8n+4)) &= \Lambda_{KO^*(+)}(\beta(\lambda^1 \rho_{8n+4}), \cdots, \beta(\lambda^{4n} \rho_{8n+4}), \tilde{\kappa}^+_{8n+4}, \tilde{\kappa}^-_{8n+4}) , \\ KO^*(\mathrm{Spin}(8n+5)) &= \Lambda_{KO^*(+)}(\beta(\lambda^1 \rho_{8n+5}), \cdots, \beta(\lambda^{4n+1} \rho_{8n+5}), \tilde{\kappa}_{8n+5}) \end{aligned}$$

as $KO^*(+)$-modules where there hold the relations:

$$\beta(\lambda^i \rho_l)^2 = \eta_1(\beta(\lambda^2(\lambda^i \rho_l)) + \binom{l}{i} \beta(\lambda^i \rho_l))$$

for $1 \leqq i \leqq l$ *and* $l = 8n+3,\ 8n+4,\ 8n+5$,

$$\tilde{\kappa}_l^2 = \eta_1 \beta(\lambda_C^2 \Delta_l)$$

for $l = 8n+3,\ 8n+5$ *and*

$$(\tilde{\kappa}^+_{8n+4})^2 = \eta_1 \beta(\lambda_C^2 \Delta^+_{8n+4}) , \quad (\tilde{\kappa}^+_{8n+4})^2 = \eta_1 \beta(\lambda_C^2 \Delta^-_{8n+4}) .$$

Proof. See Theorem 5.6 in [15] for the additive structures and [7] (also (1.7) in [14]) and Corollary 2.3 for the relations.

We now show how to express the right sides of relations in Proposition 3.5 in terms of the basic generators. First we recall that for base point preserving maps $f \colon X \to GL(p, \boldsymbol{R})$ and $g \colon X \to GL(q, \boldsymbol{R})$ there hold the following formulas in $\widetilde{KO}^{-1}(X)$

$$(3.6)\qquad \beta(f\oplus g)=\beta(f)+\beta(g)\,,\quad \beta(f\otimes g)=q\beta(f)+p\beta(g)$$

(cf. [10; I, § 4]). Here $f\oplus g$ and $f\otimes g$ are maps from X to $GL(p+q, \boldsymbol{R})$ and $GL(pq, \boldsymbol{R})$ given by $(f\oplus g)(x)=f(x)\oplus g(x)$ and $(f\otimes g)(x)=f(x)\otimes g(x)$ for $x\in X$.

We consider the case of $\mathrm{Spin}(8n+3)$. Because $\lambda^k\rho_{8n+3}=\lambda^{8n+3-k}\rho_{8n+3}$ for $0\leqq k\leqq 8n+3$, $\lambda^2(\lambda^i\rho_{8n+3})$ and $\lambda_C^2\Delta_{8n+3}$ are polynomials of $\lambda^k\rho_{8n+3}$, $0\leqq k\leqq 4n+1$ clearly. Using (3.6) we hence see that $\beta(\lambda^2(\lambda^i\rho_{8n+3}))$ and $\beta(\lambda_C^2\Delta_{8n+3})$ are expressed in the form of linear combinations of $\beta(\lambda^k\rho_{8n+3})$, $0\leqq k\leqq 4n+1$. On the other hand, from Theorem 10.3 in [11; § 13] it follows that

$$\Delta_{8n+3}\otimes_{\boldsymbol{C}}\Delta_{8n+3}=\lambda^{4n+1}\rho_{8n+3}+\lambda^{4n}\rho_{8n+3}+\cdots+1$$

as a real representation, so that

$$\beta(\lambda^{4n+1}\rho_{8n+3})=\beta(\Delta_{8n+3}\otimes_{\boldsymbol{C}}\Delta_{8n+3})-\beta(\lambda^{4n}\rho_{8n+3})-\cdots-\beta(\lambda^1\rho_{8n+3})\,.$$

Hence we see that it suffices only to describe $\beta(\Delta_{8n+3}\otimes_{\boldsymbol{C}}\Delta_{8n+3})$ in terms of the basic generators.

Let ι_l denote the constant map from X to $GL(l, \boldsymbol{H})$. Then

$$(\Delta_{8n+3}\otimes_{\boldsymbol{C}}\Delta_{8n+3})(x)=(\Delta_{8n+3}\otimes_{\boldsymbol{C}}\iota_{2^{4n}})(x)(\iota_{2^{4n}}\otimes_{\boldsymbol{C}}\Delta_{8n+3})(x)$$

for $x\in X$. Therefore it follows that

$$\begin{aligned}\beta(\Delta_{8n+3}\otimes_{\boldsymbol{C}}\Delta_{8n+3})&=\beta(\Delta_{8n+3}\otimes_{\boldsymbol{C}}\iota_{2^{4n}})+\beta(\iota_{2^{4n}}\otimes_{\boldsymbol{C}}\Delta_{8n+3})\\&=2^{4n}(\beta(\Delta_{8n+3}\otimes_{\boldsymbol{C}}\iota_1)+\beta(\iota_1\otimes_{\boldsymbol{C}}\Delta_{8n+3}))\,.\end{aligned}$$

Because $\beta(\iota_1\otimes_{\boldsymbol{C}}\Delta_{8n+3})=\varepsilon_H\wedge_C\beta(\Delta_{8n+3})$ by construction, applying $\sigma\wedge_C\sigma$ to both sides of this we have

$$(\sigma\wedge_C\sigma)\wedge_C\beta(\iota_1\otimes_{\boldsymbol{C}}\Delta_{8n+3})=(\sigma\wedge_C\varepsilon_H)\wedge_C(\sigma\wedge_C\beta(\Delta_{8n+3}))\,.$$

By (1.5) we hence have

$$\beta(\iota_1\otimes_{\boldsymbol{C}}\Delta_{8n+3})=\eta_3\tilde{\kappa}_{8n+3}$$

since $\sigma\wedge_C\beta(\Delta_{8n+3})=\tilde{\kappa}_{8n+3}$ by definition.

Analogously we have $\beta(\Delta_{8n+3}\otimes_{\boldsymbol{C}}\iota_1)=\eta_4\tilde{\kappa}_{8n+3}$. Thus we obtain

$$\beta(\Delta_{8n+3}\otimes_{\boldsymbol{C}}\Delta_{8n+3})=2^{4n+1}\eta_4\tilde{\kappa}_{8n+3}\,,$$

so that we get

$$(3.7)\qquad \begin{aligned}\beta(\lambda^{4n+1}\rho_{8n+3})=&\,2^{4n+1}\eta_4\tilde{\kappa}_{8n+3}-\beta(\lambda^{4n}\rho_{8n+3})\\&-\beta(\lambda^{4n-1}\rho_{8n+3})-\cdots-\beta(\lambda^1\rho_{8n+3})\,.\end{aligned}$$

Arguing as above we have

$$\beta(\Delta^+_{8n+4}\otimes_C\Delta^+_{8n+4}) = 2^{4n+1}\eta_4\tilde{\kappa}^+_{8n+4}\,,$$
$$\beta(\Delta^-_{8n+4}\otimes_C\Delta^-_{8n+3}) = 2^{4n+1}\eta_4\tilde{\kappa}^-_{8n+4}\,,$$
$$\beta(\Delta^+_{8n+4}\otimes_C\Delta^-_{8n+4}) = 2^{4n}\eta_4(\tilde{\kappa}^+_{8n+4}+\tilde{\kappa}^-_{8n+4})\,,$$
$$\beta(\Delta_{8n+5}\otimes_C\Delta_{8n+5}) = 2^{4n+2}\eta_4\tilde{\kappa}_{8n+5}\,,$$

so that

(3.8)
$$\begin{aligned}\beta(\lambda^{4n+1}\rho_{8n+4}) &= 2^{4n}\eta_4(\tilde{\kappa}^+_{8n+4}+\tilde{\kappa}^-_{8n+4})-\beta(\lambda^{4n-1}\rho_{8n+4})\\ &\quad -\beta(\lambda^{4n-3}\rho_{8n+4})-\cdots-\beta(\lambda^1\rho_{8n+4})\,,\\ \beta(\lambda^{4n+2}\rho_{8n+4}) &= 2^{4n+1}\eta_4(\tilde{\kappa}^+_{8n+4}+\tilde{\kappa}^-_{8n+4})-2\beta(\lambda^{4n}\rho_{8n+4})\\ &\quad -2\beta(\lambda^{4n-2}\rho_{8n+4})-\cdots-2\beta(\lambda^2\rho_{8n+4})\,,\\ \beta(\lambda^{4n+2}\rho_{8n+4}) &= 2^{4n+2}\eta_4\tilde{\kappa}_{8n+5}-\beta(\lambda^{4n+1}\rho_{8n+5})\\ &\quad -\beta(\lambda^{4n}\rho_{8n+5})-\cdots-\beta(\lambda^1\rho_{8n+5})\,.\end{aligned}$$

Using these formulas we can similarly express the relations in the other cases in terms of the basic generators.

4. $KO^*(SO(l))$ for $l\equiv 3, 4, 5 \bmod 8$

In this section, for a compact free G-space X we identify $KO^*_G(X)$ and $K^*_G(X)$ with $KO^*(X/G)$ and $K^*(X/G)$ via natural isomorphisms.

Let G act diagonally on $S^{l,0}\times \mathrm{Spin}(l)$. Then we have a homeomorphism

(4.1) $$S^{l,0}\times_G \mathrm{Spin}(l)\approx P^{l-1}\times \mathrm{Spin}(l) \quad [9]$$

which is induced by the map

$$(x, g)\mapsto (\pi(x), \bar{\iota}(x)g)$$

for $x\in S^{l,0}$, $g\in \mathrm{Spin}(l)$. According to Proposition 3.5, when $l\equiv 3$, 4 or 5 mod 8 $KO^*(\mathrm{Spin}(l))$ is a free module over $KO^*(+)$ and therefore the above hoemomorphism yields an isomorphism

$$KO^*_G(S^{l,0}\times \mathrm{Spin}(l))\cong KO^*(P^{l-1})\otimes_{KO^*(+)}KO^*(\mathrm{Spin}(l))\,.$$

Viewing this as an equality and applying KO_G-functor to the cofibering $S^{l,0}\times \mathrm{Spin}(l)\to B^{l,0}\times \mathrm{Spin}(l)\to \Sigma^{l,0}\wedge \mathrm{Spin}(l)$, we have an exact sequence

(4.2)
$$\begin{aligned}\cdots &\leftarrow \widetilde{KO}^*_G(\Sigma^{l,0}\wedge \mathrm{Spin}(l)_+)\xleftarrow{\delta} KO^*(P^{l-1})\otimes_{KO^*(+)}KO^*(\mathrm{Spin}(l))\\ &\xleftarrow{I} KO^*(SO(l))\xleftarrow{J}\widetilde{KO}^*_G(\Sigma^{l,0}\wedge \mathrm{Spin}(l)_+)\leftarrow\cdots\end{aligned}$$

for $l\equiv 3$, 4, 5 mod 8 *provided with the formula*

(4.3) $$\delta(xI(y)) = \delta(x)y\,.$$

a) $KO^*(SO(8n+3))$

Let us define a map

$$\varepsilon\colon SO(8n+3) \to GL(2^{4n}, \boldsymbol{H})$$

by $\varepsilon(\pi(g))=\Delta_{8n+3}(g)^2$ for $g\in \mathrm{Spin}(8n+3)$ and set

$$\kappa_{8n+3} = t(\beta(\varepsilon))\in KO^{-5}(SO(8n+3))\,.$$

We consider (4.2) when $l=8n+3$.

Lemma 4.4. i) $I(\xi_{8n+3})=\gamma_{8n+2}\otimes 1$,

ii) $I(\beta(\lambda^k\rho_{8n+3}))=1\otimes\beta(\lambda^k\rho_{8n+2})+\binom{8n+1}{k-1}\eta_1\gamma_{8n+2}\otimes 1$ $(1\leqq k\leqq 8n+3)$,

iii) $I(\kappa_{8n+3})=(\gamma_{8n+2}+2)\otimes\tilde{\kappa}_{8n+3}$.

Proof. i) Immediate from definition.

ii) Consider the map from $P^{8n+2}\times \mathrm{Spin}(8n+3)$ to $GL\left(2\binom{8n+3}{k}, \boldsymbol{R}\right)$ given by

$$(\pi(x), g) \mapsto \lambda^k\rho_{8n+3}(\pi(g))\oplus\lambda^k\rho_{8n+3}(\iota\pi(x))^{-1}$$

for $x\in S^{8n+3,0}$, $g\in\mathrm{Spin}(8n+3)$. By observing (4.1) we then see that this map represents $I(\beta(\lambda^k\rho_{8n+3}))$, so that

$$I(\beta(\lambda^k\rho_{8n+3})) = 1\otimes\beta(\lambda^k\rho_{8n+3})-\beta(\lambda^k\rho_{8n+3}\cdot\iota)\otimes 1\,.$$

Since $\widetilde{KO}^{-1}(P^{8n+2})=Z_2\cdot\eta_1\gamma_{8n+2}$ by Proposition 3.4, we can write

$$\beta(\lambda^k\rho_{8n+3}\cdot\iota) = \varepsilon\eta_1\gamma_{8n+3}\,, \qquad \varepsilon\in Z\,.$$

Let denote by j the inclusions $P^1\subset P^{8n+2}$ and $SO(2)\subset SO(8n+3)$ such that $j\iota=\iota j$. Then obviously $j^*(\beta(\lambda^k\rho_{8n+3}))=\binom{8n+1}{k-1}\beta(\rho_2)$ and because ρ_2 viewed as a map from $SO(2)$ to $U(1)$ in a natural way defines a generator of $\tilde{K}^{-1}(SO(2))=Z\cdot\mu$ and $r(\mu)=\eta_1^2$, we have $j^*(\beta(\lambda^k\rho_{8n+3}))=\binom{8n+1}{k-1}\eta_1^2$ in $\widetilde{KO}^{-1}(SO(2))=Z_2\cdot\eta_1^2$. Also $\iota\colon P^1\subset SO(2)$ becomes a homeomorphism and hence we have $i^*j^*(\beta(\lambda^k\rho_{8n+3}))=\binom{8n+1}{k-1}\eta_1^2$ in $\widetilde{KO}^{-1}(P^1)=Z_2\cdot\eta_1^2$. On the other hand, $j^*(\eta_1\gamma_{8n+3})=\eta_1^2$. Therefore we get

$$\varepsilon=\binom{8n+1}{k-1}\,.$$

iii) Define maps

$$f\colon P^{8n+2}\to GL(2^{4n}, \boldsymbol{H})$$

and

$$h\colon P^{8n+2}\times\mathrm{Spin}(8n+3)\to GL(2^{4n}, \boldsymbol{H})$$

by $f(\pi(x))=\Delta_{8n+3}(\bar{\iota}(x))^{-2}$ and $h(\pi(x), g)=\Delta_{8n+3}(\bar{\iota}(x))\Delta_{8n+3}(g)\Delta_{8n+3}(\bar{\iota}(x))^{-1}$ for $x\in S^{8n+3,0}$, $g\in \mathrm{Spin}(8n+3)$. Then it is verified by observing (4.1) again that

$$I(\kappa_{8n+3}) = t(\beta(f))\times 1+t(\beta(h))+\tilde{\kappa}_{8n+3}$$

in $KO^{-1}(P^{8n+2}\times \mathrm{Spin}(8n+3))$ under our identification.

Since $\widetilde{KO}^{-5}(P^{8n+2})=0$ by Proposition 3.4

$$t(\beta(f)) = 0\,.$$

Next we consider $t(\beta(h))$. Let $CX=[0, 1]\times X/\{1\}\times X$ for a space X and define isomorphisms of vector bundles

$$a\colon\ (S^{8m+3,0}\times_G H)\otimes_{\boldsymbol{R}}(C\,\mathrm{Spin}(8n+3)\times \boldsymbol{H}^{2^{4n}})\cong P^{8n+2}\times C\,\mathrm{Spin}(8n+3)\times \boldsymbol{H}^{2^{4n}},$$
$$b\colon\ P^{8n+2}\times \{0\}\times \mathrm{Spin}(8n+3)\times \boldsymbol{H}^{2^{4n}}\cong P^{8n+2}\times \{0\}\times \mathrm{Spin}(8n+3)\times \boldsymbol{H}^{2^{4n}},$$
$$c\colon\ (S^{8n+3,0}\times_G H)\otimes_{\boldsymbol{R}}(\{0\}\times \mathrm{Spin}(8n+3)\times \boldsymbol{H}^{2^{4n}})$$
$$\cong(S^{8n+3,0}\times_G H)\otimes_{\boldsymbol{R}}(\{0\}\times \mathrm{Spin}(8n+3)\times \boldsymbol{H}^{2^{4n}})$$

by

$$a([x, \lambda]\otimes([t, g], v)) = (\pi(x), [t, g], \Delta_{8n+3}(\bar{\iota}(x))\lambda v)\,,$$
$$b(\pi(x), [0, g], v) = (\pi(x), [0, g], \Delta_{8n+3}(\bar{\iota}(x))\Delta_{8n+3}(g)\Delta_{8n+3}(\bar{\iota}(x))^{-1}v)\,,$$
$$c([x, \lambda]\otimes([0, g], v)) = [x, \lambda]\otimes([0, g], \Delta_{8n+3}(g)v)$$

for $x\in S^{8n+3,0}$, $g\in \mathrm{Spin}(8n+3)$, $\lambda\in H$, $v\in \boldsymbol{H}^{2^{4n}}$ respectively. Here we denote by [ ] the equivalence classes. Let ξ_b and ξ_c be the quaternionic vector bundles over $P^{8n+2}\times\Sigma\,\mathrm{Spin}(8n+3)$ with b and c as clutching functions where ΣX denotes an unreduced cone of X. Then clearly we have

$$\xi_b\cong\xi_c$$

through the isomorphism a and since $\xi_b-2^{4n}\varepsilon_H$ can be viewed as an element of $KH^{-1}(P^{8n+2}\times \mathrm{Spin}(8n+3))$ we then see by construction that $\beta(h)=\xi_b-2^{4n}\varepsilon_H$, so that

$$\beta(h) = \xi_c-2^{4n}\varepsilon_H\,.$$

From this it follows that

$$t(\beta(h)) = (\gamma_{8n+2}+1)\otimes\tilde{\kappa}_{8n+3}+2^{4n}\eta_4\gamma_{8n+2}\otimes 1$$

since $\sigma\wedge\varepsilon_H=\eta_4$ by (1.5). However $2^{4n}\eta_4\gamma_{8n+2}=0$ by Proposition 3.4. Hence we have

$$t(\beta(h)) = (\gamma_{8n+2}+1)\otimes\tilde{\kappa}_{8n+2}\,.$$

which proves iii).

Let i: $\Sigma^{0,0}\in\Sigma^{3,0}$ be the inclusion. Then i^*: $\widetilde{KO}_G^{-4}(\Sigma^{3,0})\to\widetilde{KO}_G^{-4}(\Sigma^{0,0})=RO(G)\cdot\eta_4$ is injective because of $\widetilde{KO}^{-5}(P^2)=0$. We can choose a generator $\omega_{3,4}$ of $\widetilde{KO}_G^{-4}(\Sigma^{3,0})\cong Z$ so that

$$i^*(\omega_{3,4})=(1-H)\eta_4\,.$$

Let set

$$\delta=\phi_{8n,0}^{-1}\bar{\delta}\colon KO^*(P^{8n+2})\otimes_{KO^*(+)}KO^*(\mathrm{Spin}(8n+2))\\ \to\widetilde{KO}_G^*(\Sigma^{3,0}\wedge\mathrm{Spin}(8n+3)_+)\,.$$

Then we have

Lemma 4.5. i) $\delta(1\otimes\tilde{\kappa}_{8n+3})=-\omega_{3,4}\wedge 1$,
ii) $\delta(\mu_{8n+2}\otimes 1)=\eta_1\omega_{3,4}\wedge 1$.

Proof. i) Let a be the automorphism of the vector bundle $S^{8n+3,0}\times\mathrm{Spin}(8n+3)\times\boldsymbol{H}^{2^{4n}}$ given by $a(x,g,v)=(x,g,\Delta_{8n+2}(\bar{\iota}(x)g)v)$ for $x\in S^{8n+3,0}$, $g\in\mathrm{Spin}(8n+3)$, $v\in\boldsymbol{H}^{2^{4n}}$ and ξ_a denote the quaternionic vector bundle over $\Sigma S^{8n+3,0}\times\mathrm{Spin}(8n+3)$ with a as a clutching function. We view $\xi_a-2^{4n}\varepsilon_H$ as an element of $\widetilde{KH}(\Sigma^{8n+3,0}\wedge\mathrm{Spin}(8n+3)_+)$ in a natural manner. Then by considering (4.1) we obtain

$$\bar{\delta}(1\otimes\tilde{\kappa}_{8n+3})=t(\xi_a-2^{4n}\varepsilon_H)$$

in $\widetilde{KO}_G^{-4}(\Sigma^{8n+3,0}\wedge\mathrm{Spin}(8n+3)_+)$.

Define the isomorphism of vector bundles

$$b\colon CS^{8n+3,0}\times\mathrm{Spin}(8n+3)\times H\otimes_{\boldsymbol{R}}\boldsymbol{H}^{2^{4n}}\cong CS^{8n+3,0}\times\mathrm{Spin}(8n+3)\times\boldsymbol{H}^{2^{4n}}$$

by $b([t,x],g,1\otimes v)=([t,x],g,\Delta_{8n+3}(g)^{-1}v)$ for $t\in[0,1]$, $x\in S^{8n+3,0}$, $g\in\mathrm{Spin}(8n+3)$, $v\in\boldsymbol{H}^{2^{4n}}$ and denote by $\bar{b}$ the restriction of b to $S^{8n+3,0}\times\mathrm{Spin}(8n+3)\times H\otimes_{\boldsymbol{R}}\boldsymbol{H}^{2^{4n}}$ where we take $S^{8n+3,0}=\{0\}\times S^{8n+3,0}$. Also let $\xi_{a\bar{b}}$ be the vector bundle with $a\bar{b}$ as a clutching function. Then clearly $\xi_{a\bar{b}}\cong\xi_a$. Hence

$$\bar{\delta}(1\otimes\tilde{\kappa}_{8n+3})=t(\xi_{a\bar{b}}-2^{4n}\varepsilon_H)\wedge 1$$

in $\widetilde{KO}_G^{-4}(\Sigma^{8n+3,0}\wedge\mathrm{Spin}(8n+3)_+)$.

We consider $t(\xi_{a\bar{b}}-2^{4n}\varepsilon_H)\in\widetilde{KO}_G^{-4}(\Sigma^{8n+3,0})$. Let j: $\Sigma^{0,0}\subset\Sigma^{8n+3,0}$ be the inclusion. By construction we then have

$$cj^*t(\xi_{a\bar{b}}-2^{4n}\varepsilon_H)=2^{4n+1}(L-1)\mu^2$$

since $\sigma\wedge\varepsilon_H=\eta_4$ by (1.5) and $c(\eta_4)=2\mu^2$, so that

$$j^*t(\xi_{a\bar{b}}-2^{4n}\varepsilon_H)=2^{4n}(H-1)\eta_4\,.$$

Here j^* is injective because $\widetilde{KO}_G^{-4}(\Sigma^{8n+3,0}/\Sigma^{0,0})\cong KO^{-5}(P^{8n+2})=0$ by Proposition 3.4. Therefore by our choice of $\omega_{3,4}$ and (1.1) we see that

$$t(\xi_{a\bar{b}}-2^{4n}\varepsilon_H) = -\omega_{8n,0}\wedge\omega_{3,4}\,,$$

so that we get

$$\delta(1\otimes\kappa_{8n+3}) = -\omega_{8n,0}\wedge\omega_{3,4}$$

which proves i).

ii) Consider the exact sequence for $(B^{8n+3,0}, S^{8n+3,0})$ in KO_G-theory. Then

$$\begin{array}{ccc} \delta\colon KO^{-6}(P^{8n+2}) = KO_G^{-6}(S^{8n+3,0}) & \cong & \widetilde{KO}_G^{-5}(\Sigma^{8n+3,0})\,. \\ \| & & \| \\ Z_2\cdot\mu_{8n+2} & & Z_2\cdot\eta_1\omega_{8n,0}\wedge\omega_{3,4} \end{array}$$

Hence clearly $\delta(\mu_{8n+2})=\eta_1\omega_{8n,0}\wedge\omega_{3,4}$. From this and an inspection of (4.1) immediately we have $\delta(\mu_{8n+2}\otimes 1)=\eta_1\omega_{3,4}\wedge 1$, which completes the proof.

From Lemma 4.5 it follows that $\delta(\mu_{8n+2}\otimes 1+1\otimes\eta_1\tilde{\kappa}_{8n+3})=0$. Hence by exactness we see

(4.6) *There exists an element* $\nu_{8n+3}\in KO^{-6}(SO(8n+3))$ *such that*

$$I(\nu_{8n+3}) = \mu_{8n+2}\otimes 1+1\otimes\eta_1\tilde{\kappa}_{8n+3}\,.$$

Using this we determine the order of $\eta_4\xi_{8n+3}$. Observe the exact sequence (3.4) in [3]:

$$\begin{array}{ccccccc} \to \widetilde{KO}^{-6}(P^{8n+2}) & \xrightarrow{c} & \tilde{K}^{-6}(P^{8n+2}) & \xrightarrow{\delta} & \widetilde{KO}^{-4}(P^{8n+2}) & \to \\ \| & & \| & & \| & \\ Z_2\cdot\mu_{8n+2} & & Z_{2^{4n+1}}\cdot\mu^3c(\gamma_{8n+2}) & & Z_{2^{4n}}\cdot\eta_4\gamma_{8n+2} & \end{array}$$

Since $\delta(\mu^3c(\gamma_{8n+2}))=\eta_4\gamma_{8n+2}$, we have $\delta(2^{4n}\mu^3c(\gamma_{8n+2}))=0$. Evidently this implies

$$c(\mu_{8n+2}) = 2^{4n}\mu^3c(\gamma_{8n+2})\,.$$

Let us write I_c for I in complex case. Then it follows from (4.6) that $I_c(c(\nu_{8n+3}))=2^{4n}\mu^3c(\gamma_{8n+2})$, while $I_c(2^{4n}\mu^3c(\xi_{8n+3}))=2^{4n}\mu^3c(\gamma_{8n+2})$ and I_c is injective by Lemmas 3.8 and 3.7 in [14]. Hence we have

$$c(\nu_{8n+3}) = 2^{4n}\mu^3c(\xi_{8n+3})\,.$$

Consider again the exact sequence (3.4) when $X=SO(8n+3)$ in [3]. Then analogously we have $\delta(\mu^3c(\xi_{8n+3}))=\eta_4\xi_{8n+3}$ and therefore

$$\delta c(\nu_{8n+3}) = 2^{4n}\eta_4\xi_{8n+3}\,,$$

so that by exactness we obtain

$$2^{4n}\eta_4\xi_{8n+3} = 0\,.$$

On the other hand, $i^*(\eta_4\xi_{8n+3})=\eta_4\gamma_{8n+2}$, which has order 2^{4n} by Proposition 3.4. Hence we see

(4.7) *$\eta_4\xi_{8n+3}$ has order 2^{4n}.*

Lemma 4.8.

$$0 \leftarrow \widetilde{KO}_G^*(\Sigma^{3,0}\wedge \mathrm{Spin}(8n+3)_+) \overset{\delta}{\leftarrow} KO^*(P^{8n+2})\otimes_{KO^*(+)}KO^*(\mathrm{Spin}(8n+3))$$
$$\overset{I}{\leftarrow} KO^*(SO(8n+3)) \leftarrow 0$$

ix exact and there holds

$$\delta(xI(y)) = \delta(x)y\,.$$

Proof. The equality follows from (4.3) immediately and so it suffices to show

$$J=0\colon \widetilde{KO}_G^*(\Sigma^{8n+3,0}\wedge \mathrm{Spin}(8n+3)_+) \to KO^*(SO(8n+3))$$

in (4.2) when $l=8n+3$.

Consider the exact sequence

$$\cdots \to \tilde{K}_G^*(\Sigma^{1,0}\wedge \mathrm{Spin}(8n+3)_+) \overset{\chi}{\to} K^*(SO(8n+3))$$
$$\overset{\pi^*}{\to} K^*(\mathrm{Spin}(8n+3)) \to \cdots$$

which arises from the cofibering $S^{1,0}\to B^{1,0}\to \Sigma^{1,0}$ under obvious identifications. Then by Theorem 3.10 and Proposition 2.3 in [14] we see that $\mathrm{Im}\,\chi$ is additively generated by

$$\mu^i c(\xi_{8n+3}m(b_1, \cdots, b_{4n}))$$

for $0\leqq i\leqq 3$, $b_1, \cdots, b_{4n}=0, 1$ where $m(b_1, \cdots, b_{4n})=\beta(\lambda^1\rho_{8n+3})^{b_1}\cdots \beta(\lambda^{4n}\rho_{8n+3})^{b_{4n}}$.

Let $h=K, KO$ and let χ_h denote the homomorphism $\tilde{h}_G^*(\Sigma^{3,0}\wedge \mathrm{Spin}(8n+3)_+)$ $\to h^*(SO(8n+3))$ induced by an evident inclusion $\Sigma^{0,0}\subset\Sigma^{3,0}$. Using (1.3) and (3.2) we can verify from the result above that $\mathrm{Im}\,\chi_K$ is additively generated by

$$2\mu^i c(\xi_{8n+3}m(b_1, \cdots, b_{4n}))$$

for $0\leqq i\leqq 3$, $b_1, \cdots, b_{4n}=0, 1$, so that we see that $2\,\mathrm{Im}\,\chi_{KO}$ is additively generated by

$$2^2\xi_{8n+3}m(b_1, \cdots, b_{4n}) \quad\text{and}\quad 2\eta_4\xi_{8n+3}m(b_1, \cdots, b_{4n})$$

for $b_1, \cdots, b_{4n}=0, 1$.

Thus for $x\in\widetilde{KO}_G^*(\Sigma^{3,0}\wedge \mathrm{Spin}(8n+3)_+)$ we can write

$$2\chi_{KO}(x) = 2^2\xi_{8n+3}\Sigma\alpha(b_1, \cdots, b_{4n})m(b_1, \cdots, b_{4n})$$
$$+2\eta_4\xi_{8n+3}\Sigma\alpha(c_1, \cdots, c_{4n})m(c_1, \cdots, c_{4n})$$

for $\alpha(x_1, \cdots, x_{4n})\in Z$. Using (1.1) we therefore have

$$J(\omega_{8n,0}\wedge x) = -2^{4n-1}\xi_{8n+3}\chi_{KO}(x)$$
$$= 2^{4n+1}\xi_{8n+3}\Sigma\alpha(b_1, \cdots, b_{4n})m(b_1, \cdots, b_{4n})$$

because $2^{4n}\eta_4\xi_{8n+3}=0$ by (4.7). Applying I to this formula, by Lemma 4.4 we have

$$IJ(\omega_{8n,0}\wedge x) = 2^{4n+1}\gamma_{8n+2}\otimes\Sigma\alpha(b_1, \cdots, b_{4n})m(b_1, \cdots, b_{4n}),$$

which equals zero by exactness. Moreover because by Propositions 3.4 and 3.5 the order of γ_{8n+2} is 2^{4n+2} and $\{m(b_1, \cdots, b_{4n})\}$ is a basis of a free submodule of $KO^*(\mathrm{Spin}(8n+3))$ we see that $\alpha(b_1, \cdots, b_{4n})$ are divisible by 2, so that since by (3.3) ξ_{8n+3} has order 2^{4n+2} we obtain

$$J(\omega_{8n,0}\wedge x) = 0,$$

which completes the proof.

Theorem 4.9.

$$KO^*(SO(8n+3)) = \Lambda_{KO^*(+)}(\beta(\lambda^1\rho_{8n+3}), \cdots, \beta(\lambda^{4n}\rho_{8n+3}), \kappa_{8n+3})$$
$$\otimes_Z(Z\cdot 1\oplus Z_{2^{4n+2}}\cdot\xi_{8n+2}\oplus Z_2\cdot\nu_{8n+3})$$

as a $KO^*(+)$-module and the following relations hold:

$$\xi_{8n+3}^2 = -2\xi_{8n+3}, \quad 2^{4n}\eta_4\xi_{8n+3} = 0,$$
$$\beta(\lambda^k\rho_{8n+3})^2 = \eta_1\Big(\beta(\lambda^2(\lambda^k\rho_{8n+3}))+\binom{8n+3}{k}\beta(\lambda^k\rho_{8n+3})\Big) \qquad (1\leqq k\leqq 4n),$$
$$\kappa_{8n+3}^2 = \nu_{8n+3}^2 = \kappa_{8n+3}\xi_{8n+3} = \eta_4\nu_{8n+3} = 0,$$
$$\kappa_{8n+3}\nu_{8n+3} = \eta_1^2\beta(\lambda^2\Delta_{8n+3})\xi_{8n+3},$$
$$\eta_1^2\nu_{8n+3} = 2^{4n+1}\xi_{8n+3},$$
$$\eta_1\kappa_{8n+3} = \xi_{8n+3}\nu_{8n+3}$$

in which $\otimes_Z$ is left out.

Proof. We begin with the relations. The first two ones are already shown in (3.2) and [7; p. 67] (also (1.7) in [14]). According to Lemma 4.8, I is an injection of algebras. Hence using the formulas in Lemma 4.4 and (4.6) the others follows from the relations in Propositions 3.4, 3.5, (3.3) and (3.7).

Let R_{8n+3} denote the right side of the above equality together with the relations. Then by a similar argument it is verified that R_{8n+3} is a subalgebra of $KO^*(SO(8n+3))$.

Moreover, in virtue of the surjectivity of δ and the formulas of Lemmas 4.5 and 4.8 it follows that $\widetilde{KO}_G^*(\Sigma^{3,0}\wedge \mathrm{Spin}(8n+3)_+)$ is generated by $\omega_{3,4}\wedge x$ for $x\in R_{8n+3}$ as a $KO^*(+)$-module.

Let X be a G-space with base point. We consider the exact sequence in KO_G-theory associated with a cofibering $X=\Sigma^{0,0}\wedge X\xrightarrow{i}\Sigma^{1,0}\wedge X\xrightarrow{p}\Sigma^{1,0}/\Sigma^{0,0}\wedge X$ where i, p are evident maps. Since $\Sigma^{1,0}/\Sigma^{0,0}\wedge X\approx S_+^{0,1}\wedge\Sigma^{0,1}\wedge X$ there is an isomorphism $\widetilde{KO}_G^*(\Sigma^{1,0}/\Sigma^{0,0}\wedge X)\cong\widetilde{KO}^*(\Sigma^{0,1}\wedge X)$. By observing the above homeomorphism we can check that the composite of this isomorphism with p^* agrees with a transfer

$$\tau\colon \widetilde{KO}^*(\Sigma^{0,1}\wedge X)\to\widetilde{KO}_G^*(\Sigma^{1,0}\wedge X),$$

so that we have an exact sequence

$$\cdots\to\widetilde{KO}^*(\Sigma^{0,1}\wedge X)\xrightarrow{\tau}\widetilde{KO}_G^*(\Sigma^{1,0}\wedge X)\xrightarrow{i^*}\widetilde{KO}_G^*(X)\to\cdots.$$

Here take X to be $\Sigma^{3,4}\wedge\mathrm{Spin}(8n+3)_+$. Then this yields the following exact sequence:

$$\begin{array}{cccc}\cdots\to\widetilde{KO}^*(\Sigma^{0,8}\wedge\mathrm{Spin}(8n+3)_+) & \xrightarrow{\tau} & \widetilde{KO}_G^*(\Sigma^{4,4}\wedge\mathrm{Spin}(8n+3)_+) & \xrightarrow{i^*} \\ \cong & & \cong & \\ KO^*(\mathrm{Spin}(8n+3))\cdot\psi(\omega_{4,4}) & & KO^*(SO(8n+3))\cdot\omega_{4,4} & \\ & & \to\widetilde{KO}_G^*(\Sigma^{3,4}\wedge\mathrm{Spin}(8n+3)_+)\to\cdots & \end{array}$$

with Thom isomorphisms. From (1.2) and the definition of $\omega_{3,4}$ it follows that $i^*(\omega_{4,4})=\omega_{3,4}\wedge 1$. Therefore by the result above we see that

$$i^*(R_{8n+3}\cdot\omega_{4,4})=\widetilde{KO}_G^*(\Sigma^{3,4}\wedge\mathrm{Spin}(8n+3)_+) \qquad \text{(a).}$$

By construction we obtain

$$\tau(\tilde{\kappa}_{8n+3})=\kappa_{8n+2}\,.$$

Using this and the properties of τ such that

$$\tau\psi(y)=(1+H)y \quad\text{and}\quad \tau(x\psi(y))=\tau(x)y$$

we see by Proposition 3.5 that

$$\mathrm{Im}\,\tau\subset R_{8n+3}\cdot\omega_{4,4} \qquad \text{(b).}$$

From (a) and (b) it follows that $KO^*(SO(8n+3))\subset R_{8n+3}$, that is,

$$KO^*(SO(8n+3))=R_{8n+3}$$

which completes the proof.

To express $\beta(\lambda^2(\lambda^k\rho_{8n+3}))$ and $\beta(\lambda^2\Delta_{8n+3}))$ appeared in Theorem 4.9 in terms of the generators of $KO^*(SO(8n+3))$ it is sufficient to know about $\beta(\lambda^{4n+1}\rho_{8n+3})$ as in $KO^*(\mathrm{Spin}(8n+3))$. In virtue of the injectivity of I and Lemma 4.4, this is easily obtained from (3.7) as follows:

$$(4.10)\qquad \beta(\lambda^{4n+1}\rho_{8n+3}) = 2^{4n}\eta_4\kappa_{8n+3}-\beta(\lambda^{4n}\rho_{8n+3}) \\ -\beta(\lambda^{4n-1}\rho_{8n+3})-\cdots-\beta(\lambda^1\rho_{8n+3})\,.$$

b) $KO^*(SO(8n+4))$

Let us define maps

$$\delta,\ \varepsilon\colon SO(8n+4)\to GL(2^{4n},\boldsymbol{H})$$

by $\delta(\pi(g))=\Delta^-_{8n+4}(g)^{-1}\Delta^+_{8n+4}(g)$, $\varepsilon(\pi(g))=\Delta^+_{8n+4}(g)^2$ for $g\in\mathrm{Spin}(8n+4)$ and put

$$\kappa_{8n+4}=t(\beta(\delta)),\ \theta=t(\beta(\varepsilon))\in KO^{-5}(SO(8n+4))\,.$$

We consider (4.2) when $l=8n+4$.

Lemma 4.11. i) $I(\xi_{8n+4})=\gamma_{8n+3}\otimes 1$,

ii) $I(\beta(\lambda^k\rho_{8n+4}))=1\otimes\beta(\lambda^k\rho_{8n+4})+\binom{8n+2}{k-1}\eta_1\gamma_{8n+3}\otimes)1\quad(1\leqq k\leqq 8n+4)$,

iii) $I(\kappa_{8n+4})=1\otimes(\tilde{\kappa}^+_{8n+4}-\tilde{\kappa}^-_{8n+4})-\bar{\nu}_{8n+3}\otimes 1$,

iv) $I(\theta_{8n+4})=(\gamma_{8n+3}+2)\otimes\tilde{\kappa}^+_{8n+4}-\bar{\nu}_{8n+3}\otimes 1$.

Proof. i), ii) Similar to the proofs of i), ii) of Lemma 4.4.

iii) Considering (4.1) we see that $I(\kappa_{8n+4})$ is represented by the map

$$f\colon P^{8n+3}\times\mathrm{Spin}(8n+4)\to GL(2^{4n},\boldsymbol{H})$$

given by $f(\pi(x),g)=\Delta^-_{8n+4}(g)^{-1}\Delta^-_{8n+4}(\bar{\iota}(x))\Delta^+_{8n+4}(\bar{\iota}(x))^{-1}\Delta^+_{8n+4}(g)$ for $x\in S^{8n+4,0}$, $g\in\mathrm{Spin}(8n+4)$. But the composite of f with an inclusion $GL(2^{4n},\boldsymbol{H})\subset GL(3\cdot 2^{4n},\boldsymbol{H})$ is evidently homotopic to the map from $P^{8n+3}\times\mathrm{Spin}(8n+4)$ to $GL(3\cdot 2^{4n},\boldsymbol{H})$ given by

$$(\pi(x),g)\mapsto\Delta^-_{8n+4}(g)^{-1}\oplus\Delta^+_{8n+4}(g)\oplus\Delta^-_{8n+4}(\bar{\iota}(x))\Delta^+_{8n+4}(\bar{\iota}(x))^{-1}$$

for $x\in S^{8n+4,0}$, $g\in\mathrm{Spin}\ (8n+4)$. iii) is immediate from this.

iv) Define a map

$$f\colon P^{8n+3}\to GL(2^{4n},\boldsymbol{H})$$

by $f(\pi(x))=\Delta^+_{8n+4}(\bar{\iota}(x))^2$ for $x\in S^{8n+4,0}$. Then by arguments parallel to iii) of Lemma 4.4 we have

$$I(\theta_{8n+3})=(\gamma_{8n+3}+2)\otimes\tilde{\kappa}^+_{8n+4}+t(\beta(f))\otimes 1$$

where $t(\beta(f))\in\widetilde{KO}^{-5}(P^{8n+3})=Z\cdot\bar{\nu}_{8n+3}$.

Because

$$\tilde{K}^{-5}(S^{8n+3}) = \tilde{K}^{-4}(\mathrm{Spin}(8n+5)/\mathrm{Spin}(8n+4)) = Z\cdot\mu^2 s(\beta(\Delta^+_{8n+4}))$$

(see e.g. Theorem 13.3 in [11; § 13]), observing the composite

$$\widetilde{KO}^{-5}(P^{8n+3}) \xrightarrow{c} \tilde{K}^{-5}(P^{8n+3}) \xrightarrow{\pi^*} \tilde{K}^{-5}(S^{8n+3})$$

we have by definition

$$\pi^* c(\bar{\nu}_{8n+3}) = 2\mu^2 s(\beta(\Delta^+_{8n+4})) \quad \text{and} \quad \pi^* c(t(\beta(f)) = -2\mu^2 s(\beta(\Delta^+_{8n+4}))\,.$$

This shows

$$t(\beta(f)) = -\bar{\nu}_{8n+3}\,,$$

which completes the proof.

In proving Proposition 3.4 we showed that $\delta'(\mu_{8n+3})=\varepsilon\eta_1\omega_{8n,0}\wedge\omega_{4,4}$, $\varepsilon=1$ or H for the coboundary homomorphism δ': $KO^*(P^{8n+3})=KO^*_G(S^{8n+4,0})\to\widetilde{KO}^*_G(\Sigma^{8n+4,0})$. However by definition we may assume that

$$\delta'(\mu_{8n+3}) = \eta_1\omega_{8n,0}\wedge\omega_{4,4}\,.$$

Take $\omega_{8n+4,4}$ to be $\omega_{8n,0}\wedge\omega_{4,4}$ and set

$$\delta = \phi^{-1}_{8n+4,4}\bar{\delta}\colon KO^*(\mathrm{P}^{8n+3})\otimes_{KO^*(+)}KO^*(\mathrm{Spin}(8n+4)) \to KO^*(SO(8n+4))\,.$$

Then we have

Lemma 4.12. i) $\delta(1\otimes\tilde{\kappa}^+_{8n+4})=\xi_{8n+4}+1$, $\delta(1\otimes\tilde{\kappa}^-_{8n+4})=-1$,
ii) $\delta(\bar{\nu}_{8n+3}\otimes 1)=\xi_{8n+4}+2$,
iii) $\delta(\nu_{8n+3}\otimes 1)=\eta_1$.

Proof. i) As in the proof of Lemma 4.5, i), by using the half spin representations Δ^+_{8n+4}, Δ^-_{8n+4} we can define quaternionic vector bundles $\xi^+_{a\bar{b}}$, $\xi^-_{a\bar{b}}$ over $\Sigma^{8n+4,0}$ similar to $\xi_{a\bar{b}}$ such that

$$\bar{\delta}(1\otimes\tilde{\kappa}^+_{8u+4}) = t(\xi^+_{a\bar{b}}-2^{4n}\varepsilon_H)\wedge 1,\ \bar{\delta}(1\otimes\tilde{\kappa}^-_{8n+4}) = t(\xi^-_{a\bar{b}}-2^{4n}\varepsilon_H)\wedge 1$$

and

$$t(\xi^+_{a\bar{b}}-2^{4n}\varepsilon_H) = H\omega_{8n,0}\wedge\omega_{4,4},\ t(\xi^-_{a\bar{b}}-2^{4n}\varepsilon_H) = -H\omega_{8n,0}\wedge\omega_{4,4}\,.$$

Hence i) is clear.

ii) Immediate from (c) in the proof of Proposition 3.4, 2), that is,

$$\delta'(\bar{\nu}_{8n+3}) = (H+1)\omega_{8n,0}\wedge\omega_{4,4}\,.$$

iii) Similarly immediate from the formula for μ_{8n+3} above.

From Lemma 4.12 it follows that $\delta(\mu_{8n+3}\otimes 1+1\otimes\eta_1\tilde{\kappa}^-_{8n+3})=0$ and hence by exactness we have

(4.13) *There exists an element* $\nu_{8n+4}\in KO^{-6}(SO(8n+3))$ *such that*

$$I(\nu_{8n+4})=\mu_{8n+3}\otimes 1+1\otimes\tilde{\kappa}_{8n+4}.$$

Making use of (4.13), by arguments parallel to (4.6) we get

(4.14) $\eta_4\xi_{8n+4}$ *has order* 2^{4n}.

Observe J in (4.2) when $l=8n+4$. Similarly using (1.1) and (1.2) we then have

$$J(\omega_{8n,0}\wedge\omega_{4,4}\wedge x) = -2^{4n}\eta_4\xi_{8n+4}x$$

for $x\in KO^*(SO(8n+4))$, so that by (4.14) we see that

$$J=0\,.$$

Clearly from this and (4.3) we get

Lemma 4.15.

$$0\leftarrow KO^*(SO(8n+4))\overset{\delta}{\leftarrow} KO^*(P^{8n+3})\otimes_{KO^*(+)}KO^*(\mathrm{Spin}(8n+4))$$
$$\overset{I}{\leftarrow} KO^*(SO(8n+4))\leftarrow 0$$

is exact and there holds

$$\delta(xI(y))=\delta(x)y\,.$$

Using Lemmas 4.11, 4.12, 4.15, (4.13), Propositions 2.4, 3.5, (3.3) and (4.14), by arguments similar to Theorem 4.9 we obtain

Theorem 4.16.

$$KO^*(SO(8n+4))=\Lambda_{KO^*(+)}(\beta(\lambda^1\rho_{8n+4}),\cdots,\beta(\lambda^{4n}\rho_{8n+4}),\kappa_{8n+4},\theta_{8n+4})$$
$$\otimes_Z(Z\cdot 1\oplus Z_{2^{4n+2}}\cdot\xi_{8n+4}\oplus Z_2\cdot\nu_{8n+4})$$

as a $KO^*(+)$*-module and the following relatoins holds*:

$$\xi_{8n+4}^2=-2\xi_{8n+4}\,,\quad 2^{4n}\eta_4\xi_{8n+4}=0\,,$$
$$\beta(\lambda^k\rho_{8n+4})^2=\eta_1\left(\beta(\lambda^2(\lambda^k\rho_{8n+4}))+\binom{8n+4}{k}\beta(\lambda^k\rho_{8n+4})\right)\quad(1\leqq k\leqq 8n+4)\,,$$
$$\kappa_{8n+4}^2=\theta_{8n+4}^2=\nu_{8n+4}^2=\theta_{8n+4}\xi_{8n+4}=\eta_4\nu_{8n+4}=\eta_1\kappa_{8n+4}\theta_{8n+4}\nu_{8n+4}=0$$
$$\eta_1^2\nu_{8n+4}=2^{4n+1}\xi_{8n+4}$$

in which $\otimes_Z$ *is lefto out.*

Arguing as in the case a) and noticing that $\binom{8n+2}{4n+1}\equiv 0 \bmod 2$ we have

$$(4.17)\qquad \begin{aligned}\beta(\lambda^{4n+1}\rho_{8n+4}) &= 2^{4n}\eta_4(\theta_{8n+4}-\kappa_{8n+4})-\beta(\lambda^{4n-1}\rho_{8n+4})\\ &\quad -\beta(\lambda^{4n-3}\rho_{8n+4})-\cdots-\beta(\lambda^{1}\rho_{8n+4}),\\ \beta(\lambda^{4n+2}\rho_{8n+4}) &= 2^{4n+1}\eta_4(\theta_{8n+4}-\kappa_{8n+4})-2\beta(\lambda^{4n}\rho_{8n+4})\\ &\quad -2\beta(\lambda^{4n-2}\rho_{8n+4})-\cdots-2\beta(\lambda^{2}\rho_{8n+4}),\end{aligned}$$

from which we can express $\beta(\lambda^2(\lambda^k\rho_{8n+4}))$ in the form desired.

c) $KO^*(SO(8n+5))$

This case is discussed exactly as in the case a). Define a map

$$\varepsilon\colon SO(8n+5)\to GL(2^{4n+1}, \boldsymbol{H})$$

by $\varepsilon(\pi(g))=\Delta_{8n+5}(g)^2$ for $g\in \mathrm{Spin}(8n+5)$ and set

$$\kappa_{8n+5}=t(\beta(\varepsilon))\in KO^{-5}(SO(8n+5)).$$

We consider (4.2) when $l=8n+5$.

Lemma 4.18. i) $I(\xi_{8n+5})=\gamma_{8n+4}\otimes 1$,

ii) $I(\beta(\lambda^k\rho_{8n+5}))=1\otimes\beta(\lambda^k\rho_{8n+5})+\binom{8n+3}{k-1}\eta_1\gamma_{8n+4}\otimes 1\quad(1\leqq k\leqq 8n+5)$,

iii) $I(\kappa_{8n+5})=(\gamma_{8n+4}+2)\otimes\tilde{\kappa}_{8n+5}$.

Proof. Similar to the proof of Lemma 4.4.

Take $\omega_{8n+4,4}$ to be $\omega_{8n,8}\wedge\omega_{4,4}$ as in the case b) and set

$$\delta=\phi^{-1}_{8n+4,4}\bar{\delta}\colon KO^*(P^{8n+4})\otimes_{KO^*(+)}KO^*(\mathrm{Spin}(8n+5))\to \widetilde{KO}{}^*_G(\Sigma^{1,0}\wedge\mathrm{Spin}(8n+5)_+).$$

Let $\omega_{1,0}\in\widetilde{KO}{}^*_G(\Sigma^{1,0})\cong Z$ be a generator such that

$$i^*(\omega_{1,0})=1-H\in\widetilde{KO}_G(\Sigma^{0,0})=R(G)$$

where $i\colon\Sigma^{0,0}\subset\Sigma^{1,0}$ be the inclusion. Then we have

Lemma 4.19. i) $\delta(1\otimes\tilde{\kappa}_{8n+5})=-\omega_{1,0}\wedge 1$,

ii) $\delta(\mu_{8n+4}\otimes 1)=\eta_1\omega_{1,0}\wedge 1$.

Proof. Also similar to the proof of Lemma 4.5.

By this lemma we see

(4.20) *There exisits an element* $\nu_{8n+5}\in KO^{-6}(SO(8n+5))$ *such that*

$$I(\nu_{8n+5})=\mu_{8n+4}\otimes 1+1\otimes\eta_1\tilde{\kappa}_{8n+5}.$$

Analogously from this we have

(4.21) $\eta_4\xi_{8n+5}$ *has order* 2^{4n+1}

and hence it follows that J in (4.2) when $l=8n+5$ is a zero map. Consequently we obtain

Lemma 4.22.

$$0 \leftarrow \widetilde{KO}_G^*(\Sigma^{1,0}\wedge \mathrm{Spin}(8n+5)_+) \overset{\delta}{\leftarrow} KO^*(P^{8n+4})\otimes_{KO^*(+)}KO^*(\mathrm{Spin}(8n+5))$$
$$\overset{I}{\leftarrow} KO^*(SO(8n+5)) \leftarrow 0$$

is exact and there holds

$$\delta(xI(y)) = \delta(x)y\,.$$

Theorem 4.23.

$$KO^*(SO(8n+5)) = \Lambda_{KO^*(+)}(\beta(\lambda^1\rho_{8n+5}), \cdots, \beta(\lambda^{4n+1}\rho_{8n+5}), \kappa_{8n+5})$$
$$\otimes_Z(Z\cdot 1\oplus Z_{2^{4n+3}}\cdot\xi_{8n+5}\oplus Z_2\cdot\nu_{8n+5})$$

as a $KO^*(+)$-module and the following relations hold:

$$\xi_{8n+5}^2 = -2\xi_{8n+5}\,,\quad 2^{4n+1}\eta_4\xi_{8n+5} = 0\,,$$
$$\beta(\lambda^k\rho_{8n+5})^2 = \eta_1\Big(\beta(\lambda^2(\lambda^k\rho_{8n+5})) + \binom{8n+5}{k}\beta(\lambda^k\rho_{8n+5})\Big) \quad (1\leqq k\leqq 8n+5)\,,$$
$$\kappa_{8n+5}^2 = \nu_{8n+5}^2 = \kappa_{8n+5}\xi_{8n+5} = \eta_4\nu_{8n+5} = 0\,,$$
$$\kappa_{8n+5}\nu_{8n+5} = \eta_1^2\beta(\lambda^2\Delta_{8n+5})\xi_{8n+5}\,,$$
$$\eta_1^2\nu_{8n+5} = 2^{4n+2}\xi_{8n+5}\,,$$
$$\eta_1\kappa_{8n+5} = \xi_{8n+5}\nu_{8n+5}$$

where $\otimes_Z$ is left out.

Proof. We write R_{8n+5} for the right side of the equality together with the relations above. As in proving Theorem 4.9 we can then verify that R_{8n+5} is a subalgebra of $KO^*(SO(8n+5))$.

To prove that $R_{8n+5}\supset KO^*(SO(8n+5))$ we make use of exactness at $\widetilde{KO_G^*(X)}$ of the exact sequence stated in the proof of Theorem 4.9. By definition we see that the homomorphism next to i^* coincides with a forgetful homomorphism under an obvious identification. Hence taking X to be $\mathrm{Spin}(8n+5)_+$, we have an exact sequence

$$\widetilde{KO}_G^*(\Sigma^{1,0}\wedge \mathrm{Spin}(8n+5)_+) \to KO^*(SO(8n+5)) \overset{\pi^*}{\to} KO^*(\mathrm{Spin}(8n+5))\,.$$

Clearly we have

$$\pi^*(\beta(\lambda^k\rho_{8n+5})) = \beta(\lambda^k\rho_{8n+5}),\ \pi^*(\xi_{8n+5}) = 0$$

and

$$\pi^*(\kappa_{8n+5}) = 2\tilde{\kappa}_{8n+5}\,.$$

Moreover considering π^* in the complex case and using the results for $K^*(SO(8n+5))$ and $K^*(\mathrm{Spin}(8n+5))$ [14], we see that $\tilde{\kappa}_{8n+5}$ does not lie in $\mathrm{Im}\,\pi^*$, so that we have

$$\mathrm{Im}\,\pi^* = \pi^*(R_{8n+5})\,.$$

In virtue of the surjectivity of δ; by the same argument as in the proof of Theorem 4.9 it is verified that $\tilde{K}_G^*(\Sigma^{1,0}\wedge \mathrm{Spin}(8n+5)_+)$ is generated by $\omega_{1,0}\wedge x$, $x\in R_{8n+5}$. Therefore by exactness we obtain

$$KO^*(SO(8n+5)) = R_{8n+5}\,,$$

which completes the proof.

Furthermore we have a relation similar to (4.10):

$$\beta(\lambda^{4n+2}\rho_{8n+5}) = 2^{4n+1}\eta_4\kappa_{8n+5}-\beta(\lambda^{4n+1}\rho_{8n+5}) -\beta(\lambda^{4n}\rho_{8n+5})-\cdots-\beta(\lambda^{1}\rho_{8n+5})\,. \tag{4.24}$$

Similarly the calculation of $\beta(\lambda^2(\lambda^k\rho_{8n+5}))$ and $\beta(\lambda^2\Delta_{8n+5})$ is reduced to this formula.

REMARK TO THE PREVIOUS PAPER [14]. Analogously we see that ε's stated in Lemmas 4.3, 4.10 and 4.14 of [14] are equal to $\binom{8n-2}{k-1}$, $\binom{8n-3}{k-1}$ and $\binom{8n-1}{k-1}$ respectively and hence we obtain

$$\beta(\lambda^{4n-1}\rho_{8n-1}) = 2^{4n-1}\beta(\varepsilon_{8n-1})-\beta(\lambda^{4n-2}\rho_{8n-1})-\beta(\lambda^{4n-3}\rho_{8n-1})-\cdots-\beta(\lambda^{1}\rho_{8n-1})\,,$$
$$\beta(\lambda^{4n-1}\rho_{8n}) = 2^{4n-1}(\beta(\varepsilon_{8n})-\beta(\delta_{8n}))-\beta(\lambda^{4n-3}\rho_{8n})-\beta(\lambda^{4n-5}\rho_{8n})-\cdots-\beta(\lambda^{1}\rho_{8n})\,,$$
$$\beta(\lambda^{4n}\rho_{8n}) = 2^{4n}(\beta(\varepsilon_{8n})-\beta(\delta_{8n}))-2\beta(\lambda^{4m-2}\rho_{8n})-2\beta(\lambda^{4n-4}\rho_{8n})-\cdots-2\beta(\lambda^{2}\rho_{8n})\,,$$
$$\beta(\lambda^{4n}\rho_{8n+1}) = 2^{4n}\beta(\varepsilon_{8n+1})-\beta(\lambda^{4n-1}\rho_{8n+1})-\beta(\lambda^{4n-3}\rho_{8n+1})-\cdots-\beta(\lambda^{1}\rho_{8n+1})$$

with the notations as in [14].

Using these formulas it is also possible to express $\beta(\lambda^2\Delta_{8n-1})$, $\beta(\lambda^2\Delta_{8n+1})$ and $\beta(\lambda^2(\lambda^k\rho_l))$, $l=8n-1$, $8n$, $8n+1$, in terms of the generators.

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