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A GROUP-THEORETIC CHARACTERIZATION OF THE SPACE OBTAINED BY OMITTING THE COORDINATE HYPERPLANES FROM THE COMPLEX EUCLIDEAN SPACE

Dedicated to Professor Makoto Namba on his sixtieth birthday

AKIO KODAMA and SATORU SHIMIZU

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Introduction

In the study of the holomorphic automorphism group $\mathrm{Aut}(M)$ of a complex manifold M, it seems to be natural to direct our attention not only to the abstract group structure of $\mathrm{Aut}(M)$ but also to its topological group structure equipped with the compact-open topology. In fact, a well-known theorem of H. Cartan says that the topological group of the holomorphic automorphisms of a bounded domain in $\mathbf{C}^n$ has the structure of a Lie group, and this result enables us to make various kinds of detailed studies of bounded domains in $\mathbf{C}^n$. On the other hand, in contrast to the case of bounded domains, the holomorphic automorphism group $\mathrm{Aut}(\mathbf{C}^k \times (\mathbf{C}^*)^l)$ of the unbounded domain $\mathbf{C}^k \times (\mathbf{C}^*)^l$ is terribly big when $k+l \geq 2$, and cannot have the structure of a Lie group. But, by looking at topological subgroups of $\mathrm{Aut}(\mathbf{C}^k \times (\mathbf{C}^*)^l)$ with Lie group structures, we can find a lead to apply the Lie group theory to the investigation of the problems related to the structure of $\mathrm{Aut}(\mathbf{C}^k \times (\mathbf{C}^*)^l)$. In the present paper, we try to approach from this standpoint to the fundamental problem of what complex manifold has the holomorphic automorphism group isomorphic to $\mathrm{Aut}(\mathbf{C}^k \times (\mathbf{C}^*)^l)$ as topological groups. Namely, we prove the following result with the aid of the theory of Reinhardt domains developed in Shimizu [8], [9] (cf. Kruzhilin [6]).

Main Theorem. *Let M be a connected Stein manifold of dimension n. Assume that $\mathrm{Aut}(M)$ is isomorphic to $\mathrm{Aut}(\mathbf{C}^k \times (\mathbf{C}^*)^{n-k})$ as topological groups. Then M is biholomorphically equivalent to $\mathbf{C}^k \times (\mathbf{C}^*)^{n-k}$.*

As a consequence of the above theorem, we can obtain the fundamental result on the topological group structure of $\mathrm{Aut}(\mathbf{C}^k \times (\mathbf{C}^*)^l)$.

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Corollary. *If two pairs* (k, l) *and* (k', l') *of nonnegative integers do not coincide, then the topological groups* $\mathrm{Aut}(\mathbf{C}^k \times (\mathbf{C}^*)^l)$ *and* $\mathrm{Aut}(\mathbf{C}^{k'} \times (\mathbf{C}^*)^{l'})$ *are not isomorphic.*

It should be remarked that, as shown in Ahern and Rudin [1], the groups $\mathrm{Aut}(\mathbf{C}^n)$ and $\mathrm{Aut}(\mathbf{C}^m)$ are isomorphic as abstract groups precisely when $n = m$. Also, as a consequence of the study of $U(n)$-actions on complex manifolds of dimension n, a related result to our Main Theorem has been obtained by Isaev and Kruzhilin [4].

This paper is organized as follows. In Section 1, we collect some preliminary facts. In particular, two main tools for our study are given. One is a tool to obtain the normal form of some compact group action on a Reinhardt domain, and the other is a tool for the standardization of torus actions on complex manifolds. Section 2 is devoted to the proof of the Main Theorem and its corollary. Our method used in Section 2 has interesting applications. As one of such applications, we discuss in Section 3 a new approach to the study of $U(n)$-actions on complex manifolds of dimension n.

1. Lie group actions, Reinhardt domains and torus actions

We begin with a basic fact on Lie group actions on complex manifolds. Let M be a complex manifold. An *automorphism of* M means a biholomorphic mapping of M onto itself. We denote by $\mathrm{Aut}(M)$ the topological group of all automorphisms of M equipped with the compact-open topology. Let G be a Lie group and consider a continuous group homomorphism $\rho\colon G \to \mathrm{Aut}(M)$. Then the mapping

$$G \times M \ni (g, p) \longmapsto (\rho(g))(p) \in M$$

is continuous. It follows from Akhiezer [2] that this mapping is actually of class C^ω, and therefore G acts on M as a Lie transformation group. In view of this, when a continuous group homomorphism $\rho\colon G \to \mathrm{Aut}(M)$ is given, we say that *G acts on M as a Lie transformation group through* ρ. Also, the action of G on M is called *effective* if ρ is injective.

We now recall basic concepts and results on Reinhardt domains (cf. [8], [9]). We denote by $U(k)$ the *unitary group of degree* k. Write $T^n = (U(1))^n$. The n-dimensional torus T^n acts as a group of automorphisms on $\mathbf{C}^n$ by the standard rule

$$\alpha \cdot z = (\alpha_1 z_1, \ldots, \alpha_n z_n) \quad \text{for } \alpha = (\alpha_1, \ldots, \alpha_n) \in T^n \text{ and } z = (z_1, \ldots, z_n) \in \mathbf{C}^n.$$

By definition, a *Reinhardt domain* D in $\mathbf{C}^n$ is a domain in $\mathbf{C}^n$ which is stable under this action of T^n. Each element α of T^n then induces an automorphism π_α of D given by $\pi_\alpha(z) = \alpha \cdot z$, and the mapping ρ_D sending α to π_α is an injective continuous group homomorphism of T^n into $\mathrm{Aut}(D)$. The subgroup $\rho_D(T^n)$ of $\mathrm{Aut}(D)$ is denoted by $T(D)$.

Let f be a holomorphic function on a Reinhardt domain D in $\mathbf{C}^n$. Then f can be expanded uniquely into a "Laurent series"

$$f(z) = \sum_{\nu \in \mathbf{Z}^n} a_\nu z^\nu,$$

which converges absolutely and uniformly on any compact set in D, where $z = (z_1, \dots, z_n)$, $\nu = (\nu_1, \dots, \nu_n)$, and $z^\nu = z_1^{\nu_1} \cdots z_n^{\nu_n}$. The following lemma is a consequence of the uniqueness of the Laurent series expansion.

Lemma 1.1. *If f satisfies the condition that, for some $\nu_0 \in \mathbf{Z}^n$,*

$$f(\alpha \cdot z) = \alpha^{\nu_0} f(z) \quad \textit{for all } \alpha \in T^n \textit{ and all } z \in D,$$

then f has the form $f(z) = a_{\nu_0} z^{\nu_0}$.

Proof. Since we have

$$f(\alpha \cdot z) = \sum_{\nu \in \mathbf{Z}^n} \alpha^\nu a_\nu z^\nu \quad \text{and} \quad \alpha^{\nu_0} f(z) = \sum_{\nu \in \mathbf{Z}^n} \alpha^{\nu_0} a_\nu z^\nu,$$

it follows from the assumption that, for every $\nu \in \mathbf{Z}^n$, we have

$$\alpha^\nu a_\nu = \alpha^{\nu_0} a_\nu \quad \text{for all } \alpha \in T^n.$$

This implies that if $a_\nu \neq 0$, then $\nu = \nu_0$, and our lemma is proved. □

We denote by $\Pi(\mathbf{C}^n)$ the group of all automorphisms of $\mathbf{C}^n$ of the form

$$\mathbf{C}^n \ni (z_1, \dots, z_n) \longmapsto (\alpha_1 z_1, \dots, \alpha_n z_n) \in \mathbf{C}^n,$$

where $(\alpha_1, \dots, \alpha_n) \in (\mathbf{C}^*)^n$. For a Reinhardt domain D in $\mathbf{C}^n$, we denote by $\Pi(D)$ the subgroup of $\Pi(\mathbf{C}^n)$ consisting of all elements of $\Pi(\mathbf{C}^n)$ leaving D invariant. Identifying $\Pi(\mathbf{C}^n)$ with the multiplicative group $(\mathbf{C}^*)^n$, we see that, when $\Pi(D)$ is regarded as a topological subgroup of $\mathrm{Aut}(D)$, it is isomorphic to a closed Lie subgroup of $(\mathbf{C}^*)^n$. Using Lemma 1.1, we obtain a characterization of $\Pi(D)$ as a subgroup of $\mathrm{Aut}(D)$.

Lemma 1.2. *Let D be a Reinhardt domain in $\mathbf{C}^n$. Then $\Pi(D)$ is the centralizer $C_{\mathrm{Aut}(D)}(T(D))$ of $T(D)$ in $\mathrm{Aut}(D)$.*

Proof. It is immediate that $\Pi(D) \subset C_{\mathrm{Aut}(D)}(T(D))$. To prove the reverse inclusion, let φ be any element of $C_{\mathrm{Aut}(D)}(T(D))$ and write $\varphi = (\varphi_1, \dots, \varphi_n)$, where $\varphi_1, \dots, \varphi_n$ are holomorphic functions on D. Then, for every $i = 1, \dots, n$, we have

$$\varphi_i(\alpha \cdot z) = \alpha_i \varphi_i(z) = \alpha^{e_i} \varphi_i(z) \quad \text{for all } \alpha \in T^n \text{ and all } z \in D,$$

where each e_i denotes the element of $\mathbf{Z}^n$ whose i-th component is equal to 1 and whose components except the i-th are all equal to 0. By Lemma 1.1, it follows from this property that every function $\varphi_i(z)$ has the form

$$\varphi_i(z) = a_{e_i} z^{e_i} = a_{e_i} z_i .$$

This implies that $\varphi \in \Pi(D)$, and the reverse inclusion $C_{\mathrm{Aut}(D)}(T(D)) \subset \Pi(D)$ is shown, as desired. □

The argument used in Shimizu [9] for determining the automorphisms of bounded Reinhardt domains has the following consequence, which plays a crucial role in our study.

Proposition 1.1. *Let D be a bounded Reinhardt domain in $\mathbf{C}^n$ and suppose that*

$$D \cap \{z_i = 0\} \neq \emptyset, \quad 1 \leq i \leq m,$$
$$D \cap \{z_i = 0\} = \emptyset, \quad m+1 \leq i \leq n.$$

If G is a connected compact subgroup of Aut(D) *containing $T(D)$, then there exists a transformation*

$$\varphi\colon \mathbf{C}^m \times (\mathbf{C}^*)^{n-m} \ni (z_1, \dots, z_n) \longmapsto (w_1, \dots, w_n) \in \mathbf{C}^m \times (\mathbf{C}^*)^{n-m},$$
$$\begin{cases} w_i = r_i z_{\sigma'(i)} (z'')^{\nu_i''}, & 1 \leq i \leq m, \\ w_i = r_i z_{\sigma''(i)}, & m+1 \leq i \leq n, \end{cases}$$

such that, for $\tilde{D} = \varphi(D)$ and $\tilde{G} = \varphi G \varphi^{-1} \subset$ Aut$(\tilde{D})$, *one has*

$$\tilde{G} = U(k_1) \times \cdots \times U(k_s) \times U(k_{s+1}) \times \cdots \times U(k_t),$$
$$k_1 + \cdots + k_s + k_{s+1} + \cdots + k_t = n,$$
$$k_1 + \cdots + k_s = m,$$
$$k_{s+1} = \cdots = k_t = 1,$$

where $r_1, \dots, r_n$ are positive constants, σ' and σ'' are permutations of $\{1, \dots, m\}$ and $\{m+1, \dots, n\}$, respectively, z'' denotes the coordinates $(z_{m+1}, \dots, z_n)$, and $\nu_1'', \dots, \nu_m''$ are elements of $\mathbf{Z}^{n-m}$.

We give a useful form of this proposition as a corollary.

Corollary. *In the above proposition, if G is isomorphic to $U(k) \times (U(1))^{n-k}$ as topological groups and if $k \geq 2$, then $m \geq k$.*

Proof. Since G is necessarily isomorphic to $U(k) \times (U(1))^{n-k}$ as Lie groups, we have $\dim G = k^2 + (n-k)$. On the other hand, Proposition 1.1 implies that $\dim G = \dim \tilde{G} = k_1^2 + \cdots + k_s^2 + (n-m)$. Therefore, if $m < k$, then it follows that

$$k^2 = k_1^2 + \cdots + k_s^2 + (k-m) \quad \text{and} \quad k = k_1 + \cdots + k_s + (k-m).$$

By noting that $k \geq 2$ and $k - m > 0$, this is a contradiction. Thus we obtain $m \geq k$. □

We recall the fundamental result on torus actions on complex manifolds, which is a part of Barrett, Bedford and Dadok [3, Theorem 1].

Standardization Theorem. *Let M be a connected Stein manifold of dimension n. Assume that T^n acts effectively on M as a Lie transformation group through ρ. Then there exist a biholomorphic mapping F of M into $\mathbf{C}^n$ and a continuous group automorphism θ of T^n such that*

$$F\big((\rho(\alpha))(p)\big) = \theta(\alpha) \cdot F(p) \quad \text{for all } \alpha \in T^n \text{ and all } p \in M.$$

Consequently, $D := F(M)$ is a Reinhardt domain in $\mathbf{C}^n$, and one has $F\rho(T^n)F^{-1} = T(D)$.

To apply the Standardization Theorem to our study, we need a lemma.

Lemma 1.3. *In the Standardization Theorem, if $M = \mathbf{C}^k \times (\mathbf{C}^*)^{n-k}$, then we have $D = F(M) = \mathbf{C}^k \times (\mathbf{C}^*)^{n-k}$ after a suitable permutation of coordinates, if necessary.*

Proof. We first show that $D \cap (\mathbf{C}^*)^n = D - \{z_1 \cdots z_n = 0\} = (\mathbf{C}^*)^n$. Suppose contrarily that $D \cap (\mathbf{C}^*)^n \neq (\mathbf{C}^*)^n$. Since $D \cap (\mathbf{C}^*)^n$ is a Stein manifold, the logarithmic image of the Reinhardt domain $D \cap (\mathbf{C}^*)^n$ is a convex domain contained in a half space of $\mathbf{R}^n$. Hence, there exists a nonconstant bounded plurisubharmonic function u on $D \cap (\mathbf{C}^*)^n$. Since u extends to the whole of D, we have a nonconstant bounded plurisubharmonic function on D. This contradicts the fact that D is biholomorphically equivalent to $M = \mathbf{C}^k \times (\mathbf{C}^*)^{n-k}$. Thus we obtain $D \cap (\mathbf{C}^*)^n = (\mathbf{C}^*)^n$.

Since D is a Stein manifold, it follows from what we have shown above that, after a suitable permutation of coordinates, D has the form $D = \mathbf{C}^h \times (\mathbf{C}^*)^{n-h}$ (cf. [7, p. 46, Theorem 1.5]). Note that $\mathbf{C}^k \times (\mathbf{C}^*)^{n-k}$ and $\mathbf{C}^h \times (\mathbf{C}^*)^{n-h}$ are homeomorphic precisely when $k = h$. Therefore we have $D = \mathbf{C}^k \times (\mathbf{C}^*)^{n-k}$, because D and M are biholomorphically equivalent. □

2. The characterization of $\mathbf{C}^k \times (\mathbf{C}^*)^l$: Proof of the Main Theorem and its corollary

For brevity, we write $X_{k,l} = \mathbf{C}^k \times (\mathbf{C}^*)^l$ and $\Omega_k = X_{k,n-k}$.

Now, as in the Main Theorem stated in the introduction, let M be a connected Stein manifold of dimension n and assume that there exists an isomorphism $\Phi\colon \mathrm{Aut}(\Omega_k) \to \mathrm{Aut}(M)$. Since Ω_k is a Reinhardt domain in $\mathbf{C}^n$, we have the injective continuous group homomorphism $\rho_{\Omega_k}\colon T^n \to \mathrm{Aut}(\Omega_k)$. Thus we obtain an injective continuous group homomorphism $\Phi \circ \rho_{\Omega_k}\colon T^n \to \mathrm{Aut}(M)$. Hence, by the Standardization Theorem, there exists a biholomorphic mapping F of M into $\mathbf{C}^n$ such that $D := F(M)$ is a Reinhardt domain in $\mathbf{C}^n$ and we have $F(\Phi \circ \rho_{\Omega_k})(T^n)F^{-1} = T(D)$. Therefore we may assume that M is a Reinhardt domain D in $\mathbf{C}^n$ and we have an isomorphism $\Phi\colon \mathrm{Aut}(\Omega_k) \to \mathrm{Aut}(D)$ such that $\Phi(T(\Omega_k)) = T(D)$.

We show that $(\mathbf{C}^*)^n \subset D$. Since $\Phi\colon \mathrm{Aut}(\Omega_k) \to \mathrm{Aut}(D)$ is a group isomorphism and since $\Phi(T(\Omega_k)) = T(D)$, we see that Φ gives rise to a topological group isomorphism $\Phi\colon C_{\mathrm{Aut}(\Omega_k)}(T(\Omega_k)) \to C_{\mathrm{Aut}(D)}(T(D))$ between the centralizers. Moreover, by Lemma 1.2 we have $C_{\mathrm{Aut}(\Omega_k)}(T(\Omega_k)) = \Pi(\Omega_k)$, and it is immediate that $\Pi(\Omega_k) = \Pi(\mathbf{C}^n)$. On the other hand, again by Lemma 1.2 we have $C_{\mathrm{Aut}(D)}(T(D)) = \Pi(D)$. Therefore we obtain

$$2n = \dim \Pi(\mathbf{C}^n) = \dim C_{\mathrm{Aut}(\Omega_k)}(T(\Omega_k)) = \dim C_{\mathrm{Aut}(D)}(T(D)) = \dim \Pi(D).$$

Since $\Pi(D)$ is a closed Lie subgroup of $\Pi(\mathbf{C}^n)$, it follows that $\Pi(D) = \Pi(\mathbf{C}^n)$. By taking a point z_0 in $D \cap (\mathbf{C}^*)^n$, this shows that

$$(\mathbf{C}^*)^n = \Pi(\mathbf{C}^n) \cdot z_0 = \Pi(D) \cdot z_0 \subset D,$$

as required.

Since D is a Stein manifold by assumption, we see from the result of the preceding paragraph that D has the form $D = \Omega_h$ after a suitable permutation of coordinates.

When $n = 1$, we have $D = \Omega_0 = \mathbf{C}^*$ or $D = \Omega_1 = \mathbf{C}$. Moreover, since $\mathrm{Aut}(\mathbf{C}^*)$ and $\mathrm{Aut}(\mathbf{C})$ are not isomorphic, the condition that $\mathrm{Aut}(\Omega_k)$ and $\mathrm{Aut}(D)$ are isomorphic implies that, according to the cases of $k = 0$ and $k = 1$, we must have $D = \Omega_0$ and $D = \Omega_1$. This proves the Main Theorem when $n = 1$. Therefore, in what follows, we assume that $n \geq 2$.

We show that $h \geq k$. When $k = 0$, there is nothing to prove. To prove our assertion when $k \neq 0$, we divide into the two cases of $k = 1$ and $k \geq 2$.

First consider the case of $k \geq 2$. Noting that $\mathrm{Aut}(\Omega_k)$ contains the subgroup $U(k) \times (U(1))^{n-k}$, we set $G = \Phi(U(k) \times (U(1))^{n-k})$, which is a connected compact subgroup of $\mathrm{Aut}(D)$ containing $T(D)$, because $U(k) \times (U(1))^{n-k} \supset T(\Omega_k)$ and $\Phi(T(\Omega_k)) = T(D)$. Take a relatively compact subdomain U of D and put

$$D_0 = \{g(z) \in D \mid g \in G,\ z \in U\} = \bigcup_{g \in G} g(U) = \bigcup_{z \in U} G \cdot z.$$

Then D_0 is a bounded Reinhardt domain contained in D and G can be regarded as a connected compact subgroup of the Lie group $\mathrm{Aut}(D_0)$ containing $T(D_0)$. Recalling that G is isomorphic to $U(k) \times (U(1))^{n-k}$ and $k \geq 2$, we can apply the corollary to Proposition 1.1 to D_0 and $G \subset \mathrm{Aut}(D_0)$. Therefore, after a suitable permutation of coordinates, we have for some $m \geq k$,

$$\emptyset \neq D_0 \cap \{z_i = 0\} \subset D \cap \{z_i = 0\}, \quad 1 \leq i \leq m.$$

This implies that $\Omega_m \subset D$, and, when we write $D = \Omega_h$, we must have $h \geq m \geq k$, as required.

Now consider the case of $k = 1$. It suffices to show that $\mathrm{Aut}(\Omega_1)$ and $\mathrm{Aut}(\Omega_0)$ are not isomorphic. Suppose contrarily that we have an isomorphism $\Phi\colon \mathrm{Aut}(\Omega_1) \to \mathrm{Aut}(\Omega_0)$. Then, by the Standardization Theorem and Lemma 1.3, we may assume that we have an isomorphism $\Phi\colon \mathrm{Aut}(\Omega_1) \to \mathrm{Aut}(\Omega_0)$ such that $\Phi(T(\Omega_1)) = T(\Omega_0)$. For $s = 0, 1$, let us set

$$T'(\Omega_s) = \{(1, \alpha_2, \ldots, \alpha_n) \in T(\Omega_s) \mid \alpha_2, \ldots, \alpha_n \in U(1)\}.$$

Then $\Phi(T'(\Omega_1))$ is an $(n-1)$-dimensional subtorus of $T(\Omega_0)$, and, after a suitable change of coordinates by a transformation of the form

$$\begin{aligned} \Omega_0 &= (\mathbf{C}^*)^n \ni (z_1, \ldots, z_n) \longmapsto (w_1, \ldots, w_n) \in (\mathbf{C}^*)^n = \Omega_0, \\ w_i &= z^{\nu_i}, \quad 1 \leq i \leq n, \end{aligned}$$

where $\nu_1, \ldots, \nu_n$ are elements of $\mathbf{Z}^n$, we have $\Phi(T'(\Omega_1)) = T'(\Omega_0)$. Since $\Phi\colon \mathrm{Aut}(\Omega_1) \to \mathrm{Aut}(\Omega_0)$ is a group isomorphism, we see that Φ maps the centralizer Z_1 of $T'(\Omega_1)$ in $\mathrm{Aut}(\Omega_1)$ onto the centralizer Z_0 of $T'(\Omega_0)$ in $\mathrm{Aut}(\Omega_0)$. Therefore, for the groups Z_0 and Z_1, their commutator groups $[Z_0, Z_0]$ and $[Z_1, Z_1]$ must be isomorphic. To derive a contradiction, it is sufficient to see that $[Z_0, Z_0]$ is an abelian group, while $[Z_1, Z_1]$ is not an abelian group. We verify this only in the case of $n = 2$, because the verification in the case of $n > 2$ is almost identical. Using a method similar to that in the proof of Lemma 1.2, we can show that Z_1 and Z_0 are the groups of all elements

$$g_1 \in \mathrm{Aut}(\Omega_1) = \mathrm{Aut}(\mathbf{C} \times \mathbf{C}^*) \quad \text{and} \quad g_0 \in \mathrm{Aut}(\Omega_0) = \mathrm{Aut}((\mathbf{C}^*)^2)$$

having the forms

$$g_1(z) = \big(\alpha z_1 + \beta,\ f(z_1) z_2\big) \tag{$*$}$$

and

$$g_0(z) = \big(\alpha z_1,\ f(z_1) z_2\big),$$

respectively, where $\alpha \in \mathbf{C}^*$, $\beta \in \mathbf{C}$, and $f(z_1)$ is a nowhere vanishing holomorphic function that is defined on $\mathbf{C}$ for g_1 and on $\mathbf{C}^*$ for g_0. Take any two transformations $K_{\alpha,\beta,f}$ and $K_{\alpha',\beta',f'}$ of the form $(*)$ given by

$$K_{\alpha,\beta,f}(z) = \big(\alpha z_1 + \beta,\, f(z_1)z_2\big) \quad \text{and} \quad K_{\alpha',\beta',f'}(z) = \big(\alpha' z_1 + \beta',\, f'(z_1)z_2\big)$$

and write $[K_{\alpha,\beta,f}, K_{\alpha',\beta',f'}](z) = (K_1(z), K_2(z))$ in terms of the coordinates in $\mathbf{C}^2$, where $[\varphi, \psi] := \varphi^{-1} \circ \psi^{-1} \circ \varphi \circ \psi$ denotes the commutator of transformations φ and ψ. Then we have

$$\begin{aligned} K_1(z) &= \frac{\alpha\alpha' z_1 + \alpha\beta' - \beta\alpha' + \beta - \beta'}{\alpha\alpha'}, \\ K_2(z) &= \frac{f(\alpha' z_1 + \beta')f'(z_1)z_2}{f((\alpha\alpha' z_1 + \alpha\beta' - \beta\alpha' + \beta - \beta')/\alpha\alpha')f'((\alpha\alpha' z_1 + \alpha\beta' + \beta - \beta')/\alpha')}. \end{aligned}$$

As a consequence, considering the case of $(\beta, \beta') = (0, 0)$, we have

$$(**) \qquad [K_{\alpha,0,f}, K_{\alpha',0,f'}](z) = \left(z_1, \frac{f(\alpha' z_1)f'(z_1)z_2}{f(z_1)f'(\alpha z_1)}\right).$$

Now it follows immediately from $(**)$ that $[Z_0, Z_0]$ is abelian. On the other hand, consider three elements

$$P(z) = (\alpha z_1 + \beta, z_2), \;\; Q(z) = (z_1, z_2 \exp z_1), \;\text{ and }\; R(z) = (\gamma z_1, z_2 \exp z_1)$$

in Z_1. Then, using the computation result above, we obtain

$$\begin{aligned} [P, Q](z) &= (z_1, z_2 \exp\{(1-\alpha)z_1 - \beta\}), \\ [P, R](z) &= \left(\frac{\alpha\gamma z_1 + \beta(1-\gamma)}{\alpha\gamma}, z_2 \exp\left\{(1-\alpha)z_1 - \frac{\beta}{\gamma}\right\}\right), \end{aligned}$$

and therefore $[[P, Q], [P, R]]$ is not the identity mapping whenever $\beta(\alpha-1)(\gamma-1) \neq 0$. This implies that $[Z_1, Z_1]$ is not abelian, and our assertion that $\mathrm{Aut}(\Omega_1)$ and $\mathrm{Aut}(\Omega_0)$ are not isomorphic is shown.

Summarizing our results obtained so far, we have shown that if M is a connected Stein manifold of dimension n and if the topological groups $\mathrm{Aut}(M)$ and $\mathrm{Aut}(\Omega_k)$ are isomorphic, then M is biholomorphically equivalent to Ω_h with $h \geq k$.

To complete the proof of our Main Theorem, it is sufficient to see $h = k$. Suppose contrarily that $h \neq k$. Then, for the connected Stein manifold Ω_k of dimension n, we have that $\mathrm{Aut}(\Omega_k)$ and $\mathrm{Aut}(\Omega_h)$ are isomorphic. By letting $M = \Omega_k$, an application of what we have shown just above yields that Ω_k is biholomorphically equivalent to Ω_p with $p \geq h$. Since $k < h \leq p$, this contradicts the fact that Ω_s and Ω_t are not homeomorphic when $s \neq t$. We thus obtain $h = k$, and our Main Theorem is proved. □

It remains to prove the corollary to the Main Theorem. If $k + l = k' + l'$, then it is immediate from the Main Theorem that $\mathrm{Aut}(X_{k,l})$ and $\mathrm{Aut}(X_{k',l'})$ are isomorphic precisely when $(k, l) = (k', l')$. To prove the corollary in the case of $k + l \neq k' + l'$, we need the following lemma.

Lemma 2.1. *Let M be a connected Stein manifold of dimension n. If $N > n$, then there is no injective continuous group homomorphism of the torus T^N into the topological group* $\mathrm{Aut}(M)$.

Proof. Suppose contrarily that we have an injective continuous group homomorphism ρ of T^N into $\mathrm{Aut}(M)$. Choose an n-dimensional subtorus T^n of T^N. By the Standardization Theorem, there exists a biholomorphic mapping $F: M \to D$ of M onto a Reinhardt domain D in $\mathbf{C}^n$ such that $F\rho(T^n)F^{-1} = T(D)$. Set $G = F\rho(T^N)F^{-1}$ and take a relatively compact subdomain U of D. Then $D_0 := \{g(z) \in D \mid g \in G, z \in U\}$ is a bounded Reinhardt domain in $\mathbf{C}^n$ and G can be regarded as a connected compact subgroup of the Lie group $\mathrm{Aut}(D_0)$ containing $T(D_0)$. Since G is isomorphic to T^N and $N > n = \dim T(D_0)$, G is a torus in $\mathrm{Aut}(D_0)$ containing $T(D_0)$ properly. But, by [8, Section 4, Proposition 1], $T(D_0)$ is a maximal torus in $\mathrm{Aut}(D_0)$, that is, any torus in $\mathrm{Aut}(D_0)$ containing $T(D_0)$ must coincide with $T(D_0)$. This is a contradiction, and our assertion is proved. □

Suppose $k+l \neq k'+l'$, say, $k+l < k'+l'$, and write $n = k+l$, $n' = k'+l'$. If there exists an isomorphism $\Phi\colon \mathrm{Aut}(X_{k',l'}) \to \mathrm{Aut}(X_{k,l})$, then we have an injective continuous group homomorphism $\Phi \circ \rho_{X_{k',l'}}$ of $T^{n'}$ into $\mathrm{Aut}(X_{k,l})$. Since $X_{k,l}$ is a connected Stein manifold of dimension $n < n'$, this contradicts the above lemma. Therefore, $\mathrm{Aut}(X_{k,l})$ and $\mathrm{Aut}(X_{k',l'})$ are not isomorphic, and the proof of the corollary is completed. □

3. $U(n)$-actions on a Stein manifold of dimension n

The method used in the preceding section can be applied to the study of $U(n)$-actions on a complex manifold M of dimension n. The following theorem gives a different approach from Kaup [5], Isaev and Kruzhilin [4]. In the case where $\mathrm{Aut}(M)$ is not a Lie group, we cannot obtain various results on the conjugacy of subgroups of $\mathrm{Aut}(M)$ by applying the conjugacy theorems in the Lie group theory, in general. However, even when $\mathrm{Aut}(M)$ is not a Lie group, we have a conjugacy result on $\mathrm{Aut}(M)$ in a case, as is shown in our theorem below.

Theorem. *Let M be a connected Stein manifold of dimension $n \geq 2$. Assume that $U(n)$ acts effectively on M as a Lie transformation group through ρ. Then M is biholomorphically equivalent to either B^n or $\mathbf{C}^n$, where B^n denotes the unit ball in $\mathbf{C}^n$. Moreover, if we identify M with B^n or $\mathbf{C}^n$, then there exists an element ψ of* $\mathrm{Aut}(M)$ *such that $\psi\rho(U(n))\psi^{-1} = U(n)$.*

Proof. Choose a maximal torus T^n in $U(n)$. By the Standardization Theorem, there exists a biholomorphic mapping $F\colon M \to D$ of M onto a Reinhardt domain D in $\mathbf{C}^n$ such that $F\rho(T^n)F^{-1} = T(D)$. Set $G = F\rho(U(n))F^{-1}$ and take a relatively compact subdomain U of D. Then $D_0 := \{g(z) \in D \mid g \in G, z \in U\}$ is a bounded Reinhardt domain in $\mathbf{C}^n$ and G can be regarded as a connected compact subgroup of the Lie group $\mathrm{Aut}(D_0)$ containing $T(D_0)$. Recalling that G is isomorphic to $U(n)$ and $n \geq 2$, we can apply Proposition 1.1 and its corollary to D_0 and $G \subset \mathrm{Aut}(D_0)$. Therefore there exists a transformation

$$\begin{aligned} \varphi\colon \mathbf{C}^n \ni (z_1, \dots, z_n) &\longmapsto (w_1, \dots, w_n) \in \mathbf{C}^n, \\ w_i = r_i z_{\sigma(i)}, &\quad 1 \leq i \leq n, \end{aligned}$$

such that, for $\tilde{D}_0 = \varphi(D_0)$ and $\tilde{G} = \varphi G \varphi^{-1} \subset \mathrm{Aut}(\tilde{D}_0)$, we have $\tilde{G} = U(n)$, where $r_1, \dots, r_n$ are positive constants and σ is a permutation of $\{1, \dots, n\}$. Putting $\tilde{D} = \varphi(D)$, we see by the uniqueness theorem on holomorphic functions that $U(n) = \tilde{G} \subset \mathrm{Aut}(\tilde{D})$, or $g(\tilde{D}) = \tilde{D}$ for all $g \in U(n)$. Since $\tilde{D}$ is a Stein manifold, it follows that $\tilde{D}$ has the form

$$\tilde{D} = \left\{ (z_1, \dots, z_n) \in \mathbf{C}^n \;\middle|\; \sum_{i=1}^{n} |z_i|^2 < r \right\},$$

where $0 < r \leq +\infty$. This shows that $\tilde{D}$, and hence M is biholomorphically equivalent to either B^n or $\mathbf{C}^n$, proving the first assertion.

Now, let us identify M with B^n or $\mathbf{C}^n$. When $M = B^n$, the existence of $\psi \in \mathrm{Aut}(M)$ satisfying the relation $\psi\rho(U(n))\psi^{-1} = U(n)$ is a consequence of the conjugacy of maximal compact subgroups of the simple Lie group $\mathrm{Aut}(B^n)$. So, consider the case of $M = \mathbf{C}^n$. Then, by the same reasoning as above, there exist biholomorphic mappings $F\colon M = \mathbf{C}^n \to D = \mathbf{C}^n$ and $\varphi\colon \mathbf{C}^n \to \mathbf{C}^n$ such that $(\varphi \circ F)\rho(U(n))(\varphi \circ F)^{-1} = U(n)$. Therefore, the composition $\psi = \varphi \circ F$ is an element of $\mathrm{Aut}(\mathbf{C}^n)$ required in the theorem. □

Added in proof. After the submission of this paper, the authors learned in the letter of August 21, 2002, from Professor A. Isaev that, in the special case of $k = n$, the same result as our Main Theorem had been obtained independently by him (Proc. Steklov Inst. Math. **235** (2001), 103–106).

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A. Kodama
Department of Mathematics
Faculty of Science
Kanazawa University
Kanazawa, 920-1192, Japan
e-mail: kodama@kenroku.kanazawa-u.ac.jp

S. Shimizu
Mathematical Institute
Tohoku University
Sendai, 980-8578, Japan
e-mail: shimizu@math.tohoku.ac.jp