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ON MULTIPLY TRANSITIVE PERMUTATION GROUPS

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1. Introduction

In this paper we shall give some improvements of the following four results:

RESULT 1 (E. Bannai [5] Theorem 1). Let p be an odd prime. Let G be a permutation group on a set $\Omega = \{1, 2, \dots, n\}$ which satisfies the following condition: For any p^2 elements $\alpha_1, \dots, \alpha_{p^2}$ of Ω , a Sylow p -subgroup P of the stabilizer in G of the p^2 points $\alpha_1, \dots, \alpha_{p^2}$ is nontrivial and fixes p^2+r points of Ω , and moreover P is semiregular on the set $\Omega - I(P)$ of the remaining $|\Omega| - p^2 - r$ points, where r is independent of the choice of $\alpha_1, \dots, \alpha_{p^2}$ and $0 \leq r \leq p-1$. Then $n = p^2 + p + r$, and one of the following three cases holds:
(1) There exists an orbit Ω_1 of G such that $|\Omega - \Omega_1| \leq r$ and $G^{\Omega_1} \geq A^{\Omega_1}$. Moreover, $(G_{\Omega - \Omega_1})^{\Omega_1} \geq A^{\Omega_1}$.
(2) $r = p-1$, and G has just two orbits Ω and Ω_2 (with $|\Omega_1| \geq |\Omega_2| \geq p$) such that $G^{\Omega_1} \geq A^{\Omega_1}$. Moreover $(G_{\Omega_2})^{\Omega_1} \geq A^{\Omega_1}$ and G^{Ω_2} is primitive and contains an element of a p -cycle (therefore $G^{\Omega_2} \geq A^{\Omega_2}$ if $|\Omega_2| \geq p+3$).
(3) $r = p-1$, and G is imprimitive on Ω with just two blocks Ω_1 and Ω_2 . Moreover, $(G_{\Omega_1})^{\Omega_2} \geq A^{\Omega_2}$ and $(G_{\Omega_2})^{\Omega_1} \geq A^{\Omega_1}$.

RESULT 2 (E. Bannai [4] Theorem 1). Let p be an odd prime. Let G be a $2p$ -transitive permutation group such that either (i) each element in G of order p fixes at most $2p + (p-1)$ points, or (ii) a Sylow p -subgroup of $G_{1,2,\dots,2p}$ is cyclic. Then G is one of S_n ($2p \leq n \leq 4p-1$) and A_n ($2p+2 \leq n \leq 4p-1$).

RESULT 3 (D. Livingstone and A. Wanger [10] Lemma 10). If G is a k -transitive group on a set Ω of n points, with $n > k \geq 4$, then there exists a subset Π of $k+1$ points such that $G_{(\Pi)}^{\Pi} \geq A^{\Pi}$.

RESULT 4 (H. Wielandt [13] Satz B). If G is a nontrivial t -transitive group on Ω of n points, and if t is sufficiently large, then $\log(n-t) > \frac{1}{2}t$.

In § 2 and § 3, we shall prove the following two theorems which improve Result 1 and Result 2.

Theorem A. *Let p be an odd prime. Let G be a permutation group on a*

set $\Omega = \{1, 2, \dots, n\}$ which satisfies the following condition. For any $2p$ points $\alpha_1, \dots, \alpha_{2p}$ of Ω , a Sylow p -subgroup P of the stabilizer in G of the $2p$ points $\alpha_1, \dots, \alpha_{2p}$ is nontrivial and fixes exactly $2p+r$ points of Ω , and moreover P is semiregular on the set $\Omega - I(P)$ of the remaining $n-2p-r$ points, where r is independent of the choice of $\alpha_1, \dots, \alpha_{2p}$ and $0 \leq r \leq p-2$. Then $n=3p+r$, and there exists an orbit Γ of G such that $|\Gamma| \geq 3p$ and $G^\Gamma \geq A^\Gamma$.

Theorem B. Let p be an odd prime ≥ 11 . Let G be a permutation group on a set $\Omega = \{1, 2, \dots, n\}$ which satisfies the following condition. For any $2p$ points $\alpha_1, \dots, \alpha_{2p}$ of Ω , a Sylow p -subgroup P of the stabilizer in G of the $2p$ points $\alpha_1, \dots, \alpha_{2p}$ is nontrivial and fixes exactly $3p-1$ points of Ω , and moreover P is semiregular on the set $\Omega - I(P)$ of the remaining $n-3p+1$ points. Then $n=4p-1$, and one of the following two cases holds: (1) There exists an orbit Γ of G such that $|\Gamma| \geq 3p$ and $G^\Gamma \geq A^\Gamma$. (2) G has just two orbits Γ_1 and Γ_2 with $|\Gamma_1| \geq p$, $|\Gamma_2| \geq p$ and $|\Gamma_1| + |\Gamma_2| = 4p-1$, and G^{Γ_i} is $(|\Gamma_i| - p + 1)$ -transitive on Γ_i ($i=1, 2$). Moreover, $G^{\Gamma_i} \geq A^{\Gamma_i}$ if $|\Gamma_i| \geq p+3$.

REMARK. We note that T. Oyama proved:

RESULT 5 (T. Oyama [12] Theorem 1). Let G be a permutation group on $\Omega = \{1, 2, \dots, n\}$. Assume that a Sylow 2-subgroup P of the stabilizer of any four points in G satisfies the following condition: P is a nonidentity semi-regular group and P fixes exactly r points. Then (I) $r=4$, then $|\Omega|=6, 8$ or 12 , and $G=S_6, A_8$ or M_{12} respectively. (II) If $r=5$, then $|\Omega|=7, 9$ or 13 . In particular, if $|\Omega|=9$, then $G \leq A_9$, and if $|\Omega|=13$, then $G=S_1 \times M_{12}$. (III) If $r=7$ and $N_G(P)^{I(P)} \leq A_7$, then $G=M_{23}$.

Theorem A and Theorem B might look to be too technical. However they are useful in applications. In § 4, we shall prove the following two consequences of them which improve Result 3 and Result 4 respectively.

Theorem C. Let p be an odd prime. Let G be a nontrivial $2p$ -transitive group on $\Omega = \{1, 2, \dots, n\}$. Then there exists a subset Γ of Ω such that $|\Gamma| \geq 3p-1$ and $G_{(\Gamma)}^\Gamma \geq A^\Gamma$.

Theorem D. Let G be a nontrivial t -transitive group on $\Omega = \{1, 2, \dots, n\}$. If t is sufficiently large, then $\log(n-t) > \frac{3}{4}t$.

We give the outline of § 2. Let G be a group satisfying the assumption of Theorem A. Then, G has the only one orbit whose length is not less than p . So, we may assume that G is transitive on Ω . Moreover, we find that if $p \geq 5$, then G is $(p+3)$ -transitive on Ω , and that if $p=3$, then G is 5-transitive on Ω . Suppose that $G \not\geq A^\Omega$. Similarly to Bannai [4, § 1], we get a contradiction by using the idea of Miyamoto and Nago which uses the formula of

Frobenius ingeniously (cf. [11, Lemma 1.1]).

Next we give the outline of § 3. Let G be a counter-example to Theorem B with the least degree. So, we may assume that G is transitive on Ω . Moreover, we find that G is $\left(p+\frac{p+1}{2}+2\right)$ -transitive on Ω . Again by the similar argument to that of [4, § 1], we get a contradiction.

NOTATION. Our notation will be more or less standard. Let Ω be a set and Δ be a subset of Ω . If G is a permutation group on Ω , then G_Δ denotes the pointwise stabilizer of Δ in G , and $G_{(\Delta)}$ denotes the global stabilizer of Δ in G . When $\Delta=\{\alpha_1, \dots, \alpha_k\}$, we also denote G_Δ by $G_{\alpha_1, \dots, \alpha_k}$. The totality of points left fixed by a set X of permutations is denoted by $I(X)$, and if a subset Γ of Ω is fixed as a whole by X , then the restriction of X on Γ is denoted by X^Γ . For a permutation x , let $\alpha_i(x)$ denote the number of i -cycles of x and $\alpha(x)=\alpha_1(x)$. S^Ω and A^Ω denote the symmetric and alternating groups on Ω . If $|\Omega|$, the cardinality of Ω , is n , we denote them S_n and A_n instead of S^Ω and A^Ω .

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2. Proof of Theorem A

Let G be a permutation group satisfying the assumption of Theorem A.

Step 1. *G has an orbit Γ such that $|\Gamma| \geq 3p$ and $|\Omega - \Gamma| < p$.*

Proof. Since a Sylow p -subgroup of the stabilizer in G of $2p$ points is nontrivial and fixes exactly $2p+r$ points, we have $|\Omega| \geq 3p+r$ and that G has an orbit Γ whose length is at least p . Set $|\Gamma| \equiv k \pmod{p}$ with $0 \leq k \leq p-1$.

Suppose that $|\Gamma|=p+k$. We take $k+1$ points $\alpha_1, \dots, \alpha_{k+1}$ from Γ and $2p-k-1$ points $\alpha_{k+2}, \dots, \alpha_{2p}$ from $\Omega - \Gamma$. A Sylow p -subgroup of $G_{\alpha_1, \dots, \alpha_{2p}}$ fixes at least $3p-1$ points, which contradicts the assumption of Theorem A. Hence we have $|\Gamma| \geq 2p+k$.

Suppose that $|\Omega - \Gamma| \geq p$. We take $p+k+1$ points $\alpha_1, \dots, \alpha_{p+k+1}$ from Γ and $p-k-1$ points $\alpha_{p+k+2}, \dots, \alpha_{2p}$ from $\Omega - \Gamma$. A Sylow p -subgroup of $G_{\alpha_1, \dots, \alpha_{2p}}$ fixes at least $3p-1$ points, which contradicts the assumption of Theorem A. Hence we have $|\Omega - \Gamma| < p$. So, we have $|\Gamma| \geq 3p$. (q.e.d.)

By Step 1, from now on we may assume that G is transitive on Ω .

Step 2. *Let $1 \leq t \leq p+2$. If G is t -transitive on Ω , then G is t -primitive on Ω .*

Proof. Suppose, by way of contradiction, that G is t -transitive on Ω , and that $G_{1, \dots, t-1}$ is imprimitive on $\Omega - \{1, \dots, t-1\}$. Let $\Gamma_1, \dots, \Gamma_s$ be a system

of imprimitivity of $G_{1, \dots, t-1}$. Let $|\Gamma_1| \equiv k \pmod{p}$, where $0 \leq k \leq p-1$. We divide the consideration into the following two cases: (I) $2p-(t-1) > k$. (II) $2p-(t-1) \leq k$.

Suppose that Case (I) holds. First assume that $|\Gamma_1| \geq 2p$. We take $k+1$ points $\alpha_t, \dots, \alpha_{t+k}$ from Γ_1 and $2p-t-k$ points $\alpha_{t+k+1}, \dots, \alpha_{2p}$ from Γ_2 . A Sylow p -subgroup of $G_{1, \dots, t-1, \alpha_t, \dots, \alpha_{2p}}$ fixes at least $3p-1$ points, which is a contradiction. Next assume that $p \leq |\Gamma_1| < 2p$. We take $k+1$ points $\alpha_t, \dots, \alpha_{t+k}$ from Γ_1 . Moreover, we are able to take $2p-t-k$ points $\alpha_{t+k+1}, \dots, \alpha_{2p}$ from $\Omega - (\Gamma_1 \cup \{1, \dots, t-1\})$. A Sylow p -subgroup of $G_{1, \dots, t-1, \alpha_t, \dots, \alpha_{2p}}$ fixes at least $3p-1$ points, which is a contradiction. Hence we may assume that $|\Gamma_1| < p$. Let γ_i be a point of Γ_i ($i=1, \dots, s$). Assume $s \leq 2p-t+1$. Then a Sylow p -subgroup of $G_{1, \dots, t-1, \gamma_1, \dots, \gamma_s}$ is trivial, a contradiction. Hence $s > 2p-t+1$. Since a Sylow p -subgroup of $G_{1, \dots, t-1, \gamma_1, \dots, \gamma_{2p-t-1}}$ fixes at most $3p-2$ points, we have $(k-1) \leq (2p-t+1) \leq p-2$. But, since $t \leq p+2$ and $k \geq 2$, we have a contradiction.

Suppose that Case (II) holds. In this case, we have $t=p+2$ and $k=p-1$. We take a point α from Γ_1 and $p-2$ points $\beta_1, \dots, \beta_{p-2}$ from Γ_2 . A Sylow p -subgroup of $G_{1, \dots, p+1, \alpha, \beta_1, \dots, \beta_{p-2}}$ fixes at least $3p-1$ points, which is a contradiction. (q.e.d)

Step 3. *G is $(p+3)$ -transitive on Ω when $p \geq 5$, and G is 5-transitive on Ω when $p=3$.*

Proof. In order to prove Step 3, we show that if G is t -transitive on Ω then G is $(t+1)$ -transitive on Ω , where $1 \leq t \leq p+2$ when $p \geq 5$ and $1 \leq t \leq 4$ when $p=3$. Suppose, by way of contradiction, that G is t -transitive on Ω , but G is not $(t+1)$ -transitive on Ω . By Step 2, G is t -primitive on Ω . Let $\Delta_1, \dots, \Delta_s$ be the orbits of $G_{1, \dots, t}$ on $\Omega - \{1, \dots, t\}$, where $s \geq 2$. By Theorem 18.4 in [14], $|\Delta_i| \geq p$ for every Δ_i ($i=1, \dots, s$). Let $|\Delta_i| \equiv u_i \pmod{p}$, where $0 \leq u_i \leq p-1$ ($i=1, \dots, s$). By the assumption of t , we have that $p-2 \leq 2p-t \leq 2p-1$ when $p \geq 5$, and $2 \leq 2p-t \leq 5$ when $p=3$. We divide the consideration into the following two cases: (I) $2p-t \geq p$. (II) $2p-t < p$.

Suppose that Case (I) holds. First assume that $2p-t-u_1-1 \leq p$. We take u_1+1 points $\alpha_1, \dots, \alpha_{u_1+1}$ from Δ_1 and $2p-t-u_1-1$ points $\beta_1, \dots, \beta_{2p-t-u_1-1}$ from Δ_2 . A Sylow p -subgroup of $G_{1, \dots, t, \alpha_1, \dots, \alpha_{u_1+1}, \beta_1, \dots, \beta_{2p-t-u_1-1}}$ fixes at least $3p-1$ points, which is a contradiction. Next assume that $2p-t-u_1-1 > p$ and $|\Delta_1| \geq 2p$. We take u_1+p+1 points $\alpha_1, \dots, \alpha_{u_1+p+1}$ from Δ_1 and $p-t-u_1-1$ points $\beta_1, \dots, \beta_{p-t-u_1-1}$ from Δ_2 . A Sylow p -subgroup of $G_{1, \dots, t, \alpha_1, \dots, \alpha_{u_1+p+1}, \beta_1, \dots, \beta_{p-t-u_1-1}}$ fixes at least $3p-1$ points, which is a contradiction. Hence we may assume that $2p-t-u_1-1 > p$ and $|\Delta_1| < 2p$. We take u_1+1 points $\alpha_1, \dots, \alpha_{u_1+1}$ from Δ_1 . Moreover we are able to take $2p-t-u_1-1$ points $\beta_1, \dots, \beta_{2p-t-u_1-1}$ from $\Omega - (\{1, \dots, t\} \cup \Delta_1)$. A Sylow p -subgroup of $G_{1, \dots, \alpha_1, \dots, \alpha_{u_1+1}, \beta_1, \dots, \beta_{2p-t-u_1-1}}$ fixes

at least $3p-1$ points, which is a contradiction.

Suppose that Case (II) holds. In this case, we have that $2p-t=p-2$ or $p-1$ when $p \geq 5$, and $2p-t=2$ when $p=3$. Assume that there is an orbit Δ_i of $G_{1, \dots, t}$ with $u_i < 2p-t$. We take u_i+1 points $\alpha_1, \dots, \alpha_{u_i+1}$ from Δ_i and $2p-t-u_i-1$ points $\beta_1, \dots, \beta_{2p-t-u_i-1}$ from $\Omega - (\{1, \dots, t\} \cup \Delta_i)$. A Sylow p -subgroup of $G_{1, \dots, t, \alpha_1, \dots, \alpha_{u_i+1}, \beta_1, \dots, \beta_{2p-t-u_i-1}}$ fixes at least $3p-1$ points, which is a contradiction. Hence $u_i \geq 2p-t$ for every Δ_i ($i=1, \dots, s$). Assume that $s \geq 3$ or $p=3$. We take a point α_1 from Δ_1 and a point α_2 from Δ_2 . If $p=3$, then a Sylow p -subgroup of $G_{1, 2, 3, 4, \alpha_1, \alpha_2}$ fixes at least 8 points, which is a contradiction. If $p \geq 5$, we take $2p-t-2$ points $\beta_1, \dots, \beta_{2p-t-2}$ from Δ_3 . Then a Sylow p -subgroup of $G_{1, \dots, t, \alpha_1, \alpha_2, \beta_1, \dots, \beta_{2p-t-2}}$ fixes at least $3p-1$ points, which is a contradiction. Thus we have $p \geq 5$ and $s=2$. So, $\Omega = \{1, \dots, t\} \cup \Delta_1 \cup \Delta_2$. Hence $2p+r=t+\mu_1+\mu_2$. Let Q be a Sylow p -subgroup of $G_{1, \dots, t}$. Then, $N_G(Q)^{I(Q)}$ is t -transitive and has an element of order p . Since $3p-2 \geq |I(Q)| = t+u_1+u_2 \geq t+2(2p-t)=2p+(2p-t)$, we have $|I(Q)|=3p-2$, and $N_G(Q)^{I(Q)} \geq A^{I(Q)}$ by [14, Theorem 13.10]. So, $N_G(Q)^{I(Q)}_{1, \dots, t}$ has an element of order p . Hence Q is not a Sylow p -subgroup of $G_{1, \dots, t}$, a contradiction. (q.e.d)

Step 4. $G \geq A^\Omega$, or $\alpha_p(x) \geq 4$ for any element x of order p of G .

Proof. Let us assume that $\min \{\alpha_p(X) \mid x \text{ is an element of order } p \text{ of } G\} = m \leq 3$. Hence $|\Omega| \geq 2p+mp$. Since G is 5-transitive, we have $G \geq A^\Omega$ by [14, Theorem 13.10]. (q.e.d.)

From now on we assume that $G \not\geq A^\Omega$, and prove that this case does not occur.

Step 5. Let a be an element of order p of G with $\alpha(a)=2p+r$. Then there exists an orbit Δ of $C_G(a)^{I(a)}$ such that $C_G(a)^\Delta \geq A^\Delta$ and $|\Delta| \geq 2p$.

Proof. We may assume that

$$a = (1)(2) \cdots (2p+r)(2p+r+1, \dots, 3p+r) \cdots .$$

Set $T = C_G(a)^{I(a)}_{2p+r+1, \dots, 3p+r}$. For any p points $\alpha_1, \dots, \alpha_p$ of $I(a)$, a normalizes $G_{\alpha_1, \dots, \alpha_p, 2p+r+1, \dots, 3p+r}$. Hence a centralizes an element of order p of $G_{\alpha_1, \dots, \alpha_p, 2p+r+1, \dots, 3p+r}$. So, $T_{\alpha_1, \dots, \alpha_p}$ has an element of order p for any p elements $\alpha_1, \dots, \alpha_p$ of $I(a)$. Thus T has an orbit Γ with $|\Gamma| \geq p$. Let $|\Gamma|=p+k$. Suppose that $0 \leq k \leq p-1$. We take $k+1$ points $\delta_1, \dots, \delta_{k+1}$ from Γ and $p-k-1$ points $\delta_{k+2}, \dots, \delta_p$ from $I(a) - \Gamma$. Then $T_{\delta_1, \dots, \delta_p}$ has no element of order p , which is a contradiction. Therefore T has an orbit Γ whose length is at least $2p$. Since it is easily seen that T^Γ is primitive, we have $T^\Gamma \geq A^\Gamma$ by [14, Theorem 13.9]. Let Δ be an orbit of maximal length of $C_G(a)^{I(a)}$, then $C_G(a)^\Delta \geq A^\Delta$ and $|\Delta| \geq 2p$. (q.e.d.)

Step 6. *For any $2p$ points $\alpha_1, \dots, \alpha_{2p}$ of Ω , the order of a Sylow p -subgroup of $G_{\alpha_1, \dots, \alpha_{2p}}$ is p .*

Proof. Suppose, by way of contradiction, that for some $2p$ points $\alpha_1, \dots, \alpha_{2p}$, the order of a Sylow p -subgroup P of $G_{\alpha_1, \dots, \alpha_{2p}}$ is more than p . We may assume that $\{\alpha_1, \dots, \alpha_{2p}\} = \{1, \dots, 2p\}$ and $I(P) = \{1, \dots, 2p, \dots, 2p+r\}$. For any $2p$ points $\gamma_1, \dots, \gamma_{2p}$ of $I(P)$, the order of a Sylow p -subgroup of $G_{\gamma_1, \dots, \gamma_{2p}}$ is $|P|$. Let a be an element of order p of $Z(P)$. We may assume that

$$a = (1)(2) \cdots (2p+r)(2p+r+1, \dots, 3p+r) \cdots .$$

Since a normalizes $G_{1, \dots, p, 2p+r+1, \dots, 3p+r}$, $G_{1, \dots, p, 2p+r+1, \dots, 3p+r}$ has an element b of order p commuting with a . We may assume that

$$b = (1) \cdots (p)(p+1, \dots, 2p)(2p+1) \cdots (2p+r)(2p+r+1) \cdots (3p+r) \cdots .$$

Then we may assume that $P^b = P$. Since $C_P(b)$ is semiregular on $I(b) - \{1, \dots, p\} \cup \{2p+1, \dots, 2p+r\} = \{2p+r+1, \dots, 3p+r\}$, we have $|C_P(b)| = p$, and b does not centralize P . On the other hand, since $\langle P, b \rangle = P \cdot \langle b \rangle$, we have $\langle a \rangle \times \langle b \rangle \supseteq C_{\langle P, b \rangle}(b) \supseteq Z(\langle P, b \rangle)$. Hence $|Z(\langle P, b \rangle)| = |\langle a \rangle| = p$, since $[P, b] \neq 1$.

Now, since $I(a) = I(P)$, we have $C_G(a) \subseteq G_{I(P)} = N_G(G_{I(P)})$. By the Frattini-Sylow argument, $N_G(G_{I(P)}) = N_G(P) \cdot G_{I(P)}$. So, $C_G(a) \subseteq N_G(P)G_{I(P)}$. Hence $C_G(a)^{I(a)} = C_G(a)^{I(P)} \subseteq N_G(P)^{I(P)}$. Thus by Step 5, $N_G(P)^{I(P)}$ has an orbit Δ of maximal length such that $N_G(P)^\Delta \geq A^\Delta$ and $|\Delta| \geq 2p$. We may assume that $\Delta = \{1, 2, \dots, |\Delta|\}$. Set $\Gamma = \{2, 3, \dots, 2p\}$, then $N_G(P)_{(\Gamma)}^\Gamma \geq A^\Gamma$. Since $|I(P) - \Gamma| \leq p-1$, $|N_G(P)_{(\Gamma)}|_p$ (= the order of a Sylow p -subgroup of $N_G(P)_{(\Gamma)} = |P|$). Moreover since $|N_G(P)_{(\Gamma)}|_p = p$, we have $N_G(P)_{(\Gamma)}|_p = p \cdot |P|$. Thus $\langle P, b \rangle$ is a Sylow p -subgroup of $N_G(P)_{(\Gamma)}$.

Suppose that $C_G(P)_{(\Gamma)}^\Gamma = 1$. Since $N_G(P)_{(\Gamma)} / C_G(P)_{(\Gamma)} \leq \text{Aut}(P)$, A_{2p-1} is involved in $\text{Aut}(P)$. But, we can easily seen that A_{2p-1} is not involved in $\text{Aut}(P)$ (cf. [2. § 2, (3)]), which is a contradiction. Therefore we have $C_G(P)_{(\Gamma)}^\Gamma \geq A^\Gamma$. Since the center of a Sylow p -subgroup of $N_G(P)_{(\Gamma)}$ is of order p , this is a contradiction. (q.e.d.)

Step 7. $|\Omega| - (2p+r) \equiv p \pmod{p^2}$.

(The proof of this step is the same as that of [4, § 2], but we repeat it for the completeness.)

Proof. We may assume that there exist two elements a and b of order p which commute to each other such that

$a = (1) \cdots (2p)(2p+1) \cdots (2p+r)(2p+r+1, \dots, 3p+r)(3p+r+1, \dots, 4p+r) \cdots$,
and

$$b = (1, \dots, p)(p+1, \dots, 2p)(2p+1) \cdots (2p+r)(2p+r+1) \cdots \cdots (3p+r)(3p+r+1) \cdots (4p+r) \cdots .$$

Since $\langle a, b \rangle$ normalizes $G_{p+1, \dots, 2p, 2p+r+1, \dots, 3p+r}$, $G_G(\langle a, b \rangle)_{p+1, \dots, 2p, 2p+r+1, \dots, 3p+r}$ has an element c of order p . The element c must be of the form

$$c = (1, \dots, p)^a(p+1) \cdots (2p) \cdots (2p+r) \cdots (3p+r)(3p+r+1, \dots, 4p+r)^b \cdots,$$

where $1 \leq \alpha, \beta \leq p-1$. Suppose, by way of contradiction, that $|\Omega| - (2p+r) \equiv p \pmod{p^2}$. $\langle a, c \rangle$ has at least $p+2$ orbits of length p . Hence there is an integer γ ($1 \leq \gamma \leq p-1$) such that $|I(ac^\gamma)| \geq 3p$, which is a contradiction. (q.e.d)

From now on, let a be an element of order p of G such that

$$a = (1) \cdots (2p)(2p+1) \cdots (2p+r)(2p+r+1, \dots, 3p+r)(3p+r+1, \dots, 4p+r) \cdots.$$

By Step 5, $C_G(a)^{I(a)}$ has an orbit Δ such that $C_G(a)^\Delta \geq A^\Delta$ and $|\Delta| \geq 2p$. Hereafter we may assume that $\Delta = \{1, 2, \dots, |\Delta|\}$.

Step 8. Set $C_G(a)_0 = C_G(a)$. If $p \geq 5$, then there is an integer i ($0 \leq i \leq 2$) such that $C_G(a)_{0, \dots, i}$ and $C_G(a)_{0, \dots, i, i+1}$ have exactly m orbits on $\Omega - I(a)$, where m is at most three, and moreover m is at most two when $|\Omega| - (2p+r) \not\equiv 0 \pmod{p^2}$. If $p=3$, then there is an integer i ($0 \leq i \leq 1$) such that $C_G(a)_i$ and $C_G(a)_{i, i+1}$ have exactly m orbits on $\Omega - I(a)$, where m is at most two, and moreover m is one when $|\Omega| - (2p+r) \not\equiv 0 \pmod{p^2}$.

Proof. Suppose that $p \geq 5$. In order to prove Step 8 for $p \geq 5$, it is sufficient to show that $C_G(a)_{1, 2, 3}$ has at most three orbits on $\Omega - I(a)$, and that $C_G(a)_{1, 2, 3}$ has at most two orbits on $\Omega - I(a)$ when $|\Omega| - (2p+r) \not\equiv 0 \pmod{p^2}$.

Set $H = G_{1, 2, 3}$. Then H is p -transitive on $\Omega - \{1, 2, 3\}$ by Step 3. By the remark following Lemma 1.1 in [11], we get the following expression:

$$\frac{|H|}{p} = \sum_{x \in H} \alpha_p(x) \geq \sum_k \frac{|H|}{|C_H(u_k)|} \frac{1}{p} \sum_y \alpha^*(y),$$

where u_k ranges all representatives of conjugacy classes (in H) of elements of order p , and y ranges all p' -elements in $C_H(u_k)$ and $\alpha^*(y) = \alpha(y^{\Omega - I(u_k)})$. Hence,

$$\frac{|H|}{p} \geq \frac{|H|}{|C_H(a)|} \frac{1}{p} \sum_y \alpha^*(y).$$

Assume that $|\Omega| - (2p+r) \not\equiv 0 \pmod{p^2}$. Since a normalizes $G_{1, \dots, p, 2p+r+1, \dots, 3p+r}$, $G_{1, \dots, p, 2p+r+1, \dots, 3p+r}$ has an element b of order p with $ab = ba$. If $|I(X)| = 2p+r$ for any nontrivial element x of $\langle a, b \rangle$, then $\langle a, b \rangle$ has just $p-1$ orbits of length p on $\Omega - \{1, \dots, 3p+r\}$. So $|\Omega| - (2p+r) \equiv 0 \pmod{p^2}$, a contradiction. Hence $H (\supseteq \langle a, b \rangle)$ contains an element of order p which fixes less than $2p+r$ points, and so, the equality in the above expression does not hold. Now, assume that $x \in C_H(a)$ and $p \nmid |x|$. Set $|x| = p \cdot s$. Since $|I(x^s)| \leq 2p+r$, we have $\alpha^*(x^s) \leq p \cdot \alpha_p((x^s)^{I(a)})$. So, $\alpha^*(x) \leq p \cdot \alpha_p(x^{I(a)}) + 2p \cdot \alpha_{2p}(x^{I(a)})$. Hence, we have that

$\sum_y \alpha^*(y) \geq \sum_{y \in C_H(a)} (y) - p \cdot \sum_{y \in C_H(a)} \alpha_p(y^{I(a)}) - 2p \cdot \sum_{y \in C_H(a)} \alpha_{2p}(y^{I(a)})$. Since $C_H(a)^{\Delta-\{1,2,3\}} \geq A^{\Delta-\{1,2,3\}}$ and $|\Delta| \geq 2p$, we get $p \cdot \sum_{y \in C_H(a)} \alpha_p(y^{I(a)}) = p \cdot \sum_{y \in C_H(a)} \alpha_p(y^{\Delta-\{1,2,3\}}) = |C_H(a)|$ by the formula of Frobenius. Similarly, if $2p \cdot \sum_{y \in C_H(a)} \alpha_{2p}(y^{I(a)}) \neq 0$, then $2p \cdot \sum_{y \in C_H(a)} \alpha_{2p}(y^{I(a)}) = |C_H(a)|$. On the other hand, $\sum_{y \in C_H(a)} \alpha^*(y) = f \cdot |C_H(a)|$, where f is the number of orbits of $C_H(a)$ on $\Omega - I(a)$. Hence we get

$$\frac{|H|}{p} \geq \frac{|H|}{p} (f-2), \text{ and hence } f \leq 3.$$

In the above expression, if $|\Omega| - (2p+r) \not\equiv 0 \pmod{p^2}$, the equality does not hold.

Suppose that $p=3$. Then $r=0$ or 1 . If $r=0$, then G is 6-transitive on Ω by [10, Lemma 6]. So, we have $G \geq A^{\Omega}$ by [4, Theorem 1]. But this contradicts our assumption. Hence $r=1$. Since $\langle a \rangle \in \text{Syl}_3(G_{1,2,3,4,5})$, we have $N_G(\langle a \rangle)^{I(a)} \geq A_7$ by Step 3. Hence $C_G(a)^{I(a)} \geq A_7$. Set $H=G_{1,2}$. Then H is 3-transitive on $\Omega - \{1, 2\}$, and $C_H(a)^{I(a)-\{1,2\}} \geq A_5$. By the similar argument as in the case $p \geq 5$, we have that $C_H(a)$ has at most two orbits on $\Omega - I(a)$, and that $C_H(a)$ is transitive on $\Omega - I(a)$ when $|\Omega| - 7 \not\equiv 0 \pmod{9}$. Therefore, the consequences of Step 8 hold. (q.e.d.)

Step 9. $C_G(a)_{1,2,\dots,|\Delta|}$ has at most $2m$ orbits on $\Omega - I(a)$. Moreover $C_G(a)_{1,\dots,p,(p+1,p+2),p+3,\dots,|\Delta|} (=C_{G_{(p+1,p+2)}}(a)_{1,\dots,p,p+3,\dots,|\Delta|})$ has exactly m orbits on $\Omega - I(a)$.

Proof. By Step 8, $C_G(a)_{0,\dots,i}$ has exactly m orbits on $\Omega - I(a)$. Let $\Gamma_1, \dots, \Gamma_m$ be the orbits. We take an arbitrarily fixed orbit Γ_j . Let $\Sigma_1, \dots, \Sigma_k$ be the orbits of $C_G(a)_{1,\dots,|\Delta|}$ on Γ_j . Since $C_G(a)_{0,\dots,i} \triangleright C_G(a)_{1,\dots,|\Delta|}$ and Γ_j is an orbit of $C_G(a)_{0,\dots,i}$, $C_G(a)_{0,\dots,i}^{\Delta-\{1,\dots,i\}}$ acts on the set $\{\Sigma_1, \dots, \Sigma_k\}$ transitively. Let $Y = C_{G_{0,\dots,i}}(a)_{\Sigma_1}$. Then $|C_G(a)_{0,\dots,i}^{\Delta-\{1,\dots,i\}} : Y^{\Delta-\{1,\dots,i\}}| = k$. Similarly, we have $|C_G(a)_{0,\dots,i,i+1}^{\Delta-\{1,\dots,i\}} : Y_{i+1}^{\Delta-\{1,\dots,i\}}| = k$. Hence, $|C_G(a)_{0,\dots,i}^{\Delta-\{1,\dots,i\}} : C_G(a)_{0,\dots,i,i+1}^{\Delta-\{1,\dots,i\}}| = |Y^{\Delta-\{1,\dots,i\}} : Y_{i+1}^{\Delta-\{1,\dots,i\}}| = |\Delta| - i$. Therefore Y is transitive on $\Delta - \{1, \dots, i\}$. Let $(\beta_1, \dots, \beta_p)$ be a p -cycle of a such that $\{\beta_1, \dots, \beta_p\} \subseteq \Sigma_1$. For any $p-i$ elements $\alpha_1, \dots, \alpha_{p-i}$ of $\Delta - \{1, \dots, i\}$, $G_{0,\dots,i,\alpha_1,\dots,\alpha_{p-i},\beta_1,\dots,\beta_p}$ has an element b of order p commuting with a . Then $b \in Y$ and b^{Δ} is a p -cycle, and so, $Y_{\alpha_1,\dots,\alpha_{p-i}}^{\Delta-\{1,\dots,i\}}$ has the p -cycle. Since $\alpha_1, \dots, \alpha_{p-i-1}, \alpha_{p-i}$ are any $p-i$ elements of $\Delta - \{1, \dots, i\}$, we have $Y^{\Delta-\{1,\dots,i\}} \geq A^{\Delta-\{1,\dots,i\}}$ (cf. [14, Theorem 8.4, Theorem 13.9]). Therefore $k \leq 2$. If $k=2$, then $Y^{\Delta-\{1,\dots,i\}} = A^{\Delta-\{1,\dots,i\}}$ and $C_G(a)_{0,\dots,i}^{\Delta-\{1,\dots,i\}} = S^{\Delta-\{1,\dots,i\}}$. Therefore Γ_j is an orbit of $C_G(a)_{1,\dots,p,(p+1,p+2),p+3,\dots,|\Delta|}$ on $\Omega - I(a)$, even if $k=2$. (q.e.d.)

Step 10. $|\Omega| - (2p+r) \equiv 2p \pmod{p^2}$ and $p \geq 5$.

Proof. Since a is an element of order p of the form

$$a = (1) \cdots (p)(p+1) \cdots (2p)(2p+1) \cdots (2p+r)(2p+r+1, \dots, 3p+r) \\ (3p+r+1, \dots, 4p+r) \cdots,$$

we may assume that $C_G(a)_{p+1, \dots, 2p, 2p+r+1, \dots, 3p+r}$ has an element b of order p . By Step 7, we may assume that

$$b = (1, \dots, p)(p+1) \cdots (2p)(2p+1) \cdots (2p+r)(2p+r+1) \cdots \\ (3p+r)(3p+r+1, \dots, 4p+r) \cdots.$$

Let $K = G_{1, \dots, p(p+1, \dots, p+2)p+3, \dots, |\Delta|}$ and $L = \langle b \rangle \cdot K$. Then $|C_L(a) : C_K(a)| = p$. By Step 9, $C_K(a)$ and $C_L(a)$ have exactly m orbits on $\Omega - I(a)$. Since $m | C_K(a) | = \sum_{y \in C_K(a)} \alpha^*(y)$ and $m | C_L(a) | = \sum_{y \in C_L(a)} \alpha^*(y)$, we have

$$m \frac{p-1}{p} |C_L(a)| = \sum_{y \in C_L(a) - C_K(a)} \alpha^*(y).$$

Next we show that the elements of order p of $\langle a, b \rangle$ are not conjugate to each other in $C_L(a)$. Suppose $a^i b^j$ and $a^{i'} b^{j'}$ are conjugate to each other, where $0 \leq i, j, i', j' \leq p-1$. If $j \neq j'$, then $(a^i b^j)^{(1, \dots, p)} \neq (a^{i'} b^{j'})^{(1, \dots, p)}$, which is a contradiction. Hence $j = j'$. Assume $i \neq i'$. There exists an element x in $C_L(a)$ such that $(a^i b^j)^x = a^{i'} b^{j'}$. Then $(b^j)^x = a^{i'-i} b^j$. Since $(b^j)^x = a^{(i'-i)p} b^j = b^j$, we have $p | |x|$. Hence there exists a p -element x_0 in $C_L(a) \cap N_L(\langle a, b \rangle)$ such that $x_0 \notin C_L(\langle a, b \rangle)$. Since $\langle a, b \rangle \in \text{Syl}_p(C_L(a))$, this is a contradiction. Thus $i = i'$ and $j = j'$.

Let s be the number of orbits of length p of $\langle a, b \rangle$ on $\Omega - I(a)$. For each fixed j ($1 \leq j \leq p-1$), there are s elements i_1, \dots, i_s of $\{0, 1, \dots, p-1\}$ such that $|I(a^{i_k} b^j)| = |I(a)|$ ($k = 1, \dots, s$). Let i be an arbitrarily fixed element of $\{i_1, \dots, i_s\}$, and let $\{\gamma_1, \dots, \gamma_p\} = I(a^{i_k} b^j) \cap (\Omega - I(a))$. Since $\langle a, b \rangle$ is a Sylow P -subgroup of $C_L(\langle a, b \rangle)$, $C_L(\langle a, b \rangle)$ has the normal subgroup Y such that $C_L(\langle a, b \rangle) = \langle a, b \rangle \times Y$, where $(|Y|, p) = 1$, and $Y \subseteq C_K(a)$. Since Y acts on $I(\langle a, b \rangle) = \{p+1, \dots, 2p, 2p+1, \dots, 2p+r\}$, Y acts on $\{\gamma_1, \dots, \gamma_p\}$. Since $a^{(\gamma_1 \cdots \gamma_p)}$ is a p -cycle and $[Y, a] = 1$, we have $Y^{(\gamma_1 \cdots \gamma_p)} = 1$. Hence any element of $a^i b^j \cdot Y$ fixes at least p points of $\Omega - I(a)$. Moreover, it is clear that $a^i b^j \cdot Y \cap C_K(a) = \phi$. Therefore

$$\sum_{y \in C_L(\langle a, b \rangle) - C_K(a)} \alpha^*(y) \geq s(p-1)p |C_L(\langle a, b \rangle) : \langle a, b \rangle|.$$

Let d be any element of $C_L(a)$ such that d is conjugate to b in $C_L(a)$ and $d \neq b$. Then $\langle a, b \rangle \cap \langle a, d \rangle = \langle a \rangle$. Hence $C_L(\langle a, b \rangle) \cap C_L(\langle a, d \rangle) \subseteq C_K(a)$.

Therefore, we have

$$\sum_{y \in C_L(a) - C_K(a)} \alpha^*(y) \geq s(p-1)p |C_L(a) : C_{C_L(a)}(b)| |C_L(\langle a, b \rangle) : \langle a, b \rangle| \\ = \frac{s(p-1)}{p} |C_L(a)|.$$

Hence, $\frac{m(p-1)}{p}|C_L(a)| \geq \frac{s(p-1)}{p}|C_L(a)|$. Then $m \geq s$. On the other hand, if $|\Omega| - (2p+r) \equiv hp \pmod{p^2}$, where $2 \leq h \leq p$, then we have $s=h$. Therefore, we have that $|\Omega| - (2p+r) \equiv 2p \pmod{p^2}$ and $p \geq 5$, by Step 8. (q.e.d.)

Step 11. *We complete the proof.*

Proof. By Step 10, $\{2p+r+1, \dots, 3p+r\}$ and $\{3p+r+1, \dots, 4p+r\}$ are the orbits of length p of $\langle a, b \rangle$ on $\Omega - I(a)$, and $m=2$ and $p \geq 5$. By Step 4 we have $\alpha_p(a) \geq 4$, hence $|\Omega - I(a)| \geq p^2 + 2p$. Let $\Gamma_1, \dots, \Gamma_l$ be the orbits of $C_G(a)_{1,2,\dots,|\Delta|}$ on $\Omega - I(a)$, where $2 \leq l \leq 4$ by Step 9. Since $|b|=p$, b acts on the set $\{\Gamma_1, \dots, \Gamma_l\}$ trivially. If $l=2$, then Γ_1 and Γ_2 are the orbits of $C_G(a)_{1,\dots,p(p+1,p+2)p+3,\dots,|\Delta|}$ on $\Omega - I(a)$ by Step 9, and one of the following three cases holds: (i) $|\Gamma_1| \equiv 2p \pmod{p^2}$, $|\Gamma_2| \equiv 0 \pmod{p^2}$. (ii) $|\Gamma_1| \equiv 0 \pmod{p^2}$, $|\Gamma_2| \equiv 2p \pmod{p^2}$. (iii) $|\Gamma_1| \equiv |\Gamma_2| \equiv p \pmod{p^2}$. If $l=3$, then we may assume that $\Gamma_1 \cup \Gamma_2$ and Γ_3 are the orbits of $C_G(a)_{1,\dots,p(p+1,p+2)p+3,\dots,|\Delta|}$ on $\Omega - I(a)$, and one of the following two cases holds: (i) $|\Gamma_1|=|\Gamma_2| \equiv 0 \pmod{p^2}$, $|\Gamma_3| \equiv 2p \pmod{p^2}$. (ii) $|\Gamma_1|=|\Gamma_2| \equiv p \pmod{p^2}$, $|\Gamma_3| \equiv 0 \pmod{p^2}$. If $l=4$, then we may assume that $\Gamma_1 \cup \Gamma_2$ and $\Gamma_3 \cup \Gamma_4$ are the orbits of $C_G(a)_{1,\dots,p(p+1,p+2)p+3,\dots,|\Delta|}$ on $\Omega - I(a)$, and one of the following two cases holds: (i) $|\Gamma_1|=|\Gamma_2| \equiv 0 \pmod{p^2}$, $|\Gamma_3|=|\Gamma_4| \equiv p \pmod{p^2}$. (ii) $|\Gamma_1|=|\Gamma_2| \equiv p \pmod{p^2}$, $|\Gamma_3|=|\Gamma_4| \equiv 0 \pmod{p^2}$. We have the following for any value of l : There is a Γ_j ($1 \leq j \leq 4$) such that $|\Gamma_j| \equiv 0$ or $p \pmod{p^2}$ and $|\Gamma_j| \geq p^2$. Let $(\beta_1, \dots, \beta_p)$ and $(\gamma_1, \dots, \gamma_p)$ be two p -cycles of a such that $\{\beta_1, \dots, \beta_p, \gamma_1, \dots, \gamma_p\} \subseteq \Gamma_j$. $C_G(a)_{\beta_1, \dots, \beta_p, \gamma_1, \dots, \gamma_p}$ has an element c of order p . Hereafter we examine the relation between a and c . We may assume that

$$c = (1, \dots, p)(p+1, \dots, 2p)(2p+1) \cdots (2p+r)(\beta_1) \cdots (\beta_p)(\gamma_1) \cdots (\gamma_p) \cdots.$$

Since $|\Gamma_j| \not\equiv 2p \pmod{p^2}$, $\langle a, c \rangle$ has at least $p+2$ orbits of length p on $\Omega - I(a)$. Let $K = G_{1,2,\dots,|\Delta|}$, and $L = \langle c \rangle \cdot K$. By the same argument as in the proof of Step 10, we have that $l \cdot \frac{p-1}{p}|C_L(a)| = \sum_{y \in C_L(a) - C_K(a)} \alpha^*(y)$, and that the elements of $\langle a, c \rangle - \{1\}$ are not conjugate to each other in $C_L(a)$. For each fixed j ($1 \leq j \leq p-1$), there are at least $\frac{p+3}{2}$ elements $i_1, \dots, i_{(p+3)/2}$ of $\{0, 1, \dots, p-1\}$ such that $|I(a^{i_k}c^j)| \geq p+r$ ($k=1, \dots, \frac{p+3}{2}$). Let i be an arbitrarily fixed element of $\{i_1, \dots, i_{(p+3)/2}\}$. Since $\langle a, c \rangle$ is a Sylow p -subgroup of $C_L(\langle a, c \rangle)$ there exists the normal subgroup M of $C_L(\langle a, c \rangle)$ such that $C_L(\langle a, c \rangle) = \langle a, c \rangle \times M$. First assume that $a^i c^j$ fixes exactly p points $\delta_1, \dots, \delta_p$ in $\Omega - I(a)$. Then, by the same argument as in the proof of Step 10, any element of $a^i c^j \cdot M$ fixes $\{\delta_1, \dots, \delta_p\}$ pointwise. Next assume that $a^i c^j$ fixes exactly $2p$ points η_1, \dots, η_{2p} in $\Omega - I(a)$

and a fixes $\{\beta_1, \dots, \beta_p\}$ and $\{\gamma_1, \dots, \gamma_p\}$ with $\{\beta_1, \dots, \beta_p\} \cup \{\gamma_1, \dots, \gamma_p\} = \{\eta_1, \dots, \eta_{2p}\}$. If M fixes $\{\beta_1, \dots, \beta_p\}$ and $\{\gamma_1, \dots, \gamma_p\}$, then any element of $a^i c^j \cdot M$ fixes $\{\eta_1, \dots, \eta_{2p}\}$ pointwise. And if M transposes $\{\beta_1, \dots, \beta_p\}$ and $\{\gamma_1, \dots, \gamma_p\}$ then there exists the subgroup M_0 of index two of M such that any element of $a^i c^j \cdot M_0$ fixes $\{\eta_1, \dots, \eta_{2p}\}$ pointwise. Therefore, by the same argument as in the proof of Step 10, we have that

$$\begin{aligned} \sum_{y \in C_L(a) \cap C_K(a)} \alpha^*(y) &\geq \frac{p+3}{2} \cdot (p-1) \cdot p |C_L(a)| |C_{C_L(a)}(c)| |C_L(\langle a, c \rangle)| |C_L(\langle a, c \rangle)| \\ &= \frac{(p+3)(p-1)}{2p} \cdot |C_L(a)|. \end{aligned}$$

Hence $l \geq \frac{p+3}{2}$. So, we have $p=5$ and $l=4$.

We may assume that $|\Gamma_1|=|\Gamma_2| \equiv 0 \pmod{5^2}$. Let $(\delta_1, \dots, \delta_5)$ and (η_1, \dots, η_5) be two 5-cycles of a such that $\{\delta_1, \dots, \delta_5\} \subseteq \Gamma_1$ and $\{\eta_1, \dots, \eta_5\} \subseteq \Gamma_2$. $C_G(a)_{\delta_1, \dots, \delta_5, \eta_1, \dots, \eta_5}$ has an element d of order 5. Since d acts on the set $\{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4\}$ trivially, $\langle a, d \rangle$ has at least $2 \cdot 5 + 2$ orbits of length 5 on $\Omega - I(a)$. Hence, there exists an element x of order 5 of $\langle a, d \rangle$ such that $|I(x)| \geq 3 \cdot 5 + r$, which is a contradiction. (q.e.d.)

3. Proof of Theorem B

In the proof of Theorem B, we shall use the following Lemma.

Lemma. *There is no group satisfying the following condition: Let G be a 3-transitive group on Ω . Let α and β be two points of Ω . $G_{\alpha, \beta}$ is an imprimitive group on $\Omega - \{\alpha, \beta\}$ with two blocks Δ_1, Δ_2 of length $\frac{|\Omega|}{2} - 1$, and moreover, for any point γ of Δ_1 and any point δ of Δ_2 , $G_{\alpha, \beta, \gamma, \delta}^{\Delta_1 - \{\gamma\}}$ and $G_{\alpha, \beta, \gamma, \delta}^{\Delta_2 - \{\delta\}}$ are 2-transitive groups.*

(I think that this lemma is essentially known already in [7, § 1, Proof of Theorem 1])

Proof of Lemma (cf. [7, § 1, Proof of Theorem 1]). Let G be a group satisfying the above condition.

Set $|\Omega|=n$ and $|\Delta_i|=v+1$ ($i=1, 2$). Then $G_{\alpha, \beta}$ has just two orbits Σ_1 and Σ_2 on $\Omega - \{\alpha, \beta\}$ such that $|\Sigma_1|=v+1$ and $|\Sigma_2|=v$.

For any subset Δ of Ω with $|\Delta|=4$, G_Δ has two orbits Π_1 and Π_2 on $\Omega - \Delta$ such that $|\Pi_1|=|\Pi_2|$ or $|\Pi_1| - |\Pi_2| = 2$. In either case, G_Δ is a subgroup of $G_{\alpha_1, \alpha_2, \alpha_3}$ which satisfies the assumption of the Witt's Lemma [14, Theorem 9.4], where $\alpha_1, \alpha_2, \alpha_3$ are three elements of Δ . Hence $G_{(\Delta)}^\Delta$ is a 3-transitive group. Thus, $G_{(\Delta)}^\Delta = S_4$. Therefore, G acts on $\Omega^{(2)}$, the set of unordered pairs of elements of Ω , as a transitive permutation group of rank 4, where the orbitals, $\Gamma_0, \Gamma_1, \Gamma_2$ and Γ_3 of this permutation group are defined as follows: for $\{\alpha, \beta\} \in$

$$\Omega^{(2)}, \Gamma_0(\{\alpha, \beta\}) = \{\alpha, \beta\}$$

$$\Gamma_1(\{\alpha, \beta\}) = \{\{\gamma, \delta\} \in \Omega^{(2)} \mid |\{\alpha, \beta\} \cap \{\gamma, \delta\}| = 1\}$$

$$\Gamma_2(\{\alpha, \beta\}) = \{\{\gamma, \delta\} \in \Omega^{(2)} \mid \{\alpha, \beta\} \cap \{\gamma, \delta\} = \emptyset\}.$$

δ is in the orbit of length v of $G_{\alpha\beta\gamma}$ on $\Omega - \{\alpha, \beta, \gamma\}$

$$\Gamma_3(\{\alpha, \beta\}) = \{\{\gamma, \delta\} \in \Omega^{(2)} \mid |\{\alpha, \beta\} \cap \{\gamma, \delta\}| = 2\}.$$

δ is in the orbit of length $v+1$ of $G_{\alpha\beta\gamma}$ on $\Omega - \{\alpha, \beta, \gamma\}$.

The degrees corresponding to Γ_i ($i=0, 1, 2, 3$) are respectively

$$1, 2(n-2) = 4(v+1), \quad \frac{(n-2)v}{2} = v(v+1), \quad \frac{(n-2)(v+1)}{2} = (v+1)^2.$$

Moreover, these orbitals Γ_i ($i=0, 1, 2, 3$) are all self-paired.

Let us define the intersection matrices M_i ($i=0, 1, 2, 3$) for the permutation group G on $\Omega^{(2)}$ as follows:

$$M_i = (\mu_{jk}^{(i)}) \text{ with } 0 \leq j \leq 3, 0 \leq k \leq 3, \text{ where}$$

$$\mu_{jk}^{(i)} = |\Gamma_j(x) \cap \Gamma_i(y)| \text{ with } y \in \Gamma_k(x)$$

$$(\text{where } x, y \in \Omega^{(2)}).$$

Now we can obtain the intersection matrix M_2 (cf. [9, §4]). This is,

$$M_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & v & 2v-2 & 2v \\ v(v+1) & \frac{v(v-1)}{2} & -v+2 & v(v-1) \\ 0 & \frac{v(v+1)}{2} & v^2-1 & 0 \end{pmatrix}$$

By direct calculations, we obtain the eigenvalues $\theta_0, \theta_1, \theta_2$ and θ_3 of M_2 .

$$\theta_0 = v(v+1), \quad \theta_1 = -v, \quad \theta_2 = \frac{-v^2+2+\sqrt{v^4+4v+4}}{2} \text{ and}$$

$$\theta_3 = \frac{-v^2+2-\sqrt{v^4+4v+4}}{2}.$$

Since $(v^2)^2 < v^4+4v+4 < (v^2+2)^2$, it is clear that θ_2 and θ_3 are irrational numbers.

Let us denote by $\pi^{(2)}$ the permutation character of G on $\Omega^{(2)}$. Then $\pi^{(2)}$ is multiplicity free and $\pi^{(2)} = 1 + X_1 + X_2 + X_3$, where $X_1 = X^{(n-1,1)}|G$ and X_2 and X_3 are irreducible characters appearing in $X^{(n-2,2)}|G$ corresponding to θ_2 and θ_3 respectively. Since θ_2 and θ_3 are irrational, X_2 and X_3 are not rational characters (cf. [6, Lemma 1]), so X_2 and X_3 are algebraic conjugate

and especially of the same degree. Therefore $X_2(1)=X_3(1)=n(n-3)/4$ and $X_1(1)=n-1$. By a theorem of Frame [14, Theorem 30.1 (A)], we obtain that the number

$$q = \left\{ \frac{n(n-1)}{2} \right\}^2 \frac{2(n-2) \cdot v(n-2)/2 \cdot (n-2)(v+1)/2}{(n-1) \cdot n(n-3)/4 \cdot n(n-3)/4}$$

must be an integer. But, since $n=2v+4$, we have a contradiction. (q.e.d.)

Proof of Theorem B. Let G be a counter-example to the theorem with the least possible degree.

Step 1. *The number of orbits of G on Ω is at most two.*

Proof. By Theorem A and the assumption for G , G has no orbit on Ω whose length is less than p .

Suppose, by way of contradiction, that G has three orbits Δ_1 , Δ_2 and Δ_3 with $|\Delta_i| \geq p$ ($i=1, 2, 3$). Set $|\Delta_i| \equiv k_i \pmod{p}$, where $0 \leq k_i \leq p-1$ ($i=1, 2, 3$). Assume that $2p-(k_1+k_2+2) \geq p$. We take k_1+p-1 points $\alpha_1, \dots, \alpha_{k_1+p-1}$ from Δ_1 , k_2+1 points $\beta_1, \dots, \beta_{k_2+1}$ from Δ_2 and $p-k_1-k_2$ points $\gamma_1, \dots, \gamma_{p-k_1-k_2}$ from Δ_3 . A Sylow p -subgroup of $G_{\alpha_1, \dots, \alpha_{k_1+p-1}, \beta_1, \dots, \beta_{k_2+1}, \gamma_1, \dots, \gamma_{p-k_1-k_2}}$ fixes at least $3p$ points, which contradicts the assumption of Theorem B. Hence $2p-(k_1+k_2+2) < p$. We take k_1+1 points $\alpha_1, \dots, \alpha_{k_1+1}$ from Δ_1 , k_2+1 points $\beta_1, \dots, \beta_{k_2+1}$ from Δ_2 and $2p-k_1-k_2-2$ points $\gamma_1, \dots, \gamma_{2p-k_1-k_2-2}$ from Δ_3 . A Sylow p -subgroup of $G_{\alpha_1, \dots, \alpha_{k_1+1}, \beta_1, \dots, \beta_{k_2+1}, \gamma_1, \dots, \gamma_{2p-k_1-k_2-2}}$ fixes at least $3p$ points, which is a contradiction. (q.e.d.)

Step 2. *We may assume that G is transitive on Ω . ($|\Omega| \equiv p-1 \pmod{p}$.)*

Proof. Suppose that G is not transitive on Ω . By Step 1, G has two orbits Δ_1 and Δ_2 such that $\Delta_1 \cup \Delta_2 = \Omega$ and $|\Delta_i| \geq p$ ($i=1, 2$). Set $|\Delta_i| = s_i p + k_i$, where $0 \leq k_i \leq p-1$ ($i=1, 2$). In this case $k_1+k_2=p-1$. By the assumption of Theorem B, $s_1 \geq 2$ or $s_2 \geq 2$. We may assume that $s_1 \geq 2$ and $s_1 \geq s_2$. We divide the consideration into the following three cases: (I) $s_1 \geq 3$. (II) $s_1 = s_2 = 2$. (III) $s_1 = 2, s_2 = 1$.

Suppose that Case (I) holds. By Theorem A and the assumption for G , $G^{\Delta_1} \geq A^{\Delta_1}$, and so, $s_1=3$. For k_2+1 points $\alpha_1, \dots, \alpha_{k_2+1}$ of Δ_2 , $G_{\alpha_1, \dots, \alpha_{k_2+1}}^{\Delta_1}$ is $(p+k_1)$ -transitive by [10, Lemma 6]. Since $G_{\alpha_1, \dots, \alpha_{k_2+1}}^{\Delta_1}$ has an element x of order p with $\alpha_p(x)=2$, we have $G_{\alpha_1, \dots, \alpha_{k_2+1}}^{\Delta_1} \geq A^{\Delta_1}$ by [14, Theorem 13.10]. This is a contradiction.

Suppose that Case (II) holds. We may assume that $k_1 \geq k_2$. For $p+k_2+1$ points $\alpha_1, \dots, \alpha_{p+k_2+1}$ of Δ_2 , $G_{\alpha_1, \dots, \alpha_{p+k_2+1}}^{\Delta_1}$ has an element of order p , and moreover $G_{\alpha_1, \dots, \alpha_{p+k_2+1}}^{\Delta_1}$ is k_1 -transitive by [10, Lemma 6]. Since $k_1 \geq 5$, $G_{\alpha_1, \dots, \alpha_{p+k_2+1}}^{\Delta_1} \geq A^{\Delta_1}$ by [14, Theorem 13.10]. This is a contradiction.

Suppose that Case (III) holds. By [10, Lemma 6] and [14, Theorem 13.10], G is a group satisfying the consequence (2) of Theorem B. This is a contradiction. (q.e.d.)

Step 3. G is primitive on Ω . For any element x of order p of G , $\alpha_p(x) \geq 8$ holds.

Proof. Suppose, by way of contradiction, that G is imprimitive on Ω . Let $\Delta_1, \dots, \Delta_s$ be a system of imprimitivity of G . Set $|\Delta_i| \equiv k \pmod{p}$, where $0 \leq k \leq p-1$. First assume that $|\Delta_i| \leq p$. Then $s > 2p$ and we are able to take $2p$ points $\delta_1, \dots, \delta_{2p}$ from Ω such that $\delta_i \in \Delta_i$ ($i = 1, \dots, 2p$). A Sylow p -subgroup of $G_{\delta_1, \dots, \delta_{2p}}$ fixes at least $4p$ points, which is a contradiction. Next assume that either $p < |\Delta_i| < 2p$, or $|\Delta_i| \geq 2p$ and $s \geq 3$. We take $k+1$ points $\alpha_1, \dots, \alpha_{k+1}$ from Δ_1 and $k+1$ points $\beta_1, \dots, \beta_{k+1}$ from Δ_2 . We are able to take $2p-2k-2$ points $\gamma_1, \dots, \gamma_{2p-2k-2}$ from $\Omega - (\Delta_1 \cup \Delta_2)$. A Sylow p -subgroup of $G_{\alpha_1, \dots, \alpha_{k+1}, \beta_1, \dots, \beta_{k+1}, \gamma_1, \dots, \gamma_{2p-2k-2}}$ fixes at least $3p$ points, which is a contradiction.

Therefore, we have that $|\Delta_i| \geq 2p$ and $s=2$. Then $\Omega = \Delta_1 \cup \Delta_2$ and $k = \frac{p-1}{2}$.

By Theorem A, $|\Delta_i| = 3p + \frac{p-1}{2}$ or $2p + \frac{p-1}{2}$. By the similar argument to that of Case (II) of Step 2, we have a contradiction. Thus G is primitive on Ω . By [14, Theorem 13.10], for any element x of order p of G , we have $\alpha_p(x) \geq 8$. (q.e.d.)

Step 4. Let $2 \leq t \leq p + \frac{p-1}{2} + 2$. If G is t -transitive on Ω , then G is t -primitive on Ω .

Proof. Suppose, by way of contradiction, that G is t -transitive on Ω and $G_{1, \dots, t-1}$ is imprimitive on $\Omega - \{1, \dots, t-1\}$. Let $\Delta_1, \dots, \Delta_s$ be a system of imprimitivity of $G_{1, \dots, t-1}$ on $\Omega - \{1, \dots, t-1\}$. Set $|\Delta_i| \equiv k \pmod{p}$ and $|\Delta_i| = lp + k$, where $0 \leq k \leq p-1$. In this case, $(t-1) + sk \equiv p-1 \pmod{p}$. We divide the consideration into the following two cases: (I) $2p-t+1 \geq p$. (II) $2p-t+1 < p$.

Suppose that Case (I) holds. First assume that $l=0$. Then $s > 2p-t+1$ and we are able to take $2p-t+1$ points $\delta_1, \dots, \delta_{2p-t+1}$ of Ω such that $\delta_i \in \Delta_i$ ($i = 1, \dots, 2p-t+1$). A Sylow p -subgroup of $G_{1, \dots, t-1, \delta_1, \dots, \delta_{2p-t+1}}$ fixes at least $3p$ points, which is a contradiction. Secondly assume that $l=1$. By Step 3, we get $s \geq 8$. Assume that $k \geq \frac{p-1}{2}$. We take a point α from Δ_1 , a point β from Δ_2 , a point γ from Δ_3 and $2p-t-2$ points $\delta_1, \dots, \delta_{2p-t-2}$ from $\Delta_4 \cup \Delta_5$. A Sylow p -subgroup of $G_{1, \dots, t-1, \alpha, \beta, \gamma, \delta_1, \dots, \delta_{2p-t-2}}$ fixes at least $3p$ points, which is a contradiction. Hence we have $k \leq \frac{p-3}{2}$ when $l=1$. We take $k+1$ points $\alpha_1, \dots, \alpha_{k+1}$

from Δ_1 , $k+1$ points $\beta_1, \dots, \beta_{k+1}$ from Δ_2 and $2p-t-2k-1$ points $\gamma_1, \dots, \gamma_{2p-t-2k-1}$ from $\Delta_3 \cup \Delta_4$. A Sylow p -subgroup of $G_{1, \dots, t-1, \alpha_1, \dots, \alpha_{k+1}, \beta_1, \dots, \beta_{k+1}, \gamma_1, \dots, \gamma_{2p-t-2k-1}}$ fixes at least $3p$ points, which is a contradiction. Thirdly assume that $l \geq 2$ and $2p-t-k \neq k, k+p$. We take $k+1$ points $\alpha_1, \dots, \alpha_{k+1}$ from Δ_1 and $2p-t-k$ points $\beta_1, \dots, \beta_{2p-t-k}$ from Δ_2 . A Sylow p -subgroup of $G_{1, \dots, t-1, \alpha_1, \dots, \alpha_{k+1}, \beta_1, \dots, \beta_{2p-t-k}}$ fixes at least $3p$ points, which is a contradiction. Fourthly assume that $l \geq 2$ and $2p-t-k = k+p$. Assume that $s \geq 3$. We take $k+1$ points $\alpha_1, \dots, \alpha_{k+1}$ from Δ_1 , $k+1$ points $\beta_1, \dots, \beta_{k+1}$ from Δ_2 and $p-1$ points $\gamma_1, \dots, \gamma_{p-1}$ from Δ_3 . A Sylow p -subgroup of $G_{1, \dots, t-1, \alpha_1, \dots, \alpha_{k+1}, \beta_1, \dots, \beta_{k+1}, \gamma_1, \dots, \gamma_{p-1}}$ fixes at least $3p$ points, which is a contradiction. Hence we have $\Omega = \{1, \dots, t-1\} \cup \Delta_1 \cup \Delta_2$ when $l \geq 2$ and $2p-t-k = k+p$. Since $k = \frac{p-t}{2}$ and $t \geq 2$, we get $t \geq 3$. Let γ be any

point of Δ_1 , and δ be any point of Δ_2 . By [10, Lemma 6], it is easily seen that $G_{1, \dots, t-1, \gamma, \delta}^{\Delta_1}$ and $G_{1, \dots, t-1, \gamma, \delta}^{\Delta_2}$ are $(k-1+p)$ -transitive. By Lemma, we have a contradiction. Fifthly assume that $l \geq 2$ and $2p-t-k = k$. In this case, $k = \frac{2p-t}{2} \geq \frac{p-1}{2}$. Assume that $s \geq 3$. We take $k+1$ points $\alpha_1, \dots, \alpha_{k+1}$ from Δ_1 , $k-1$ points $\beta_1, \dots, \beta_{k-1}$ from Δ_2 and a point γ from Δ_3 . A Sylow p -subgroup of $G_{1, \dots, t-1, \alpha_1, \dots, \alpha_{k+1}, \beta_1, \dots, \beta_{k-1}, \gamma}$ fixes at least $3p$ points, which is a contradiction. Hence, we have $\Omega = \{1, \dots, t-1\} \cup \Delta_1 \cup \Delta_2$ when $l \geq 2$ and $2p-t-k = k$. Let Q be a Sylow p -subgroup of $G_{1, \dots, t}$. Then $N_G(Q)^{I(Q)}$ is a t -transitive group and $|I(Q)| \geq t-1+2k=2p-1$. Let x be an element of order p of Q with $|I(x)| = 3p-1$, and $(\gamma_1, \dots, \gamma_p)$ be a p -cycle of x . Let $\{\delta_1, \dots, \delta_p\}$ be a subset of Ω such that if $|I(Q)| = 2p-1$, then $\{\delta_1, \dots, \delta_p\} = I(x) - I(Q)$, and if $|I(Q)| = 3p-1$, then $x^{\{\delta_1, \dots, \delta_p\}}$ is a p -cycle of x different from $(\gamma_1, \dots, \gamma_p)$. $C_G(x)_{\gamma_1, \dots, \gamma_p, \delta_1, \dots, \delta_p}$ has an element y of order p . Since y fixes $I(Q)$, we may assume that $y \in N_G(Q)$. Then $y^{I(Q)}$ is an element of order p of $N_G(Q)^{I(Q)}$ which is 2-transitive on $I(Q)$ and we have $N_G(Q)^{I(Q)} \geq A^{I(Q)}$. Since $G_{1, \dots, t-1}$ is imprimitive on $\Omega - \{1, \dots, t-1\}$, this is a contradiction.

Suppose that Case (II) holds. In this case, $p+2 \leq t \leq p + \frac{p-1}{2} + 2$. Let Q be a Sylow p -subgroup of $G_{1, \dots, t}$. Then $N_G(Q)^{I(Q)}$ is t -transitive on $I(Q)$. Since $|\Omega| \equiv p-1 \pmod{p}$, we have $|I(Q)| \equiv p-1 \pmod{p}$, and so, $|I(Q)| = 2p-1$ or $3p-1$. Since $t \geq p+2$, $N_G(Q)^{I(Q)}$ has an element of order p , and so, we get $N_G(Q)^{I(Q)} \geq A^{I(Q)}$. We may assume that $\{\Delta_1, \dots, \Delta_u\}$ is the subset of $\{\Delta_1, \dots, \Delta_s\}$ such that $I(Q) \cap \Delta_i \neq \emptyset$ for $1 \leq i \leq u$ and $I(Q) \cap \Delta_i = \emptyset$ for $u < i \leq s$. Since $G_{1, \dots, t-1}$ is imprimitive on $\Omega - \{1, \dots, t-1\}$, we have that $k \leq 1$ or $u = 1$. Assume that $k \geq 2$. Then $u = 1$, and so, $(t-1)+k \equiv p-1 \pmod{p}$. Hence $t-1+k=2p-1$. Then $p - \frac{p-1}{2} - 2 \leq k \leq p-2$. On the other hand, $(t-1)+sk \equiv p-1 \pmod{p}$. Then $(t+k)+(s-1)k \equiv 0 \pmod{p}$, and so, $p \mid s-1$. Hence

$s \geq p+1$. Let α_i be a point of Δ_i ($i=1, \dots, s$). A Sylow p -subgroup of $G_{1, \dots, t-1, \alpha_1, \dots, \alpha_{k+1}}$ fixes at least $2p+(k+1)(k-1)$ points. But, $(k+1)(k-1) \geq \left(p - \frac{p-1}{2} - 1\right)\left(p - \frac{p-1}{2} - 3\right) \geq p$, which is a contradiction. Therefore $k=0$ or

1. We take two points α_1, α_2 from Δ_1 and $2p-t-1$ points $\beta_1, \dots, \beta_{2p-t-1}$ from Δ_2 . A Sylow p -subgroup of $G_{1, \dots, t-1, \alpha_1, \alpha_2, \beta_1, \dots, \beta_{2p-t-1}}$ fixes at least $3p$ points, which is a contradiction. (q.e.d.)

Step 5. G is $\left(p + \frac{p+1}{2} + 2\right)$ -transitive on Ω .

Proof. By Step 3 and Step 4, in order to prove Step 5 we show that if G is t -primitive on Ω then G is $(t+1)$ -transitive on Ω , where $1 \leq t \leq p + \frac{p-1}{2} + 2$.

Suppose, by way of contradiction, that G is t -primitive on Ω , but G is not $(t+1)$ -transitive on Ω . Let $\Delta_1, \dots, \Delta_s$ be the orbits of $G_{1, \dots, t}$ on $\Omega - \{1, \dots, t\}$, where $s \geq 2$. We may assume that $|\Delta_1| \geq |\Delta_2| \geq \dots \geq |\Delta_s| \geq p$ (cf. [14, Theorem 18.4]). Set $|\Delta_i| \equiv k_i \pmod{p}$ ($i=1, \dots, s$), then $t+k_1+\dots+k_s \equiv p-1 \pmod{p}$. We divide the consideration into the following two cases: (I) $2p-t \geq p+1$. (II) $2p-t \leq p$.

Suppose that Case (I) holds. First assume that $|\Delta_1| = p$ or $p+1$. We take two points α_1, α_2 from Δ_1 and two points β_1, β_2 from Δ_2 . We are able to take $2p-t-4$ points $\gamma_1, \dots, \gamma_{2p-t-4}$ from $\Delta_3 \cup \dots \cup \Delta_s$. A Sylow p -subgroup of $G_{1, \dots, t-1, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \dots, \gamma_{2p-t-4}}$ fixes at least $3p$ points, which is a contradiction. Therefore $|\Delta_1| \geq p+2$. Secondly assume that $2p-t-k_1 \geq p$ and $|\Delta_1| \geq 2p+k_1$. We take $p-t-k_1$ points $\beta_1, \dots, \beta_{p-t-k_1}$ from $\Delta_2 \cup \dots \cup \Delta_s$. By [10, Lemma 6], $G_{1, \dots, t, \beta_1, \dots, \beta_{p-t-k_1}}^{\Delta_1}$ is $(p+k_1)$ -transitive, which contradicts Theorem 17.7 in [14]. If $k_1=0$ or 1 then our assumptions are satisfied. Therefore $k_1 \geq 2$. Thirdly assume that either $2p-t-k_1 \geq p$ and $|\Delta_1| = p+k_1$, or $2p-t-k_1 < p$. We are able to take $2p-t-k_1$ points $\beta_1, \dots, \beta_{2p-t-k_1}$ from $\Delta_2 \cup \dots \cup \Delta_s$. By [10, Lemma 6], $G_{1, \dots, t, \beta_1, \dots, \beta_{2p-t-k_1}}^{\Delta_1}$ is k_1 -transitive, which contradicts Theorem 17.7 in [14].

Suppose that Case (II) holds. In this case, $p \leq t \leq p + \frac{p-1}{2} + 2$. Let Q be

a Sylow p -subgroup of $G_{1, \dots, t}$, then $N_G(Q)^{I(Q)}$ is t -transitive, and $|I(Q)| = 2p-1$ or $3p-1$. Since $t \geq p$, we have $N_G(Q)^{I(Q)} \geq A^{I(Q)}$. Hence, there is a unique orbit Δ_j such that $k_j \neq 0$. Since $t+k_j \equiv p-1 \pmod{p}$, we have that $k_j = 2p-1-t \geq 3$. By [10, Lemma 6], $G_{1, \dots, t}^{\Delta_j}$ is k_j -transitive, and so, we have $j \neq 1$ by [14, Theorem 17.7]. Assume that $s \geq 3$. We take a point α from Δ_1 , $2p-t-2$ points $\beta_1, \dots, \beta_{2p-t-2}$ from Δ_j and a point γ from Δ_i where $1 < i \leq s$ and $i \neq j$. A Sylow p -subgroup of $G_{1, \dots, t-1, \alpha, \beta_1, \dots, \beta_{2p-t-2}, \gamma}$ fixes at least $3p$ points, which is a contradiction. Therefore $s=j=2$. If $p \geq 13$, then $k_j = 2p-1-t \geq 4$. This is a contradiction by [1]. Hence, we have $p=11$. Moreover, we have

$k_j=2p-1-t=3$ by [1]. By [8, Theorem 5], we have that either (i) $|\Delta_1|+|\Delta_2|+1=\frac{1}{2}(|\Delta_1|^2+|\Delta_2|^2+2)$, or (ii) $|\Delta_1|+|\Delta_2|+1=(\lambda+1)^2(\lambda+4)^2$, $|\Delta_2|=(\lambda+1)(\lambda^2+5\lambda+5)$, for some positive integer λ . Case (i) does not hold, since $3+1 \not\equiv \frac{1}{2}(3^2+3+2) \pmod{11}$. Moreover Case (ii) does not hold, since for every λ ($\lambda=0, 1, \dots, 10$), we have $3+1 \not\equiv (\lambda+1)^2(\lambda+4)^2 \pmod{11}$ or $3 \not\equiv (\lambda+1) \cdot (\lambda^2+5\lambda+5) \pmod{11}$. (q.e.d.)

Step 6. Let a be an element of order p of the form

$$a = (1) \cdots (p) \cdots (2p) \cdots (3p-1)(3p, \dots, 4p-1) \cdots$$

Then one of the following holds for $C=C_G(a)_{3p, \dots, 4p-1}^{I(a)}$.

- (i) C has an orbit Δ such that $C^\Delta \geq A^\Delta$ and $|\Delta| \geq 2p$.
- (ii) There exist two orbits Δ_1 and Δ_2 of C such that $|\Delta_i| \geq p$ and C^{Δ_i} is $(|\Delta_i|-p+1)$ -transitive ($i=1, 2$), and $\Delta_1 \cup \Delta_2 = I(a)$. Moreover, if $|\Delta_i| \geq p+3$, then $C^{\Delta_i} \geq A^{\Delta_i}$.

- (iii) C is an imprimitive group with two blocks Γ_1 and Γ_2 of length $p+\frac{p-1}{2}$ such that $C^{\Gamma_i} \geq A^{\Gamma_i}$ ($i=1, 2$).

Proof. For any p points $\alpha_1, \dots, \alpha_p$ of $I(a)$, $C_{\alpha_1, \dots, \alpha_p}$ has an element of order p . Since C has an element of order p , it has an orbit whose length is at least p . Assume that C has two orbits Δ_1 and Δ_2 with $|\Delta_i| \geq p$ ($i=1, 2$). Set $|\Delta_i|=p+k_i$ ($i=1, 2$). If $\Delta_1 \cup \Delta_2 = I(a)$, then $k_1+k_2+2 \leq p$. We take k_1+1 points $\alpha_1, \dots, \alpha_{k_1+1}$ from Δ_1 and k_2+1 points $\beta_1, \dots, \beta_{k_2+1}$ from Δ_2 , so $C_{\alpha_1, \dots, \alpha_{k_1+1}, \beta_1, \dots, \beta_{k_2+1}}$ has no element of order p , a contradiction. Hence $\Delta_1 \cup \Delta_2 = I(a)$. By [10, Lemma 6], we have that C is a group satisfying (ii). Assume that C has a unique orbit Δ with $|\Delta| \geq p$. Then we have $|\Delta| \geq 2p$. If C^Δ is primitive, by [14, Theorem 13.9] we have that C^Δ is a group satisfying (i). Assume that C^Δ is imprimitive. Let $\Gamma_1, \dots, \Gamma_s$ be a system of imprimitivity of C^Δ . If $|\Gamma_1| < p$, then $|\Gamma_1| = 2$. We take p points $\alpha_1, \dots, \alpha_p$ with $\alpha_i \in \Gamma_i$ ($i=1, \dots, p$), so $C_{\alpha_1, \dots, \alpha_p}$ has no element of order p , a contradiction. Hence $|\Gamma_1| \geq p$, and so we have $s=2$ and $|\Gamma_1|=|\Gamma_2|=p+\frac{p-1}{2}$. By [10, Lemma 6], we have that C is a group satisfying (iii). (q.e.d.)

Step 7. For any $2p$ points $\alpha_1, \dots, \alpha_{2p}$ of Ω , the order of a Sylow p -subgroup of $G_{\alpha_1, \dots, \alpha_{2p}}$ is p .

Proof. Suppose, by way of contradiction, that for some $2p$ points $\alpha_1, \dots, \alpha_{2p}$, the order of a Sylow p -subgroup P of $G_{\alpha_1, \dots, \alpha_{2p}}$ is more than p . We may assume that $\{\alpha_1, \dots, \alpha_{2p}\} = \{1, \dots, 2p\}$ and $I(P) = \{1, \dots, 2p, \dots, 3p-1\}$. Let a be an element of order p of $Z(P)$. We may assume that

$$a = (1) \cdots (3p-1)(3p, \dots, 4p-1) \cdots .$$

Since $C_{G_1}(a)^{I(a)-\{1\}}$ is a permutation group of degree $3p-2$, one of the following two cases holds:

- (I) $C_{G_1}(a)^{I(a)-\{1\}}$ has an orbit Δ such that $C_{G_1}(a)^\Delta \geq A^\Delta$ and $|\Delta| \geq 2p-1$.
- (II) $C_{G_1}(a)^{I(a)-\{1\}}$ has two orbits Δ_1, Δ_2 such that $|\Delta_i| \geq p$ and $C_{G_1}(a)^{\Delta_i}$ is $(|\Delta_i|-p+1)$ -transitive ($i=1, 2$), and $\Delta_1 \cup \Delta_2 = I(a) - \{1\}$. Moreover, if $|\Delta_i| \geq p+3$, then $C_{G_1}(a)^{\Delta_i} \geq A^{\Delta_i}$.

Suppose that Case (I) holds. We may assume that $\Delta = \{2, 3, \dots, |\Delta|, |\Delta|+1\}$. Let $\Gamma = \{2, 3, \dots, 2p\}$, then $\Gamma \subseteq \Delta$. Since $C_{G_1}(a)^\Delta \geq A^\Delta$, we have $G_{1(\Gamma)}^r \geq A^\Gamma$. On the other hand, by the Frattini-Sylow argument, $G_{1(\Gamma)} = N_{G_1(\Gamma)}(G_{1(\Gamma)}) = N_{G_1(\Gamma)}(P) \cdot G_{1(\Gamma)}$. Hence, $N_{G_1}(P)_{(\Gamma)}^r = G_{1(\Gamma)}^r \geq A^\Gamma$, so we have $|N_{G_1}(P)_{(\Gamma)}|_p$ (=the order of a Sylow p -subgroup of $N_{G_1}(P)_{(\Gamma)} = |P| \cdot p$). $C_G(a)_{1, 2p+1, \dots, 3p-1, 3p, \dots, 4p-1}$ has an element b of order p . Since $|\Gamma| < 2p$, b^r is a p -cycle. Since b normalizes $G_{1, \dots, 3p-1}$, we may assume that $P^b = P$. Then $\langle b, P \rangle \in \text{Syl}_p(N_{G_1}(P)_{(\Gamma)})$. Since $C_p(b)$ is semiregular on $(\Omega - I(P)) \cap I(b) = \{3p, \dots, 4p-1\}$, we have $|C_p(b)| = p$. Hence, since $[P, b] \neq 1$ we have $|Z(\langle P, b \rangle)| = p$. Assume that $C_{G_1}(P)_{(\Gamma)}^r = 1$. Since $N_{G_1}(P)/C_{G_1}(P)_{(\Gamma)} \leq \text{Aut}(P)$, A_{2p-1} is involved in $\text{Aut}(P)$. But, we can easily seen that A_{2p-1} is not involved in $\text{Aut}(P)$ (cf. [2, §2. (3)]), which is a contradiction. Hence $C_{G_1}(P)_{(\Gamma)}^r \geq A^\Gamma$. Since the center of a Sylow p -subgroup of $N_{G_1}(P)_{(\Gamma)}$ is of order p , this is a contradiction.

Suppose that Case (II) holds. Then, one of the following two cases holds:

- (i) $N_{G_1}(P)^{I(P)-\{1\}} \geq A^{I(P)-\{1\}}$.
- (ii) Δ_1 and Δ_2 are the orbits of $N_{G_1}(P)^{I(P)-\{1\}}$. $N_{G_1}(P)^{\Delta_i}$ is $(|\Delta_i|-p+1)$ -transitive ($i=1, 2$), and if $|\Delta_i| \geq p+3$, then $N_{G_1}(P)^{\Delta_i} \geq A^{\Delta_i}$.

If Case (i) holds, then we have a contradiction by the similar argument to that of Case (I). Hence we assume that Case (ii) holds. We may assume that $|\Delta_1| > |\Delta_2|$ and $\Delta_1 = \{2, 3, \dots, |\Delta_1|, |\Delta_1|+1\}$. Let $\Gamma = \{2, 3, \dots, 2p\}$. Since $|\Gamma \cap \Delta_2| \leq \frac{p-1}{2}$, we have $(C_{G_1}(a)_{\Gamma \cap \Delta_2})^{\Delta_1} \geq A^{\Delta_1}$ by [10, Lemma 6]. Then $N_{G_1}(P)_{(\Gamma)}^{\Delta_1} \geq A^{\Delta_1}$, and so, $|N_{G_1}(P)_{(\Gamma)}|_p = |P| \cdot p$. $C_G(a)_{1, 2p+1, \dots, 3p-1, 3p, \dots, 4p-1}$ has an element b of order p . Then b^{Δ_1} is a p -cycle, and we may assume that $P^b = P$. So $\langle b, P \rangle \in \text{Syl}_p(N_{G_1}(P)_{(\Gamma)})$. By the same argument as in Case (I), we have $|Z(\langle b, P \rangle)| = p$. Assume that $C_{G_1}(P)_{(\Gamma)}^{\Delta_1} = 1$. Then $C_{G_1}(a)_{\Delta_1} \geq C_{G_1}(a)_{(\Gamma)}$. Since $N_{G_1}(P)/C_{G_1}(P)_{(\Gamma)} \leq \text{Aut}(P)$ and $N_{G_1}(P)/N_{G_1}(P)_{\Delta_1} \cong N_{G_1}(P)_{(\Gamma)}^{\Delta_1} \geq A^{\Delta_1}$, we have that $A_{(3p-1)/2}$ is involved in $\text{Aut}(P)$. But, we can easily seen that $A_{(3p-1)/2}$ is not involved in $\text{Aut}(P)$ (cf. [2, §2. (3)]), which is a contradiction. Hence $C_{G_1}(P)_{(\Gamma)}^{\Delta_1} \geq A^{\Delta_1}$. Since the center of a Sylow p -subgroup of $N_{G_1}(P)_{(\Gamma)}$ is of order p , this is a contradiction. (q.e.d.)

By the same argument as in Step 7 in the proof of Theorem A, we have

Step 8. $|\Omega| - (3p-1) \equiv p \pmod{p^2}$.

From now on, let a be an element of order p of the form

$$a = (1) \cdots (2p)(2p+1) \cdots (3p-1)(3p, \dots, 4p-1)(4p, \dots, 5p-1) \cdots.$$

We divide the consideration into the following two cases:

- (α) $C_G(a)^{I(a)}$ has an orbit Δ such that $|\Delta| \geq 2p$ and $C_G(a)^\Delta \geq A^\Delta$;
- (β) otherwise.

When Case (α) holds, we may assume that $\Delta = \{1, \dots, |\Delta|\}$. When Case (β) holds, we may assume that $\Delta_1 = \{1, \dots, w\}$ and $\Delta_2 = \{w+1, \dots, 3p-1\}$ are the orbits or the blocks of $C_G(a)^{I(a)}$, and that $|\Delta_1| \geq |\Delta_2| \geq p$.

By the same argument as in Step 8, Step 9, Step 10 and Step 11 in the proof of Theorem A, we have

Step 9. *Case (α) does not hold.*

Hereafter we assume that Case (β) holds.

Step 10. Set $C_G(a)_{w+1, w+2, \dots, 2p, 0} = C_G(a)_{w+1, w+2, \dots, 2p}$. There is an integer i ($0 \leq i \leq 1$) such that $C_G(a)_{w+1, w+2, \dots, 2p, i}$ and $C_G(a)_{w+1, w+2, \dots, 2p, i, i+1}$ have exactly m orbits on $\Omega - I(a)$, where m is at most two, and moreover $m=1$ when $|\Omega| - (3p-1) \not\equiv 0 \pmod{p^2}$.

Proof. In order to prove Step 10, it is sufficient to show that $C_G(a)_{w+1, \dots, 2p, 1, 2}$ has at most two orbits on $\Omega - I(a)$, and is transitive on $\Omega - I(a)$ when $|\Omega| - (3p-1) \not\equiv 0 \pmod{p^2}$.

Set $H = G_{w+1, \dots, 2p, 1, 2}$. Then H is p -transitive on $\Omega - \{w+1, \dots, 2p, 1, 2\}$ by Step 5. By the remark following Lemma 1.1 in [11], we get the following expression:

$$\frac{|H|}{p} \geq \frac{|H|}{|C_H(a)|} \frac{1}{p} \sum_y \alpha^*(y),$$

where y ranges all p' -elements in $C_H(a)$ and $\alpha^*(y) = \alpha(y^{\Omega - I(a)})$. Here the equality does not hold when $|\Omega| - (3p-1) \not\equiv 0 \pmod{p^2}$ (cf. Step 8 in the proof of Theorem A). Now, $\sum_y \alpha^*(y) \geq \sum_{y \in \Omega - I(a)} \alpha^*(y) - p \cdot \sum_{y \in \Omega - I(a)} \alpha_p(y^{I(a)})$. Since

$|\Delta_1 - \{1, 2\}| \geq p + \frac{p-1}{2} - 2 \geq p+3$, we have $C_H(a)^{\Delta_1 - \{1, 2\}} \geq A^{\Delta_1 - \{1, 2\}}$ by Step 6.

Hence, $p \cdot \sum_{y \in \Omega - I(a)} \alpha_p(y^{I(a)}) = p \cdot \sum_{y \in \Omega - I(a)} \alpha_p(y^{\Delta_1 - \{1, 2\}}) = |C_H(a)|$ by the formula of Frobenius. On the other hand, $\sum_{y \in \Omega - I(a)} \alpha^*(y) = f \cdot |C_H(a)|$, where f is the number of orbits of $C_H(a)$ on $\Omega - I(a)$. Hence we get

$$\frac{|H|}{p} \geq \frac{|H|}{|C_H(a)|} (f-1), \quad \text{and hence } f \leq 2.$$

In the above expression, if $|\Omega| - (3p-1) \not\equiv 0 \pmod{p^2}$, the equality does not hold. (q.e.d.)

Step 11. $C_G(a)_{1,2,\dots,2p}$ has at most $2m$ orbits on $\Omega - I(a)$. Moreover, $C_G(a)_{1,\dots,p,(p+1,p+2)p+3,\dots,2p} (= C_{G((p+1,p+2))}(a)_{1,\dots,p,p+3,\dots,2p})$ has exactly m orbits on $\Omega - I(a)$.

Proof. By Step 10, $C_G(a)_{w+1,\dots,2p,i}$ has exactly m orbits on $\Omega - I(a)$. Let $\Gamma_1, \dots, \Gamma_m$ be the orbits. We take an arbitrarily fixed orbit Γ_j of $C_G(a)_{w+1,\dots,2p,i}$ on $\Omega - I(a)$. Let $\Sigma_1, \dots, \Sigma_k$ be the orbits of $C_G(a)_{1,2,\dots,2p}$ on Γ_j . Since $C_G(a)_{w+1,\dots,2p,i} \triangleright C_G(a)_{1,2,\dots,2p}$ and Γ_j is an orbit of $C_G(a)_{w+1,\dots,2p,i}$, $C_G(a)_{w+1,\dots,2p,i}^{\Delta_1 - \{i\}}$ acts on the set $\{\Sigma_1, \dots, \Sigma_k\}$ transitively. Let $Y = C_{G(\Sigma_1)}(a)_{w+1,\dots,2p,i}$, then $|C_G(a)_{w+1,\dots,2p,i}^{\Delta_1 - \{i\}} : Y^{\Delta_1 - \{i\}}| = k$. Similarly we have that $|C_G(a)_{w+1,\dots,2p,i,i+1}^{\Delta_1 - \{i\}} : Y_{i+1}^{\Delta_1 - \{i\}}| = k$. Hence, $|C_G(a)_{w+1,\dots,2p,i}^{\Delta_1 - \{i\}} : C_G(a)_{w+1,\dots,2p,i,i+1}^{\Delta_1 - \{i\}}| = |Y^{\Delta_1 - \{i\}} : Y_{i+1}^{\Delta_1 - \{i\}}| = |\Delta_1| - i$. Therefore Y is transitive on $\Delta_1 - \{i\}$. Let $(\beta_1, \dots, \beta_p)$ be a p -cycle of a such that $\{\beta_1, \dots, \beta_p\} \subseteq \Sigma_1$. For any $w-p-i$ elements $\alpha_1, \dots, \alpha_{w-p-i}$ of $\Delta_1 - \{i\}$, $C_G(a)_{i,\alpha_1,\dots,\alpha_{w-p-i},w+1,\dots,2p,\beta_1,\dots,\beta_p}$ has an element b of order p . Then $b \in Y$ and b^{Δ_1} is a p -cycle, and so, $Y_{\alpha_1,\dots,\alpha_{w-p-i}}^{\Delta_1 - \{i\}}$ has the p -cycle. Since $\alpha_1, \dots, \alpha_{w-p-i-1}, \alpha_{w-p-i}$ are any $w-p-i$ points of $\Delta_1 - \{i\}$, we have $Y^{\Delta_1 - \{i\}} \geq A^{\Delta_1 - \{i\}}$ (cf. [14, Theorem 13.9]). Therefore $k \leq 2$. If $k = 2$, then $Y^{\Delta_1 - \{i\}} = A^{\Delta_1 - \{i\}}$ and $C_G(a)_{w+1,\dots,2p,i}^{\Delta_1 - \{i\}} = S^{\Delta_1 - \{i\}}$. Therefore Γ_j is an orbit of $C_G(a)_{1,\dots,p,(p+1,p+2)p+3,\dots,2p}$ on $\Omega - I(a)$, even if $k = 2$. (q.e.d.)

Step 12. We complete the proof.

Proof. Since a is an element of order p of the form

$$a = (1) \cdots (p)(p+1) \cdots (3p-1)(3p, \dots, 4p-1)(4p, \dots, 5p-1) \cdots ,$$

$C_G(a)_{p+1,\dots,2p,3p,\dots,4p-1}$ has an element b of order p . By Step 8, we may assume that

$$b = (1, \dots, p)(p+1) \cdots (3p-1)(3p) \cdots (4p-1)(4p, \dots, 5p-1) \cdots .$$

Let $K = G_{1,\dots,p,(p+1,p+2)p+3,\dots,2p}$ and $L = \langle b \rangle \cdot K$. By the same argument as Step 10 in the proof of Theorem A, we have a contradiction. (q.e.d.)

4. Proofs of Theorem C and Theorem D

Proof of Theorem C. Let G be a nontrivial $2p$ -transitive group on $\Omega = \{1, \dots, n\}$. Let P be a Sylow p -subgroup of $G_{1,\dots,2p}$, then $P \neq 1$ and P is not semiregular on $\Omega - I(P)$ by [3] and [4]. Moreover, $N_G(P)^{I(P)}$ is S_m ($2p \leq m \leq 3p-1$) or A_m ($2p+2 \leq m \leq 3p-1$). Hence, if $n \equiv |I(P)| \pmod{p}$, then Theorem C holds. Suppose that $n \not\equiv p-1 \pmod{p}$. Let Q be a subgroup of P such that the order of Q is maximal among all subgroups of P fixing more than $|I(P)|$ points. Set $N = N_G(Q)^{I(Q)}$, then N has an orbit Γ such that $N^\Gamma \geq A^\Gamma$ and $|\Gamma| \geq 3p$, by Theorem A. (q.e.d.)

Proof of Theorem D. Let G be a nontrivial t -transitive group on $\Omega =$

$\{1, \dots, n\}$. Suppose that t is sufficiently large. By Satz B in [13], $\log(n-t) > \frac{t}{2}$. By the proof of [13, Satz B], we can see that $\log(n-t) > \left(\frac{1}{2} + \varepsilon_0\right)t$ for some $\varepsilon_0 > 0$. Moreover, we can see that, in the proof of [13, Satz B], it was only used that for any k -transitive group H on Σ , there exists a subset Π of Σ such that $|\Pi| = k$ and $H_{(\Pi)}^{\Pi} \geq A^{\Pi}$.

Let $p_1 = 2, p_2 = 3, \dots$, and p_i be the i -th prime number. Then $\lim_{i \rightarrow \infty} \frac{p_{i+1}}{p_i} \rightarrow 1$. (This result is well known in the theory of numbers.)

Since t is sufficiently large, by the above remark and Theorem C, there exists a positive number ε which is sufficiently close to 0, and exists a subset Δ of Ω such that $|\Delta| \geq \left(\frac{3}{2} - \varepsilon\right)t$ and $G_{(\Delta)}^{\Delta} \geq A^{\Delta}$. Therefore we have $\log(n-t) > \frac{3}{4}t$. (q.e.d.)

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