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ON CLASSICAL SOLUTIONS IN THE LARGE IN TIME OF TWO-DIMENSIONAL VLASOV'S EQUATION

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1. Introduction

In this paper we study the initial value problem to Vlasov's equation,

$$(1.1)\qquad \begin{cases} \dfrac{\partial f^{\pm}}{\partial t}+\xi\cdot\nabla_x f^{\pm}+\alpha^{\pm}\nabla_x\phi\cdot\nabla_\xi f^{\pm}=0,\ (t,x,\xi)\in[0,\infty)\times R^n\times R^n\,, \\ \Delta_x\phi=\beta\displaystyle\int_{R^n}(f^+(t,x,\xi)-f^-(t,x,\xi))d\xi,\ (t,x)\in[0,\infty)\times R^n\,, \\ f|_{t=0}=f_0(x,\xi),\ (x,\xi)\in R^n\times R^n\,, \\ \nabla_x\phi \text{ is uniformly bounded and } \nabla_x\phi\to 0\ (|x|\to\infty)\,. \end{cases}$$

Here the unknowns are the functions $f^{\pm}=f^{\pm}(t,x,\xi)$ and $\phi=\phi(t,x)$ where $t\geqq 0$, $x=(x_1,x_2,\cdots,x_n)\in R^n$, $\xi=(\xi_1,\xi_2,\cdots,\xi_n)\in R^n$, and$\nabla_x=(\partial/\partial x_1,\partial/\partial x_2,\cdots,\partial/\partial x_n)$, $\nabla_\xi=(\partial/\partial\xi_1,\partial/\partial\xi_2,\cdots,\partial/\partial\xi_n)$, $\Delta_x=\partial^2/\partial x_1^2+\partial^2/\partial x_2^2+\cdots+\partial^2/\partial x_n^2$, while $\cdot$ denotes the inner product in R^n and $\alpha^{\pm},\beta\in\boldsymbol{R}$. Physically, (1.1) describes the evolution of a rarefied plasma in self-consistent field approximation, where $f^{\pm}$ are respectively the densities of ions (+) and electrons (−) of a plasma at time t in the space of position x and velocity ξ, and ϕ is the potential of electric field of the plasma[1).

If we assume that $f^+\equiv 0$, (1.1) reduces to the initial value problem to the Liouville-Newton equation,

$$(1.2)\qquad \begin{cases} \dfrac{\partial f}{\partial t}+\xi\cdot\nabla_x f+\alpha\nabla_x\phi\cdot\nabla_\xi f=0,\ (t,x,\xi)\in[0,\infty)\times R^n\times R^n\,, \\ \Delta_x\phi=\beta\displaystyle\int_{R^n}f(t,x,\xi)d\xi,\ (t,x)\in[0,\infty)\times R^n\,, \\ f|_{t=0}=f_0(x,\xi),\ (x,\xi)\in R^n\times R^n\,, \\ \nabla_x\phi \text{ is uniformly bounded and } \nabla_x\phi\to 0\ (|x|\to\infty)\,, \end{cases}$$

where $f=f(t,x,\xi)$, $\phi=\phi(t,x)$, and $\alpha,\beta\in\boldsymbol{R}$. (1.2) is a special case of (1.1), but has an independent physical interest in connection with the dynamics of steller

1) $\alpha^{\pm}=\mp e/m^{\pm}$, $\beta=-4\pi e$ where e is the unit of electric charge, and $m^{\pm}$ the masses of the ion (+) and the electron (−), and $\alpha=-1$, $\beta=4\pi\gamma m$ where γ is the gravitational constant and m the mass of the particle.

systems and with the birth of stars where f and ϕ mean the density of mass particles and the potential of gravitational field respectively[1)].

The initial value problem (1.1) has been studied first by Iordanskii [7] for the one-dimensional case $n=1$. Assuming that f^+ is known, he proved that (1.1) has a unique classical solution in the large in time. His method of the proof, however, takes advantages of the peculiarity of $n=1$ and can not be applied to higher dimensional cases.

The three-dimensional case has been solved by Arsen'ev, on the existence of weak solutions in the large in time ([1]), and also on the existence of classical solutions which are, however, local in time ([2])[2)].

The main purpose of this paper is to show that the two-dimensional case admits a unique classical solution in the large in time. We also discuss local classical solutions for $n\geqq 3$ under weaker conditions on the initial data than those imposed in [2].

Our method of the existence proof is elementary; it is based on Schauder's fixed point theorem and has bearings upon the arguments in [8]. In the sequel, we will study only (1.2) since (1.1) can be investigated essentially in the same way.

2. Scheme for the construction of a solution

Our plan for solving (1.2) is as follows. Given $g=g(t,x,\xi)$, we first seek a solution $\phi=\phi(t,x)$ of Poisson's equation

$$(2.1)\qquad \begin{cases} \Delta_x\phi = \beta\int_{R^n} g(t,x,\xi)d\xi\,, \\ \nabla_x\phi \text{ is uniformly bounded and } \nabla_x\phi\to 0\ (|x|\to\infty)\,. \end{cases}$$

Let $K(x)$ be the fundamental solution of Δ_x in R^n given as

$$(2.2)\qquad K(x) = \begin{cases} \dfrac{1}{(2-n)\omega_n}\dfrac{1}{|x|^{n-2}},\ n\geqq 3\,, \\ \dfrac{1}{2\pi} ln|x|\,,\ n=2\,, \end{cases}$$

where ω_n is the surface area of the unit sphere in R^n. It is expected that

$$(2.3)\qquad \phi(t,x) = \beta\int_{R^n} K(x-x')\left\{\int_{R^n} g(t,x',\xi)d\xi\right\}dx'$$

is a solution of (2.1). With this ϕ, we then solve the initial value problem of the first order partial differential equation

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2) (1.2) has been discussed also in [3] for $n=3$, whose proof, however, contains an elementary error which assures no longer the existence even of a local solution ([3], p. 47).

$$(2.4)\qquad \begin{cases} \dfrac{\partial f}{\partial t}+\xi\cdot\nabla_x f+\alpha\nabla_x\phi\cdot\nabla_\xi f=0\,, \\ f|_{t=0}=f_0\,. \end{cases}$$

Clearly it is equivalent to solving the ordinary differential equations (characteristic equations to (2.4))

$$(2.5)\qquad \frac{dX}{ds}=\Xi,\ \frac{d\Xi}{ds}=\alpha(\nabla_x\phi)(s,X)$$

for $(X,\Xi)\in R^n\times R^n$. Denote by $X(s;t,x,\xi)$, $\Xi(s;t,x,\xi)$ the solutions of (2.5) satisfying the initial conditions

$$(2.6)\qquad X|_{s=t}=x,\ \Xi|_{s=t}=\xi\,.$$

Put $X_0(t,x,\xi)=X(0;t,x,\xi)$, $\Xi_0(t,x,\xi)=\Xi(0;t,x,\xi)$. Then the solution to (2.4) is given formally as

$$(2.7)\qquad f(t,x,\xi)=f_0(X_0(t,x,\xi),\Xi_0(t,x,\xi))\,.$$

In this way we shall have assigned a function f to a given function g which we will denote as $f=V[g]$. Thus we shall specify a set S of functions g in such a way that the map V defined on this S can be shown to have a fixed point with the aid of Schauder's fixed point theorem, and that any fixed point of V in S is a classical solution of (1.2). We will describe our choice of S in the following section, but the precise definition of S will be made in §7 after the study of (2.1), (2.4) and (2.5) in §4 to §6.

3. Classes of functions

Let $Q_T=[0,T]\times R^n\times R^n$ and $\Omega_T=[0,T]\times R^n$ with some $T>0$. In general, if Ω is an unbounded closed domain in R^m, $B^{l+\sigma}(\Omega)$, $l=$integer$\geqq 0$, $0\leqq\sigma\leqq 1$, will denote the set of all continuous and bounded functions defined on Ω having continuous and bounded l-th derivatives which are uniformly Hölder-continuous in Ω with exponent σ if $\sigma>0$. It is a Banach space with the usual norm denoted as $\|\ \|_{B^{l+\sigma}(\Omega)}$. Thus we will use $B^{l+\sigma}(Q_T)$, $B^{l+\sigma}(\Omega_T)$, $B^l(R^n\times R^n)$ etc. Moreover we will need $B^{\sigma_1,l+\sigma_2}(\Omega_T)$, $l=$integer$\geqq 0$, $0\leqq\sigma_1,\sigma_2\leqq 1$, which is the class of continuous and bounded functions $\phi(t,x)$ on Ω_T having continuous and bounded l-th derivatives in x which are uniformly Hölder continuous in t with exponent σ_1 if $\sigma_1>0$, in x with exponent σ_2 if $\sigma_2>0$, or in both if $\sigma_1,\sigma_2>0$.

The set S on which the map V is to be defined is a subset of $B^0(Q_T)$ consisting of all the functions $g=g(t,x,\xi)$ which satisfy the following conditions.

(i) $g\in B^\delta(Q_T)$,

(ii) $\|g\|_{B^\delta(Q_T)}\leqq M_1$,

$$(3.1)\qquad \text{(iii)}\quad |g(t,x,\xi)|\leqq M_2(1+|x|)^{-\gamma}(1+|\xi|)^{-\gamma},\ (t,x,\xi)\in Q_T\,,$$

$$\text{(iv)}\quad \int_{R^n\times R^n}|g(t,x,\xi)|\,dxd\xi\leqq M_3,\ t\in[0,T]\,,$$

$$\text{(v)}\quad \int_{R^n}|g(t,x,\xi)|\,d\xi\leqq M_0(t),\ (t,x)\in\Omega_T\,.$$

Here δ, γ, M_1, M_2 and M_3 are positive constants with $\delta\in(0,1)$, and $M_0(t)$ is a positive nondecreasing function of t on $[0,T]$. All of these constants as well as T and the function $M_0(t)$ will be specified in §**7**. In this and the following three sections, however, they are assumed to be arbitrarily fixed with $\gamma>n$ and $\delta\in(0,1)$. Since it is necessary to make clear the dependence on them of various nonnegative constants which we shall meet in the below, we use two symbols C and M for the constants, where C stands for the constants depending only on n, and α,β in (1.2), and M for those depending on the above-mentioned quantities defining the set S in (3.1).

In the rest of this section, we state fundamental properties of S.

Proposition 3.1. *S is a compact convex subset of $B^0(Q_T)$.*

Proof. The convexity is easy to see and the compactness follows from (3.1) (ii) (iii) and the Ascoli-Arzela theorem.

Define the operater Λ as

$$(3.2)\qquad (\Lambda g)(t,x)=\beta\int_{R^n}g(t,x,\xi)d\xi\,.$$

By virtue of (3.1) (iii) and the assumption $\gamma>n$, the integral converges for each $g\in S$, and

$$(3.3)\qquad |(\Lambda g)(t,x)|\leqq M(1+|x|)^{-\gamma}\,.$$

Lemma 3.1. *Suppose $g\in S$ and $\lambda\in\left(0,1-\frac{n}{\gamma}\right)$, then $\Lambda g\in B^{\lambda\delta}(\Omega_T)$ and*

$$(3.4)\qquad |(\Lambda g)(t,x)-(\Lambda g)(t',x')|\leqq M|(t,x)-(t',x')|^{\lambda\delta}(1+|x|)^{-(1-\lambda)\gamma}$$

for any (t,x), $(t',x')\in\Omega_T$ if $|x-x'|<1$. Here $|(t,x)|=|t|+|x|$.

Proof. Using the inequality

$$(3.5)\qquad (1+|x|)(1+|x'|)^{-1}\leqq 1+|x-x'|,\ x,x'\in R^n\,,$$

we get from (3.1) (iii),

$$|g(t,x,\xi)-g(t',x',\xi)|\leqq M_2((1+|x|)^{-\gamma}+(1+|x'|)^{-\gamma})(1+|\xi|)^{-\gamma}$$

$$\leqq M_2(1+2^\gamma)(1+|x|)^{-\gamma}(1+|\xi|)^{-\gamma}$$

for $|x-x'|<1$. This and (3.1) (ii) then yield (3.4) with

$$M=M_1^\lambda M_2^{1-\lambda}(1+2^\gamma)^{1-\lambda}\int_{R^n}(1+|\xi|)^{-(1-\lambda)\gamma}d\xi<\infty,\ \lambda\in\left(0,1-\frac{n}{\gamma}\right).$$

4. Poisson's equation (2.1)

In this section we consider ϕ given by (2.3) or

$$\phi(t,x)=\int_{R^n}K(x-x')(\Lambda g)(t,x')dx' \tag{4.1}$$

assuming $g\in S$. We note

$$\left|\frac{\partial K(x)}{\partial x_i}\right|\leqq\frac{C}{|x|^{n-1}},\ \left|\frac{\partial^2K(x)}{\partial x_i\partial x_j}\right|\leqq\frac{C}{|x|^n}. \tag{4.2}$$

Porposition 4.1. *Let $g\in S$. Then ϕ given by* (4.1) *is continuously differentiable in x in Ω_T with $\partial\phi/\partial x_i\in B^{\lambda\delta,1+\lambda\delta}(\Omega_T)$ for any $\lambda\in\left(0,1-\frac{n}{\gamma}\right)$ (we may have $\phi\in B^{\lambda\delta,2+\lambda\delta}(\Omega_T)$ if $n\geqq 3$) and is a solution of* (2.1) *which is unique except for an additive function of t.*

Proof. That $\partial\phi/\partial x_i\in B^{1+\lambda\delta}(R^n)$ for each fixed t is a classical result if $\Lambda g\in B^{\lambda\delta}(R^n)$ and is of compact support in x for each t (see e.g. [9], p. 126). In view of (3.3) and (3.4), this can be extended to the case $g\in S$ and we obtain

$$\begin{aligned}|\phi(t,x)|&\leqq M(1+ln(1+|x|)) && \text{if } n=2,\\ &\leqq M(1+|x|)^{-(n-2)} && \text{if } n\geqq 3,\end{aligned} \tag{4.3}$$

$$\left|\frac{\partial\phi}{\partial x_i}\right|\leqq M(1+|x|)^{-(n-1)}, \tag{4.4}$$

$$\left\|\frac{\partial\phi(t,\cdot)}{\partial x_i}\right\|_{B^{1+\sigma}(R^n)}\leqq M,\ 0<\sigma<\lambda\delta. \tag{4.5}$$

In fact, noting that (4.1) is differentiable under integral sign, and using (3.3) and (4.2), we get

$$\begin{aligned}\left|\frac{\partial\phi}{\partial x_i}\right|&\leqq CM\int_{R^n}\frac{1}{|x'|^{n-1}}\frac{1}{(1+|x-x'|)^\gamma}dx'\\ &=CM\left\{\int_{|x'|\leqq 1/2|x|}+\int_{|x'|\geqq 1/2|x|}\right\}\\ &\leqq M\left\{\frac{1}{(1+|x|)^{\gamma-1}}+\frac{1}{|x|^{n-1}}\right\},\end{aligned}$$

whence (4.4) follows since $\partial\phi/\partial x_i$ is at the same time uniformly bounded as will be seen in the following lemma. In a similar manner, we obtain (4.3), and using the explicit formula for $\partial^2\phi/\partial x_i\partial x_j$ (see e.g. [9], p. 127), we obtain (4.5). Moreover the constant M of (4.5) is found to be majorized as

$$M\leqq C(\|(\Lambda g)(t,\cdot)\|_{B^\sigma(R^n)}+\|(\Lambda g)(t,\cdot)\|_{L^1(R^n)}). \tag{4.6}$$

This implies, together with (3.4), that $\partial\phi/\partial x_i\in B^{\sigma,1+\sigma}(\Omega_T)$ for any $0<\sigma<\lambda\delta$ (cf. [8], Lemma 1.3). Thus we have proved first half of the proposition. It is now clear that this ϕ is a classical solution of (2.1): $\nabla_x\phi\rightarrow 0$ ($|x|\rightarrow\infty$) in virtue of (4.4). For the proof of the uniqueness, it suffices to prove that a regular harmonic function $u(x)$ in R^n having bounded derivatives which vanish at infinity is constant in the whole of R^n. And this is easily seen upon applying the mean value theorem on harmonic functions ([5], p. 275) to $\partial u/\partial x_i$ since $\partial u/\partial x_i$ is also regular harmonic in R^n in virtue of Weyl's Lemma ([11], p. 80).

Let $\|\cdot\|_p$, $p=1,\infty$, be the norms of $L^p(R^n)$ and define

$$\|u\|_{1,\infty}=\|u\|_1+\|u\|_\infty \tag{4.7}$$

for $u\in L^1\cap L^\infty$. We shall need the estimates of $\nabla_x\phi$ depending only on these norms of Λg.

Lemma 4.1. *Let g and ϕ be as in Proposition 4.1, and put $w(t,x)=\alpha\nabla_x\phi$. We have*

$$|w(t,x)|\leqq C\|(\Lambda g)(t,\cdot)\|_1^{1/n}\|(\Lambda g)(t,\cdot)\|_\infty^{(n-1)/n} \tag{4.8}$$

for any $(t,x)\in\Omega_T$, and, if $\chi(\rho)=\rho(1-ln\rho)$ for $0\leqq\rho\leqq 1$ and $\chi(\rho)=\rho$ for $\rho>1$,

$$|w(t,x)-w(t,x')|\leqq C\|(\Lambda g)(t,\cdot)\|_{1,\infty}\chi(|x-x'|) \tag{4.9}$$

for any $t\in[0,T]$ and $x,x'\in R^n$.

Proof. For brevity, fix t and write $\rho(x)=\Lambda g$. Using (4.2), we get

$$\begin{aligned}|w(t,x)|&\leqq C\int_{R^n}\frac{1}{|x-x'|^{n-1}}|\rho(x')|dx'\\&=C\left\{\int_{|x-x'|\leqq r}+\int_{|x-x'|>r}\right\}\\&\leqq C\left\{r\|\rho\|_\infty+\frac{1}{r^{n-1}}\|\rho\|_1\right\}\end{aligned}$$

for any $r>0$, whence we obtain (4.8) with $r=(\|\rho\|_1/\|\rho\|_\infty)^{1/n}$.

To prove (4.9) we proceed as in [4]. Put $d=|x-x'|$ and $\sum=\{y\in R^n; |y-x|\leqq 2d\}$. We have

$$|w(t,x)-w(t,x')|\leqq\int_{R^n}|(\nabla_xK)(x-y)-(\nabla_xK)(x'-y)| \tag{4.10}$$

$$\times|\rho(y)|dy=\int_{\Sigma}+\int_{R^n\setminus\Sigma}.$$

Noting that $|x'-y|\leqq 3d$ if $y\in\sum$, we get by (4.2),

$$\int_{\Sigma}\leqq C\|\rho\|_\infty\Big(\int_{|x-y|\leqq 2d}\frac{dy}{|x-y|^{n-1}}+\int_{|x'-y|\leqq 3d}\frac{dy}{|x'-y|^{n-1}}\Big)$$
$$\leqq Cd\|\rho\|_\infty.$$

On the other hand, if $y\in R^n\setminus\sum$ and if x'' is any point on the segment $[x,x']$, we see that $|x''-y|\geqq|x-y|-|x-x''|\geqq|y-x|-d\geqq\frac{1}{2}|y-x|$, and hence by the mean value theorem and (4.2), that

$$|(\nabla_xK)(x-y)-(\nabla_xK)(x'-y)|$$
$$\leqq|x-x'||(\nabla_x\nabla_xK)(x''-y)|$$
$$\leqq\frac{Cd}{|x''-y|^n}\leqq\frac{2^nCd}{|x-y|^n}.$$

Therefore the integral in $R^n\setminus\sum$ in (4.10) can be estimated as

$$\int_{R^n\setminus\Sigma}\leqq Cd\int_{|x-y|\geqq 2d}\frac{|\rho(y)|}{|x-y|^n}dy$$
$$=Cd\Big(\int_{2d\leqq|x-y|<r}+\int_{r\leqq|x-y|}\Big)$$
$$\leqq Cd\Big(\|\rho\|_\infty ln\frac{r}{d}+\frac{1}{r^n}\|\rho\|_1\Big)$$

with any $r>2d$. Thus (4.9) was proved.

5. Characteristic equations (2.5)

In this section we solve the differential equations (2.5) associated with the initial conditions (2.6).

Proposition 5.1. *Let g and ϕ be as in Proposition* 4.1. *Then there exist unique solutions $X=X(s;t,x,\xi)$, $\Xi=\Xi(s;t,x,\xi)$ of* (2.5) *and* (2.6) *in the interval $0\leqq s\leqq T$ for any $(t,x,\xi)\in Q_T$. X and Ξ are continuously differentiable in all variables. For any fixed s and t, (X,Ξ) is one to one and measure preserving map of $R^n\times R^n$ onto itself, with the Jacobian*

$$(5.1)\qquad\left|\frac{\partial(X,\Xi)}{\partial(x,\xi)}\right|=1.$$

$(X(t;t,x,\xi),\Xi(t;t,x,\xi))$ is the identity map, and $(X(t;s,x,\xi),\Xi(t;s,x,\xi))$ is the inverse map of $(X(s;t,x,\xi),\Xi(s;t,x,\xi))$.

Proof. Since $\nabla_x\phi \in B^{\lambda\delta,1+\lambda\delta}(\Omega_T)$, $\lambda \in \left(0, 1-\frac{n}{\gamma}\right)$ by Proposition 4.1, these are well known results in the theory of ordinary differential equations. (5.1) is a consequence of $\nabla_x\xi = \nabla_\xi \nabla_x \phi = 0$ (cf. [4], Chap. I. Th. 7.2.).

We now derive estimates for X and Ξ. In what follows, $L_i(\eta)$, $i=1,2,\cdots$ will denote nondecreasing positive finctions of $\eta \geqq 0$ depending possibly on T but not on the other quantities defining the set S in (3.1). Put

$$\|\Lambda g\| = \sup_{0 \leqq t \leqq T} \|(\Lambda g)(t, \cdot)\|_{1,\infty} . \tag{5.2}$$

We begin with

Lemma 5.1. *For any $s \in [0, T]$ and $(t, x, \xi) \in Q_T$, we have*

$$|X - x - \xi(s-t)|, \ |\Xi - \xi| \leqq L_1(\|\Lambda g\|), \tag{5.3}$$

$$\left|\frac{\partial X}{\partial t}\right|, \ \left|\frac{\partial \Xi}{\partial t}\right| \leqq (1+|\xi|)M, \tag{5.4}$$

$$|\partial X|, \ |\partial \Xi| \leqq M, \tag{5.5}$$

where ∂ stands for $\partial/\partial x_i$, and $\partial/\partial \xi_i$, $1 \leqq i \leqq n$.

Proof. Integration of (2.5) with respect to s gives rise to the integral equations

$$\begin{cases} X(s;t,x,\xi) = x - \int_s^t \Xi(\tau;t,x,\xi)d\tau , \\ \Xi(s;t,x,\xi) = \xi - \int_s^t w(\tau, X(\tau;t,x,\xi))d\tau . \end{cases} \tag{5.6}$$

Now (5.3) can be easily deduced from (5.6) with the aid of (4.8). Similarly, we obtain (5.4) and (5.5) by use of (4.5) and the integral equations for the derivatives of X and Ξ obtained upon the differentiation of (5.6).

The following lemma gives a Hölder estimate of (X, Ξ) which do not depend on the Hölder continuity of Λg.

Lemma 5.2. *Suppose $|(x,\xi)-(x',\xi')| < 1$, then*

$$\begin{aligned} &|(X(s;t,x,\xi),\ \Xi(s;t,x,\xi)) - (X(s';t',x',\xi'),\ \Xi(s';t',x',\xi'))| \\ &\leqq L_2(\|\Lambda g\|)(1+|\xi|)|(s,t,x,\xi)-(s',t',x',\xi')|^{\delta_1} \end{aligned} \tag{5.7}$$

holds with $\delta_1 = L_3(\|\Lambda g\|)^{-1}$ for any $s, s' \in [0, T]$ and (t,x,ξ), $(t',x',\xi') \in Q_T$.

Proof. We follow the argument of [8]. It suffices to consider the special cases

(i) $t=t', s=s'$, (ii) $t=t', x=x', \xi=\xi'$,
(iii) $s=s', x=x', \xi=\xi'$.

(i) Put $\rho_1(s)=|X(s;t,x,\xi)-X(s;t,x',\xi')|$, $\rho_2(s)=|\Xi(s;t,x,\xi)-\Xi(s;t,x',\xi')|$. It is easily seen from (5.6) and (4.9) that

$$(5.8)\quad \begin{aligned}\rho_1(s)&\leqq|x-x'|+T|\xi-\xi'|+CT\|\Lambda g\|\left|\int_s^t \chi(\rho_1(\tau))d\tau\right|,\\ \rho_2(s)&\leqq|\xi-\xi'|+C\|\Lambda g\|\left|\int_s^t \chi(\rho_1(\tau))d\tau\right|.\end{aligned}$$

Put $a=2(|x-x'|+T|\xi-\xi'|)$, $b=CT\|\Lambda g\|$, and consider the integral equation

$$(5.9)\quad \rho(s)=a\pm b\int_s^t \rho(\tau)(1-1n\rho(\tau))d\tau,\ s\gtrless t,$$

which can be eaisly solved as

$$\rho(s)=a^{e^{-b|t-s|}}e^{1-e^{-b|t-s|}},\ 0\leqq s,t\leqq T.$$

Suppose that $0<a<e^{1-e^{bT}}$, then $0<\rho(s)<1$ for $0\leqq s\leqq T$ and hence (5.9) can be written as

$$(5.10)\quad \rho(s)=a+b\left|\int_s^t \chi(\rho(\tau))d\tau\right|.$$

Since $\chi(\rho)\geqq 0$ and is strictly monotone increasing in $0\leqq\rho\leqq 1$, the comparison theorem (e.g. [10], p. 315) can be applied to (5.8) and (5.10), to conclude that $\rho_1(s)\leqq\rho(s)$ if $0<a<e^{1-e^{bT}}$. Thus we have proved that if $|(x,\xi)-(x',\xi')|<1$,

$$(5.11)\quad \rho_1(s),\ \rho_2(s)\leqq L_4(\|\Lambda g\|)|(x,\xi)-(x',\xi')|^{\delta_1}$$

with $\delta_1=e^{-CT^2\|\Lambda g\|}$. This suffices to prove the lemma for the case (i), but later, we shall need estimates which are valid also for $|(x,\xi)-(x',\xi')|\geqq 1$. To this end, we note that (5.3) gives the estimates

$$\begin{aligned}\rho_1(s)&\leqq|x-x'|+T|\xi-\xi'|+2L_1(\|\Lambda g\|),\\ \rho_2(s)&<|\xi-\xi'|+2L_1(\|\Lambda g\|),\end{aligned}$$

whence, together with (5.11),

$$(5.12)\quad \begin{aligned}&\rho_1(s),\ \rho_2(s)\\ &\leqq L_5(\|\Lambda g\|)(1+|(x,\xi)-(x',\xi')|)^{1-\delta_1}|(x,\xi)-(x',\xi')|^{\delta_1},\end{aligned}$$

holds good for any $t,s\in[0,T]$, and $x,x',\xi,\xi'\in R^n$.

(ii) By virtue of (4.8), (5.3), (5.6), we obtain

$$|X(s;t,x,\xi)-X(s';t,x,\xi)|$$

$$\leqq \left|\int_s^{s'} |\Xi(\tau;t,x,\xi)|\,d\tau\right| \leqq (|\xi|+L_1(\|\Lambda g\|))|s-s'|\ ,$$

$$|\Xi(s;t,x,\xi)-\Xi(s';t,x,\xi)|$$

$$\leqq \left|\int_s^{s'} |w(\tau;X(\tau;t,x,\xi))|\,d\tau\right| \leqq C\|\Lambda g\|\,|s-s'|\ .$$

(iii) Put $X'=X(t';t,x,\xi)$, $\Xi'=\Xi(t';t,x,\xi)$, and note that $X(s;t,x,\xi)=X(s;t',X',\Xi')$, $\Xi(s;t,x,\xi)=\Xi(s;t',X',\Xi')$ in virtue of Proposition 5.1. Hence by (5.12)

$$|X-X(s;t',x,\xi)| = |X(s;t',X',\Xi')-X(s;t',x,\xi)|$$
$$\leqq L_5(\|\Lambda g\|)\,(1+|(X',\Xi')-(x,\xi)|)^{1-\delta_1}|(X',\Xi')-(x,\xi)|^{\delta_1}\ ,$$

while by (ii)

$$|(X',\Xi')-(x,\xi)| \leqq (1+|\xi|)L_6(\|\Lambda g\|)\,|t-t'|\ ,$$

since $X'|_{t=t'}=x$, $\Xi'|_{t=t'}=\xi$. $|\Xi(s;t',x,\xi)|$ can be estimated similarly and thus (5.7) was proved for the case (iii).

6. Initial value problem (2.4)

From now on, we shall impose the following condition on the initial f_0.

$$\text{(6.1)}\qquad \begin{aligned}&\text{(i)}\quad f_0\in B^1(R^n\times R^n)\ ,\\ &\text{(ii)}\quad |f_0(x,\xi)|\leqq \kappa_0(1+|x|)^{-2\gamma}(1+|\xi|)^{-2\gamma}\ ,\end{aligned}$$

where $\kappa_0\geqq 0$ and $\gamma>n$. The condition (6.1) is much weaker than that in [2].

Proposition 6.1. *Let X,Ξ be as in Proposition 5.1 and assume (6.1) for f_0. Then $f=f(t,x,\xi)$ given by (2.7) is in $B^0(Q_T)$, continuously differentiable in Q_T and satisfies (2.4) in the classical sense. Moreover $(1+|\xi|)^{-1}\partial f/\partial t, \nabla_x f, \nabla_\xi f\in B^0(Q_T)$.*

Proof. The first assertion is due to Proposition 5.1 and the assumption (6.1) (i), and the second assertion to (5.4) and (5.5).

Let us denote by $\kappa_i, i=1,2,\cdots$, positive constants depending only on κ_0,γ, $\|f_0\|_{L^1}$ and $\|f_0\|_{B^1}$.

Lemma 6.1. *For f given in the preceding Proposition, we get*

$$\text{(6.2)}\qquad \|f\|_{B^0(Q_T)} = \|f_0\|_{B^0(R_n\times R_n)}\ ,$$

$$\text{(6.3)}\qquad \|f(t,\cdot,\cdot)\|_{L^1(R^n\times R^n)} = \|f_0\|_{L^1(R^n\times R^n)},\ 0\leqq t\leqq T\ ,$$

$$\text{(6.4)}\qquad |f(t,x,\xi)|\leqq \kappa_0 L_6(\|\Lambda g\|)\,(1+|x|)^{-\gamma}(1+|\xi|)^{-\gamma}\ ,$$

$$\text{(6.5)}\qquad \|f\|_{B^{\delta_2}(Q_T)}\leqq \kappa_1 L_7(\|\Lambda g\|)\ ,$$

where $\delta_2=L_8(\|\Lambda g\|)^{-1}$.

Proof. In view of the definition (2.7) of f, (6.2) is obvious and (6.3) is a consequence of (5.1). Write $L_i=L_i(\|\Lambda g\|)$, $i=1,2,\cdots$. By virtue of (5.3) and (6.1) (ii), we have by repeated use of (3.5),

$$
\begin{aligned}
(6.6)\qquad |f(t,x,\xi)| &= |f_0(X_0,\Xi_0)| \\
&\leqq \kappa_0(1+|X_0|)^{-2\gamma}(1+|\Xi_0|)^{-2\gamma} \\
&\leqq \kappa_0(1+||x-\xi t|-L_1|)^{-2\gamma}(1+||\xi|-L_1|)^{-2\gamma} \\
&\leqq \kappa_0(1+L_1)^{4\gamma}(1+|x-\xi t|)^{-2\gamma}(1+|\xi|)^{-2\gamma} \\
&\leqq \kappa_0(1+L_1)^{4\gamma}(1+T)^{\gamma}(1+|x|)^{-\gamma}(1+|\xi|)^{-\gamma},
\end{aligned}
$$

which verifies (6.4). To prove (6.5), put $\kappa_2=\|f_0\|_{B^1(R^n\times R^n)}$ and write $X_0'=X_0(t',x',\xi')$, $\Xi_0'=\Xi_0(t',x',\xi')$. From (5.7), it follows that, for $|(t,x,\xi)-(t',x',\xi')|<1$,

$$
\begin{aligned}
|f(t,x,\xi)-f(t',x',\xi')| &\leqq \kappa_2|(X_0,\Xi_0)-(X_0',\Xi_0')| \\
&\leqq \kappa_2(1+|\xi|)L_2|(t,x,\xi)-(t',x',\xi')|^{\delta_1},
\end{aligned}
$$

while from (6.6) and (3.5),

$$|f(t,x,\xi)-f(t',x',\xi')|\leqq \kappa_0 L_6(1+2^{2\gamma})\,(1+|\xi|)^{-2\gamma},$$

for $|\xi-\xi'|<1$. These two inequalities yield the estimate

$$
\begin{aligned}
&|f(t,x,\xi)-f(t',x',\xi')| \\
&\leqq \{\kappa_0\kappa_2L_2L_6(1+2^{2\gamma})\}^{1/2}(1+|\xi|)^{1-\gamma}|(t,x,\xi)-(t',x',\xi')|^{\delta_1/2}
\end{aligned}
$$

for $|(x,\xi)-(x',\xi')|<1$, which, together with (6.2), proves (6.5).

Lemma 6.2. *Let f be as above. Then*

$$(6.7)\qquad \|(\Lambda f)\,(t,\cdot)\|_\infty\leqq \kappa_3+\kappa_4\Big(\int_0^t\|(\Lambda g)\,(\tau,\cdot)\|_1^{1/n}\|(\Lambda g)\,(\tau,\cdot)\|_\infty^{(n-1)/n}d\tau\Big)^n$$

for $0\leqq t\leqq T$.

Proof. In view of the first inequality of (6.6),

$$|(\Lambda f)\,(t,x)|\leqq \kappa_0\int_{R^n}(1+|\Xi_0(t,x,\xi)|)^{-2\gamma}d\xi$$

holds for $(t,x)\in\Omega_T$. We divide the region of integration as

$$\text{(I)}\quad |\xi|\geqq 2\int_0^t\|w(\tau,\cdot)\|_\infty d\tau, \qquad \text{(II)}\quad |\xi|\leqq 2\int_0^t\|w(\tau,\cdot)\|_\infty d\tau,$$

where $w=\alpha\nabla_x\phi$ is given in Proposition 4.1. In the region (I), it is easily found by (5.6) that

$$|\Xi_0|\geq|\xi|-\int_0^t\|w(\tau,\cdot)\|_\infty d\tau\geqq\frac{1}{2}|\xi|,$$

and thereby that

$$\int_{(I)}(1+|\Xi_0|)^{-2\gamma}d\xi \leqq \int_{R^n}\Big(1+\frac{1}{2}|\xi|\Big)^{-2\gamma}d\xi < +\infty .$$

In the region (II), we simply estimate as

$$\int_{(II)}(1+|\Xi_0|)^{-2\gamma}d\xi \leqq \text{mes}\,(II) = \frac{\omega_n}{n}\Big(2\int_0^t ||w(\tau,\cdot)||_\infty d\tau\Big)^n .$$

Now (6.7) follows from these two inequalities with (4.8) taken into account.

Finally set $\kappa_5=\kappa_4|\beta|\,||f_0||_{L^1(R^n\times R^n)}$ and consider the integral equation

$$(6.8)\qquad \rho_0(t) = \kappa_3+\kappa_5\Big(\int_0^t \rho_0(\tau)^{(n-1)/n}d\tau\Big)^n .$$

Reducing this to an ordinary differential equation, we can easily prove,

Lemma 6.3. *Assume* (6.1) *for* f_0. *Then,*

(i) *for the case* $n=2$, (6.8) *possesses a unique positive solution* $\rho_0(t)$ *in any interval* $[0, T]$, *and*

(ii) *for the case* $n\geqq 3$, *there exists a constant* $T_0>0$ *depending only on* f_0 *and* γ, *and* (6.8) *possesses a unique positive solution* $\rho_0(t)$ *in the interval* $[0,T_0)$.

In both cases, $\rho_0(t)$ *is monotone increasing in* t, *and* $\rho_0(t)\to\infty\,(t\nearrow T_0)$ *if* $n\geqq 3$.

7. Construction of a solution

We are now in a position to prove the existence theorem for (1.2). We have to specify the quantities defining the set S in (3.1) as follows. We start by choosing freely a $\gamma>n$ and assuming that an f_0 is given which satisfies (6.1). We then choose a $T>0$ arbitrarily for the case $n=2$ and in such a way as $T<T_0$ for the case $n\geqq 3$ where T_0 is given in Lemma 6.3. Recall that $T_0>0$ and depends only on γ and f_0. Put

$$(7,.1)\qquad \eta_0 = |\beta|\,||f_0||_{L^1(R^n\times R^n)}+\rho_0(T) ,$$

where β is the constant in (1.2), while $\rho_0(t)$ is the function given in Lemma 6.3. η_0 is determined uniquely by γ and f_0. We shall now choose all the remaining quantities in (3.1) as follows.

$$(7.2)\qquad \begin{aligned} &M_1 = \kappa_1 L_7(\eta_0),\ M_2 = \kappa_0 L_6(\eta_0),\ M_3 = ||f_0||_{L^1(R^n\times R^n)} ,\\ &\delta = L_8(\eta_0)^{-1},\ M_0(t) = \rho_0(t) .\end{aligned}$$

Denote again by S the class of functions $g=g(t,x,\xi)$ satisfying (3.1) with the quantities thus specified. Thus S is determined uniquely by γ, f_0 and T ($T<T_0$ for $n\geqq 3$).

Let us recall the definition of the map V of §2: $f=V[g]$ is defined through (2.3), (2.5) and (2.7).

Proposition 7.1. *V maps S into itself and is continuous in the topology of* $B^0(Q_T)$.

Proof. We shall first prove $f=V[g]\in S$ for each $g\in S$. That f satisfies (3.1) (iv) with $M_0=\|f_0\|_{L^1}$ is obvious from (6.3). Since $g\in S$, it satisfies (3.1) with the choice (7.2). In particular,

$$\text{(7.3)}\qquad \begin{aligned}&\|(\Lambda g)(t,\cdot)\|_1\leqq|\beta|\,\|g(t,\cdot,\cdot)\|_{L^1}\leqq|\beta|\,\|f_0\|_1\,,\\ &\|(\Lambda g)(t,\cdot)\|_\infty\leqq\rho_0(t)\,.\end{aligned}$$

Substitution of (7.3) into the right hand side of (6.7) shows, together with (6.8), that $f=V[g]$ satisfies (3.1) (v) with $M_0(t)=\rho_0(t)$. Moreover, since $\rho_0(t)$ is monotone increasing in $t\geqq0$, we see from (7.3) that

$$\begin{aligned}\|\Lambda g\|&=\sup_{0\leqq t\leqq T}(\|(\Lambda g)(t,\cdot)\|_1+\|(\Lambda g)(t,\cdot)\|_\infty)\\ &\leqq|\beta|\,\|f_0\|_1+\rho_0(T)=\eta_0\,.\end{aligned}$$

Therefore $L_i(\|\Lambda g\|)\leqq L_i(\eta_0)$ for each $g\in S$ and $i=1,2,\cdots$, because $L_i(\eta)$ is monotone increasing in $\eta\geqq0$. This and Lemma 6.1 imply that f satisfies also (3.1) (i) (ii) (iii) with the choice (7.2). Thus V maps S into itself.

To prove the continuity of V, let us consider a sequence $\{g^n\}\subset S$ and a $g^0\in B^0(Q_T)$ such that $\|g^n-g^0\|_{B^0(Q_T)}\to0$ as $n\to\infty$. Clearly $g^0\in S$ since S is closed (Proposition 3.1). Thus, (3.1) holds for all g^n, $n\geqq0$, and we see that

$$\text{(7.4)}\qquad \|\Lambda(g^n-g^0)\|_{B^0(\Omega_T)}\to0\ (n\to\infty)\,.$$

Define $w^n(t,x)=\alpha\nabla_x\phi^n$ with

$$\text{(7.5)}\qquad \phi^n(t,x)=\int_{R^n}K(x-x')(\Lambda g^n)(t,x')dx',\ n\geqq0\,.$$

We find from (4.8), (6,3) and (7.4), that

$$\text{(7.6)}\qquad \begin{aligned}&\|w^n-w^0\|_{B^0(\Omega_T)}\leqq C(2\|f_0\|_{L^1})^{1/n}\|\Lambda(g^n-g^0)\|_{B^0(\Omega_T)}{}^{(n-1)/n}\\ &\quad\to0\ (n\to\infty)\,.\end{aligned}$$

Further we see from (4.5) that

$$\text{(7.7)}\qquad \|w^n\|_{B^1(\Omega_T)}\leqq M$$

uniformly for $n\geqq0$.

Let $X^n(s)=X^n(s;t,x,\xi)$, $\Xi^n(s)=\Xi^n(s;t,x,\xi)$ be the solutions of (2.5), (2.6) with $\phi=\phi^n$ given by (7.5). Then (5.5) and (7.7) lead to

$$|X^n(s)-X^0(s)|\leqq T^2\|w^n-w^0\|_{B^0(\Omega_T)}+MT\left|\int_s^t|X^n(\tau)-X^0(\tau)|d\tau\right|\,,$$

$$|\Xi^n(s)-\Xi^0(s)| \leqq T\|w^n-w^0\|_{B^0(\Omega_T)}+M\left|\int_s^t |X^n(\tau)-X^0(\tau)|d\tau\right| ,$$

whence we get with the aid of Gronwall's inequality,

$$(7.8) \qquad \begin{aligned} &|X^n(s)-X^0(s)|, \ |\Xi^n(s)-\Xi^0(s)| \\ &\leqq (1+T)T\, e^{MT|t-s|}\|w^n-w^0\|_{B^0(\Omega_T)} . \end{aligned}$$

Finally we put $f^n=V[g^n]$. Since $f^n(t,x,\xi)=f_0(X^n(0), \Xi^n(0))$ by (2.7) and since $f_0 \in B^1(R^n \times R^n)$,

$$\begin{aligned} &|f^n(t,x,\xi)-f^0(t,x,\xi)| \\ &\leqq \|f_0\|_{B^1(R^n\times R^n)}|(X^n(0), \Xi^n(0))-(X^0(0), \Xi^0(0))| . \end{aligned}$$

This states, combined with (7.6) and (7.8), that $\|f^n-f^0\|_{B^0(Q_T)} \to 0$ $(n\to\infty)$, which we wished to prove.

In virtue of Proposition 3.1 and 7.1, Schauder's fixed point theorem now assures that V has a fixed point f in S; $f=V[f]$, $f\in S$. On the other hand, any fixed point of V in S is a classical solution of (1.2), which is a consequence of Propositions 4.1 and 6.1. Thus we have proved

Theorem 7.1. *Suppose that* f_0 *satisfy* (6.1). *Then the initial value problem* (1.2) *admits a classical solution* (f,ϕ) *in any time interval* $[0,T]$ *if* $n=2$, *and in the time interval* $[0,T_0)$ *if* $n\geqq 3$ *with* $T_0>0$ *determined by* f_0 *(and* γ*). Moreover*

$$(7.9) \qquad \begin{aligned} &\text{(i) } f\in S, \ \text{(ii) } \nabla_x f, \nabla_\xi f, (1+|\xi|)^{-1}\partial f/\partial t \in B^0(Q_T), \\ &\textit{and} \ \text{(iii) } \nabla_x \phi \in B^{\delta',1+\delta'}(\Omega_T) \textit{ for any } T(T<T_0 \textit{ if } n\geqq 3) \\ &\textit{where } \delta'<L_8(\eta_0)^{-1} . \end{aligned}$$

REMARK 7.1. $f\geqq 0$ in Q_T if $f_0\geqq 0$ in R^{2n} by (2.7), and $\|f(t,\cdot,\cdot)\|_{L^p(R^n\times R^n)} =\|f_0\|_{L^p(R^n\times R^n)}$ for any $p\in[1,\infty]$ by (5.1).

REMARK 7.2. In the above, $f\in B^1(Q_T)$ if we impose on the initial f_0 the additional condition

$$(7.10) \qquad |\nabla_x f_0|, \ |\nabla_\xi f_0| \leqq \kappa_6(1+|\xi|)^{-1}, \ (x,\xi)\in R^n\times R^n .$$

To prove this, it remains only to show that $\partial f/\partial t$ is uniformly bounded. Since $\nabla_\xi f\in B^0(Q_T)$, $\nabla_x\phi\in B^0(\Omega_T)$ as stated in Theorem 7.1, (2.4) implies,

$$\left|\frac{\partial f}{\partial t}\right| \leqq |\xi||\nabla_x f|+|\alpha||\nabla_x\phi||\nabla_\xi f| \leqq |\xi||\nabla_x f|+M \quad \text{in } Q_T .$$

Thus it suffices to prove

$$|\nabla_x f| \leqq M(1+|\xi|)^{-1} \quad \text{in } Q_T ,$$

and this is easily seen from (2.7), (5.3), (5.5) and (7.10) as

$$(7.11)\qquad \begin{aligned}|\nabla_x f| &\leqq M(|(\nabla_x f_0)(X_0,\Xi_0)|+|(\nabla_\xi f_0)(X_0,\Xi_0)|)\\ &\leqq M\kappa_6(1+|\Xi_0|)^{-1}\leqq M(1+||\xi|-L_1(\eta_0)|)^{-1}\\ &\leqq M(1+L_1(\eta_0))(1+|\xi|)^{-1},\end{aligned}$$

where use was made of (3.5).

REMARK 7.3. Theorem 7.1 holds also for Vlasov's equation (1.1) if $f_0^{\pm}$ is assumed to satisfy (6.1). The proof is essentially identical and is not repeated here.

REMARK 7.4. We have not been able to construct a classical solution in the large in time for $n\geqq 3$. This is because no estimates have been available for $||(f\Lambda)(t,\cdot)||_\infty$ other than $||(\Lambda f)(t,\cdot)||_\infty\leqq\rho_0(t)$ (see (3.1) (v)) which turns to be meaningless for $t\geqq T_0$ since $\rho_0(t)\to\infty(t\nearrow T_0)$ in case $n\geqq 3$ (Lemma 6.3).

However, such a solution can be constructed even for $n\geqq 3$ if (1.2) is modified as follows. We change only the Poisson's equation of (1.2) as

$$\Delta_x\phi=\beta\int_{|\xi|\leqq R} f(t,x,\xi)d\xi\,.$$

That is, we replace only the region R^n of the integration appearing in Poisson's eq. by a sphere $|\xi|\leqq R$ with some $R>0$. All other equation and conditions remain unaltered in (1.2). In this case the definition of the operater Λ of (3.2) should be replaced by

$$(\Lambda g)(t,x)=\beta\int_{|\xi|\leqq R} g(t,x,\xi)d\xi\,.$$

Clearly all the arguments of §3 to §6 still hold with this definition of Λ, and by virtue of (6.2)

$$||(\Lambda f)(t,\cdot)||_\infty\leqq|\beta|\frac{\omega_n}{n}R^n||f_0||_{B^0(R^n\times R^n)}\equiv\rho_0\,.$$

This replaces Lemma 6.2 and gives a uniform estimate in any time interval $[0,T]$, implying that the above constant ρ_0 can be taken as $\rho_0(t)$. In this way we can construct a classical solution (1.2) thus modified, in the large in time for any $R>0$ and $n\geqq 2$. The above-mentioned modification of (1.2) is also of physical interest, [6].

8. Uniqueness of a solution

Assume, in addition to (6.1), the following condition on f_0.

$$(8.1)\qquad \begin{aligned}&\text{(i)}\quad \nabla_x f_0,\ \nabla_\xi f_0\in L^1(R^n\times R^n)\,,\\ &\text{(ii)}\quad |\nabla_x f_0|,\ |\nabla_\xi f_0|\leqq\kappa_7(1+|\xi|)^{-\gamma},\ \kappa_7>0,\gamma>n\,.\end{aligned}$$

The aim of this section is to prove the uniqueness of the solution (f, ϕ) to (1.2) constructed in Theorem 7.1 in the class of such functions that

$$(8.2)\qquad \begin{aligned} &(\mathrm{i})\quad f\in B^0(Q_T)\cap C^1((0,T)\times R^n\times R^n)\\ &(\mathrm{ii})\quad \Lambda f\in B^\sigma([0,T];L^1(R^n))\cap B^\sigma(\Omega_T),\ 0<\sigma<1\,,\\ &(\mathrm{iii})\quad \nabla_x\phi\in B^{0,1}(\Omega_T)\,, \end{aligned}$$

where $C^1(\Omega)$ is the class of continuously differentiable functions on Ω and $B^\delta([0,T];X)$ with a Banach space X the class of Hölder continuous functions with values in X on $[0,T]$ in the topology of X. Clearly the class of (f, ϕ) defined by (8.2) is wider than that by (7.9).

First of all, we shall note that any solution (f, ϕ) of (1.2) satisfying (8.2) has the properties like (7.9). More precisely, f satisfies (3.1) and (7.9) (ii) with γ of (6.1) and T of (8.2) and with $M_1=\kappa L_7(||\Lambda f||)$, $M_2=\kappa_0 L_6(||\Lambda f||)$, $M_3=||f_0||_{L^1}$, $\delta=L_8{}^{-1}(||\Lambda f||)$, $M_0(t)=||\Lambda f||$, while ϕ satisfies (7.9) (iii) with $0<\delta'<\sigma$. Here $||\Lambda f||<+\infty$ in view of (8.2) (ii). In fact, (7.9) (iii) follows from (4.5) with $g=f$ since $\Delta\phi=\Lambda f$ holds by assumption and since the constant M of (4.5) remains finite in virtue of (4.6) and (8.2) (ii). Moreover, it is easily seen that Lemma 4.1 holds with $g=f$ and that since all the constants M appearing in §5 and §6 depend only on M of (4.5), all the results obtained there are valid with $g=f$, which is what was to be proved.

Let (f^i, ϕ^i), $i=1,2$, be any two solutions of (1.2) satisfying (8.2). On subtracting the two equations for (f^i, ϕ^i), $i=1,2$, and writing $f=f^1-f^2$ and $\phi=\phi^1-\phi^2$, we obtain

$$(8.3)\qquad \frac{\partial f}{\partial t}+\xi\cdot\nabla_x f+\alpha\nabla_x\phi^1\cdot\nabla_\xi f=-\alpha\nabla_x\phi\cdot\nabla_\xi f^2\equiv h(t,x,\xi),\ f|_{t=0}=0\,,$$

$$(8.4)\qquad \Delta\phi=\Lambda f\,.$$

Put $g=|f|$. As stated above, Lemma 4.1 can be applied to (8.4), giving

$$(8.5)\qquad ||(\nabla_x\phi)(t,\cdot)||_1\leqq C||(\Lambda f)(t,\cdot)||_{1,\infty}\leqq C||(\Lambda g)(t,\cdot)||_{1,\infty}\,.$$

On the other hand it is well known (see e.g. [5]) that if $h(t,x,\xi)$ is known, (8.3) is solved as

$$(8.6)\qquad f(t,x,\xi)=\int_0^t h(\tau, X^1(\tau), \Xi^1(\tau))d\tau\,,$$

where $X^i(\tau)=X^i(\tau;t,x,\xi)$, $\Xi^i(\tau)=\Xi^i(\tau;t,x,\xi)$ are the solutions of (2.5) and (2.6) for $\phi=\phi^i$, $i=1,2$. Applying (5.1) and (5.5) to them, we get

$$(8.7)\qquad \int_{R^n\times R^n}|(\nabla_\xi f^2)(\tau, X^1(\tau), \Xi^1(\tau))|\,dxd\xi$$

$$\leqq \int_{R^n \times R^n} |(\nabla_\xi f^2)(\tau, x, \xi)| dx d\xi$$
$$\leqq M(||\nabla_x f_0||_{L^1} + ||\nabla_\xi f_0||_{L^1}),$$

while, in a similar way as in (7.11), we get by (8.1) (ii),

$$|(\nabla_\xi f^2)(\tau, X^1(\tau), \Xi^1(\tau))| \leqq M\kappa_7(1+|\Xi^2(\tau)|)^{-\gamma} \leqq M\kappa_7(1+|\xi|)^{-\gamma}, \tag{8.8}$$

where M depend only on that of (4.5) and $L_1(||\Lambda g||)$ for $g=f^i$, $i=1,2$, and hence is finite as stated above. By means of (8.5) to (8.8), we finally obtain

$$||(\Lambda g)(t, \cdot)||_{1,\infty} \leqq \kappa_7 M \int_0^t ||(\Lambda g)(\tau, \cdot)||_{1,\infty} d\tau .$$

Since $f|_{t=0}=0$, (8.6) implies that $||(\Lambda g)(t, \cdot)||_{1,\infty}=0$, and consequently $f \equiv \vec{0}$ in Q_T and $\nabla_x \phi \equiv 0$ in Ω_T by (8.5). To summarize, we have proved

Theorem 8.1. *Assume* (6.1) *and* (8.1) *on* f_0. *Then the solution of* (1.2) *constructed in Theorem* 7.1 *is unique in the class of functions defined by* (8.2) *up to an additive function of t to* ϕ.

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