



Title	A Remark on finite point transitive affine planes with two orbits on l_∞
Author(s)	Suetake, Chihiro
Citation	Osaka Journal of Mathematics. 1992, 29(3), p. 433-434
Version Type	VoR
URL	https://doi.org/10.18910/6069
rights	
Note	

The University of Osaka Institutional Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

The University of Osaka

A REMARK ON FINITE POINT TRANSITIVE AFFINE PLANES WITH TWO ORBITS ON l_∞

CHIHIRO SUETAKE

(Received October 4, 1991)

In this note for the most part we shall use the notations of [1]. Let $\mathcal{P} = \pi \cup l_\infty$ be the projective extension of an affine plane and G a collineation group of \mathcal{P} . If p is a point of \mathcal{P} and l is a line of \mathcal{P} , then $G(p, l)$ is the subgroup G consisting of all perspectivities in G with center p and axis l . If m is a line of \mathcal{P} , then $G(m, m)$ is the subgroup of all elations in G with axis m .

In [3] the author proved the following theorem on Kallaher's conjecture (See [1]).

Theorem 1. *Let π be a finite affine plane of order n with a collineation group G which is transitive on the affine points of π . Suppose that G has two orbits of length 2 and $n-1$ on l_∞ . Then π is a translation plane and the group G contains the group of translations of π , except in the following case:*

(*) $|G(l_\infty, l_\infty)| = n = 2^m$ for some $m \geq 1$, $G(p_1, l_\infty) = G(p_2, l_\infty) = 1$ and $|G(p, l_\infty)| = 2$ for all $p \in l_\infty - \{p_1, p_2\}$, where $\{p_1, p_2\}$ is a G -orbit of length 2 on l_∞ .

The case (*) actually occurs when π is a desarguesian plane of order 2. Maharjan [2] studied the planes with property (*) under the condition that $n \leq 4$. The purpose of this note is to prove the following.

Proposition 2. *Assume (*). Then $n=2$ and G is a cyclic group of order 4.*

Maharjan [2] proved the proposition under the condition that $n \leq 4$.

The proposition, together with Theorem 1, gives the following.

Theorem 3. *Let π be a finite affine plane of order n with a collineation group G which is transitive on the affine points of π . If G has two orbits of length 2 and $n-1$ on l_∞ , then one of the following statements holds:*

(i) *The plane π is a translation plane and the group G contains the group of translations of π .*

(ii) *$n=2$ and G is a cyclic group of order 4.*

In the rest of the note, we prove Proposition 2. Set $T = G(l_\infty, l_\infty)$ and $L = G_{P_1, P_2}$. Then $|G:L| = 2$. Let O be an affine point of π . Set $l = P_1 O$. Sup-

pose that G_l is transitive on the points of l . Since $2|n$, there exists an involution σ in the center of a Sylow 2-subgroup of G_l . By Corollary 3.6.1 of [1], σ is a perspectivity. Therefore σ is an elation with the center P_1 . On the other hand $G_l \cap T = 1$. Thus $G(P_1, l) \neq 1$. This yields $P_2^G \neq P_2$, a contradiction. Hence we have the following.

(1) G_l is not transitive on the points on l .

As G leaves $\{P_1, P_2\}$ invariant, $G_{P_1} = L$. Hence, by Theorem 4.3 of [1], we have

(2) L is transitive on the lines through P_1 .

Since $|G:L| = 2$, (1) and (2) imply that L has exactly two orbits Ω_1 and Ω_2 on the points of π such that $|\Omega_1| = |\Omega_2|$. Hence,

(3) $|\Omega_1| = |\Omega_2| = n^2/2$.

Therefore $L_l (= G_l)$ has exactly two orbits Γ_1 and Γ_2 on the points of l . It follows from (2) and (3) that $\Gamma_1 = \Omega_1 \cap l$, $\Gamma_2 = \Omega_2 \cap l$ and $|\Gamma_1| = |\Gamma_2| = n/2$. Also we get $L_0 \leq L_l$, $|G:L_0| = |G:L| \times |L:L_l| \times |L_l:L_0| = 2 \cdot n \cdot n/2 = n^2$.

Suppose that $n-1 \neq 1$. Let p be a prime such that $p|(n-1)$ and A a p -Sylow subgroup of L_0 . Then since $n-1 = |l_\infty - \{P_1, P_2\}| |G|$ and $|G:L_0| = n^2 = 2^{2m}$, A also is a Sylow p -subgroup of G and $A \neq 1$. Since $(n-1, n/2) = 1$, $|Fix(A) \cap l| \geq 2$. Assume that $N_G(A) \leq L$. Then since $L \trianglelefteq G$ and A is a p -Sylow subgroup of L , $LN_G(A) = G$ and so $L = G$, a contradiction. Therefore $N_G(A) \not\leq L$. Let $\tau \in N_G(A) - L$. Then l^τ is through P_2 . Hence there exists a point Q on l such that A fixes Q . Since A fixes P_1, O, Q, P_2, O^τ and Q^τ , A is a planar collineation group. In particular, $Fix(A) \cap (l_\infty - \Delta) \neq \emptyset$. This yields that G is not transitive on $l_\infty - \Delta$ by Theorem 3.6 of [1], a contradiction. Thus $n-1 = 1$ and so $n = 2$.

We may assume that $\pi = PG(2, 2)^{l_\infty}$, where $l_\infty = \langle(1, 0, 0)\rangle \langle(0, 1, 0)\rangle$. Let $P_1 = \langle(1, 0, 0)\rangle$ and $P_2 = \langle(0, 1, 0)\rangle$. Then, by direct computation, $G = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \right\rangle$. Clearly $|G| = 4$. Thus Proposition 2 follows.

References

[1] M.J. Kallaher: Affine planes with transitive collineation groups, North Holland, New York-Amsterdam-Oxford, 1982.
 [2] H.B. Maharjan: Personal communication.
 [3] C. Suetake: On finite point transitive affine planes with two orbits on l_∞ , Osaka J. Math. 27 (1990), 271-276.

Amagakiminami High School
 Amagasaki, Hyogo 660
 Japan