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Osaka J. Math.
7 (1970), 77-80

ON SPLITTING OF A FACTOR SET IN A RING

Dedicated to Professor Keizo Asano on his 60th birthday

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(Received October 6, 1969)

1. Introduction

In [1] and [2], the present author developed the theory of crossed products, especially, the splitting property of factor sets in division and simple rings. If one takes a close look at the proofs given in these works, one can find a few simple principles making all the results obtainable. The purpose of this note is to give a simpler proof of a generalized theorem in case of a general ring. Let R be a ring with unity 1 and G a finite automorphism group of R. A factor set $\{c_{\sigma,\tau}\}$ is defined to be a system of units $c_{\sigma,\tau}$ $(\sigma, \tau \in G)$ in the center of R such that

$$(1) \qquad c_{\tau,\rho}c_{\sigma,\tau\rho} = c_{\sigma,\tau\rho}c^{\rho}_{\sigma,\tau} \qquad (\sigma, \tau, \rho \in G).$$

The factor set $\{c_{\sigma,\tau}\}$ is called splitting if one can find d_σ in the center of R such that

$$(2) \qquad d^{\tau}_{\sigma} = d^{-1}_{\tau} d_{\sigma\tau} c_{\sigma,\tau}.$$

A theorem we want to establish is that there exist a subring B' in R containing the fixed subring S and a (skew-) Kronecker product of R and B' over S so that $\{c_{\sigma,\tau}\}$ becomes splitting, provided R satisfies some Galois conditions which we shall discuss in 2.

2. Galois conditions

Denote $G=\{\sigma_1(=\text{the identity}), \sigma_2, \cdots, \sigma_n\}$. S denotes a subring of R consisting of all elements t in R such that $t^\sigma = t$ for all σ in G. Consider the following conditions.

[I] There exist $u_1, \cdots, u_n, v_1, \cdots, v_n$ in R such that $\sum_i v^{\sigma}_i u_i = 0$ unless $\sigma=\sigma_1$, and $=1$ in the latter case.

[II] The elements u_i and v_j in [I] satisfy $\sum_\sigma (u_i v_j)^\sigma = \delta_{i,j}$.

The conditions [I] and [II] are used, in the following, to prove the main theorem in a very effective way. But the true meaning of them lies in that R satisfies [I] and [II] if

[I′] $R = Su_1 \oplus \cdots \oplus Su_n$ (direct),

[II′] $R_rG = \mathrm{Hom}_{S_l}(R, R)$,

[III′] $R_rG = R_r\sigma_1 \oplus \cdots \oplus R_r\sigma_n$ (direct).

Here R_r stands for the ring of right multiplication by elements of R, and S_l the ring of left multiplication by elements of S. In this note, we apply operators from right. For example, $t \cdot a_r\sigma = (ta)^\sigma$. Now let us prove that [I′], [II′] and [III′] imply [I] and [II]. Due to [II′], every S_l homomorphism ϕ of R to R is in R_rG, and hence $\phi = \sum_i a_{ir}\sigma_i$ with a_i in R. Moreover, if ϕ maps R to S, then $t\phi$ is in S, i.e., $(t\phi)^\sigma = t\phi$ for all t in R. But this implies $(\sum a_{ir}\sigma_i)\sigma = \sum a_{ir}\sigma_i$. From the condition [III′], we have $a_1 = \cdots = a_n$. Therefore, $\phi = a_r(\sum_\sigma \sigma)$ with an element a. Especially, S_l homomorphisms which map u_i to 1 and u_j $(j \neq i)$ to 0 (which are possible because of [I′]) are expressed as $v_{ir}(\sum \sigma)$ with v_i in R. Now [I] follows, since $\sum_i v_{ir}(\sum \sigma)u_{ir} = 1$ in GR_r and the left hand term is $\sum_\sigma \sigma \sum_i (v_i)^\sigma_r u_{ir}$ and then we use [III′]. [II] is an immediate consequence of the definition of $v_{ir}(\sum \sigma)$, because then $u_i \cdot v_{jr}(\sum \sigma) = \delta_{i,j}$ which implies $\sum_\sigma (u_iv_j)^\sigma = \delta_{i,j}$. Conversely, suppose [I] and [II]. Set $s_i = \sum_\sigma (tv_i)^\sigma$ for an element t in R. We have $\sum_i s_iu_i = \sum_i \sum_\sigma (tv_i)^\sigma u_i = \sum_\sigma t^\sigma(\sum_i v_i^\sigma u_i) = t$ by [I]. On the other hand, if $\sum s_i'u_i = 0$ for s_i' in S, then $0 = \sum_\sigma \sum_i (s_i'u_i)^\sigma v_j = \sum_i s_i' \sum_\sigma (u_i^\sigma v_j) = s_j'$ for every j, which shows the condition [I′] is satisfied. [II] also implies that $v_{ir}(\sum \sigma)$ map u_i to 1 and u_j to 0, so that under the assumption [I′] the condition [II′] is satisfied.

3. Polynomial ring $R[x_1, \cdots, x_n, x_1^{-1}, \cdots, x_n^{-1}]$

Let $x_2, \cdots, x_n$ be $n-1$ variables. For the sake of convenience, we set $x_1 = 1$. We consider a polynomial ring $A = R[x_1, \cdots, x_n, x_1^{-1}, \cdots, x_n^{-1}]$, where $x_2, \cdots, x_n$ are supposed to be in the center of the ring. Every element of A is a sum of a finite number of monomials $a(i_2, \cdots, i_n)x_2^{i_2} \cdots x_n^{i_n}$ where i_j are some positive or negative integers and $a(i_2, \cdots, i_n)$ are elements in R. Now, corresponding to a given factor set $\{c_{\sigma,\tau}\}$, we shall extend the automorphism group G of R to one of A as follows. First, write $x_i = x_{\sigma_i}$. We define

$$x_\sigma^\tau = x_\tau^{-1} x_{\sigma\tau} c_{\sigma,\tau} \qquad (\sigma_i\ \tau \text{ in } G). \tag{3}$$

Without losing generality, we suppose $c_{\sigma,\tau} = 1$ if $\sigma = \sigma_1$ or $\tau = \sigma_1$. Thus $x_\sigma^{\sigma_1} = x_\sigma$. Then, in a natural way, an automorphism τ of R in G is extended to a homomorphism of A to A. If τ and ρ are two elements in G, we can show that $(x_\sigma^\tau)^\rho = x_\sigma^{\tau\rho}$ by following routine computation. $(x_\sigma^\tau)^\rho = (x_\tau^\rho)^{-1} x_{\sigma\tau}^\rho c_{\sigma,\tau}^\rho = x_\rho x_{\tau\rho}^{-1}$

$c_{\tau,\rho}^{-1}x_{\rho}^{-1}x_{\sigma\tau\rho}c_{\sigma\tau,\rho}c_{\sigma,\tau}^{\rho}=x_{\tau\rho}^{-1}x_{\sigma\tau\rho}c_{\sigma,\tau\rho}=x_{\sigma}^{\tau\rho}$ by making use of (1). Especially, for $\rho=\tau^{-1}$, we have $(x_{\sigma}^{\tau})^{\tau^{-1}}=x_{\sigma}$, showing τ^{-1}, and hence every element of G gives an automorphism of A (i.e., an onto-monomorphism). Thus G is extended to an automorphism group of A isomorphic to G, for which we use the same letter G. Denote the fixed subring of A (by G) by B. Important is that A/B is a Galois extension satisfying [I] and [II]. Therefore by the discussion in **2**, $A=Bu_1 \oplus\cdots\oplus Bu_n$ (direct). This result is a successfull consequence of rather technical conditions [I] and [II]. Note also that in the former papers [1] and [2] a quotient ring of a usual polynomial ring was used, the existence of which in general case might be a problem. Here we can avoid the use of it. Returning to A, in the following, we express elements of A by $\sum_i b_iu_i$ with b_i in B. The uniqueness of the expression has been guaranteed in the above.

4. (Skew-) Kronecker products and the final result

Set $P(B)=\{f(x_1,\cdots,x_n,x_1^{-1},\cdots,x_n^{-1})\in B\,|\,f(1,\cdots,1,1,\cdots,1)=0\}$, and $P=\{\sum b_iu_i\in A\,|\,b_i\in P(B)\}$.

Lemma. *P is an ideal of A.*

Proof. It is sufficient to show that $u_ip\in P$ for every element p of $P(B)$ $(i=1,\cdots,n)$. To do so, express $u_ip=\sum_{\kappa} b_ku_k$ with b_k in B. Then, $\sum_{\sigma}(u_ipv_j)^{\sigma}=\sum_{\sigma}\sum_{\kappa}(b_ku_kv_j)^{\sigma}=b_j$. But $\sum_{\sigma}(u_ipv_j)^{\sigma}=\sum_{\sigma}u_i^{\sigma}pv_j^{\sigma}$ become 0 if we set $x_1=\cdots=x_n=1$, showing $b_j\in P(B)$. This completes the proof.

Now, we consider the residue class ring A/P and denote it by A'. Let us investigate A' more closely. First of all, we have $R\cap P=0$. Therefore we may identify R with its isomorphic image in A'. Secondly, we see that P is invariant under G as a whole. Therefore, G induces an automorphism group of A'. Observing the effect of G on R in A', the group is seen to be isomorphic to G, so we identify both. The question is, what is the fixed subring? Before discussing that question, we investigate the homomorphic image of B in A'. Let $1=\sum_i c_iu_i$ with c_i in S. Then every element b of B is expressed as $\sum_i b_iu_i$ where $b_i=bc_i$. This implies b is contained in P if and only if $bc_i\in P(B)$, namely, $b\in P(B)$. Thus we may identify $B/P(B)$ with a homomorphic image of B in A'. We denote this by B'. In this case, every element of A' is uniquely expressed as $\sum_i b_i'u_1$ with b_i' in B'. That is, $A'=B'u_1\oplus\cdots\oplus B'u_n$ (direct). On the other hand, B' is obviously contained in the fixed ring of G in A'. Comparing with the discussion in **2**, we seet that the fixed subring coincides with B'. Here, note that even in A' the conditions [I] and [II] hold. From the above, we also have that $B'\cap R=S$. A' is, thus, a (skew-) Kronecker product of R and B' over S, (if we may give such a definition.) Now we are in a position to conclude our final goal. Recalling the definition of P, we can see that x_σ as well as x_σ^τ are not contained in P. Denote the elements of A' represented by x_σ by d_σ. From (3), we have the identities (2).

Main theorem. *Let R/S be a Galois extension satisfying* [I] *and* [II], *and let $\{c_{\sigma,\tau}\}$ be a factor set. Then there exists a subring B' in R containing S such that we can construct a (skew-) Kronecker product of B' and R over S and that this Kronecker product A' is a Galois extension over B' satisfying* [I] *and* [II] *(with the same Galois group with that of R/S). In A', the factor set $\{c_{\sigma,\tau}\}$ is splitting.*

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References

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