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ON SPLITTING OF A FACTOR SET IN A RING

Dedicated to Professor Keizo Asano on his 60th birthday

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1. Introduction

In [1] and [2], the present author developed the theory of crossed products, especially, the splitting property of factor sets in division and simple rings. If one takes a close look at the proofs given in these works, one can find a few simple principles making all the results obtainable. The purpose of this note is to give a simpler proof of a generalized theorem in case of a general ring. Let R be a ring with unity 1 and G a finite automorphism group of R . A factor set $\{c_{\sigma, \tau}\}$ is defined to be a system of units $c_{\sigma, \tau}$ ($\sigma, \tau \in G$) in the center of R such that

$$(1) \quad c_{\tau, \rho} c_{\sigma, \tau \rho} = c_{\sigma, \tau} c_{\sigma, \tau}^{\rho} \quad (\sigma, \tau, \rho \in G).$$

The factor set $\{c_{\sigma, \tau}\}$ is called splitting if one can find d_{σ} in the center of R such that

$$(2) \quad d_{\sigma}^{\tau} = d_{\tau}^{-1} d_{\sigma\tau} c_{\sigma, \tau}.$$

A theorem we want to establish is that there exist a subring B' in R containing the fixed subring S and a (skew-) Kronecker product of R and B' over S so that $\{c_{\sigma, \tau}\}$ becomes splitting; provided R satisfies some Galois conditions which we shall discuss in 2.

2. Galois conditions

Denote $G = \{\sigma_1 (= \text{the identity}), \sigma_2, \dots, \sigma_n\}$. S denotes a subring of R consisting of all elements t in R such that $t^{\sigma} = t$ for all σ in G . Consider the following conditions.

[I] There exist $u_1, \dots, u_n, v_1, \dots, v_n$ in R such that $\sum_i v_i^{\sigma} u_i = 0$ unless $\sigma = \sigma_1$, and $= 1$ in the latter case.

[II] The elements u_i and v_j in [I] satisfy $\sum_{\sigma} (u_i v_j)^{\sigma} = \delta_{i, j}$.

The conditions [I] and [II] are used, in the following, to prove the main theorem in a very effective way. But the true meaning of them lies in that R satisfies [I] and [II] if

$$\begin{aligned}
[\text{I}'] \quad & R = Su_1 \oplus \cdots \oplus Su_n \quad (\text{direct}), \\
[\text{II}'] \quad & R_r G = \text{Hom}_{S_l}(R, R), \\
[\text{III}'] \quad & R_r G = R_r \sigma_1 \oplus \cdots \oplus R_r \sigma_n \quad (\text{direct}).
\end{aligned}$$

Here R_r stands for the ring of right multiplication by elements of R , and S_l the ring of left multiplication by elements of S . In this note, we apply operators from right. For example, $t \cdot a_r \sigma = (ta)^\sigma$. Now let us prove that [I'], [II'] and [III'] imply [I] and [II]. Due to [II'], every S_l homomorphism ϕ of R to R is in $R_r G$, and hence $\phi = \sum_i a_{i,r} \sigma_i$ with a_i in R . Moreover, if ϕ maps R to S , then $t\phi$ is in S , i.e., $(t\phi)^\sigma = t\phi$ for all t in R . But this implies $(\sum a_{i,r} \sigma_i)^\sigma = \sum a_{i,r} \sigma_i$. From the condition [III'], we have $a_1 = \cdots = a_n$. Therefore, $\phi = a_r (\sum \sigma)$ with an element a . Especially, S_l homomorphisms which map u_i to 1 and u_j ($j \neq i$) to 0 (which are possible because of [I']) are expressed as $v_{i,r} (\sum \sigma)$ with v_i in R . Now [I] follows, since $\sum_i v_{i,r} (\sum \sigma) u_{i,r} = 1$ in GR_r , and the left hand term is $\sum_\sigma \sigma \sum_i (v_i)^\sigma u_{i,r}$ and then we use [III']. [II] is an immediate consequence of the definition of $v_{i,r} (\sum \sigma)$, because then $u_i \cdot v_{j,r} (\sum \sigma) = \delta_{i,j}$ which implies $\sum_\sigma (u_i v_j)^\sigma = \delta_{i,j}$. Conversely, suppose [I] and [II]. Set $s_i = \sum_\sigma (tv_i)^\sigma$ for an element t in R . We have $\sum_i s_i u_i = \sum_i \sum_\sigma (tv_i)^\sigma u_i = \sum_\sigma t^\sigma (\sum_i v_i^\sigma u_i) = t$ by [I]. On the other hand, if $\sum s'_i u_i = 0$ for s'_i in S , then $0 = \sum_\sigma \sum_i (s'_i u_i)^\sigma v_j = \sum_i s'_i \sum_\sigma (u_i^\sigma v_j) = s'_j$ for every j , which shows the condition [I'] is satisfied. [II] also implies that $v_{i,r} (\sum \sigma)$ map u_i to 1 and u_j to 0, so that under the assumption [I'] the condition [II'] is satisfied.

3. Polynomial ring $R[x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}]$

Let x_2, \dots, x_n be $n-1$ variables. For the sake of convenience, we set $x_1 = 1$. We consider a polynomial ring $A = R[x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}]$, where x_2, \dots, x_n are supposed to be in the center of the ring. Every element of A is a sum of a finite number of monomials $a(i_2, \dots, i_n) x_2^{i_2} \cdots x_n^{i_n}$ where i_j are some positive or negative integers and $a(i_2, \dots, i_n)$ are elements in R . Now, corresponding to a given factor set $\{c_{\sigma, \tau}\}$, we shall extend the automorphism group G of R to one of A as follows. First, write $x_i = x_{\sigma_i}$. We define

$$(3) \quad x_\sigma^\tau = x_\tau^{-1} x_{\sigma\tau} c_{\sigma, \tau} \quad (\sigma_i \tau \text{ in } G).$$

Without losing generality, we suppose $c_{\sigma, \tau} = 1$ if $\sigma = \sigma_1$ or $\tau = \sigma_1$. Thus $x_{\sigma_1}^\tau = x_\sigma$. Then, in a natural way, an automorphism τ of R in G is extended to a homomorphism of A to A . If τ and ρ are two elements in G , we can show that $(x_\sigma^\tau)^\rho = x_\sigma^{\tau\rho}$ by following routine computation. $(x_\sigma^\tau)^\rho = (x_\tau^\rho)^{-1} x_{\sigma\tau}^\rho c_{\sigma, \tau}^\rho = x_\rho x_\tau^{-1}$

$c_{\tau, \rho}^{-1} x_{\rho}^{-1} x_{\sigma \tau \rho} c_{\sigma \tau, \rho} c_{\sigma, \tau}^{\rho} = x_{\tau \rho}^{-1} x_{\sigma \tau \rho} c_{\sigma, \tau \rho} = x_{\sigma}^{\tau \rho}$ by making use of (1). Especially, for $\rho = \tau^{-1}$, we have $(x_{\sigma}^{\tau})^{\tau^{-1}} = x_{\sigma}$, showing τ^{-1} , and hence every element of G gives an automorphism of A (i.e., an onto-monomorphism). Thus G is extended to an automorphism group of A isomorphic to G , for which we use the same letter G . Denote the fixed subring of A (by G) by B . Important is that A/B is a Galois extension satisfying [I] and [II]. Therefore by the discussion in 2, $A = Bu_1 \oplus \dots \oplus Bu_n$ (direct). This result is a successful consequence of rather technical conditions [I] and [II]. Note also that in the former papers [1] and [2] a quotient ring of a usual polynomial ring was used, the existence of which in general case might be a problem. Here we can avoid the use of it. Returning to A , in the following, we express elements of A by $\sum_i b_i u_i$ with b_i in B . The uniqueness of the expression has been guaranteed in the above.

4. (Skew-) Kronecker products and the final result

Set $P(B) = \{f(x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}) \in B \mid f(1, \dots, 1, 1, \dots, 1) = 0\}$, and $P = \{\sum b_i u_i \in A \mid b_i \in P(B)\}$.

Lemma. P is an ideal of A .

Proof. It is sufficient to show that $u_i p \in P$ for every element p of $P(B)$ ($i = 1, \dots, n$). To do so, express $u_i p = \sum_{\kappa} b_{\kappa} u_{\kappa}$ with b_{κ} in B . Then, $\sum_{\sigma} (u_i p v_j)^{\sigma} = \sum_{\sigma} \sum_{\kappa} (b_{\kappa} u_{\kappa} v_j)^{\sigma} = b_j$. But $\sum_{\sigma} (u_i p v_j)^{\sigma} = \sum_{\sigma} u_i^{\sigma} p v_j^{\sigma}$ become 0 if we set $x_1 = \dots = x_n = 1$, showing $b_j \in P(B)$. This completes the proof.

Now, we consider the residue class ring A/P and denote it by A' . Let us investigate A' more closely. First of all, we have $R \cap P = 0$. Therefore we may identify R with its isomorphic image in A' . Secondly, we see that P is invariant under G as a whole. Therefore, G induces an automorphism group of A' . Observing the effect of G on R in A' , the group is seen to be isomorphic to G , so we identify both. The question is, what is the fixed subring? Before discussing that question, we investigate the homomorphic image of B in A' . Let $1 = \sum_i c_i u_i$ with c_i in S . Then every element b of B is expressed as $\sum_i b_i u_i$ where $b_i = b c_i$. This implies b is contained in P if and only if $b c_i \in P(B)$, namely, $b \in P(B)$. Thus we may identify $B/P(B)$ with a homomorphic image of B in A' . We denote this by B' . In this case, every element of A' is uniquely expressed as $\sum_i b'_i u_i$ with b'_i in B' . That is, $A' = B' u_1 \oplus \dots \oplus B' u_n$ (direct).

On the other hand, B' is obviously contained in the fixed ring of G in A' . Comparing with the discussion in 2, we see that the fixed subring coincides with B' . Here, note that even in A' the conditions [I] and [II] hold. From the above, we also have that $B' \cap R = S$. A' is, thus, a (skew-) Kronecker product of R and B' over S , (if we may give such a definition.) Now we are in a

position to conclude our final goal. Recalling the definition of P , we can see that x_σ as well as x_σ^τ are not contained in P . Denote the elements of A' represented by x_σ by d_σ . From (3), we have the identities (2).

Main theorem. *Let R/S be a Galois extension satisfying [I] and [II], and let $\{c_{\sigma,\tau}\}$ be a factor set. Then there exists a subring B' in R containing S such that we can construct a (skew-) Kronecker product of B' and R over S and that this Kronecker product A' is a Galois extension over B' satisfying [I] and [II] (with the same Galois group with that of R/S). In A' , the factor set $\{c_{\sigma,\tau}\}$ is splitting.*

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