Naitoh, H.
Osaka J. Math.
18 (1981), 427–464

ISOTROPIC SUBMANIFOLDS WITH PARALLEL SECOND FUNDAMENTAL FORM IN $P^m(c)$

HIROO NAITOH

(Received November 30, 1979)

Introduction. In the study of submanifolds of a riemannian manifold, as a generalization of a totally geodesic submanifold, the notion of an isotropic submanifold has been introduced by B. O'Neill [10]. On the other hand, as another generalization of a totally geodesic submanifold, there is the notion of a submanifold with parallel second fundamental form. Among submanifolds belonging to both classes, those which are not totally geodesic have the property that every geodesic in the submanifold is a circle in the ambient riemannian manifold (K. Nomizu [8]).

These submanifolds have been studied recently when the ambient riemannian manifold is a riemannian symmetric space. Among them totally umbilical submanifolds are called extrinsic spheres. It is known that an extrinsic sphere is isometric to a Euclidean sphere, a Euclidean space, or a real hyperbolic space (B.Y. Chen [2] and H. Naitoh [7]). If the ambient manifold is a Hermitian symmetric space, a Kähler submanifold belonging to both classes is congruent to the Veronese manifold of degree two (H. Naitoh [7]). Moreover K. Nomizu [8] has shown that if the ambient manifold is a complex projective space with the Fubini-Study metric, the Veronese manifold of degree two is characterized by the property that every geodesic in the submanifold is a circle in the complex projective space.

Now nonzero isotropic submanifolds with parallel second fundamental form are closedly related to planer geodesic submanifolds. When the ambient manifold is a Euclidean sphere, the submanifolds coincide with those which are planer geodesic but not totally geodesic, and they have been classified by K. Sakamoto [12]. When the ambient manifold is the complex projective space, submanifolds which are planer geodesic but not totally geodesic are nonzero isotropic and have parallel second fundamental forms. Moreover it is known that these submanifolds are compact riemannian symmetric spaces of rank one (J.S.Pak [11]).

In this paper we study nonzero isotropic submanifolds with parallel second fundamental form in a complex projective space with the Fubini-Study metric. These submanifolds can be divided into the following three types; Kählerian,

$P(\boldsymbol{R})$-totally real, or $P(\boldsymbol{C})$-totally real (Proposition 2.2 and 2.3). In the Kählerian case they are congruent to the Veronese manifolds of degree two as above. In the $P(\boldsymbol{R})$-totally real case they are planer geodesic but not totally geodesic in some real projective space. Moreover, among nonzero isotropic submanifolds with parallel second fundamental form, Kählerian and $P(\boldsymbol{R})$-totally real submanifolds exhaust all the planer geodesic submanifolds in the ambient complex projective space (Theorem 3.8). In the $P(\boldsymbol{C})$-totally real case, nonzero isotropic submanifolds with parallel second fundamental form are not planer geodesic and locally isometric to the riemannian symmetric spaces; $S^1\times S^n (n\geqq 1)$, $SU(3)/SO(3)$, $SU(3)$, $SU(6)/Sp(3)$, E_6/F_4 (Theorem 4.13).

In the section 6 we shall construct a model of imbeddings for the case the submanifold is locally isometric to the riemannian symmetric space $S^1\times S^n$ (Theorem 6.5) and in the section 7 for the case the submanifold is locally isometric to the riemannian symmetric space $SU(3)/SO(3)$ (Theorem 7.2). Moreover in the section 8 we shall show that these submanifolds have the rigidity (Theorem 8.3 and 8.6).

The author wishes to express his hearty thanks to Professor M. Takeuchi and Professor Y. Sakane for their useful comments during the preparation of the present paper.

1. Preliminaries

Let $\bar{M}^m$ be an m-dimensional riemannian manifold with a riemannian metric $\langle\ ,\ \rangle$ and M^n an n-dimensional connected riemannian submanifold in $\bar{M}^m$. Denote by $\bar{\nabla}$ (resp. ∇) the riemannian connection on $\bar{M}$ (resp. M) and by $\bar{R}$ (resp. R) the riemannian curvature tensor for $\bar{\nabla}$ (resp. ∇). Moreover we denote by σ the second fundamental form of M, by D the normal connection on the normal bundle $N(M)$ of M and by $R^{\perp}$ the curvature tensor for D. For a point $p\in M$, the tangent space $T_p(\bar{M})$ is orthogonally decomposed into the direct sum of the tangent space $T_p(M)$ and the normal space $N_p(M)$. For a vector $X\in T_p(\bar{M})$, the normal component of X will be denote by $X^{\perp}$. Put

$$N_p^1(M) = \{\sigma(X, Y)\in N_p(M);\ X, Y\in T_p(M)\}_{\boldsymbol{R}}$$

where $\{*\}_{\boldsymbol{R}}$ denotes the real vector space spanned by $*$. It is called the first normal space at p. Then we have the orthogonal decomposition

$$N_p(M) = N_p^1(M)+(N_p^1(M))^{\perp}$$

where $(N_p^1(M))^{\perp}$ denotes the orthogonal complement of $N_p^1(M)$ in $N_p(M)$. Let $S^2(T_p(M))$ be the set of all symmetric endomorphisms of $T_p(M)$. Then we define the linear mapping $A\colon N_p(M)\to S^2(T_p(M))$ by

$$\langle A_\zeta(X),\ Y\rangle = \langle\sigma(X, Y),\ \zeta\rangle$$

where $X, Y \in T_p(M)$ and $\zeta \in N_p(M)$. The symmetric endomorphism A_ζ is called the shape operator defined by ζ. By the definition of A, the restriction of A to $N_p^1(M)$ is injective.

Now we recall the following fundamental equations, which are called the equations of Gauss, Codazzi-Mainardi, and Ricci respectively.

$$(1.1)\quad \langle \bar{R}(X, Y)Z, W\rangle = \langle R(X, Y)Z, W\rangle + \langle \sigma(X, Z), \sigma(Y, W)\rangle - \langle \sigma(X, W), \sigma(Y, Z)\rangle$$

$$(1.2)\quad \{\bar{R}(X, Y)Z\}^\perp = (\nabla_X^*\sigma)(Y, Z) - (\nabla_Y^*\sigma)(X, Z)$$

$$(1.3)\quad \langle \bar{R}(X, Y)\zeta, \eta\rangle = \langle R^\perp(X, Y)\zeta, \eta\rangle - \langle [A_\zeta, A_\eta](X), Y\rangle$$

where $X, Y, Z, W \in T_p(M)$, $\zeta, \eta \in N_p(M)$ and ∇^* denotes the covariant derivation associated to the submanifold $M \subset \bar{M}$, defined by

$$(\nabla_X^*\sigma)(Y, Z) = D_X\sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)$$

for vector fields X, Y, Z of M. The second fundamental form σ is said to be *parallel* if $\nabla^*\sigma = 0$. Now for a point $p \in M$, put

$$O_p^1(M) = T_p(M) + N_p^1(M)$$

which is called the first osculating space at p. Since the second fundamental form σ is parallel, dimensions of $N_p^1(M)$ and $O_p^1(M)$ are constant on M, and hence $N^1(M) = \bigcup_p N_p^1(M)$ and $O^1(M) = \bigcup_p O_p^1(M)$ are subbundles of $T(\bar{M})|M$, the restriction to M of the tangent bundle $T(\bar{M})$ of $\bar{M}$. Moreover we have the following

Lemma 1.1 (See [7, Lemmas 1, 13]). *If σ is parallel and if $\bar{M}$ is a riemannian locally symmetric space*

a) $\bar{R}(X, Y)Z \in T_p(M)$

b) $\bar{R}(X, Y)\sigma(T, Z) \in N_p^1(M)$

c) $R^\perp(T, S)\sigma(X, Y) = \sigma(R(T, S)X, Y) + \sigma(X, R(T, S)Y)$

d) $\sigma(T, \bar{R}(X, Y)Z) = \bar{R}(\sigma(T, X), Y)Z + \bar{R}(X, \sigma(T, Y))Z + \bar{R}(X, Y)\sigma(T, Z)$

e) $\bar{R}(\sigma(T, X), Y)\sigma(S, Z) + \bar{R}(X, \sigma(T, Y))\sigma(S, Z) \in O_p^1(M)$

where $X, Y, Z, T, S \in T_p(M)$.

For a given $\lambda \geqq 0$, a riemannian submanifold M in a riemannian manifold $\bar{M}$ is called a *λ-isotropic submanifold* if $|\sigma_p(X, X)| = \lambda$ for each point $p \in M$ and every unit tangent vector $X \in T_p(M)$.

Now we recall the notion of circles in a riemannian manifold $\bar{M}$. A curve x_t of $\bar{M}$ parametrized by arc length is called a *circle*, if there exists a field of unit vectors Y_t along the curve which satisfies, together with the unit tangent vectors $X_t = \dot{x}_t$, the differential equations

$$\bar{\nabla}_t X_t = kY_t \quad\text{and}\quad \bar{\nabla}_t Y_t = -kX_t$$

where k is a positive constant, which is called the curvature of the circle x_t. Let p be an arbitrary point of $\bar{M}$. For a pair of orthonormal vectors $X, Y \in T_p(M)$ and for a given constant $k>0$, there exists a unique circle x_t, defined for t near 0, such that

$$x_0 = p\,, \quad X_0 = X\,, \quad \text{and} \quad (\bar{\nabla}_t X_t)_{t=0} = kY\,.$$

If $\bar{M}$ is complete, x_t can be defined for $-\infty < t < +\infty$.

A nonzero isotropic submanifold with parallel second fundamental form has the property as given in the following lemma.

Lemma 1.2 (K. Nomizu [8]). *If M is a $\lambda(>0)$-isotropic submanifold with parallel second fundamental form in $\bar{M}$, every geodesic in M is a circle with the curvature λ in $\bar{M}$.*

2. Submanifolds with parallel second fundamental form in $P^m(c)$

Let $P^m(c)$ be the m-dimensional complex projective space of constant holomorphic sectional curvature $c(>0)$ and M^n an n-dimensional connected complete riemannian submanifold with parallel second fundamental form in $P^m(c)$. Then M^n is a riemannian locally symmetric space since $P^m(c)$ is a riemannian symmetric space.

From now on we put $\bar{M} = P^m(c)$. Let J be the almost complex structure on $P^m(c)$. Then we have the following

Lemma 2.1 (See B.Y. Chen and K. Ogiue [4]). *If A, B, C are tangent vectors of $P^m(c)$,*

$$\bar{R}(A, B)C = \frac{c}{4}(\langle B, C\rangle A + \langle JB, C\rangle JA - \langle A, C\rangle B - \langle JA, C\rangle JB + 2\langle A, JB\rangle JC)\,.$$

Proposition 2.2 (cf. [4]). *If M^n is a riemannian submanifold with parallel second fundamental form in $P^m(c)$, it holds either*

a) $J(T_p(M)) = T_p(M)$ *for every point* $p \in M$

or

b) $J(T_p(M)) \subset N_p(M)$ *for every point* $p \in M$.

Here we note that in the case a) M is a Kähler submanifold in $P^m(c)$ and in the case b) we call M a totally real submanifold in $P^m(c)$. Moreover we have the following

Proposition 2.3. *If M^n is an $n(\geqq 2)$-dimensional totally real submanifold with parallel second fundamental form in $P^m(c)$, it holds either*

b$_1$) $J(T_p(M)) \subset (N_p^1(M))^\perp$ *for every point* $p \in M$

or

b$_2$) $J(T_p(M))\subset N_p^1(M)$ *for every point* $p\in M$.

Proof. For a vector $\zeta\in N_p(M)$, we denote by ζ_a (resp. ζ_b) the $N_p^1(M)$-component (resp. $(N_p^1(M))^\perp$-component) of ζ. Let X, Y (resp. H) be vectors in $T_p(M)$ (resp. in $N_p^1(M)$). Then we have

$$\langle JY, H\rangle JX-\langle JX, H\rangle JY\in N_p^1(M)$$

by Lemma 1.1,b) and Lemma 2.1, and thus

$$(2.1)\qquad \langle (JY)_a, H\rangle (JX)_b=\langle (JX)_a, H\rangle (JY)_b\,.$$

Assume that there exists a vector JX such that $(JX)_a\neq 0$. Then putting $H=(JX)_a$ in (2.1), we have

$$(2.2)\qquad (JY)_b=c_{XY}(JX)_b$$

where $c_{XY}=\dfrac{\langle (JY)_a, (JX)_a\rangle}{\langle (JX)_a, (JX)_a\rangle}$. Hence by (2.1) we have

$$\langle (JY)_a-c_{XY}(JX)_a, H\rangle (JX)_b=0\,.$$

If $(JX)_b\neq 0$, $(JY)_a=c_{XY}(JX)_a$ and thus together with (2.2), $JY=c_{XY}JX$ for any vector $Y\in T_p(M)$. This contradicts that $\dim M\geqq 2$. Therefore we have $(JX)_b=0$. By (2.2), $(JY)_b=0$ for any vector $Y\in T_p(M)$, which shows $J(T_p(M))\subset N_p^1(M)$. Since M is connected, we get our claim by the routine way. q.e.d.

In the case b$_1$) (resp. b$_2$)) we call the submanifold M *of type* $P(\boldsymbol{R})$ (resp. *of type* $P(\boldsymbol{C})$).

Now we discribe the relation between the almost complex structure J and the second fundamental form σ in the following

Lemma 2.4. *If* M^n *is a totally real submanifold in* $P^m(c)$, *we have*

$$\langle \sigma(T, X), JY\rangle=\langle \sigma(T, Y), JX\rangle$$

for vectors T, X, $Y\in T_p(M)$.

Proof. For vector fields T, X, Y tangent to M,

$$\begin{aligned}\langle \sigma(T, X), JY\rangle &=\langle \bar\nabla_T X, JY\rangle=-\langle J\bar\nabla_T X, Y\rangle\\ &=-\langle \bar\nabla_T JX, Y\rangle=\langle JX, \bar\nabla_T Y\rangle\\ &=\langle \sigma(T, Y), JX\rangle\,.\end{aligned}$$

q.e.d.

Now we recall the notion of Lie triple system. Fix a point $p\in P^m(c)$. Let $\bar G$ be the identity component of the group of isometries of $P^m(c)$, and set $\bar K=\{g\in\bar G;\, g(p)=p\}$. Let $\bar{\mathfrak g}$ and $\bar{\mathfrak k}$ be the Lie algebras of $\bar G$ and $\bar K$ respectively, and let

$$\bar{\mathfrak{g}} = \bar{\mathfrak{k}} + \bar{\mathfrak{p}}$$

be the associated canonical decomposition. Then the tangent space $T_p(P^m(c))$ is identified with $\bar{\mathfrak{p}}$. A subspace $\mathfrak{m}$ in $\bar{\mathfrak{p}}$ is called a *Lie triple system* if $[[X, Y], Z] \in \mathfrak{m}$ for $X, Y, Z \in \mathfrak{m}$. Since

$$\bar{R}_p(X, Y)Z = -[[X, Y], Z]$$

for $X, Y, Z \in \bar{\mathfrak{p}}$ under the above identification, we call a subspace V in $T_p(P^m(c))$ a Lie triple system if $\bar{R}_p(X, Y)Z \in V$ for $X, Y, Z \in V$. Then we know that for a given Lie triple system V in $T_p(P^m(c))$, there exists a unique complete totally geodesic submanifold N in $P^m(c)$ such that $p \in N$ and $T_p(N) = V$. If M is a submanifold with parallel second fundamental form in $P^m(c)$, the subspace $T_p(M)$ is a Lie triple system in $T_p(P^m(c))$ for every point $p \in M$. Concerning totally geodesic submanifolds in $P^m(c)$ we get easily the following

Lemma 2.5. *Let M^n be a complete totally geodesic snbmadifold in $P^m(c)$. If M^n is Kählerian, the manifold M^n is isometric to the complex projective space $P^r(c)$ $(2r=n)$ of constant holomorphic sectional curvature c. If M^n is totaly real, the manifold M^n is isometric to the real projective space $P^n(\boldsymbol{R})$ of constant sectional curvature $\frac{c}{4}$.*

3. Planer geodesic submanifolds in $P^m(c)$

In this section we consider the cases when M is Kählerian or totally real of type $P(\boldsymbol{R})$ (cf. Proposition 2.2 and 2.3). In the former case we have

Proposition 3.1 (K. Nomizu [8]). *If M^n is a complete nonzero isotropic Kähler submanifold with parallel second fundamental form in $P^m(c)$, M^n is the full Veronese submanifold of degree 2 in some totally geodesic complex projective space in $P^m(c)$.*

Note that the local version is also true by a result of Calabi [1].

In the rest of this section we exclusively study the latter case.

Lemma 3.2. *If M^n is an $n(\geqq 2)$-dimensional $P(\boldsymbol{R})$-totally real submanifold with parallel second fundamental form in $P^m(c)$, the first osculating space $O_p^1(M)$ at $p \in M$ is a Lie triple system in $T_p(P^m(c))$ and there exists a unique totally geodesic submanifold in $P^m(c)$ of constant sectional curvature $\frac{c}{4}$ whose tangent space at p is the space $O_p^1(M)$.*

Proof. At first we shall show that $O_p^1(M)$ is a Lie triple system in $T_p(P^m(c))$. Along the same line as in the proof of Lemma 13 in [7], it is sufficient to show the followings;

$$\begin{cases} \bar{R}(X, \sigma(T, Y))Z \in N_p^1(M) \subset O_p^1(M) \\ \bar{R}(X, \sigma(T, Y))\sigma(S, Z) \in T_p(M) \subset O_p^1(M) \end{cases}$$

for $X, Y, Z, T, S \in T_p(M)$. By Lemma 2.1 and the condition that $J(T_p(M)) \subset (N_p^1(M))^{\perp}$, we have

$$\bar{R}(X, \sigma(T, Y))Z = -\frac{c}{4}\langle X, Z\rangle\sigma(T, Y) \in N_p^1(M).$$

Similarly we have

$$(3.1)\quad \begin{cases} \bar{R}(X, \sigma(T, Y))\sigma(S, Z) = \frac{c}{4}(\langle\sigma(T, Y), \sigma(S, Z)\rangle X \\ \qquad\qquad +\langle J\sigma(T, Y), \sigma(S, Z)\rangle JX) \\ \bar{R}(\sigma(T, X), Y)\sigma(S, Z) = -\frac{c}{4}(\langle\sigma(T, X), \sigma(S, Z)\rangle Y \\ \qquad\qquad +\langle J\sigma(T, X), \sigma(S, Z)\rangle JY). \end{cases}$$

By Lemma 1.1,e) and (3.1), we have

$$\langle J\sigma(T, Y), \sigma(S, Z)\rangle JX - \langle J\sigma(T, X), \sigma(S, Z)\rangle JY \in O_p^1(M).$$

For any vector X there exists a vector $Y \in T_p(M)$ such that X and Y are linearly independent, since $\dim M \geqq 2$. Hence by the condition that $J(T_p(M)) \subset (N_p^1(M))^{\perp}$, we have

$$\langle J\sigma(T, X), \sigma(S, Z)\rangle = 0$$

and thus

$$(3.2)\quad \langle J(N_p^1(M)), N_p^1(M)\rangle = \{0\}.$$

Hence by (3.1) and (3.2), we get

$$\bar{R}(X, \sigma(T, Y))\sigma(S, Z) = \frac{c}{4}\langle\sigma(T, Y), \sigma(S, Z)\rangle X \in T_p(M).$$

Now noting that the first osculating space $O_p^1(M)$ is a totally real Lie triple system in $T_p(P^m(c))$ by (3.2) and the condition that $J(T_p(M)) \subset (N_p^1(M))^{\perp}$, we have the second assertion by Lemma 2.5. q.e.d.

By Lemma 2.5, Lemma 1.2 and the uniqueness of circle, we have

Proposition 3.3. *If M^n is an $n(\geqq 2)$-dimensional complete nonzero isotropic $P(\boldsymbol{R})$-totally real submanifold with parallel second fnndamental form in $P^m(c)$, there exists a unique totally geodesic submanifold $P^r(\boldsymbol{R})$ such that*

1) *M^n is a submanifold in $P^r(\boldsymbol{R})$*

and

2) $O_q^1(M)=T_q(P^r(\boldsymbol{R}))$ *for any point* $q\in M$.

Proof. Aanalogous to the proof of Proposition 15 in [7].

Now we recall the notion of planer geodesic submanifolds. A submanifold M in a riemannian symmetric space $\bar{M}$ is called a *planer geodesic submanifold* if for any maximal geodesic γ in M, there exists a 2-dimensional totally geodesic submanifold in $\bar{M}$ containing γ. K. Sakamoto [12] has studied the case that $\bar{M}$ is of constant sectional curvature. We discribe two lemmas which we use in this paper.

Lemma 3.4 (K. Sakamoto [12]). *Let M be a riemannian submanifold in a riemannian symmetric space $\bar{M}(c)$ of constant sectional curvature c. Then the following three conditions are equivalent.*

(P.G) *The submanifold M is planer geodesic and not totally geodesic.*
(I.P) *The submanifold M is nonzero isotropic and has the parallel second fundamental form.*
(G.C) *Every geodesic in M is a circle in $\bar{M}(c)$.*

Moreover K. Sakamoto [12] has shown that a complete planer geodesic and not totally geodesic submanifold M in the Euclidean sphere S^m is one of the followings;

(1) M is a totally umbilical and not totally geodesic submanifold in S^m.
(2) M is isometric to a real projective space, a complex projective space, a quatanion projective space, or a Cayley projective space. And the imbeddings are full and minimal ones constructed by S.S.Tai.
(3) M is a Tight imbedded submanifold in some totally umbilical submanifold in S^m.

Here along his argument we also have the following

Lemma 3.5 (K. Sakamoto [12]). *Without the assumption of completeness, the planer geodesic and not totally geodesic submanifold M is locally isometric to one of the compact riemannian symmetric spaces of rank one. Here the dimension of the first normal spaces equals* $1, (n-1)(n+2)/2, (n-1)(n+2)/2+1$ *if M is locally isometric to* S^n; $(n-1)(2n+1)$ *or* $(n-1)(2n+1)+1$ *if M is locally isometric to* $P^n(\boldsymbol{H})$; n^2-1 *or* n^2 *if M is locally isometric to* $P^n(\boldsymbol{C})$; 9 *or* 10 *if M is locally isometric to* $P^2(Ca)$.

Now we study complete nonzero isotropic submanifolds with parallel second fundamental form in $P^r(\boldsymbol{R})$.

Proposition 3.6. *Let M^n be a complete nonzero isotropic submanifold with* *parallel second fundamental form in $P^r(\boldsymbol{R})$ and let π: $S^r \to P^r(\boldsymbol{R})$ be the covering map. Then there exists a complete nonzero isotropic submanifold $\tilde{M}$ with parallel second fundamental form in S^r such that π: $\tilde{M} \to M$ is isometric.*

Proof. The subset $\pi^{-1}(M)$ in S^r is a submanifold since π is a covering map. Let $\tilde{M}$ be a connected component of $\pi^{-1}(M)$. Then $\tilde{M}$ is a complete nonzero isotropic submanifold with parallel second fundamental form in S^r. We shall show that π: $\tilde{M} \to M$ is isometric. Suppose that π: $\tilde{M} \to M$ is not injective. Then there exist distinct points x and y in $\tilde{M}$ such that $\pi(x)=\pi(y)$. Here we note that x and y are anti-podal in S^r. Let γ be a geodesic in $\tilde{M}$ joining x and y. By Lemma 3.4, γ is a circle in S^r. This is a contradiction. Hence π: $\tilde{M} \to M$ is injective. Since $\tilde{M}$ is compact, we get our claim. q.e.d.

Now summing up some results of J.S.Pak [11], we have the following

Lemma 3.7 (J.S. Pak [11]). *A planer geodesic and not totally geodesic submanifold M in $P^n(c)$ is either nonzero isotropic Kählerian with parallel second fundamental form or nonzero isotropic $P(\boldsymbol{R})$-totally real with parallel second fundamental from. (Here we need not assume the completeness of the submanifold M.)*

The Veronese submanifold in $P^m(c)$ of degree 2 is planer geodesic and not totally geodesic (cf. J.S. Pak [11]), and by Lemma 3.4 so are the submanifolds in Proposition 3.6. Hence together with Lemma 3.7 and Proposition 3.1, 3.6 we have the following

Theorem 3.8. *Let M^n be an $n(\geqq 2)$-dimensional complete nonzero isotropic submanifold with parallel second fundamental form in $P^m(c)$. Then the submanifold M is planer geodesic if and only if M is Kählerian or $P(\boldsymbol{R})$-totally real. Moreover such submanifolds are those given in Proposition* 3.1 *and* 3.6.

4. Non planer geodesic submanifolds in $P^m(c)$

In this section we study the case when M is totally real of type $P(\boldsymbol{C})$ (cf. Proposition 2.3).

Lemma 4.1. *If M^n is an $n(\geqq 2)$-dimensional complete $P(\boldsymbol{C})$-totally real submanifold with parallel second fundamental form in $P^m(c)$, the first osculating space $O_p^1(M)$ at $p \in M$ is a Lie triple system in $T_p(P^m(c))$ and there exists a unique totally geodesic Kähler submanifold $P^r(c)$ in $P^m(c)$ such that $p \in P^r(c)$ and $T_p(P^r(c))=O_p^1(M)$.*

Proof. At first we shall show that $O_p^1(M)$ is a Lie triple system in $T_p(P^m(c))$. Along the same line as in the proof of Lemma 13 in [7], it is sufficient to show the followings;

$$\bar{R}(X, \sigma(T, Y))Z \in O_p^1(M) \quad \text{and} \quad \bar{R}(X, \sigma(T, Y))\sigma(S, Z) \in O_p^1(M)$$

for $X, Y, Z, S, T \in T_p(M)$. By Lemma 2.1 and the condition that $J(T_p(M)) \subset N_p^1(M)$, we have

$$\bar{R}(X, \sigma(T, Y))Z = \frac{c}{4}(\langle J\sigma(T, Y), Z\rangle JX - \langle X, Z\rangle \sigma(T, Y) + 2\langle X, J\sigma(T, Y)\rangle JZ) \in N_p^1(M) .$$

Similarly we have

$$(4.1) \quad \begin{cases} \bar{R}(X, \sigma(T, Y))\sigma(S, Z) = \frac{c}{4}(\langle \sigma(T, Y), \sigma(S, Z)\rangle X \\ \qquad + \langle J\sigma(T, Y), \sigma(S, Z)\rangle JX - \langle JX, \sigma(S, Z)\rangle J\sigma(T, Y) \\ \qquad + 2\langle X, J\sigma(T, Y)\rangle J\sigma(S, Z)) \\ \bar{R}(\sigma(T, X), Y)\sigma(S, Z) = \frac{c}{4}(\langle JY, \sigma(S, Z)\rangle J\sigma(T, X) \\ \qquad - \langle \sigma(T, X), \sigma(S, Z)\rangle Y - \langle J\sigma(T, X), \sigma(S,Z)\rangle JY \\ \qquad + 2\langle \sigma(T, X), JY\rangle J\sigma(S, Z)) . \end{cases}$$

By Lemma 1.1,e), (4.1), and Lemma 2.4, we have

$$(4.2) \quad \langle JY, H\rangle J\sigma(T, X) - \langle JX, H\rangle J\sigma(T, Y) \in O_p^1(M)$$

for $X, Y, T \in T_p(M)$ and $H \in N_p^1(M)$. For any vector X there exists a nonzero vector $Y \in T_p(M)$ such that X and Y are orthogonal, since dim $M \geqq 2$. Since $J(T_p(M)) \subset N_p^1(M)$, we may substitute H for JY in (4.2). Hence we have

$$(4.3) \quad J\sigma(T, X) \in O_p^1(M) \quad \text{and thus} \quad JN_p^1(M) \subset O_p^1(M) .$$

By (4.1), we have $\bar{R}(X, \sigma(T, Y))\sigma(S, Z) \in O_p^1(M)$.

Moreover (4.3) and the condition that $J(T_p(M)) \subset N_p^1(M)$ imply that $JO_p^1(M) \subset O_p^1(M)$. Hence the second assersion is proved. q.e.d.

In the same way as Proposition 3.3, we have the following

Proposition 4.2. *If M^n is an $n(\geqq 2)$-dimensional complete nonzero isotropic $P(\boldsymbol{C})$-totally real submanifold with parallel second fundamental form in $P^m(c)$, there exists a unique totally geodesic Kähler submanifold $P^r(c)$ such that*

1) *M^n is a submanifold in $P^r(c)$*

and that

2) *$O_q^1(M) = T_q(P^r(c))$ for every point $q \in M$.*

Now we have the following fundamental

Lemma 4.3. *Let M be a totally real $\lambda(>0)$-isotropic submanifold with parallel second fundamental forms in $P^m(c)$ and let N be a totally geodesic submanifold in M. Then N is a totally real λ-isotropic submanifold with parallel second fundamental form in $P^m(c)$.*

Proof. We claim that N is a submanifold with parallel second fundamental form in $P^m(c)$. The second fundamental form for the imbedding $N \to P^m(c)$ is the restriction of σ to $T(N)\times T(N)$, so we use the same notation σ for the imbedding $N \to P^m(c)$. Denote by D^N (resp. ∇^{*N}) the normal connection on N (resp. the covariant derivation for normal bundle valued tensors on N). At first we note that the Lie triple system $T_p(M)$ in $T_p(P^m(c))$ defines a totally geodesic submanifold of constant sectional curvature $\frac{c}{4}$. Thus $T_p(N)$ is a Lie triple system in $T_p(P^m(c))$. So, by the equation of Codazzi-Mainardi, $\nabla^{*N}\sigma$ is a symmetric tensor on N. We shall show that

$$(\nabla^{*N}_X\sigma)(X, X) = 0 \quad \text{for any } X \in T_p(N)\,.$$

Let X_t be the tangent vector field of the geodesic in N starting from p with initial vector X. Then we have

$$\begin{aligned}(\nabla^{*N}_X\sigma)(X, X)) &= D^N_t\sigma(X_t, X_t)\,|_{t=0} \\ &= D_t\sigma(X_t, X_t)\,|_{t=0} - (A_{\sigma(X,X)}X)^{\perp} \\ &= -(A_{\sigma(X,X)}X)^{\perp}\end{aligned}$$

where $(*)^{\perp}$ denotes the normal component of $*$ with respect to the decomposition $T_p(M) = T_p(N) + (T_p(N))^{\perp}$. Since M is isotropic in $P^m(c)$, we have

$$\langle A_{\sigma(X,X)}X, Y\rangle = -\langle \sigma(X, X), \sigma(X, Y)\rangle = 0$$

for any vector $Y \in T_p(M)$ orthogonal to X. Hence we have $(\nabla^{*N}_X\sigma)(X, X) = 0$.
Now the other assertions are easy to see. q.e.d.

For orthonormal vectors $X, Y \in T_p(M)$, denote by $K(X, Y)$ (resp. $\bar{K}(X,Y)$) the sectional curvature of the plane spanned by X and Y for M (resp. for $\bar{M}$), and put $\Delta_{XY} = K(X, Y) - \bar{K}(X, Y)$. We call Δ the *discriminant* at $p \in M$. Then we have

Lemma 4.4 (B.O'Neill [10]). *Let M^n be a $\lambda(>0)$-isotropic submanifold in a riemannian manifold $\bar{M}$. Assume that the discriminant Δ at $p \in M$ is constant. Put $m_n = n(n+1)/2$, and $h_n = (n+2)/2(n-1)$. Then we have $-h_n\lambda^2 \leqq \Delta \leqq \lambda^2$. Furthermore, if σ_p is the second fundamental form at p, then*

(1) $\Delta = \lambda^2 \Leftrightarrow M$ *is totally umbilical at* $p \Leftrightarrow \dim N^1_p(M) = 1$
(2) $\Delta = -h_n\lambda^2 \Leftrightarrow M$ *is minimal at* $p \Leftrightarrow \dim N^1_p(M) = m_n - 1$
(3) $-h_n\lambda^2 < \Delta < \lambda^2 \Leftrightarrow \dim N^1_p(M) = m_n$.

Now we need some propositions in order to get our reduction.

Lemma 4.5. *Let T^n be an $n(\geqq 2)$-dimensional flat manifold. If T^n is a nonzero isotropic submanifold with parallel second fundamental form in $P^m(c)$, then $n\leqq 3$.*

Proof. Note that T^n is $P(\boldsymbol{C})$-totally real and that the discriminant equals negative constant $-\frac{c}{4}$. Thus by Lemma 4.4, we have

$$(4.4) \qquad \dim N_p^1(T^n) \geqq m_n - 1 .$$

On the other hand, by the equation of Ricci and Lemma 1.1, c), we have

$$\langle \bar{R}(X, Y)H, \bar{H}\rangle = -\langle [A_H, A_{\bar{H}}]X, Y\rangle$$

for $X, Y \in T_p(T^n)$ and $H, \bar{H} \in N_p^1(T^n)$. Noting that T^n is $P(\boldsymbol{C})$-totally real, by Lemma 2.1, we have

$$(4.5) \qquad [A_H, A_{\bar{H}}]X = \frac{c}{4}\langle JX, \bar{H}\rangle (JH)^T - \frac{c}{4}\langle JX, H\rangle (J\bar{H})^T$$

where $(*)^T$ denotes the $T_p(T^n)$-component of $*$. Set

$$N_p^1(T^n) = J(T_p(T^n)) + (J(T_p(T^n)))^{\perp} .$$

If $H, \bar{H} \in (J(T_p(T^n)))^{\perp}$, $[A_H, A_{\bar{H}}]=0$ by (4.5). Let $S^2(T_p(T^n))$ is the vector space of all the symmetric endomorphisms on $T_p(T^n)$ as in section 1. Then we get

$$\begin{aligned} \dim \{(J(T_p(T^n)))^{\perp}\} &\leqq \text{the dimension of a maximal abelian} \\ &\qquad \text{subspace in } S^2(T_p(T^n)) \\ &= n \end{aligned}$$

and thus

$$\dim N_p^1(T^n) \leqq 2n .$$

Together with (4.4), we see that $n=2$ or 3. q.e.d.

From now on a riemannian submanifold in a riemannian manifold is said to be *first full* if the first normal space equals the normal space at any point.

Proposition 4.6. *Let T^2 be a 2-dimensional first full λ-isotropic flat submanifold with parallel second fundamental form in $P^r(c)$. Then $r=2$, $\lambda=\frac{\sqrt{c}}{2\sqrt{2}}$. Moreover T^2 is a minimal submanifold in $P^2(c)$.*

Proof. Note that T^2 is $P(\boldsymbol{C})$-totally real. Since the discriminant Δ equals negative constant $-\frac{c}{4}$, we have $\dim N_p^1(T^2)=2$ or 3 by Lemma 4.4. If $\dim N_p^1(T^2)=3$, then $\dim P^r(c)=5$, which is a contradiction. Hence $\dim N_p^1(T^2)=2$ and thus $r=2$. Again by Lemma 4.4, T^2 is a minimal submanifold in $P^2(c)$ and $\lambda=\frac{\sqrt{c}}{2\sqrt{2}}$. q.e.d.

Proposition 4.7. *Let T^n be an $n(\geqq 2)$-dimensional flat manifold. If T^n is a nonzero isotropic submanifold with parallel second fundamental form in $P^m(c)$, then $n=2$.*

Proof. By Lemma 4.5 it is enough to see that $n\neq 3$. Suppose that $n=3$. Then by Proposition 4.2 and Lemma 4.4, T^3 is a first full minimal submanifold in $P^4(c)$. Let T^2 be a 2-dimensional totally geodesic flat submanifold in T^3. Then by Lemma 4.3 and Proposition 4.6, T^2 is a minimal submanifold in $P^4(c)$. For a point $p\in T^2$, let $\{e_1, e_2, e_3\}$ be an orthonormal basis in $T_p(T^3)$ such that $\{e_1, e_2\}$ is an orthonormal basis in $T_p(T^2)$. By the minimality of imbeddings $T^3\to P^4(c)$ and $T^2\to P^4(c)$, we have

$$\begin{cases} \sigma(e_1, e_1)+\sigma(e_2, e_2)+\sigma(e_3, e_3) = 0 \\ \sigma(e_1, e_1)+\sigma(e_2, e_2) = 0 \end{cases}$$

and thus $\sigma(e_3, e_3)=0$, which contradicts the fact that T^3 is a nonzero isotropic submanifold in $P^4(c)$. q.e.d.

Lemma 4.8 (B.Y. Chen and K. Ogiue [4]). *Let M^n be a totally real minimal submanifold immersed in $P^n(c)$. If M^n is of constant sectional curvature and has the parallel second fundamental form, then M^n is either totally geodesic or flat.*

Proposition 4.9. *Let M^n be an $n(\geqq 2)$-dimensional first full totally real nonzero isotropic submanifold with parallel second fundamental form in $P^r(c)$. Then M^n is not of rank one.*

Proof. Assume that M^n is of rank one. Let γ be a geodesic in M. Since M is of rank one, it is easy to see that there exists a 2-dimensional complete totally geodesic submanifold N^2 immersed in M which has nonzero constant sectional curvature and which contains γ (cf. [3]). By Lemma 4.3, N^2 is a nonzero isotropic totally real submanifold immersed in $P^r(c)$ with parallel second fundamental form. Suppose that the submanifold N^2 immersed in $P^r(c)$ is of type $P(\boldsymbol{C})$. Then by Lemma 4.4, N^2 is a first full minimal totally real nonzero isotropic submanifold with parallel second fundamental form in $P^2(c)$. This contradicts Lemma 4.8. Hence the submanifold N^2 immersed in $P^r(c)$ is of type $P(\boldsymbol{R})$ and thus planer geodesic. Therefore M is a planer geodesic submanifold in $P^r(c)$. By Lemma 3.7, the imbedding $M \rightarrow P^r(c)$ is of type $P(\boldsymbol{R})$. This is a contradiction. q.e.d.

Proposition 4.10. *Under the assumption of Proposition* 4.9, *a riemannian locally symmetric space M^n has not noncompact factors.*

Proof. Assume that M has a noncompact factor. Then it is easy to see that there exists a two dimensional totally geodesic submanifold N^2 in M of constant negative curvature [cf. [3]). By Lemma 4.3, N^2 is a nonzero isotropic submanifold with parallel second fundamental form in $P^r(c)$. Since N^2 is of constant negative sectional curvature, we see that the discriminant Δ of the imbedding $N^2 \rightarrow P^r(c)$ is not more than $-\frac{c}{4}$. This contradicts Lemma 4.4, since $\lambda = \frac{\sqrt{c}}{2\sqrt{2}}$ by Proposition 4.6, 4.7, 4.9. q.e.d.

Let M^n be an $n(\geqq 2)$-dimensional complete nonzere isotroplc $P(\boldsymbol{C})$-totally real submanifold with parallel second fundamental form in $P^m(c)$. Then by Proposition 4.7, 4, 9, and 4.10, M^n is a riemannian locally symmetric space of rank two and without noncompact factors. We shall consider the submanifold M^n in detail.

Proposition 4.11. *Let N be a riemannian locally symmetric space locally isometric to one of the following riemanniann symmetric spaces $S^1 \times P^2(\boldsymbol{C})$, $S^1 \times P^2(\boldsymbol{H})$, or $S^1 \times P^2(Ca)$. Then N can not be locally imbedded in $P^m(c)$ as a nonzero isotropic $P(\boldsymbol{C})$-totally real submanifold with parallel second fundmental form.*

Proof. We consider the case when N is locally isometric to $S^1 \times P^2(\boldsymbol{C})$. Suppose that N is locally imbedded in $P^m(c)$ as a nonzeao isotropic $P(\boldsymbol{C})$-totally real submanifold with parallel second fundamental form. Then we may assume that the above imbedding is first full in $P^r(c)$ by Proposition 4.2. Moreover by Lemma 4.3, the local imbedding $P^2(\boldsymbol{C}) \rightarrow S^1 \times P^2(\boldsymbol{C}) \rightarrow P^r(c)$ is nonzero isotropic totally real, and has parallel second fundamental forms. Since $P^2(\boldsymbol{C})$ is of rank one, the local imbedding is planer geodesic and not totally geodesic by Proposition 4.9. By Lemma 3.5, the dimension of the first normal space $N^1_{P^2(C)}$ of $P^2(\boldsymbol{C})$ equals either 3 or 4. Suppose that

$$\text{(4.7)} \qquad \dim N^1_{P^2(C)} = 3 \ (\text{resp.} \dim N^1_{P^2(C)} = 4)\,.$$

Then the first osculating space of the imbedding $P^2(\boldsymbol{C}) \rightarrow P^r(c)$ is a Lie triple system which defines the unique totally geodesic submanifold $P^7(\boldsymbol{R})$ (resp. $P^8(\boldsymbol{R})$), and hence we have $7 \leqq r$ (resp. $8 \leqq r$). On the other hand we have the local imbedding $P^2(\boldsymbol{C}) \rightarrow S^1 \times P^2(\boldsymbol{C}) \rightarrow P^r(c)$, and thus by (4.7) we see that $\dim O^1_{S^1 \times P^2(C)} \leqq 13$ (resp. $\dim O^1_{S^1 \times P^2(C)} \leqq 14$) and thus $r \leqq 6$ (resp. $r \leqq 7$).

This is a contradiction.

The other cases can be proved by the same way. q.e.d.

Proposition 4.12. *Let N be a riemannian locally symmetric space locally isometric to the riemannian symmetric space $S^2 \times S^2$. Then N can not be locally imbedded in $P^m(c)$ as a nonzero isotropic $P(\boldsymbol{C})$-totally real submanifold with parallel second fundmeutal from.*

Proof. Assume that N is locally imbedded in $P^m(c)$ as a nonzero isotropic $P(\boldsymbol{C})$-totally real submanifold with parallel second funpamental form. Take a point $p \in N$ and identify p with a point $(p_1, p_2) \in S^2 \times S^2$. Then the tangent space $T_p(S^2 \times S^2)$ is decomposed into two orthogonal subspaces $T_{p_1}(S^2)$ and $T_{p_2}(S^2)$. Let X (resp. Y) be a unit vector in $T_{p_1}(S^2)$ (resp. $T_{p_2}(S^2)$). Then the Lie triple system $\{X, Y\}_{\boldsymbol{R}}$ in $T_p(S^2 \times S^2)$ defines a two dimensional flat totally geodesic submanifold in N. Hence by Lemma 4.3 and Proposition 4.6, we have

$$(4.8) \qquad \sigma(X, X) + \sigma(Y, Y) = 0 .$$

We may assume that the above imbedding is first full in $P^r(c)$ since the symmetric space $S^2 \times S^2$ is of rank two. Note that $\sigma(X, X) = \sigma(Z, Z)$ for all unit vectors $X, Z \in T_{p_j}(S^2)$ $(j=1, 2)$ by (4.8). Then we have $2r = \dim O_p^1(S^2 \times S^2) \leqq 9$ and thus $r \leqq 4$. On the other hand since the Lie triple system $T_p(S^2 \times S^2)$ in $T_p(P^r(c))$ defines a unique totally geodesic submanifold $P^4(\boldsymbol{R})$ in $P^r(c)$, we have $4 \leqq r$ and thus $r=4$. Again by (4.8) there exists a nonzero vector H such that

$$\{\sigma(X, X), \sigma(Y, Y); X \in T_{p_1}(S^2), Y \in T_{p_2}(S^2)\}_{\boldsymbol{R}} = \{H\}_{\boldsymbol{R}} .$$

Since two totally geodesic submanifolds S^2 in $S^2 \times S^2$ are planer geodesic and not totally geodesic in $P^4(c)$ by Proposition 4.9, we have $J(T_{p_1}(S^2)) \perp H$ and $J(T_{p_2}(S^2)) \perp H$ and thus $J(T_p(S^2 \times S^2)) \perp H$. This is a contradiction to our assumption that N is $P(\boldsymbol{C})$-totally real. q.e.d.

Now B.Y. Chen and T. Nagano [3] have classified the maximal totally geodesic submanifolds in irreducible compact riemannian symmetric spaces of rank two. Their classification make a mistake for the riemannian symmetric space $SU(3)$. (Table VIII in [3] shows that the space $SU(2) \times SU(2)$ is totally geodesic in $SU(3)$.) But along their arguments we can see that the riemannian symmetric spaces $S^1 \times P^2(\boldsymbol{C})$, $S^1 \times P^2(\boldsymbol{H})$, $S^1 \times P^2(Ca)$, $S^2 \times S^2$ can not be locally imbedded in $SU(3)$ as totally geodesic submanifolds. Together with their classification for the other spaces we see that every space except the following spaces $SU(3)/SO(3)$, $SU(3)$, $SU(6)/Sp(3)$, E_6/F_4 contains one of the above four spaces immersed totally geodesic submanifolds. Then, by Lemma 4.3 and Proposition 4.7, 4.9, 4.10, 4.11, 4.12, we have the following

Theorem 4.13. *Let M^n be an $n(\geqq 2)$-dimensional complete $P(\boldsymbol{C})$-totally real*

$\lambda(>0)$-isotropic submanifold with parallel setond fundamental from in $P^m(c)$, Tnen the submanifold M is locally isometric to one of the riemannian symmetric spaces; $S^1\times S^{n-1}(n\geqq 2)$, $SU(3)/SO(3)$, $SU(3)$, $SU(6)/Sp(3)$, E_6/F_4. Moreover the constant λ equals $\frac{\sqrt{c}}{2\sqrt{2}}$.

Proof. The second statement follows from Proposition 4.6. q.e.d.

5. $P(C)$-totally real $\frac{\sqrt{c}}{2\sqrt{2}}$-isotropic flat submanifolds with parallel second fundamental from in $P^m(c)$

Let M^2 be a 2-dimensional complete first full $P(C)$-totally real $\frac{\sqrt{c}}{2\sqrt{2}}$-isotropic flat submanifold with parallel second fundamental form in $P^r(c)$. Then by Proposition 4.6, $r=2$. In this section we shall construct such sumbanifolds.

At first we study isometric equivariant imbeddings of riemannian locally symmetric spaces into a complex projective space $P^m(c)$.

Let $\bar{G}=SU(m+1)$ be the special unitary group and set

$$\bar{K}=\left\{\left(\begin{array}{c|c}\alpha & 0\cdots 0\\ \hline 0 & \\ \vdots & A\\ 0 & \end{array}\right)\in SU(m+1);\ \alpha\in U(1), A\in U(m)\right\}.$$

Fix a $\bar{G}$-invariant metric $\langle\ ,\ \rangle$ on the homogeneous space $\bar{G}/\bar{K}=P^m(C)$ induced from a bi-invariant metric on $\bar{G}$. Then the riemannian manifold $(P^m(C), \langle\ ,\ \rangle)$ has constant positive holomorphic sectional curvature. Let $\bar{\mathfrak{g}}$ (resp. $\bar{\mathfrak{k}}$) be the Lie algebra of $\bar{G}$ (resp. $\bar{K}$) and $\bar{\mathfrak{g}}=\bar{\mathfrak{k}}+\bar{\mathfrak{p}}$ be the canonical decomposition. Then we can identify the tangent space $T_{\bar{o}}(\bar{G}/\bar{K})$ with the vector space $\bar{\mathfrak{p}}$ canonically, where $\bar{o}=e\bar{K}$. Let G be a connected compact Lie group and ρ an injective homomorphism of G into $\bar{G}$. Then the imbedding f of the homogeneous space $M=G/K$ into $P^m(C)$ is induced as follows;

$$f(gK)=\rho(g)\bar{K}\quad\text{for any } g\in G$$

where $K=\rho^{-1}(\bar{K})$. Moreover when we take the metric on M induced from the metric on $P^m(C)$, the imbedding f is G-equivariant and isometric. Let $\mathfrak{g}$ (resp. $\mathfrak{k}$) be the Lie algebra of G (resp. K).

From now on we assume that $\mathfrak{g}$ is an orthogonal symmetric Lie algebra with the subalgebra $\mathfrak{k}$ as the fixed points of the involution. Then M is a complete riemannian locally symmetric space, and we have the canonical decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ and identify the tangent space $T_0(M)$ at $o=eK$ with the vector space $\mathfrak{p}$ canonically. Let $\bar{\mathfrak{m}}$ be a subspace in $\bar{\mathfrak{p}}$ consisting of $\bar{\mathfrak{p}}$-components of elements in $d\rho(\mathfrak{p})$ with respect to the decomposition $\bar{\mathfrak{g}}=\bar{\mathfrak{k}}+\bar{\mathfrak{p}}$ and let $\bar{\mathfrak{m}}^\perp$ be the orthogonal complement of $\bar{\mathfrak{m}}$ in $\bar{\mathfrak{p}}$. Then we may regard the second fundamental form σ_0 at o of the imbedding f as an element in $S^2(\mathfrak{p}^*)\otimes\bar{\mathfrak{m}}^\perp$.

Proposition 5.1. *For $X, Y \in \mathfrak{p}$,*

$$\sigma_0(X, Y) = ([d\rho(X)_{\bar{\mathfrak{k}}}, d\rho(Y)_{\bar{\mathfrak{p}}}])_{\bar{\mathfrak{m}}}{}^{\perp}$$

where $(*)_{\bar{\mathfrak{k}}}$ (resp. $(*)_{\bar{\mathfrak{p}}}$) denotes the $\bar{\mathfrak{k}}$-component (resp. $\bar{\mathfrak{p}}$-component) of $*$ with respect to the decomposition $\bar{\mathfrak{g}}=\bar{\mathfrak{k}}+\bar{\mathfrak{p}}$, and $(*)_{\bar{\mathfrak{m}}}{}^{\perp}$ denotes the $\bar{\mathfrak{m}}^{\perp}$-component of $*$ with respectto the decomposition $\bar{\mathfrak{p}}=\bar{\mathfrak{m}}+\bar{\mathfrak{m}}^{\perp}$.

Proof. Let A^* (resp. $\bar{B}^*$) be the Killing vector field of M (resp. of $P^m(\boldsymbol{C})$) generated by $A \in \mathfrak{p}$ (resp. $\bar{B} \in \bar{\mathfrak{g}}$). For $X, Y \in \mathfrak{p}$ and $\bar{H} \in \bar{\mathfrak{m}}^{\perp}$,

$$\begin{aligned}
&\langle \sigma_0(X, Y), \bar{H} \rangle = \langle \bar{\nabla}_{d\rho(X)^*} d\rho(Y)^*, \bar{H}^* \rangle_0 \\
&= \frac{1}{2}\{\langle [\bar{H}, d\rho(X)]^*, d\rho(Y)^* \rangle + \langle [\bar{H}, d\rho(Y)]^*, d\rho(X)^* \rangle \\
&\quad + \langle [d\rho(X), d\rho(Y)]^*, \bar{H}^* \rangle\}_0 \\
&= \frac{1}{2}\langle [d\rho(X), d\rho(Y)_{\bar{\mathfrak{p}}}] + [d\rho(Y), d\rho(X)_{\bar{\mathfrak{p}}}], \bar{H} \rangle \\
&= \frac{1}{2}\langle [d\rho(X)_{\bar{\mathfrak{k}}}, d\rho(Y)_{\bar{\mathfrak{p}}}] + [d\rho(Y)_{\bar{\mathfrak{k}}}, d\rho(X)_{\bar{\mathfrak{p}}}], \bar{H} \rangle
\end{aligned}$$

since $\mathfrak{g}$ is an orthogonal symmetric Lie algebra with the compactly imbedded subalgebra $\mathfrak{k}$. Thus we have

$$\sigma_0(X, Y) = \frac{1}{2}\{[d\rho(X)_{\bar{\mathfrak{k}}} d\rho(Y)_{\bar{\mathfrak{p}}}] + [d\rho(Y)_{\bar{\mathfrak{k}}}, d\rho(X)_{\bar{\mathfrak{p}}}]\}_{\bar{\mathfrak{m}}}{}^{\perp}.$$

Noting that $(\mathfrak{g}, \mathfrak{k})$ and $(\bar{\mathfrak{g}}, \bar{\mathfrak{k}})$ are orthogonal symmetric Lie algebras such that $d\rho(\mathfrak{k}) \subset \bar{\mathfrak{k}}$, we have $d\rho([X, Y]) \in \bar{\mathfrak{k}}$ for any $X, Y \in \mathfrak{p}$ and hence

$$[d\rho(X)_{\bar{\mathfrak{k}}}, d\rho(Y)_{\bar{\mathfrak{p}}}] + [d\rho(X)_{\bar{\mathfrak{p}}}, d\rho(Y)_{\bar{\mathfrak{k}}}] = 0.$$

Thus

$$\sigma_0(X, Y) = [d\rho(X)_{\bar{\mathfrak{k}}}, d\rho(Y)_{\bar{\mathfrak{p}}}]_{\bar{\mathfrak{m}}}{}^{\perp}.$$

q.e.d.

Proposition 5.2. *The imbedding f of M into $P^m(\boldsymbol{C})$ has the parallel second fundamental form if and only if the following conditions are satisfied;*

$$[d\rho(X)_{\bar{\mathfrak{p}}}, [d\rho(Y)_{\bar{\mathfrak{p}}}, d\rho(Z)_{\bar{\mathfrak{p}}}]] \in \bar{\mathfrak{m}}$$

and

$$[d\rho(X)_{\bar{\mathfrak{k}}}, [d\rho(X)_{\bar{\mathfrak{k}}}, d\rho(X)_{\bar{\mathfrak{p}}}]] \in \bar{\mathfrak{m}}$$

for any $X, Y, Z \in \mathfrak{p}$.

Proof. Since f is a G-equivariant, we may consider only at the point o. By the equation of Codazzi-Mainardi, the first condition implies that $(\nabla^*\sigma)_0 \in S^3(\mathfrak{p}^*) \otimes \bar{\mathfrak{m}}^{\perp}$. Since the integral curve of X^* through o for $X \in \mathfrak{p}$ is a geodesic in M, we have

$$\langle(\nabla^*\sigma)_0(X, X, X), \bar{H}\rangle = \langle D_{X^*}(\sigma(X^*, X^*)) - 2\sigma(\nabla_{X^*}X^*, X^*), \bar{H}^*\rangle_0$$
$$= \langle D_{X^*}(\sigma(X^*, X^*)), \bar{H}^*\rangle_0 = \langle\bar{\nabla}_{d\rho(X)^*}\bar{\nabla}_{d\rho(X)^*}d\rho(X)^*, \bar{H}^*\rangle_0$$
$$= \{d\rho(X)^*\langle\bar{\nabla}_{d\rho(X)^*}d\rho(X)^*, \bar{H}^*\rangle\}_0 - \langle\bar{\nabla}_{d\rho(X)^*}d\rho(X)^*, \bar{\nabla}_{d\rho(X)^*}\bar{H}^*\rangle_0$$

for $\bar{H}\in\bar{\mathfrak{m}}^\perp$. Note that $d\rho(X)^*$ and $\bar{H}^*$ are Killing vector fields of $P^m(C)$. By the same calculation as in Proposition 5.1, we have

$$\text{the first term of the right hand} = \langle[d\rho(X)_{\bar{\mathfrak{k}}}, [d\rho(X)_{\bar{\mathfrak{k}}}, d\rho(X)_{\bar{\mathfrak{k}}}]], \bar{H}\rangle$$

and

$$\text{the second term of the right hand} = -\langle[d\rho(X)_{\bar{\mathfrak{k}}}, [d\rho(X)_{\bar{\mathfrak{k}}}, d\rho(X)_{\bar{\mathfrak{p}}}]], \bar{H}\rangle\,.$$

Thus

$$(\nabla^*\sigma)_0(X, X, X) = 2[d\rho(X)_{\bar{\mathfrak{k}}}, [d\rho(X)_{\bar{\mathfrak{k}}}, d\rho(X)_{\bar{\mathfrak{p}}}]]_{\bar{\mathfrak{m}}}^\perp\,.$$

So the second condition implies that $(\nabla^*\sigma)_0(X, X, X)=0$ for any $X\in\mathfrak{p}$. The converse follows by the same way. q.e.d.

Now we consider the case of $P^2(\boldsymbol{C})$. Set

$$\bar{X}_2=\begin{pmatrix}0&i&0\\i&0&0\\0&0&0\end{pmatrix},\quad \bar{X}_3=\begin{pmatrix}0&0&i\\0&0&0\\i&0&0\end{pmatrix},\quad \bar{H}_2=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix},\quad \bar{H}_3=\begin{pmatrix}0&0&-1\\0&0&0\\1&0&0\end{pmatrix}$$
$$\bar{Q}=\begin{pmatrix}0&0&0\\0&0&i\\0&i&0\end{pmatrix},\quad \bar{R}=\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix},\quad \bar{S}=\begin{pmatrix}i&0&0\\0&i&0\\0&0&-2i\end{pmatrix},\quad \bar{T}=\begin{pmatrix}i&0&0\\0&-i&0\\0&0&0\end{pmatrix}.$$

Then $\bar{\mathfrak{k}}=\{\bar{Q}, \bar{R}, \bar{S}, \bar{T}\}_{\boldsymbol{R}}$ and $\bar{\mathfrak{p}}=\{\bar{X}_j\,\bar{H}_j;\, j=2, 3\}_{\boldsymbol{R}}$. Moreover the bracket relation $[\bar{A}, \bar{B}]$ is given by the following table 1.

Table 1.

$\bar{A}$ \ $\bar{B}$	$\bar{X}_2$	$\bar{X}_3$	$\bar{H}_2$	$\bar{H}_3$	$\bar{Q}$	$\bar{R}$	$\bar{S}$	$\bar{T}$
$\bar{X}_2$	0	$\bar{R}$	$2\bar{T}$	$-\bar{Q}$	$\bar{H}_3$	$-\bar{X}_3$	0	$-2\bar{H}_2$
$\bar{X}_3$		0	$-\bar{Q}$	$\bar{S}+\bar{T}$	$\bar{H}_2$	$\bar{X}_2$	$-3\bar{H}_3$	$-\bar{H}_3$
$\bar{H}_2$			0	$\bar{R}$	$-\bar{X}_3$	$-\bar{H}_3$	0	$2\bar{X}_2$
$\bar{H}_3$				0	$-\bar{X}_2$	$\bar{H}_2$	$3\bar{X}_3$	$\bar{X}_3$
$\bar{Q}$					0	$\bar{S}-\bar{T}$	$-3\bar{R}$	$\bar{R}$
$\bar{R}$						0	$3\bar{Q}$	$-\bar{Q}$
$\bar{S}$							0	0
$\bar{T}$								0

Note that the almost complex structure J on $\bar{\mathfrak{p}}$ is given by

$$J = ad\left(\frac{1}{6}(\bar{S}+3\bar{T})\right)$$

and that

$$J\bar{X}_2 = \bar{H}_2\,, \quad J\bar{H}_2 = -\bar{X}_2\,, \quad J\bar{X}_3 = \bar{H}_3\,, \quad J\bar{H}_3 = -\bar{X}_3\,.$$

Now we define an inner product on $\bar{\mathfrak{g}}$ by

$$\langle \bar{A}, \bar{B} \rangle = \frac{2}{c} \operatorname{trace} \bar{A} \cdot \bar{B}^*$$

for $\bar{A}, \bar{B} \in \bar{\mathfrak{g}}$. Then this inner product induces the metric of $P^2(\boldsymbol{C})$ of constant holomorphic sectional curvature c.

Now we shall find out our examples in orbit spaces of maximal tori of $\bar{G}=SU(3)$. Let $\bar{A}$ (resp. $\bar{B}$) be a unit vector in $\bar{\mathfrak{g}}$ such that

$$\bar{A}_{\bar{\mathfrak{k}}} = \alpha\bar{S}+\beta\bar{T}+\gamma\bar{R}+\delta\bar{Q} \text{ (resp. } \bar{B}_{\bar{\mathfrak{k}}} = \tilde{\alpha}\bar{S}+\tilde{\beta}\bar{T}+\tilde{\gamma}\bar{R}+\tilde{\delta}\bar{Q})$$

and

$$\bar{A}_{\bar{\mathfrak{p}}} = \bar{X}_2 \text{ (resp. } \bar{B}_{\bar{\mathfrak{p}}} = \bar{X}_3)$$

where $\alpha, \beta, \gamma, \delta, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta} \in \boldsymbol{R}$. Set $\bar{\mathfrak{a}}(\bar{A}, \bar{B}) = \{\bar{A}, \bar{B}\}_{\boldsymbol{R}}$. Then, by Table 1, we have the following

Lemma 5.3. *The followings are equivalent*:

(1) *The vector subspace $\bar{\mathfrak{a}}(\bar{A}, \bar{B})$ is an abelian subalgebra in $\bar{\mathfrak{g}}$.*

(2) $$\begin{cases} \gamma=\tilde{\gamma}=0, \ \delta=-2\tilde{\beta}, \ 3\alpha+\beta+\tilde{\delta}=0, \\ 1+3\alpha\tilde{\delta}-\beta\tilde{\delta}-3\delta\tilde{\alpha}+\delta\tilde{\beta}=0. \end{cases}$$

Let $T(\bar{A}, \bar{B})$ be the maximal torus in $\bar{G}$ with the abelian Lie algebra $\bar{\mathfrak{a}}(\bar{A}, \bar{B})$ and $\Gamma(\bar{A}, \bar{B})$ the discrete subgroup in $T(\bar{A}, \bar{B})$ defined by

$$\Gamma(\bar{A}, \bar{B}) = \{t \in T(\bar{A}, \bar{B});\, t\cdot\bar{o}=\bar{o}\}\,.$$

Then the homogeneous space $M(\bar{A}, \bar{B})=T(\bar{A}, \bar{B})/\Gamma(\bar{A}, \bar{B})$ is an abelian Lie group. Since the imbedding $f_{(\bar{A},\bar{B})}\colon M(\bar{A}, \bar{B}) \to P^2(c)$ is $T(\bar{A}, \bar{B})$-equivariant, the induced metric $\langle\ ,\ \rangle_{(\bar{A},\bar{B})}$ on $M(\bar{A}, \bar{B})$ is flat. Moreover we have the following

Lemma 5.4. *The imbedding $f_{(\bar{A},\bar{B})}$ of the compact flat manifold $M(\bar{A}, \bar{B})$ into $P^2(c)$ is minimal if and only if the following conditions are satisfied*;

$$\begin{cases} \beta = -\alpha, \quad \gamma = 0, \quad \delta = 2\tilde{\alpha} \\ \tilde{\beta} = -\tilde{\alpha}, \quad \tilde{\gamma} = 0, \quad \tilde{\delta} = -2\alpha \quad \textit{and} \quad 8\alpha^2+8\tilde{\alpha}^2 = 1\,. \end{cases}$$

Proof. Since $f_{(\bar{A},\bar{B})}$ is $T(\bar{A},\bar{B})$-equivariant, it is enough to see our claim at the point $o=e\Gamma(\bar{A},\bar{B})$. Note that vectors $\bar{A}$ and $\bar{B}$ are orthogonal and have the same length with respect to $\langle\ ,\ \rangle_{(\bar{A},\bar{B})}$, and that $\bar{\mathfrak{m}}=\{\bar{X}_2, \bar{X}_3\}_{\boldsymbol{R}}$ and $\bar{\mathfrak{m}}^{\perp}=\{\bar{H}_2, \bar{H}_3\}_{\boldsymbol{R}}$. Hence by Proposition 5.1, the imbedding $f_{(\bar{A},\bar{B})}$ is minimal if and only if the following condition is satisfied;

$$\{[\bar{A}_{\bar{\mathfrak{k}}}, \bar{X}_2]+[\bar{B}_{\bar{\mathfrak{k}}}, \bar{X}_3]\}_{\bar{\mathfrak{m}}}^{\perp} = 0\,.$$

By Table 1, we see that this is equivalent to

$$(5.1)\qquad -\tilde{\delta}+2\beta = 0 \quad\text{and}\quad -\delta+3\tilde{\alpha}+\tilde{\beta} = 0\,.$$

Now our claim follows from (5.1) and Lemma 5.3, (2). q.e.d.

Now put $\alpha=\frac{1}{2\sqrt{2}}\cos t$ and $\tilde{\alpha}=-\frac{1}{2\sqrt{2}}\sin t$, and for simplicity denote the vectors

$$\bar{A}=\begin{pmatrix} 0 & i & 0 \\ i & \frac{1}{2}i\cos t & \frac{1}{2}i\sin t \\ 0 & \frac{1}{2}i\sin t & -\frac{1}{2}i\cos t \end{pmatrix},\quad \bar{B}=\begin{pmatrix} 0 & 0 & i \\ 0 & \frac{1}{2}i\sin t & -\frac{1}{2}i\cos t \\ i & -\frac{1}{2}i\cos t & -\frac{1}{2}i\sin t \end{pmatrix}$$

by $\bar{A}_t$, $\bar{B}_t$ respectively, the abelian subalgebra $\bar{\mathfrak{a}}(\bar{A},\bar{B})$ by $\bar{\mathfrak{a}}_t$, the maximal torus $T(\bar{A},\bar{B})$ by T_t, the discrete subgroup $\Gamma(\bar{A},\bar{B})$ by Γ_t, the compact flat manifold $M(\bar{A},\bar{B})$ by M_t, the minimal imbedding $f_{(\bar{A},\bar{B})}$ by f_t, and so on.

Lemma 5.5. *The minimal imbedding f_t: $M_t\to P^2(c)$ has the parallel second fundamental form σ_t.*

Proof. Since f_t is T_t-equivariant, it is enough to see our claim at o_t. By Table 1, $\bar{\mathfrak{m}}=\{\bar{X}_2, \bar{X}_3\}_{\boldsymbol{R}}$ is a Lie triple system in $\bar{\mathfrak{p}}$ and hence the first condition of Proposition 5.2 is satisfied. Again by Table 1, we have

$$[(\bar{A})_{\bar{\mathfrak{k}}}, (\bar{A}_t)_{\bar{\mathfrak{p}}}],\ [(\bar{A}_t)_{\bar{\mathfrak{k}}}, (\bar{B}_t)_{\bar{\mathfrak{p}}}],\ [(\bar{B}_t)_{\bar{\mathfrak{k}}}, (\bar{A}_t)_{\bar{\mathfrak{p}}}]\,,\ [(\bar{B}_t)_{\bar{\mathfrak{k}}}, (\bar{B}_t)_{\bar{\mathfrak{p}}}]\in\{\bar{H}_2, \bar{H}_3\}_{\boldsymbol{R}}$$

and

$$[(\bar{A}_t)_{\bar{\mathfrak{k}}}, \bar{H}_2],\ [(\bar{A}_t)_{\bar{\mathfrak{k}}}, \bar{H}_3],\ [(\bar{B}_t)_{\bar{\mathfrak{k}}}, \bar{H}_2],\ [(\bar{B}_t)_{\bar{\mathfrak{k}}}, \bar{H}_3]\in\{\bar{X}_2, \bar{X}_3\}_{\boldsymbol{R}}\,.$$

Thus we get

$$[(\lambda\bar{A}_t+\mu\bar{B}_t)_{\bar{\mathfrak{k}}}, [(\lambda\bar{A}_t+\mu\bar{B}_t)_{\bar{\mathfrak{k}}}, (\lambda\bar{A}_t+\mu\bar{B}_t)_{\bar{\mathfrak{p}}}]]\in\bar{\mathfrak{m}}$$

for λ, $\mu\in\boldsymbol{R}$. by Proposition 5.2, σ_t is parallel. q.e.d.

Lemma 5.6. *The minimal imbedding f_t: $M_t \to P^2(c)$ is totally real first full $\frac{\sqrt{c}}{2\sqrt{2}}$-isotropic.*

Proof. Since f_t is T_t-equivariant, it is enough to see our claim at o_t. The imbedding f_t is totally real at o_t since $J\bar{\mathfrak{m}}=\bar{\mathfrak{m}}^{\perp}$.

Now by Proposition 5.1 and Table 1, we have

$$(5.2)\qquad \begin{aligned}&\sigma_t(\cos\theta\bar{A}_t+\sin\theta\bar{B}_t,\ \cos\theta\bar{A}_t+\sin\theta\bar{B}_t)\\ &= -\frac{1}{\sqrt{2}}(\cos(t-2\theta)\bar{H}_2+\sin(t-2\theta)\bar{H}_3)\,.\end{aligned}$$

Thus f_t is the first full imbedding. Moreover we have

$$|\sigma_t(\cos\theta\bar{A}_t+\sin\theta\bar{B}_t,\ \cos\theta\bar{A}_t+\sin\theta\bar{B}_t)| = \sqrt{\frac{2}{c}}$$

while

$$|\cos\theta\bar{A}_t+\sin\theta\bar{B}_t|_t = |\cos\theta\bar{X}_2+\sin\theta\bar{X}_3| = \frac{2}{\sqrt{c}}.$$

Thus the imbedding f_t is $\frac{\sqrt{c}}{2\sqrt{2}}$-isotropic. q.e.d.

Moreover, by (5.2), note that the second fundamental form σ_t at o_t is given by

$$(5.3)\qquad \begin{cases}\sigma_t(\bar{A}_t,\bar{A}_t) = -\dfrac{1}{\sqrt{2}}(\cos t\bar{H}_2+\sin t\bar{H}_3)\\[2ex] \sigma_t(\bar{B}_t,\bar{B}_t) = \dfrac{1}{\sqrt{2}}(\cos t\bar{H}_2+\sin t\bar{H}_3)\\[2ex] \sigma_t(\bar{B}_t,\bar{A}_t) = \dfrac{1}{\sqrt{2}}(\cos t\bar{H}_2-\sin t\bar{H}_3)\end{cases}$$

Lemma 5.7. *The discrete subgroup Γ_t is given by*

$$\Gamma_t = \{aE_3;\ a\in \boldsymbol{C},\ a^3=1\}$$

where E_3 denotes the unit element in $SU(3)$.

Proof. Take $g\in\Gamma_t$. Since $g\in T_t$, $Ad(g)|\bar{\mathfrak{a}}_t=id|\bar{\mathfrak{a}}_t$ and thus $Ad(g)\bar{A}_t=\bar{A}_t$ and $Ad(g)\bar{B}_t=\bar{B}_t$. Also since $g\in\bar{K}$, $Ad(g)\bar{\mathfrak{k}}\subset\bar{\mathfrak{k}}$ and $Ad(g)\bar{\mathfrak{p}}\subset\bar{\mathfrak{p}}$. Hence we have

$$(5.4)\qquad Ad(g)\bar{X}_2=\bar{X}_2 \quad\text{and}\quad Ad(g)\bar{X}_3=\bar{X}_3\,.$$

By (5.4) and the condition that $g\in\bar{K}$, we have

$$g \in \{aE_3;\ a \in \boldsymbol{C},\ a^3 = 1\}\ .$$

Conversely, noting that the subgroup $\{aE_3;\ a \in \boldsymbol{C},\ a^3=1\}$ in $\bar{K}$ is the center in $SU(3)$, we have

$$\{aE_3;\ a \in \boldsymbol{C},\ a^3 = 1\} \subset \bar{K} \cap T_t = \Gamma_t\,. \qquad \text{q.e.d.}$$

Summing up Lemma 5.3, 5.4, 5.5, 5.6, 5.7 and (5.3), we have the following

Theorem 5.8. *For a real number t, the minimal imbedding f_t of the compact flat manifold $M_t = T_t/\Gamma_t$ into $P^2(c)$ is first full $\frac{\sqrt{c}}{2\sqrt{2}}$-isotropic $P(\boldsymbol{C})$-totally real and has the parallel second fundamental form. Moreover the second fundamental form σ_t is given by* (5.3) *and the discrete subgroup Γ_t is the center of $SU(3)$.*

Now we can write down the minimal imbedding f_0: $M_0 \to P^2(c)$ explicitly. When $t=0$, we have

$$\bar{A}_0 = \begin{pmatrix} 0 & i & 0 \\ i & \frac{1}{\sqrt{2}}i & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}}i \end{pmatrix} \quad \text{and} \quad \bar{B}_0 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & -\frac{1}{\sqrt{2}}i \\ i & -\frac{1}{\sqrt{2}}i & 0 \end{pmatrix}.$$

Put

$$P = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{\sqrt{2}}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Then we have

$$^{t}P\bar{A}_0P = \begin{pmatrix} \sqrt{2}i & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}}i & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}}i \end{pmatrix}, \quad ^{t}P\bar{B}_0P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{\sqrt{2}}i & 0 \\ 0 & 0 & -\frac{3}{\sqrt{2}}i \end{pmatrix}$$

and thus

$$\exp(\lambda\bar{A}_0 + \mu\bar{B}_0) = P\begin{pmatrix} e^{\sqrt{2}\lambda i} & 0 & 0 \\ 0 & e^{-\frac{\lambda}{\sqrt{2}}i + \frac{\sqrt{3}}{\sqrt{2}}\mu i} & 0 \\ 0 & 0 & e^{-\frac{\lambda}{\sqrt{2}}i - \frac{\sqrt{3}}{\sqrt{2}}\mu i} \end{pmatrix} {}^{t}P = P\begin{pmatrix} e^{-(x+y)i} & 0 & 0 \\ 0 & e^{xi} & 0 \\ 0 & 0 & e^{yi} \end{pmatrix} {}^{t}P$$

where $x=-\frac{\lambda}{\sqrt{2}}+\frac{\sqrt{3}}{\sqrt{2}}\mu$ and $y=-\frac{\lambda}{\sqrt{2}}-\frac{\sqrt{3}}{\sqrt{2}}\mu$. Let $\pi\colon S^5\to P^2(c)$ be the Hopf fibring. Then we have

$$(5.5)\qquad f_0(M_0)=\pi\left\{\begin{pmatrix}\frac{1}{3}(e^{-(x+y)i}+e^{xi}+e^{yi})\\ \frac{2}{3\sqrt{2}}(2e^{-(x+y)i}-e^{xi}-e^{yi})\\ \frac{1}{\sqrt{6}}(e^{xi}-e^{yi})\end{pmatrix}\in \boldsymbol{C}^3;\ x, y\in\boldsymbol{R}\right\}.$$

6. $P(C)$-totally real $\frac{\sqrt{c}}{2\sqrt{2}}$-isotropic submanifolds with parallel second fuudamental form in $P^m(c)$ which are locally isometric to the riemannian symmetric space $S^1\times S^n$

In this section we construct the model of $(n+1)$-dimensional complete $P(C)$-totally real $\frac{\sqrt{c}}{2\sqrt{2}}$-isotropic submanifold with parallel second fundamental form in $P^{n+1}(c)$ which are locally isometric to the riemannian symmetric space $S^1\times S^n$.

Let $\bar{\mathfrak{g}}$ be a Lie algebra of $\bar{G}=SU(n+2)$, $\bar{\mathfrak{k}}$ be a Lie algebra of the Lie subgroup $\bar{K}=S(U(1)\times U(n+1))$ in $\bar{G}$ and $\bar{\mathfrak{g}}=\bar{\mathfrak{k}}+\bar{\mathfrak{p}}$ be the canonical decomposition. Then we have

$$\bar{\mathfrak{p}}=\left\{\begin{pmatrix}0 & a_1+ib_1\cdots a_{n+2}+ib_{n+2}\\ -a_1+ib_1 & \\ \vdots & 0\\ -a_{n+2}+ib_{n+2} & \end{pmatrix};\ a_j, b_j\in\boldsymbol{R}\right\}.$$

Moreover $\bar{\mathfrak{p}}$ is a direct sum of $\bar{\mathfrak{p}}^R$ and $\bar{\mathfrak{p}}^I$, where

$$\bar{\mathfrak{p}}^R=\left\{\begin{pmatrix}0 & a_1\cdots a_{n+2}\\ -a_1 & \\ \vdots & 0\\ -a_{n+2} & \end{pmatrix};\ a_j\in\boldsymbol{R}\right\}$$

and

$$\bar{\mathfrak{p}}^I=\left\{\begin{pmatrix}0 & b_1 i\cdots b_{n+2}i\\ b_1 i & \\ \vdots & 0\\ b_{n+2}i & \end{pmatrix};\ b_j\in\boldsymbol{R}\right\}.$$

Note that the metric on $P^{n+2}(c)$ is induced from a bi-invariant metric on $SU(n+2)$. Set

$$\bar{G}_2=\left\{\left(\begin{array}{c|c} g & 0 \\ \hline 0 & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}\end{array}\right)\in\bar{G};\ g\in SU(3)\right\}$$

and

$$\bar{K}_2=\bar{G}_2\cap\bar{K}\,.$$

Then the submanifold $P^2(c)=\bar{G}_2/\bar{K}_2$ is totally geodesic in $P^{n+1}(c)$. Set

$$\bar{X}_j=\begin{array}{c} \\ j \\ \\ \end{array}\!\!\begin{array}{c} \begin{matrix} & & & j & & & \end{matrix} \\ \begin{pmatrix} 0 \cdots 0 & i & 0 \cdots 0 \\ \vdots & & \\ 0 & & \\ i & 0 & \\ 0 & & \\ \vdots & & \\ 0 & & \end{pmatrix}\end{array}\ (2\leqq j\leqq n+2)\,.$$

Then the set $\{\bar{X}_j;\ 2\leqq j\leqq n+2\}$ is a basis of $\bar{\mathfrak{p}}'$.

Now the minimal submanifold $f_0(M_0)$ in (5.5) is identified with the set

$$\left\{\pi\begin{pmatrix} \frac{1}{3}(e^{-\frac{x+y}{3}i}+e^{\frac{x}{3}i}+e^{\frac{y}{3}i}) \\ \frac{1}{3\sqrt{2}}(2e^{-\frac{x+y}{3}i}-e^{\frac{x}{3}i}-e^{\frac{y}{3}i}) \\ \frac{1}{\sqrt{6}}(e^{\frac{y}{3}i}-e^{\frac{y}{3}i}) \\ 0 \\ \cdots \\ 0 \end{pmatrix};\ x,y\in\boldsymbol{R}\right\}.$$

By putting $e^{xi}=e^{\theta i}(\zeta+i\eta)$ and $e^{yi}=e^{\theta i}(\zeta-i\eta)$ $\left(\theta=\frac{x+y}{2}\right)$, we have

$$f_0(M_0)=\left\{\pi\begin{pmatrix} \frac{1}{3}(e^{-\frac{2\theta}{3}i}+2\zeta e^{\frac{\theta}{3}i}) \\ \frac{\sqrt{2}}{3}(e^{-\frac{2\theta}{3}i}-\zeta e^{\frac{3}{\theta}i}) \\ \frac{2}{\sqrt{6}}i\eta e^{\frac{\theta}{3}i} \\ 0 \\ \cdots \\ 0 \end{pmatrix};\ \theta,\zeta,\eta\in\boldsymbol{R},\ \zeta^2+\eta^2=1\right\}.$$

Set

$$\bar{g}=\begin{pmatrix}1&0&0\cdots0\\0&1&0\cdots0\\0&0&\\\vdots&\vdots&g\\0&0&\end{pmatrix}\in\bar{K}$$

for $g\in SO(n)$. Then the subset $L^{n+1}=\bigcup_{g\in SO(n)}\bar{g}(f_0(M))$ is given by

$$\left\{\pi\begin{pmatrix}\frac{1}{3}(e^{-\frac{2}{3}\theta i}+2\zeta e^{\frac{1}{3}\theta i})\\\frac{\sqrt{2}}{3}(e^{-\frac{2}{3}\theta i}-\zeta e^{\frac{1}{3}\theta i})\\\frac{2}{\sqrt{6}}\eta\gamma_1 ie^{\frac{1}{3}\theta i}\\\frac{2}{\sqrt{6}}\eta\gamma_2 ie^{\frac{1}{3}\theta i}\\\vdots\\\frac{2}{\sqrt{6}}\eta\gamma_n ie^{\frac{1}{3}\theta i}\end{pmatrix}\ ;\ \theta,\zeta,\eta,\gamma_j\in\boldsymbol{R},\ \begin{matrix}\zeta^2+\eta^2=1\\\sum_{j=1}^{n}\gamma_j^2=1\end{matrix}\right\},$$

and thus putting $\zeta=a_0$, $\eta\gamma_j=a_j$ $(1\leqq j\leqq n)$, we have

$$L^{n+1}=\left\{\pi\begin{pmatrix}\frac{1}{3}(e^{-\frac{2}{3}\theta i}+2a_0 e^{\frac{1}{3}\theta i})\\\frac{\sqrt{2}}{3}(e^{-\frac{2}{3}\theta i}-a_0 e^{\frac{1}{3}\theta i})\\\frac{2}{\sqrt{6}}ia_1e^{\frac{1}{3}\theta i}\\\vdots\\\frac{2}{\sqrt{6}}ia_ne^{\frac{1}{3}\theta i}\end{pmatrix}\ ;\ \theta,a_j\in\boldsymbol{R},\ \sum_{j=0}^{n}a_j^2=1\right\}.$$

Now we define a mapping $\hat{h}\colon S^1\times S^n\to P^{n+1}(c)$ as follows;

$$\hat{h}(e^{\theta i};a_0,\cdots,a_n)=\pi\begin{pmatrix}\frac{1}{3}(e^{-\frac{2}{3}\theta i}+2a_0 e^{\frac{1}{3}\theta i})\\\frac{\sqrt{2}}{3}(e^{-\frac{2}{3}\theta i}-a_0 e^{\frac{1}{3}\theta i})\\\frac{2}{\sqrt{6}}ia_1e^{\frac{1}{3}\theta i}\\\vdots\\\frac{2}{\sqrt{6}}ia_ne^{\frac{1}{3}i\theta}\end{pmatrix}$$

for $e^{\theta i}\in S^1$ and $(a_0, \cdots a_n)\in S^n\subset \boldsymbol{R}^{n+1}$. By the easy calculation, we see that the mapping $\hat{h}$ is well-defined. Let ϕ be a diffeomorphism of $S^1\times S^n$ defined by

$$\phi\{(e^{\theta i};\ a_0, \cdots a_n)\} = (-e^{\theta i};\ -a_0, \cdots -a_n)\,.$$

Since ϕ has no fixed points on $S^1\times S^n$, the quotient space $S^1\times S^n/\phi$ is a differentiable manifold. Since $\hat{h}\circ\phi=\hat{h}$, the mapping $\hat{h}$ induces the mapping h: $S^1\times S^n/\phi\to P^{n+1}(c)$.

Now we shall show that h is an imbedding into $P^{n+1}(c)$, and that the imbedded submanifold L^{n+1} is nonzero isotropic and has the parallel second fundamental form. At first we recall the Hopf fibring π: $S^{2n+3}\to P^{n+1}(c)$. Note that the metric $\langle\ ,\ \rangle_S$ on S^{2n+3} is given by

$$\langle A, B\rangle_S = \frac{4}{c}(A, B)$$

where $(\ ,\ )$ is the canonical Euclidean metric. Then the fibring π: $S^{2n+3}\to P^{n+1}(c)$ is a riemannian submersion. Denote by ∇^S (resp. $\bar{\nabla}$) the riemannian connection on S^{2n+3} (resp. on $P^{n+1}(c)$). For a point $p\in S^{2n+3}$, let V_p be the subspace given by

$$V_p = \{A\in \boldsymbol{C}^{n+2};\ (A, p) = (A, ip) = 0\}\,.$$

Then V_p is the horizontal subspace of the connection of the principal S^1-bundle π: $S^{2n+3}\to P^{n+1}(c)$ and $\pi_*|V_p$: $V_p\to T_{\pi(p)}(P^{n+1}(c))$ is isometric. Moreover we have the following

Lemma 6.1 (K. Nomizu [8]). *Let p_t be a horizontal curve in S^{2n+3} and $u_t=\pi_*(p_t)$. If Z_t is a horizontal vector field along p_t and if $W_t=\pi_*(Z_t)$, then $\bar{\nabla}_tW_t=\pi_*(\nabla^S_tZ_t)$. Moreover $\nabla^S_t\dot{p}_t$ is horizontal.*

Let $\hat{\hat{h}}$: $\boldsymbol{R}\times S^n\to S^{2n+3}$ be the differentiable map given by

$$\hat{\hat{h}}(\theta, a_0 \cdots a_n) = \begin{pmatrix} \frac{1}{3}(e^{-\frac{2}{3}\theta i}+2a_0e^{\frac{1}{3}\theta i}) \\ \frac{\sqrt{2}}{3}(e^{-\frac{2}{3}\theta i}-a_0e^{\frac{1}{3}\theta i}) \\ \frac{2}{\sqrt{6}}ia_1e^{\frac{1}{3}\theta i} \\ \vdots \\ \frac{2}{\sqrt{6}}ia_ne^{\frac{1}{3}\theta i} \end{pmatrix}.$$

Then we have

$$
(6.1)\quad \begin{cases} \hat{\hat{h}}_*\left(\dfrac{\partial}{\partial\theta}\right) = \begin{pmatrix} \dfrac{2}{9} i(-e^{-\frac{2}{3}\theta i} + a_0 e^{\frac{1}{3}\theta i}) \\ -\dfrac{\sqrt{2}}{9} i(2e^{-\frac{2}{3}\theta i} + a_0 e^{\frac{1}{3}\theta i}) \\ -\dfrac{2}{3\sqrt{6}} a_1 e^{\frac{1}{3}\theta i} \\ \vdots \\ -\dfrac{2}{3\sqrt{6}} a_n e^{\frac{1}{3}\theta i} \end{pmatrix} \\ \hat{\hat{h}}_*(\xi) = \begin{pmatrix} \dfrac{2}{3}\xi_0 e^{\frac{1}{3}\theta i} \\ -\dfrac{\sqrt{2}}{3}\xi_0 e^{\frac{1}{3}\theta i} \\ \dfrac{2}{\sqrt{6}} i\xi_1 e^{\frac{1}{3}\theta i} \\ \vdots \\ \dfrac{2}{\sqrt{6}} i\xi_n e^{\frac{1}{3}\theta i} \end{pmatrix} \end{cases}
$$

where $\xi=(\xi_j)\in T(S^n)$, that is; $\sum_{j=0}^{n}\xi_j a_j=0$. Hence by easy calculations, the differential $\hat{\hat{h}}_*$ is injective into $V_{\hat{\hat{h}}(\theta a_0\cdots a_n)}$.

Now we define the metric $\langle\ ,\ \rangle$ on $S^1\times S^n$ as follows;

$$\langle A+\xi, B+\eta\rangle = \frac{8}{9c}\langle A, B\rangle_{S^1} + \frac{8}{3c}\langle\xi, \eta\rangle_{S^n}$$

for $A, B\in T(S^1)$ and $\xi, \eta\in T(S^n)$, where $\langle\ ,\ \rangle_{S^1}$ (resp. $\langle\ ,\ \rangle_{S^n}$) is the canonical metric on S^1 (resp. S^n). Since ϕ is an isometry of $S^1\times S^n$ with respect to this metric, this metric induces the metric on $S^1\times S^n/\phi$. Then we have the following

Lemma 6.2. *The mapping h: $S^1\times S^n/\phi\to P^{n+1}(c)$ is an isometric imbedding.*

Proof. We shall show that h is injective. Suppose that $h(e^{i\theta}, a_0\cdots a_n)=h(e^{i\bar{\theta}}, \bar{a}_0\cdots\bar{a}_n)$. Then there exists $e^{i\alpha}\in \boldsymbol{C}$ such that $\hat{\hat{h}}(\theta, a_0\cdots a_n)=e^{i\alpha}\hat{\hat{h}}(\bar{\theta}, \bar{a}_0\cdots\bar{a}_n)$. Thus we have

$$
\begin{cases} e^{-\frac{2}{3}\theta i}+2a_0 e^{\frac{1}{3}\theta i} = e^{(\alpha-\frac{2}{3}\bar{\theta})i}+2\bar{a}_0 e^{(\alpha+\frac{1}{3}\bar{\theta})i} \\ e^{-\frac{2}{3}\theta i}-a_0 e^{\frac{1}{3}\theta i} = e^{(\alpha-\frac{2}{3}\bar{\theta})i}-\bar{a}_0 e^{(\alpha+\frac{1}{3}\bar{\theta})i} \\ a_1 e^{\frac{1}{3}\theta i} = \bar{a}_1 e^{(\alpha+\frac{1}{3}\bar{\theta})i} \\ \vdots \\ a_n e^{\frac{1}{3}\theta i} = \bar{a}_n e^{(\alpha+\frac{1}{3}\bar{\theta})i} \end{cases}
$$

and consequently

$$\begin{cases} e^{-\frac{2}{3}\theta i} = e^{(\alpha-\frac{2}{3}\bar{\theta})i} \\ a_0 e^{\frac{1}{3}\theta i} = \bar{a}_0 e^{(\alpha+\frac{1}{3}\bar{\theta})i} \\ \qquad \vdots \\ a_n e^{\frac{1}{3}\theta i} = \bar{a}_n e^{(\alpha+\frac{1}{3}\bar{\theta})i} \end{cases}$$

and moreover

$$\begin{cases} a_0 e^{(\theta-\bar{\theta})i} = \bar{a}_0 \\ \qquad \vdots \\ a_n e^{(\theta-\bar{\theta})i} = \bar{a}_n \end{cases}$$

Since a_j, $\bar{a}_j$ are real numbers, we have $e^{(\theta-\bar{\theta})i}=\pm 1$. If $e^{(\theta-\bar{\theta})i}=1$, $(e^{\theta i}, a_0\cdots a_n)=(e^{\bar{\theta}i}, \bar{a}_0\cdots\bar{a}_n)$. If $e^{(\theta-\bar{\theta})i}=-1$, $(e^{\theta i}, a_0\cdots a_n)=(-e^{\bar{\theta}i}, -\bar{a}_0\cdots -\bar{a}_n)$. Hence the mapping h is injective.

The other assersions follow from the following diagram.

$$\begin{array}{ccc} \boldsymbol{R}\times S^n & \xrightarrow{\hat{\hat{h}}} & S^{2n+3} \\ \hat{\hat{}}\big\downarrow & & \big\downarrow \pi \\ S^1\times S^n/\phi & \xrightarrow{h} & P^{n+1}(c) \end{array}$$

q.e.d.

Lemma 6.3. *The isometric imbedding h is $\frac{\sqrt{c}}{2\sqrt{2}}$-isotropic.*

Proof. Note that $\frac{3\sqrt{c}}{2\sqrt{2}} i\hat{\hat{h}}_*\left(\frac{\partial}{\partial\theta}\right)$ and $i\hat{\hat{h}}_*(\xi)\left(\sum \xi_j^2=\frac{3c}{8}\right)$ are orthonormal vectors and that the normal component of $\hat{\hat{h}}_*(T(S^1\times S^n))$ in V is given by the vector space

$$\left\{ i\hat{\hat{h}}_*\left(\frac{\partial}{\partial\theta}\right),\ i\hat{\hat{h}}_*(\xi);\ \xi\in T(S^n)\right\}_{\boldsymbol{R}}.$$

Then by Lemma 6.1 we have

$$\begin{cases} \sigma\left(\left(\frac{3\sqrt{c}}{2\sqrt{2}}\right)\left(\frac{\partial}{\partial\theta}\right), \frac{3\sqrt{c}}{2\sqrt{2}}\left(\frac{\partial}{\partial\theta}\right)\right) = -\frac{\sqrt{c}}{2\sqrt{2}}\pi_*\left(\frac{3\sqrt{c}}{2\sqrt{2}} i\hat{\hat{h}}_*\left(\frac{\partial}{\partial\theta}\right)\right) \\ \sigma\left(\frac{3\sqrt{c}}{2\sqrt{2}}\left(\frac{\partial}{\partial\theta}\right), \xi\right) = \frac{\sqrt{c}}{2\sqrt{2}}\pi_*(i\hat{\hat{h}}_*(\xi)) \qquad \left(\sum \xi_j^2 = \frac{3c}{8}\right) \\ \sigma(\zeta, \eta) = \langle\zeta, \eta\rangle \frac{\sqrt{c}}{2\sqrt{2}}\pi_*\left(\frac{3\sqrt{c}}{2\sqrt{2}} i\hat{\hat{h}}_*\left(\frac{\partial}{\partial\theta}\right)\right) \end{cases} \tag{6.2}$$

for $\xi, \zeta, \eta\in T(S^n)$. Hence by the easy computation σ is $\frac{\sqrt{c}}{2\sqrt{2}}$-isotropic.

q.e.d.

Lemma 6.4. *The isometric imbedding h has the parallel second fundamental form.*

Proof. Let X, Y, Z, W be either of $\left(\frac{\partial}{\partial\theta}\right)$ and $\xi(\in T(S^n))$. Then by the definition we have

$$
\begin{aligned}
(6.3)\qquad & \langle(\nabla^*_X\sigma)(Y, Z), \pi_*(i\hat{\hat{h}}_*W)\rangle \\
&= \langle D_X(\sigma(Y, Z)), \pi_*(i\hat{\hat{h}}_*W)\rangle-\langle\sigma(\nabla_X Y, Z), \pi_*(i\hat{\hat{h}}_*W)\rangle \\
&\quad -\langle\sigma(Y, \nabla_X Z), \pi_*(i\hat{\hat{h}}_*W)\rangle\,.
\end{aligned}
$$

Now the first term of (6.3) is calculated by Lemma 6.1 and (6.2) as follows:

$$
\begin{aligned}
& \langle D_X(\sigma(Y, Z)), \pi_*(i\hat{\hat{h}}_*W)\rangle = \langle\nabla_{\pi_*(\hat{\hat{h}}_*X)}(\sigma(Y, Z)), \pi_*(i\hat{\hat{h}}_*W)\rangle \\
&= \langle\pi_*(\nabla^S_{(\hat{\hat{h}}_*X)}(\sigma^H(Y, Z))), \pi_*(i\hat{\hat{h}}_*W)\rangle \\
&= \langle\nabla^S_{(\hat{\hat{h}}_*X)}(\sigma^H(Y, Z)), i\hat{\hat{h}}_*W\rangle = \langle X(\sigma^H(Y, Z)), i\hat{\hat{h}}_*W\rangle
\end{aligned}
$$

where $\sigma^H(T, S)$=the horizontal lift of $\sigma(T, S)$ for $T, S\in T(\boldsymbol{R}\times S^n)$. Hence the first term of (6.3) is calculated by (6.1), (6.2) and the above formula. Note that

$$
(6.4)\qquad \nabla_{(\partial/\partial\theta)}\left(\frac{\partial}{\partial\theta}\right)=0,\ \nabla_{(\partial/\partial\theta)}\xi=\nabla_\xi\left(\frac{\partial}{\partial\theta}\right)=0,\ \nabla_\zeta\xi=\nabla^{S^n}_\zeta\xi
$$

where ζ, ξ are $T(S^n)$-valued vector field on $\boldsymbol{R}\times S^n$, and ∇^{S^n} denotes the riemannian connection on S^n. Then the second and the third terms are calculated by (6.1), (6.2), (6.4) and Lemma 6.1. By the above explicit computation we have $\nabla^*\sigma=0$. q.e.d.

Summing up our results in this section, we have the following

Theorem 6.5. *The isometric imbedding h: $S^1\times S^n/\phi\rightarrow P^{n+1}(c)$ is first full $P(\boldsymbol{C})$-totally real $\frac{\sqrt{c}}{2\sqrt{2}}$-isotropic and has the parallel second fundamental form.*

7. $P(\boldsymbol{C})$-totally real $\frac{\sqrt{c}}{2\sqrt{2}}$-isotropic submanifolds with parallel second fundamental form in $P^m(c)$ which are locally isometric to the riemannian symmetric space $SU(3)/SO(3)$

In this section we shall construct the model of the $P(\boldsymbol{C})$-totally real $\frac{\sqrt{c}}{2\sqrt{2}}$-isotropic submanifold with parallel second fundamental form in $P^5(c)$ which is locally isometric to the riemannian symmetric space $SU(3)/SO(3)$.

Let $S^3(\boldsymbol{C})$ (resp. $S^3(\boldsymbol{R})$) be the complex (resp. real) vector space of all the complex (resp. real) symmetric matrices of degree three, and $S^{11}(1)$ be the unit spherte of $S^3(\boldsymbol{C})$ with respect to the canonical Euclidean metric $(A, B) = Re(TrAB^*)$. Then we have the Hopf fibring π: $S^{11}(1) \rightarrow P^5(c)$. We retain the same notations as in the section 6. Moreover giving a metric $\langle\ ,\ \rangle_S$ on $S^{11}(1)$ by $\langle\ ,\ \rangle_S = \frac{4}{c}(\ ,\ )$, we have the riemannian submersion $(S^{11}(1), \langle\ ,\ \rangle_S) \rightarrow P^5(c)$.

Now we construct the equivariant imbedding $\hat{g}$ of the homogeneous space $M^5 = SU(3)/SO(3)$ into $S^{11}(1)$ as follows;

$$\hat{g}(hSO(3)) = \frac{1}{\sqrt{3}} h^t h$$

for $h \in SU(3)$. Then we can check that $\hat{g}$ is a well-defined imbedding. And since the $SU(3)$-action on M^5 and the $Ad(SU(3))$-action on $S^{11}(1)$ are compatible each other for the imbedding $\hat{g}$, the homogeneous space M^5 is a riemannian symmetric space with respect to the metric induced from that on $S^{11}(1)$. Moreover we may check easily that $\hat{g}_*(T_p(M^5)) \subset V_{\hat{g}(p)}$ for any point $p \in M^5$, using the fact that the $Ad(SU(3))$-action on $S^{11}(1)$ is compatible with the complex structure on $S^3(\boldsymbol{C})$.

Now we consider the isometric immersion $g = \pi \circ \hat{g}$: $M^5 \rightarrow P^5(c)$. We can show that the manifold M^5 is the 3-sheeted covering of $g(M^5)$. Then since the tangent space $g_*(T_{eSO(3)}(M^5))$ is totally real and the $Ad(SU(3))$-action on $S^{11}(1)$ induces holomorphic isometries on $P^5(c)$, the imbedding g is totally real. On the other hand, along the discussion in [8] we have the following

Lemma 7.1. *Let g: $M^n \rightarrow P^m(c)$ be a totally real isometric imbedding. Then the imbedding g is nonzero isotropic and has the parallel second fundamental form if and only if the imbedding g transfers geodesics in M^n into circles in $P^m(c)$.*

Now we have the following

Theorem 7.2. *The isometric imbedding g: $M^5 \rightarrow P^5(c)$ is $\frac{\sqrt{c}}{2\sqrt{2}}$-isotropic and has the parallel second fundamental form.*

Proof. By the virture of Lemma 7.1, it is enough to see that geodesics in M^5 are circles in $P^5(c)$. Set

$$\mathfrak{a} = \left\{ i \begin{pmatrix} -x-y & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{pmatrix};\ x, y \in \boldsymbol{R} \right\}.$$

Then the subspace $\mathfrak{a}$ is maximal abelian in $\mathfrak{p} = \{iA;\ A \in S^3(\boldsymbol{R}),\ Tr\,A = 0\}$. Since g is an equivariant imbedding, it is enough to see that the geodesic parameterized by arc-length

$$\gamma(t)=\begin{pmatrix} e^{-(x+y)ti} & 0 & 0 \\ 0 & e^{xti} & 0 \\ 0 & 0 & e^{yti} \end{pmatrix}\cdot SO(3)$$

is a circle, where $x^2+xy+y^2=\frac{3c}{32}$. Then the curve $g(\gamma(t))$ in $P^5(c)$ is given by

$$g(\gamma(t))=\frac{1}{\sqrt{3}}\pi\begin{pmatrix} e^{-2(x+y)ti} & 0 & 0 \\ 0 & e^{2xti} & 0 \\ 0 & 0 & e^{2yti} \end{pmatrix}$$

and the tangent vector field $X(t)$ along $g(\gamma(t))$ is given by

$$X(t)=\frac{1}{\sqrt{3}}\pi_*\begin{pmatrix} -2(x+y)ie^{-2(x+y)ti} & 0 & 0 \\ 0 & 2xie^{2xti} & 0 \\ 0 & 0 & 2yie^{2yti} \end{pmatrix}.$$

Then using Lemma 6.1, we have easily

$$|\nabla_t X_t|=\frac{\sqrt{c}}{2\sqrt{2}} \quad \text{and} \quad \nabla_t^2 X_t=-\frac{c}{8}X_t$$

and thus $g(\gamma(t))$ is a circle in $P^5(c)$. q.e.d.

8. The rigidity of $P(C)$-totally real $\frac{\sqrt{c}}{2\sqrt{2}}$-isotropic submanifolds with parallel second fundamental form in $P^m(c)$

At first let M^{n+1} be an $(n+1)$-dimensional complete first full $P(C)$-totally real $\frac{\sqrt{c}}{2\sqrt{2}}$-isotropic submanifold with parallel second fundamental form in $P^r(c)$ which is locally isometric to the riemannian symmetric space $S^1\times S^n$. Then we have the following

Lemma 8.1. *The integer r equals $n+1$.*

Proof. Fix a point $o\in M^{n+1}$. Note that the tangent space $T_0(M)$ is decomposed into $T_0(S^1)$ and $T_0(S^n)$. For unit vectors $Y\in T_0(S^1)$ and $X\in T_0(S^n)$, the Lie triple system $\{Y, X\}_R$ in $T_0(M)$ defines a unique totally geodesic flat submanifold in M of 2-dimensional. By Lemma 4.3 and Proposition 4.6, we have

$$\sigma(Y, Y)+\sigma(X, X)=0\,.$$

This implies that dim $O_0^1(M)\leqq 2n+2$, and hence $r\leqq n+1$ by Proposition 4.2. On the other hand a unique totally geodesic submanifold defined by the Lie triple system $T_0(M)$ in $T_0(P^r(c))$ is $P^{n+1}(\boldsymbol{R})$ and thus $n+1\leqq r$. Hence we have $r=n+1$. q.e.d.

Since the group $SU(n+2)$ acts transitively on the bundle of all the unitary frames of $P^{n+1}(c)$, we may assume that the submanifold M^{n+1} contains $\bar{o}=e\bar{K} \in P^{n+1}(c)$ and identify the tangent space $T_{\bar{o}}(M)$ (resp. the first normal space $N^1_{\bar{o}}(M)$) with the subspace $\bar{\mathfrak{p}}^I$ (resp. $\bar{\mathfrak{p}}^R$). Moreover we may identify the subspace $T_{\bar{o}}(S^1)$ (resp. $T_{\bar{o}}(S^n)$) in $T_{\bar{o}}(M)$ with the subspace $\{\bar{X}_2\}_R$ (resp. $\{\bar{X}_j(3 \leqq j \leqq n+2)\}_R$). Set $\bar{H}=S(O(1)\times O(n+1))\subset\bar{K}$. Then we have the following

Lemma 8.2. *There exists $\bar{h}\in\bar{K}$ such that*

$$Ad(\bar{h})\bar{\mathfrak{p}}^I = \bar{\mathfrak{p}}^I$$

and that the second fundamental form $\sigma^{\bar{h}}$ at $\bar{o}$ of the submanifold $\bar{h}(M)$ is given by

$$(8.1)\qquad \sigma^{\bar{h}}(\bar{X}_2, \bar{X}_2) = -\frac{1}{\sqrt{2}}J\bar{X}_2,\ \sigma^{\bar{h}}(\bar{Y}, \bar{Y}) = \frac{1}{\sqrt{2}}J\bar{X}_2,$$
$$\sigma^{\bar{h}}(\bar{X}_2, \bar{Y}) = \frac{1}{\sqrt{2}}J\bar{Y}$$

for any vector $\bar{Y}\in\{\bar{X}_j(3\leqq j\leqq n+2)\}_R$ of length $\frac{2}{\sqrt{c}}$.

Proof. At first we consider the case when $n=1$. Then the submanifold M^2 is minimal and $\frac{\sqrt{c}}{2\sqrt{2}}$-isotropic in $P^2(c)$.

Now we know the following

Lemma (B. O'Neill [10]). *If a riemannian manifold M is λ-isotropic in another riemannian manifold $\bar{M}$, then*

$$\Delta_{ZW}+3|\sigma(Z, W)|^2 = \lambda^2$$

for orthonormal vectors Z, W of M.

Then, since M^2 is flat minimal $\frac{\sqrt{c}}{2\sqrt{2}}$-isotropic in $P^2(c)$, together with the above lemma, we have

$$(8.2)\qquad \begin{cases} |\sigma(Z, Z)| = |\sigma(Z, W)| = \dfrac{\sqrt{c}}{2\sqrt{2}} \\ \langle\sigma(Z, Z), \sigma(Z, W)\rangle = 0 \\ \sigma(Z, Z)+\sigma(W, W) = 0 \end{cases}$$

for orthonormal vectors Z, W of M. Setting $Z=\frac{\sqrt{c}}{2}\bar{X}_2$ and $W=\frac{\sqrt{c}}{2}\bar{X}_3$ we have

$$\begin{cases} \sigma(\bar{X}_2, \bar{X}_2) = \dfrac{1}{\sqrt{2}}(\cos\theta_0 J\bar{X}_2 + \sin\theta_0 J\bar{X}_3) \\ \sigma(\bar{X}_2, \bar{X}_3) = \pm\dfrac{1}{\sqrt{2}}(\sin\theta_0 J\bar{X}_2 + \cos\theta_0 J\bar{X}_3) \end{cases}$$

for some θ_0 in $\boldsymbol{R}$.

Now for $\theta \in \boldsymbol{R}$, put

$$\bar{h}(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \in \bar{H}.$$

Then we have

$$(8.3) \qquad Ad(\bar{h}(\theta))\begin{pmatrix} \bar{X}_2 \\ \bar{X}_3 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \bar{X}_2 \\ \bar{X}_3 \end{pmatrix}.$$

Since the second fundamental form $\sigma^{\bar{h}(\theta)(M^2)}$ at $\bar{o}$ of the submanifold $\bar{h}(\theta)(M^2)$ is given by

$$\sigma^{\bar{h}(\theta)(M^2)}(\bar{Z}, \bar{W}) = Ad(\bar{h}(\theta))(\sigma(Ad(\bar{h}(\theta)^{-1})\bar{Z}, Ad(\bar{h}(\theta)^{-1})\bar{W}))$$

for $\bar{Z}, \bar{W} \in \bar{m} = \{\bar{X}_2, \bar{X}_3\}_{\boldsymbol{R}}$, we have

$$\sigma^{\bar{h}(\theta)(M^2)}(\bar{X}_2, \bar{X}_2) = \frac{1}{\sqrt{2}}(\cos(\theta_0 \mp 2\theta - \theta)J\bar{X}_2 + \sin(\theta_0 \mp 2\theta - \theta)J\bar{X}_3)$$

by (8.3). Hence, taking θ suitably, we may assume that

$$\sigma^{\bar{h}(\theta)(M^2)}(\bar{X}_2, \bar{X}) = -\frac{1}{\sqrt{2}}J\bar{X}_2.$$

Now we get our claim by (8.2) and Lemma 2.4.

Next we consider the case when $n \geqq 2$. For any vector $\bar{Y} \in \{\bar{X}_j (3 \leqq j \leqq n+2)\}_{\boldsymbol{R}}$ of length $\dfrac{2}{\sqrt{c}}$, the Lie triple system $\{\bar{X}_2, \bar{Y}\}_{\boldsymbol{R}}$ in $T_{\bar{o}}(M)$ defines a unique flat totally geodesic submanifold in M. Hence we have $\sigma(\bar{X}_2, \bar{X}_2)$, $\sigma(\bar{X}_2, \bar{Y})$, $\sigma(\bar{Y}, \bar{Y}) \in \{J\bar{X}_2, J\bar{Y}\}_{\boldsymbol{R}}$. Since $n \geqq 2$, we have

$$\bigcap_{\bar{Y} \in \{\bar{X}_j (3 \leqq j \leqq n+2)\}} \{J\bar{X}_2, J\bar{Y}\}_{\boldsymbol{R}} = \{J\bar{X}_2\}_{\boldsymbol{R}}$$

and thus

$$\sigma(\bar{X}_2, \bar{X}_2) \in \{J\bar{X}_2\}_{\boldsymbol{R}}.$$

Moreover we have $\sigma(\bar{X}_2, \bar{X}_2) = \pm\dfrac{1}{\sqrt{2}}J\bar{X}_2$ by (8.2). Now note that the involution $\bar{h}$ at $\bar{o}$ is an element in $\bar{K}$. Then, if nessesarily, taking the submanifold $\bar{h}(M)$ for the submanifold M we may assume that

$$\sigma(\bar{X}_2, \bar{X}_2) = -\frac{1}{\sqrt{2}} J\bar{X}_2 .$$

Hence we have $\sigma(\bar{X}_2, \bar{Y}) = \frac{1}{\sqrt{2}} J\bar{Y}$ and $\sigma(\bar{Y}, \bar{Y}) = \frac{1}{\sqrt{2}} J\bar{X}_2$ by (8.2) and Lemma 2.4. q.e.d.

Note that the conditions (8.1) determine the second fundamental form $\sigma^{\bar{h}}$ uniquely. By the uniqueness of circle, we have the following

Theorem 8.3. *Let M^{n+1} be a complete $P(\boldsymbol{C})$-totally real $\frac{\sqrt{c}}{2\sqrt{2}}$-isotropic submanifold with parallel second fundamental form in $P^{n+1}(c)$ which is locally isometric to the riemannian symmetric space $S^1 \times S^n$. Then the submanifold M^{n+1} is congruent to the model in Theorem 6.5 by some isometry of $P^{n+1}(c)$.*

Next let M^5 be a complete first full $P(\boldsymbol{C})$-totally real $\frac{\sqrt{c}}{2\sqrt{2}}$-isotropic submanifold with parallel second fundamental form in $P^r(c)$ which is locally isometric to the riemanian symmetric space $SU(3)/SO(3)$. Set

$$\mathfrak{g} = \mathfrak{su}(3), \quad \mathfrak{k} = \mathfrak{so}(3)$$

and

$$\mathfrak{p} = \{iX;\ X \in S^3(\boldsymbol{R}),\quad Tr\, X = 0\} .$$

Then we have the canonical decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and identify $\mathfrak{p}$ with the tangent space $T_0(M^5) = T_0(SU(3)/SO(3))$ at $o = eSO(3)$. Put

$$I_2 = \frac{1}{\sqrt{6}} i \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad I_3 = \frac{1}{\sqrt{2}} i \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad I_4 = \frac{1}{\sqrt{2}} i \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$I_5 = \frac{1}{\sqrt{2}} i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad I_6 = \frac{1}{\sqrt{2}} i \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then $\{I_j (2 \leqq j \leqq 6)\}$ is an orthogonal basis in $T_0(M^5)$ of the same length. that the subspace $\{I_2, I_3, I_4\}_{\boldsymbol{R}}$ is a Lie triple system in $T_0(M^5)$ which defines the totally geodesic submanifold $S^1 \times S^2$ locally (cf. See [3]), we have

$$(8.4) \quad \begin{cases} \sigma(I_2, I_2) = -\sigma(I_3, I_3) = -\sigma(I_4, I_4) \\ \sigma(I_3, I_4) = 0 \end{cases}$$

by Proposition 4.6. Now put

$$Q(\theta)=\begin{pmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{pmatrix} \text{ and } R(\theta)=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{pmatrix}.$$

Then we have

$$\begin{cases} Ad(Q(\theta))I_2=\left(1-\frac{3}{2}\sin^2\theta\right)I_2+\frac{\sqrt{3}}{2}\sin^2\theta I_3-\sqrt{3}\sin\theta\cos\theta I_6 \\ Ad(Q(\theta))I_3=\frac{\sqrt{3}}{2}\sin^2\theta I_2+\left(1-\frac{\sin^2\theta}{2}\right)I_3+\sin\theta\cos\theta I_6 \\ Ad(Q(\theta))I_4=\cos\theta I_4-\sin\theta I_5\,. \end{cases}$$

Since the subspace $\{Ad(Q(\theta))I_2, Ad(Q(\theta))I_3, Ad(Q(\theta))I_4\}_{\boldsymbol{R}}$ is a Lie triple system in $T_0(M^5)$ which also defines the totally geodesic submanifold $S^1\times S^2$ locally, we have

$$(8.5)\quad \begin{cases} \sigma(Ad(Q(\theta))I_2, Ad(Q(\theta)I_2)=-\sigma(Ad(Q(\theta))I_3, Ad(Q(\theta))I_3) \\ \qquad\qquad =-\sigma(Ad(Q(\theta))I_4, AdfQ(\theta))I_4) \\ \sigma(Ad(Q(\theta))I_3, Ad(Q(\theta)I_4)=0 \end{cases}$$

as (8.4). By the last equation in (8.5), we have

$$\frac{\sqrt{3}}{2}\cos\theta\sin^2\theta\sigma(I_2, I_4)+\sin\theta\cos^2\theta\sigma(I_4, I_6)-\frac{\sqrt{3}}{2}\sin^3\theta\sigma(I_2, I_5)$$
$$-\sin\theta\left(1-\frac{\sin^2\theta}{2}\right)\sigma(I_3, I_5)-\sin^2\theta\cos\theta\sigma(I_5, I_6)=0$$

and thus

$$\frac{\sqrt{3}}{2}\cos\theta\sin\theta\sigma(I_2, I_4)+\cos^2\theta\sigma(I_4, I_6)-\frac{\sqrt{3}}{2}\sin^2\theta\sigma(I_2, I_5)$$
$$-\left(1-\frac{\sin^2\theta}{2}\right)\sigma(I_3, I_5)-\sin\theta\cos\theta\sigma(I_5, I_6)=0\,.$$

Here putting $\theta=0$ $\left(\text{resp.} =\frac{\pi}{2}\right)$, we get

$$(8.6)\quad \sigma(I_3, I_5)=\sigma(I_4, I_6)\ (\text{resp. } \sqrt{3}\sigma(I_2, I_5)+\sigma(I_3, I_5)=0)\,.$$

Moreover by (8.6) the above equation implies

$$\frac{\sqrt{3}}{2}\cos\theta\sin\theta\sigma(I_2, I_4)+\sin\theta\cos\sigma(I_5, I_6)=0$$

and thus

$$\frac{\sqrt{3}}{2}\sigma(I_2, I_4)+\sigma(I_5, I_6) = 0 .$$

Using the other equations in (8.5) and the equations for $R(\theta)$, we have the following Table 2 by the same calculation as above.

Table 2 $(\sigma(T, S))$

$S \backslash T$	I_2	I_3	I_4	I_5	I_6
I_2	A	B	C	D	E
I_3		$-A$	0	$-\sqrt{3}D$	$\sqrt{3}E$
I_4			$-A$	$-\sqrt{3}E$	$-\sqrt{3}D$
I_5				$\frac{1}{2}(\sqrt{3}B+A)$	$\frac{\sqrt{3}}{2}C$
I_6					$-\frac{1}{2}(\sqrt{3}B-A)$

Proposition 8.4. *The integer r equals* 5.

Proof. By the above Table 2, we have $r\leqq 5$. On the other hand the Lie triple system $T_0(M^5)$ in $T_{\bar{o}}(P^r(c))$ defines a totally geodesic submanifold $P^5(\boldsymbol{R})$. Hence we have $r\geqq 5$ and thus $r=5$. q.e.d.

Now we may identify the point o in M^5 with the point $\bar{o}=e\bar{K}$ in $P^5(c)$ and the tangent space $T_0(M^5)$ with the totally real subspace $\bar{\mathfrak{p}}^I$ in $\bar{\mathfrak{p}}$. Moreover by taking a suitable real number t, we may identify $t\bar{X}_j$ with I_j for each $j=2\cdots 6$. Then we have the following

Lemma 8.5. *There exists $\bar{g}\in\bar{K}$ such that*

$$Ad(\bar{g})\bar{\mathfrak{p}}^I = \bar{\mathfrak{p}}^I$$

and that the second fundamental form $\sigma^{\bar{g}}$ at $\bar{o}$ of the submanifold $\bar{g}(M^5)$ is given by

$$\sigma^{\bar{g}}(\bar{X}_2, \bar{X}_2) = -\frac{1}{\sqrt{2}}J\bar{X}_2,\ \sigma^{\bar{g}}(\bar{X}_2, \bar{X}_j) = \frac{1}{\sqrt{2}}J\bar{X}_j \quad (j = 3, 4)\,,$$

$$\sigma^{\bar{g}}(\bar{X}_2, \bar{X}_j) = -\frac{1}{2\sqrt{2}}J\bar{X}_j \quad (j = 5, 6)\,.$$

Proof. The Lie triple system $\{I_2, I_3, I_4\}_{\boldsymbol{R}}$ defines a totally geodesic submanifold $S^1\times S^2$ locally such that $T_0(S^1)=\{I_2\}_{\boldsymbol{R}}$. Note in Lemma 8.2 that the isometry $\bar{h}$ is the involution at $\bar{o}$ or identity map when $n\geqq 2$. Hence we may assume that

$$\sigma(\bar{X}_2, \bar{X}_2) = -\frac{1}{\sqrt{2}} J\bar{X}_2 \text{ and } \sigma(\bar{X}_2, \bar{X}_j) = \frac{1}{\sqrt{2}} J\bar{X}_j \quad (j = 3, 4).$$

By Table 2 and Lemma 2.4 we have

$$\langle \sigma(\bar{X}_2, \bar{X}_5), J\bar{X}_j \rangle = 0 \quad (j = 2, 3, 4, 6)$$

and

$$\langle \sigma(\bar{X}_2, \bar{X}_5), J\bar{X}_5 \rangle = -\frac{2}{c\sqrt{2}}.$$

Thus $\sigma(\bar{X}_2, \bar{X}_5) = -\frac{1}{2\sqrt{2}} J\bar{X}_5$. Similarly we have $\sigma(\bar{X}_2, \bar{X}_6) = -\frac{1}{2\sqrt{2}} J\bar{X}_6$.

q.e.d.

Note that Lemma 8.5 and Table 2 determine the second fundamental form $\sigma^{\bar{g}}$ uniquely. By the uniqueness of circle, we have the following

Theorem 8.6. *Let M^5 be a complete $P(\boldsymbol{C})$-totally real $\frac{\sqrt{c}}{2\sqrt{2}}$-isotropic submanifoldt with parallel second fundamental form in $P^5(c)$ which is locally isometric to the riemannian symmetric space $SU(3)/SO(3)$. Then the submanifold M^5 is congruent to the model in Theorem 7.2 by some isometry of $P^5(c)$.*

REMARK 8.7. In the next paper we shall give examples of $P(\boldsymbol{C})$-totally real $\frac{\sqrt{c}}{2\sqrt{2}}$-isotropic isometric immersions with parallel second fundamenal form of the other spaces; $SU(3)$, $SU(6)/Sp(3)$, E_6/F_4.

References

[1] E. Calabi: *Isometric imbedding of complex manifolds,* Ann. of Math. **58** (1953), 1–23.

[2] B.Y. Chen: *Extrinsic spheres in riemannian manifolds,* Houston J. Math. to appear.

[3] B.Y. Chen and T. Nagano: *Totally geodesic submanifolds of symmetric spaces* II, Duke Math. J. **45** (1978), 405–425.

[4] B.Y. Chen and K. Ogiue: *On totally real submanifolds,* Trans. Amer. Math. Soc. **193** (1974), 257–266.

[5] S. Helgason: Differential geometry, and symmetric spaces, Academic Press, 1962.

[6] S. Kobayashi and K. Nomizu: Foundations of differential geometry, Vol. I, II, Interscience Tracts in Pure and Applied Math. 1963, 1969.

[7] H. Naitoh: *Isotropic submanifolds with parallel second fundamental forms in symmetric spaces,* Osaka J. Math. **17** (1980), 95–110.

[8] K. Nomizu: *A characterization of the Veronese varieties,* Nagoya Math. J. **60** (1976), 81–188.

[9] K. Nomizu and K. Yano: *On circles and spheres in riemannian geometry,* Math. Ann. **210** (1974), 163–170.

[10] B.O'Neill: *Isotropic and Kahier immersions,* Canad. J. Math. **17** (1965), 905–915.

[11] J.S. Pak: *Planer geodesic submanifolds in complex space forms,* Kodai Math. J. **1** (1978), 187–196.

[12] K. Sakamoto: *Planer geodesic immersions,* Tohoku Math. J. **29** (1977), 25–56.

Department of Mathematics
Yamaguchi University
Yamaguchi 753, Japan