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THE BRAUER GROUPS OF SOME SEPARABLY CLOSED RINGS

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M. Auslander and O. Goldman in [1] laid the foundation for the study of separable extensions of a commutative ring. All unexplained conventions, terminology and notation in this note are as in [1]. One of the most important and difficult problems in this theory, even when the commutative ring is a field, is the explicit computation of the Brauer group. Let K be a commutative ring. Following [2] we call a commutative K -algebra S strongly separable in case it is separable, finitely generated, and projective over K ; and we call a commutative ring Ω separably closed in case the only strongly separable Ω algebras are direct sums of copies of Ω . In [2] it is shown that if K is any commutative ring with no idempotents but 0 and 1 then there is a separably closed K -algebra Ω with no idempotents but 0 and 1, called the separable closure of K , which contains an isomorphic copy of every strongly separable K -algebra with no idempotents but 0 and 1, and with the property that any finite subset is contained in a strongly separable K -algebra.

In analogy with the situation in fields, one would expect the Brauer group of a separably closed ring to be trivial. However M. Auslander has discovered a separably closed principal ideal domain whose Brauer group has order two. If K is a commutative ring let $B(K)$ denote the Brauer group of K and if S is a strongly separable K -algebra let $B(S/K)$ denote the kernel of map induced by the correspondence $A \rightarrow S \otimes_K A$ where A is a central separable K -algebra. If K is a semi-local noetherian ring we prove that $B(K) = \cup B(S/K)$ where S ranges over all strongly separable K -algebras. If moreover, K has no idempotents but 0 and 1 and Ω is the separable closure of K then $B(\Omega)$ is trivial.

I would like to acknowledge my indebtedness to E. C. Ingraham who provided the proof of Proposition 3.

Let K be a semi-local ring (commutative ring with a finite number of maximal ideals). If $\{M_i\}_{i=1}^n$ are the maximal ideals of K then $N = \cap_{i=1}^n M_i$ is the radical of K . One can make K into a topological space and complete the space where a neighborhood base for 0 consists of the powers of N . We refer the reader to Chapter II of [4] for the facts about completeness we employ here. We will also frequently make use of the fact that any strongly separable algebra

over a semi-local ring is semi-local. Also we will employ the well known Nakayama's Lemma which asserts that if R is any ring with identity and I is a left ideal of R then I is contained in the radical of R if and only if for all finitely generated left R modules M with N a submodule of M , if $IM+N=M$ then $N=M$. Our first result generalizes Theorem 6.3 of [1].

Theorem 1. *Let A be a central separable algebra over the noetherian semi-local ring K , then there is a strongly separable K -algebra S with $S \otimes_K A$ in the zero class of $B(S)$. Moreover, if K has no idempotents but 0 and 1, we may choose S with no idempotents but 0 and 1.*

Proof. Any commutative semi-local ring is a finite direct sum of semi-local rings without idempotents but 0 and 1. Because of this observation we may assume K has no idempotents but 0 and 1.

Let $\bar{A} = A/NA = \bar{A}_1 \oplus \cdots \oplus \bar{A}_n$ with the center of \bar{A}_i equal to K/M_iK which we denote \bar{K}_i . Since K has no idempotents but 0 and 1, A is a free K -module (Page 377 of [1]). Thus $\text{Rank}_{\bar{K}_i}(\bar{A}_i) = \text{Rank}_K(A) = m^2$ for some integer m . There is a commutative separable algebra F_i over \bar{K}_i of dimension m which is a maximal commutative subalgebra of \bar{A}_i . Moreover, there is a $\theta_i \in \bar{A}_i$ so that $F_i = \bar{K}_i(\theta_i)$ and θ_i satisfies a monic polynomial $P_i(x)$ over \bar{K}_i of degree m . The existence of such an F_i and θ_i can be seen in the following way. If \bar{K}_i is finite, then there is a field extension of \bar{K}_i (necessarily separable) which can be imbedded in \bar{A}_i as a maximal commutative subring. There is always a maximal commutative separable subfield of the division algebra component of \bar{A}_i of the form $\bar{K}_i(\beta)$ and when \bar{K}_i is infinite, we can let F_i be the diagonal matrix with entries in $\bar{K}_i(\beta)$, F_i is generated over \bar{K}_i by a matrix θ_i with distinct scalar multiples of β along the main diagonal.

Let $\theta = \theta_1 + \cdots + \theta_n$. $\bar{K}(\theta)$ is a maximal commutative separable subalgebra of \bar{A} and θ satisfies the monic polynomial $p(x) \in \bar{K}[x]$ of degree m given by $p(x) = \sum_i p_i(x)$ where $p_i(x)$ are the minimum polynomials over \bar{K}_i of the θ_i . Let $S = K[x]/(P(x))$ where $P(x)$ is a monic polynomial in $K[x]$ which maps onto $p(x) \bmod N$. Since K has no idempotents but 0 and 1, S is a finitely generated free K -algebra and $S/NS = \bar{K}(\theta)$ is separable so S is separable over K . Moreover $S \otimes_K A$ is in the kernel of the map from $B(S) \rightarrow B(S/NS)$, so we may from now on assume that A is in the kernel of the map from $B(K) \rightarrow B(K/NK)$.

With all notation as before we find an element $\theta \in \bar{A}$ with $\bar{K}(\theta)$ a maximal commutative separable subalgebra of \bar{A} and satisfying a monic polynomial of degree m over \bar{K} . Let $\beta \in A$ with β mapping to θ under the map from A to \bar{A} . Let $S = K \cdot 1 + K \cdot \beta + \cdots + K \cdot \beta^{m-1}$. Since $\{1, \theta, \dots, \theta^{m-1}\}$ is a free basis for a \bar{K} direct summand of \bar{A} , Nakayama's lemma (pg. 377 of [1]) implies that $\{1, \beta, \dots, \beta^{m-1}\}$ is a free set of generators of S over K which extends to a free set of generators of A over K so S is a K -direct summand of A .

We now show S is a subring of A by showing $\beta^m \in S$. Let \hat{K} be the completion of K and let $\hat{A} = \hat{K} \otimes_K \hat{A}$. Then $A/NA \simeq \hat{A}/N\hat{A}$ so since \hat{K} is a finite direct sum of complete local rings by Corollary 6.2 of [1], A is in the zero class of $B(\hat{K})$. Since A is a free \hat{K} -module of rank m^2 , $\hat{A} \simeq \text{Hom}_{\hat{K}}(P, P)$ with P a free \hat{K} -module of rank m . Since K is noetherian, we may assume A is a subring of \hat{A} so that $\beta \in \hat{A}$. By the Cayley-Hamilton Theorem, β satisfies a monic polynomial of degree m over \hat{K} . But S is a K -direct summand of A and since $\beta^m \in \hat{K} \otimes_K S$, $\beta^m \in S$.

S is separable over K since $S \cap NS = S \cap NA$ and thus $S/NS = \bar{K}(\theta)$ which is separable over \bar{K} . Let S^* be the commutant of S in A , by Theorem 2 of [3] S^* is a separable finitely generated projective K -algebra and the commutant in A of S^* is S . But $S^* \cap NA$ is a two-sided ideal in S^* so by Corollary 3.2 of [1] there is an ideal $M \subseteq S$ with $MS^* = S^* \cap NA$. But $MS^* \cap S^* = M = (S^* \cap NA) \cap S = NS$ so $M = NS$ and $S^* = NS^* + S$. Therefore by Nakayama's lemma, $S^* = S$, and by Theorem 5.6 of [1], $S \otimes_K A$ is in the zero class of $B(S)$.

For the last statement of the theorem assume K has no idempotents but 0 and 1, then $S \simeq Se_i \oplus \dots \oplus Se_n$ with Se_i a commutative finitely generated projective separable extension of $Ke_i \simeq K$ and with no idempotents but 0 and 1. Since $S \otimes_K A$ is in the zero class of $B(S)$, there is a finitely generated projective S -module P with $\text{Hom}_S(P, P) \simeq S \otimes_K A$. Then $Se_i \otimes_K A \simeq \text{Hom}_{Se_i}(Pe_i, Pe_i)$ and $Se_i \otimes_K A$ is in the zero class of $B(Se_i)$. This proves the theorem.

Corollary 2. *If K is a semi-local noetherian ring then $B(K) = UB(S/K)$ where S ranges over all strongly separable K -algebras.*

If K has no idempotents but 0 and 1 then by Corollary 2, Theorem A. 15 of [1] and the fact that any strongly separable K -algebra without idempotents but 0 and 1 is contained in a Galois extension of K [2]; the computation of $B(K)$ is reduced to the computation of $H^2(G, U(S))$ for each Galois extension S of K . Here $U(S)$ denotes the multiplicative units in S , G is the Galois group of S over K , and $H^2(G, U(S))$ is the second cohomology group of G acting on $U(S)$.

We now give E. Ingraham's proof of a module theoretic fact about the separable closure of a semi-local ring which has no idempotents but 0 and 1.

Proposition 3. *Let K be a semi-local ring with no idempotents but 0 and 1 and let Ω be the separable closure of K , then every finitely generated projective Ω -module is Ω -free.*

Proof. Let E be a finitely generated projective Ω -module. Then there exists an Ω -free module F of rank n , and a finitely generated projective Ω -module L with $E \oplus L = F$. Let $x_1 \dots x_n$ be a free bases for F over Ω , let y_1, \dots, y_m be a minimal set of generators for E over Ω , and let y_{m+1}, \dots, y_t be a minimal set of generators for L over Ω .

To prove the Proposition we will show that if $\sum_{i=1}^m \rho_i y_i = 0$ with $\rho_i \in \Omega$ then $\rho_i = 0$ for all i .

Now there are $\alpha_{ij} \in \Omega$ with $y_i = \sum_{j=1}^n \alpha_{ij} x_j$ and $\beta_{ij} \in \Omega$ with $x_i = \sum_{j=1}^t \beta_{ij} y_j$ since the $y_1 \cdots y_t$ generate all of F . Let $T = \{\alpha_{ij}, \beta_{ij}, \rho_i\}$. This is a finite set so there is a strongly separable K -subalgebra S of Ω with $T \subseteq S$. Let $F_S = Sx_1 + \cdots + Sx_n$, $E_S = Sy_1 + \cdots + Sy_m$ and $L_S = Sy_{m+1} + \cdots + Sy_t$. F_S is a free S -module since the x_i are linearly independent over Ω and hence over S . F_S and L_S are submodules of F_S since $T \subseteq S$. Since $F_S \subseteq E$ and $L_S \subseteq L$, $E_S \cap L_S = 0$. Again since $T \subseteq S$, $E_S + L_S = F_S$ so $E_S \oplus L_S = F_S$. Thus E_S is a finitely generated projective S -module. But S is a strongly separable K -subalgebra of Ω so S is a semi-local ring without idempotents but 0 and 1. Also $y_1 \cdots y_m$ is a minimal set of generators for E_S over S since they were minimal for E over Ω .

It is not a hard exercise using Nakayama's lemma to show that in this situation $y_1 \cdots y_m$ must be free basis for E_S over S . (see for example pg. 377 of [1]). Thus $y_1 \cdots y_m$ are linearly independent over S so all the $\rho_i = 0$ which is what we wanted to show.

We can now prove our final result.

Theorem 4. *If K is a semi-local noetherian ring without idempotents but 0 and 1 and Ω is the separable closure of K , then $B(\Omega)$ is trivial.*

Proof. Let A be a central separable Ω algebra. A is free as an Ω -module by Proposition 3, so let x_1, \dots, x_n be a free Ω -basis for A . Let $\{c_{ij}^k\}$ be the multiplication constants for the algebra A with respect to the basis x_1, \dots, x_n . That is, $x_i x_j = \sum_k c_{ij}^k x_k$ with $c_{ij}^k \in \Omega$. Let S be a strongly separable K -subalgebra of Ω containing $\{c_{ij}^k\}$. Define the central separable S -algebra A_S by letting A_S be the free S -module $Sx_1 + \cdots + Sx_n$ with multiplication constants $\{c_{ij}^k\}$. Since K is a semi-local noetherian ring S is, so by theorem 1 there is a strongly separable S -algebra T with no idempotents but 0 and 1 so that $T \otimes_S A_S$ is in the zero class of $B(T)$. But T is strongly separable over K so we may identify T with a K -subalgebra of Ω . Since $A \simeq \Omega \otimes_T (T \otimes_S A_S)$, we conclude A is in the zero class of $B(\Omega)$.

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