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ON NONEXISTENCE OF GLOBAL SOLUTIONS FOR SOME NONLINEAR INTEGRAL EQUATIONS

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1. Statement of the problem

Let $a(x)$ be a nonnegative continuous function defined on the m-dimensional Euclidean space R^m and let Δ be the Laplacian. Consider the following semilinear parabolic equation

$$(1.1)_1 \qquad \frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u^{1+\alpha},$$

with the initial condition

$$(1.1)_2 \qquad u(0, x) = a(x),$$

and be concerned with non-negative solutions.

H. Fujita [1] has proved that equation (1.1) has a global solution $u(t, x)$ for sufficiently small $a(x)$ when $m\alpha>2$ but (1.1) *has no global solution for any* $a(x)\not\equiv 0$ *when* $m\alpha<2$. Recently, K. Hayakawa [2] has proved that (1.1) *has no global solution even in the critical case* $m\alpha=2$ *if the dimension m equals* 1 *or* 2 (*and hence* $\alpha=2$ *or* 1, *respectively*).

In this paper we shall treat this kind of blowing-up problem for a more general equation as follows. Let $0<\beta\leqq 2$. Let $F(u)$ be a nonnegative continuous function with $F(0)=0$, defined on $[0, \infty)$, satisfying the following conditions:

(F.1) F is increasing and convex.

(F.2) There exists some $\alpha\in\left[0, \frac{\beta}{m}\right]$ and $c'\in(0, \infty)$, such that

$$\lim_{u\downarrow 0}\frac{F(u)}{u^{1+\alpha}} = c'.$$

(F.3) $\int_1^\infty \frac{du}{F(u)} < \infty$.

It is obvious that, for $0<m\alpha\leqq\beta$, $u^{1+\alpha}$ satisfies the above conditions.

Here and hereafter, u denotes a single variable as well as function in obvious contexts.

For $0<\beta\leqq 2$, let $\left(-\frac{\Delta}{2}\right)^{\beta/2}$ denote the fractional power of the operator $-\frac{\Delta}{2}$. As a generalization of (1.1), we consider the equation

$$\text{(1.2)}\qquad \begin{cases} \dfrac{\partial u}{\partial t} = -\left(-\dfrac{\Delta}{2}\right)^{\beta/2} u+F(u)\,, \\ u(0,\, x) = a(x)\,. \end{cases}$$

Let $p(t, x)$ be the fundamental solution of (1.2) for $F(u)\equiv 0$, i.e., the density of the semigroup of (m-dimensional) symmetric stable process with index β. It is well known that $p(t, x)$ is given by

$$\text{(1.3)}\qquad \int_{R^m} e^{iz\cdot x} p(t,\, x)dx = e^{-t/2|z|^\beta} \qquad 0<\beta\leqq 2\,.$$

Using this $p(t, x)$, we can transform (1.2) into the integral equation

$$\text{(A)}\qquad u(t,\, x) = \int_{R^m} p(t,\, x-y)a(y)dy + \int_0^t ds \int_{R^m} p(t-s,\, x-y)F[u(s,\, y)]dy\,, \qquad t>0, \quad x\in R^m\,.$$

What we are going to prove is the following.

Theorem. *Let* $0<\beta\leqq 2$. *Suppose that* $a(x)$ *is a nontrivial* ($\not\equiv 0$), *nonnegative, and continuous function on* R^m, *that* $F(u)$ *satisfies* (F.1), (F.2), (F.3), *and that* $p(t, x)$ *is defined by* (1.3). *Then the nonnegative solution* $u(t, x)$ *of the integral equation* (*A*) *blows up, i.e., there exists some* $t_0>0$ *such that* $u(t, x)=\infty$ *for every* $t\geqq t_0$ *and* $x\in R^m$.

2. Some properties of $p(t, x)$

We here collect some properties of $p(t, x)$ which are required to show our Theorem. By (1.3), we have

$$\text{(2.1)}\qquad p(t,\, x) = t^{-m/\beta} p(1,\, t^{-1/\beta}x)\,,$$

$$\text{(2.2)}\qquad p(ts,\, x) = t^{-m/\beta} p(s,\, t^{-1/\beta}x)\,.$$

Note that $p(t, 0)$ is a decreasing function of t. It is known (see [3; pp. 259–268.]) that

$$\begin{aligned} p(t,\, x) &= \int_0^\infty f_{t,\beta/2}(s) T(s,\, x)ds \qquad \text{for } 0<\beta<2\,, \\ &= T(t,\, x) \qquad \text{for } \beta=2\,, \end{aligned}$$

where $$f_{t,\beta/2}(s) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{zs-tz^{\beta/2}} dz \geqq 0,\ \sigma>0,\ s>0\,,$$

$$T(s, x) = \left(\frac{1}{2\pi s}\right)^{m/2} \exp\left(-\frac{|x|^2}{2s}\right).$$

The above relation implies that $p(t, x)$ is a decreasing function of $|x|$, i.e.,

$$(2.3) \qquad p(t, x) \leqq p(t, y) \qquad \text{whenever} \quad |x| \geqq |y| .$$

We sometimes write $p(t, |x|)$ for $p(t, x)$. Combining (2.1) and (2.3),

$$(2.4) \qquad p(t, x) \geqq \left(\frac{s}{t}\right)^{m/\beta} p(s, x) \qquad \text{for } t \geqq s .$$

Finally, it follows that

(2.5) if $p(t, 0) \leqq 1$ and $\tau \geqq 2$, then $p\left(t, \frac{1}{\tau}(x-y)\right) \geqq p(t, x)p(t, y)$.

Because $\frac{1}{\tau}|x-y| \leqq \frac{2}{\tau}|x| \vee \frac{2}{\tau}|y| \leqq |x| \vee |y|$, and hence $p\left(t, \frac{1}{\tau}(x-y)\right)$ $\geqq p(t, |x| \vee |y|) \geqq p(t, |x|) \wedge p(t, |y|) \geqq p(t, x)p(t, y)$.

3. Preliminary lemmas

Lemma 1. *If F satisfies* (F.1) *and* (F.3), *then*

$$(3.1) \qquad \lim_{u \to \infty} \frac{1}{u} F(u) = \infty .$$

Proof. Since F is convex, it is obvious that $\frac{1}{u}(F(u) - F(0))$ is a monotone increasing function. If $\lim_{u \to \infty} \frac{1}{u}(F(u) - F(0)) = M < \infty$, then $\frac{1}{u}(F(u) - F(0)) \leqq M$ for all $u > 0$, i.e., $\frac{1}{Mu} \leqq \frac{1}{F(u) - F(0)}$. This contradicts assumption (F.3).

If $F(u)$ is increasing, $F(\infty)$ is defined by

$$(3.2) \qquad F(\infty) = \lim_{u \to \infty} F(u) \leqq \infty .$$

Lemma 2. (Jensen's inequality) *Let ρ be a probability measure on R^m and $u(x)$ a nonnegative function. Suppose that $F(u)$ satisfies* (F.1). *Then we have*

$$(3.3) \qquad F\left(\int_{R^m} u\, d\rho\right) \leqq \int_{R^m} F \circ u\, d\rho .$$

Note that this inequality is valid even when $\int_{R^m} u\, d\rho = \infty$.

Lemma 3. *Suppose that $F(u)$ ($\not\equiv 0$) satisfies* (F.1). *Let $u(t, x)$ be a nonnegative solution of (A) and let*

$$(3.4) \qquad f(t) = \int_{R^m} p(t, x) u(t, x)\, dx .$$

Then the following two conditions are equivalent:
(a) $u(t, x)$ *blows up.*
(b) $f(t)$ *blows up, i.e., there exists some* $t_1>0$ *such that* $f(t)=\infty$ *whenever* $t\geqq t_1$.

Proof. It is enough to show that (b) implies (a). We may assume $p(t_1, 0)\leqq 1$, so that $p(t, 0)\leqq 1$ for any $t\geqq t_1$. If $t_1\leqq t$, $t\leqq s\leqq \frac{8}{2^\beta+1}t$, then

$$\begin{aligned} p(8t-s, x-y) &= p\left(s\left(\frac{8t-s}{s}\right), x-y\right) \\ &= \left(\frac{s}{8t-s}\right)^{m/\beta} p\left(s, \left(\frac{s}{8t-s}\right)^{1/\beta}(x-y)\right) \quad \text{by (2.1)} \\ &\geqq \left(\frac{s}{8t-s}\right)^{m/\beta} p(s, x)p(s, y) \quad \text{by (2.5)}. \end{aligned}$$

Therefore,

$$\int_{R^m} p(8t-s, x-y)u(s, y)dy \geqq \left(\frac{s}{8t-s}\right)^{m/\beta} p(s, x)f(s) = \infty .$$

Finally, applying Jensen's inequality to (A) and noting that $F(\infty)=\infty$, we have $u(8t, x)\geqq \int_t^{(8/2^\beta+1)t} ds F[\int_{R^m} p(8t-s, x-y)u(s, y)dy]=\infty$, so that $u(t, x)=\infty$ for any $t\geqq 8t_1$ and $x\in R^m$.

4. Proof of the theorem

Let $u(t, x)$ be a nonnegative solution of (A), then we can find $t_0>0$, $c>0$, $\gamma>0$ such that $u(t_0, x)\geqq cp(\gamma, x)$. In fact, if we choose $t_0>0$ such that $p(t_0, 0)\leqq 1$, we have

$$\begin{aligned} p(t_0, x-y) &= p\left(t_0, \frac{1}{2}(2x-2y)\right) \\ &\geqq p(t_0, 2x)p(t_0, 2y) \quad \text{by (2.5)} \\ &= 2^{-m} p\left(\frac{t_0}{2^\beta}, x\right) p(t_0, 2y) \quad \text{by (2.2)}. \end{aligned}$$

Therefore, $u(t_0, x)\geqq \int_{R^m} p(t_0, 2y)a(y)dy\cdot 2^{-m}\cdot p\left(\frac{t_0}{2^\beta}, x\right)$. But $u(t+t_0, x)$ satisfies

$$\begin{aligned} (4.1) \qquad u(t+t_0, x) &= \int_{R^m} p(t, x-y)u(t_0, y)dy \\ &\quad + \int_0^t ds \int_{R^m} p(t-s, x-y)F[u(s+t_0, y)]dy \\ &\qquad\qquad t>0, x\in R^m , \end{aligned}$$

so that

$$(4.2)\qquad u(t+t_0, x) \geqq cp(t+\gamma, x) + \int_0^t ds \int_{R^m} p(t-s, x-y) F[u(s+t_0, y)] dy\,.$$

Hence, by the comparison theorem, it is enough to show that the solution $v(t, x)$ of the equation

$$(\mathrm{B})\qquad v(t, x) = cp(t+\gamma, x) + \int_0^t ds \int_{R^m} p(t-s, x-y) F[v(s, y)] dy$$

blows up, or by virtue of Lemma 3, that

$$(4.3)\qquad f(t) = \int_{R^m} p(t, x) v(t, x) dx$$

blows up. Multiplying both sides of (B) by $p(t,x)$, and integrating, we have

$$\begin{aligned}(4.4)\qquad f(t) &= cp(2t+\gamma, 0) + \int_0^t ds \int_{R^m} p(2t-s, y) F[v(s, y)] dy \\ &\geqq cp(1, 0)(2t+\gamma)^{-m/\beta} + \int_0^t ds \left(\frac{s}{2t-s}\right)^{m/\beta} \int_{R^m} p(s, y) F[v(s, y)] dy \\ &\qquad\qquad \text{(by (2.1), (2.4))} \\ &\geqq cp(1, 0)(2t+\gamma)^{-m/\beta} + \int_0^t ds \left(\frac{s}{2t-s}\right)^{m/\beta} F\left[\int_{R^m} p(s, y) v(s, y) dy\right] \\ &\qquad\qquad \text{(by Jensen's inequality)} \\ &\geqq cp(1, 0)(2t+\gamma)^{-m/\beta} + \int_0^t ds \left(\frac{s}{2t}\right)^{m/\beta} F[f(s)]\,.\end{aligned}$$

Let $\delta > 0$ be a fixed positive constant. Hereafter we always assume $t \geqq \delta$. Put $f_1(t) = t^{m/\beta} f(t)$, then by (4.4),

$$(4.5)\qquad f_1(t) \geqq cp(1, 0)\left(\frac{\delta}{2\delta+\gamma}\right)^{m/\beta} + \int_\delta^t ds \left(\frac{s}{2}\right)^{m/\beta} F[f_1(s) s^{-m/\beta}]\,.$$

Let $f_2(t)$ be the solution of

$$(4.6)\qquad f_2(t) = cp(1, 0)\left(\frac{\delta}{2\delta+\gamma}\right)^{m/\beta} + \int_\delta^t ds \left(\frac{s}{2}\right)^{m/\beta} F[f_2(s) s^{-m/\beta}]\,.$$

By assumption (F.2) and Lemma 1, there exists $a > 0$ such that $\max\left(\frac{F(u)}{u}, \frac{F(u)}{u^{1+\alpha}}\right) \geqq a$ for all $u > 0$. Since

$$s^{m/\beta} F(f_2(s) s^{-m/\beta}) = \frac{F(f_2(s) s^{-m/\beta})}{f_2(s) s^{-m/\beta}} \cdot f_2(s) = \frac{F(f_2(s) s^{-m/\beta})}{(f_2(s) s^{-m/\beta})^{1+\alpha}} \cdot f_2(s)^{1+\alpha} s^{-m/\beta\,\alpha}$$

it follows that

$$s^{m/\beta} F(f_2(s) s^{-m/\beta}) \geqq a \cdot \min\,(f_2(s), f_2(s)^{1+\alpha} s^{-m/\beta\,\alpha})\,.$$

Therefore,

$$f_2(t)\geqq cp(1,0)\Big(\frac{\delta}{2\delta+\gamma}\Big)^{m/\beta}+\int_\delta^t ds\Big(\frac{1}{2}\Big)^{m/\beta}\cdot a\cdot\min\,(f_2(s), f_2(s)^{1+\alpha}s^{-m/\beta\,\alpha})\,.$$

Let $f_3(t)$ be the solution of the integral equation

$$(4.7)\quad f_3(t)=cp(1,0)\Big(\frac{\delta}{2\delta+\gamma}\Big)^{m/\beta}+\int_\delta^t ds\Big(\frac{1}{2}\Big)^{m/\beta}a\cdot\min\,(f_3(s), f_3(s)^{1+\alpha}s^{-m/\beta\,\alpha})\,,$$

or, equivalently, the ordinary differential equation

$$(4.8)\qquad \begin{cases} \dfrac{df_3(t)}{dt}=\Big(\dfrac{1}{2}\Big)^{m/\beta}a\cdot\min\,(f_3(t), f_3(t)^{1+\alpha}t^{-m/\beta\,\alpha})\,,\\[2mm] f_3(\delta)=cp(1,0)\Big(\dfrac{\delta}{2\delta+\gamma}\Big)^{m/\beta}\,. \end{cases}$$

We shall show that $f_3(t)$ increases exponentially fast. This is obvious if $\alpha=0$. Next we consider the case $\alpha>0$. By the comparison theorem, c can be chosen arbitrarily small. We choose c, if necessary, satisfying the following three conditions (4.9), (4.10) and (4.11).

$$(4.9)\qquad f_3(\delta)<\delta^{m/\beta}\,.$$

Put $\theta(c)=\inf\{t\geqq\delta; f_3(t)=t^{m/\beta}\}$. For $t\in[\delta,\theta(c)]$, $\min\{f_3(t), f_3(t)^{1+\alpha}t^{-m/\beta\,\alpha}\}$ $=f_3(t)^{1+\alpha}t^{-m/\beta\,\alpha}$ by (4.9). Therefore, $f_3(t)$ satisfies the equation $\dfrac{df_3(t)}{dt}=\Big(\dfrac{1}{2}\Big)^{m/\beta}$ $af_3(t)^{1+\alpha}t^{-(m/\beta)\alpha}$, which implies that $\theta(c)<\infty$. (We here use the condition $m\alpha\leqq\beta$ in (F.2)). On the other hand $\lim_{c\downarrow 0}\theta(c)=\infty$. Hence, if c is small enough, we have

$$(4.10)\qquad \frac{\exp\Big[\Big(\frac{1}{2}\Big)^{m/\beta}at\Big]}{t^{m/\beta}}\geqq\frac{\exp\Big[\Big(\frac{1}{2}\Big)^{m/\beta}a\theta(c)\Big]}{\theta(c)^{m/\beta}}\qquad t\geqq\theta(c)\,,$$

$$(4.11)\qquad \theta(c)^{-m/\beta\,\alpha}\leqq\alpha\Big(\frac{1}{2}\Big)^{m/\beta}a\int_{\theta(c)}^t s^{-m/\beta\,\alpha}ds+t^{-m/\beta\,\alpha}\qquad t\geqq\theta(c)\,.$$

For $t\geqq\theta(c)$,

$$\begin{cases} f_3(\theta(c))=\theta(c)^{m/\beta}\,,\\[1mm] \dfrac{df_3(t)}{dt}=\Big(\dfrac{1}{2}\Big)^{m/\beta}a\cdot\min\,(f_3(t), f_3(t)^{1+\alpha}t^{-m/\beta\,\alpha})\,. \end{cases}$$

Let $x_1(t)$ and $x_2(t)$ be the solutions of the following equations;

$$(4.12)\qquad \begin{cases} x_1(\theta(c))=\theta(c)^{m/\beta}\,,\\[1mm] \dfrac{dx_1}{dt}=\Big(\dfrac{1}{2}\Big)^{m/\beta}ax_1\,, \end{cases}$$

$$(4.13)\qquad \begin{cases} x_2(\theta(c)) = \theta(c)^{m/\beta}, \\ \dfrac{dx_2}{dt} = \left(\dfrac{1}{2}\right)^{m/\beta} a x_2^{1+\alpha} t^{-m/\beta\,\alpha}. \end{cases}$$

Then it follows that, for $t \geqq \theta(c)$, $x_1(t) \geqq t^{m/\beta}$ by (4.10) and $x_2(t) \geqq t^{m/\beta}$ by (4.11). From this, it is not difficult to see that $f_3(t) = x_1(t)$ for $t \geqq \theta(c)$. Thus $f_3(t)$ increases exponentially fast. Hence there exists $b > 0$ such that

$$(4.14)\qquad f_3(t) \geqq be^{bt}.$$

By the comparison theorem, $f_1 \geqq f_2 \geqq f_3 \geqq be^{bt}$. Put $h(t) = t^{-m/\beta} f_2(t)$. Then, since $f(t) \geqq h(t)$, it is sufficient to show that $h(t) = \infty$ if t is large enough. Suppose that $h(t) < \infty$ for every $t > \delta$. Noting that $h(t) \to \infty$ as $t \to \infty$ and using Lemma 1, we have

$$(4.15)\qquad \sup_{t \geqq t'} \frac{m}{\beta t} \frac{h(t)}{F(h(t))} \leqq \left(\frac{1}{2}\right)^{m/\beta+1} \qquad \text{for some } t' > 0.$$

By (4.6), (4.15), we have for $t \geqq t'$

$$\begin{aligned} \frac{dh(t)}{dt} &= -\frac{m}{\beta t} t^{-m/\beta} f_2(t) + t^{-m/\beta} \frac{df_2(t)}{dt} \\ &= -\frac{m}{\beta t} t^{-m/\beta} f_2(t) + \left(\frac{1}{2}\right)^{m/\beta} F(f_2(t) t^{-m/\beta}) \\ &= \left(\frac{1}{2}\right)^{m/\beta} F(h(t)) - \frac{m}{\beta t} h(t) \\ &\geqq \left(\frac{1}{2}\right)^{m/\beta+1} F(h(t)). \end{aligned}$$

It then follows that

$$\left(\frac{1}{2}\right)^{m/\beta+1} (t - t') \leqq \int_{h(t')}^{h(t)} \frac{dx}{F(x)} \leqq \int_{h(t')}^{\infty} \frac{dx}{F(x)} < \infty$$

for any $t \geqq t'$, which is a contradiction.

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References

[1] H. Fujita: *On the blowing up of solutions of the Cauchy problem for* $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo Sect I, **13** (1966), 109–124.

[2] K. Hayakawa: *On nonexistence of global solutions of some semilinear parabolic differential equations*, Proc. Japan Acad. **49** (1973), 503–505.

[3] K. Yosida: Functional Analysis, Springer, 1966.