

Title	On nonexistence of global solutions for some nonlinear integral equations
Author(s)	Sugitani, Sadao
Citation	Osaka Journal of Mathematics. 12(1) P.45-P.51
Issue Date	1975
Text Version	publisher
URL	<a href="https://doi.org/10.18910/6078">https://doi.org/10.18910/6078</a>
DOI	10.18910/6078
rights	
Note	

*Osaka University Knowledge Archive : OUKA*

<https://ir.library.osaka-u.ac.jp/>

Osaka University

## ON NONEXISTENCE OF GLOBAL SOLUTIONS FOR SOME NONLINEAR INTEGRAL EQUATIONS

SADAO SUGITANI

(Received February 1, 1974)

### 1. Statement of the problem

Let  $a(x)$  be a nonnegative continuous function defined on the  $m$ -dimensional Euclidean space  $R^m$  and let  $\Delta$  be the Laplacian. Consider the following semi-linear parabolic equation

$$(1.1)_1 \quad \frac{\partial u}{\partial t} = \frac{1}{2} \Delta u + u^{1+\alpha},$$

with the initial condition

$$(1.1)_2 \quad u(0, x) = a(x),$$

and be concerned with non-negative solutions.

H. Fujita [1] has proved that equation (1.1) has a global solution  $u(t, x)$  for sufficiently small  $a(x)$  when  $m\alpha > 2$  but (1.1) has no global solution for any  $a(x) \not\equiv 0$  when  $m\alpha < 2$ . Recently, K. Hayakawa [2] has proved that (1.1) has no global solution even in the critical case  $m\alpha = 2$  if the dimension  $m$  equals 1 or 2 (and hence  $\alpha = 2$  or 1, respectively).

In this paper we shall treat this kind of blowing-up problem for a more general equation as follows. Let  $0 < \beta \leq 2$ . Let  $F(u)$  be a nonnegative continuous function with  $F(0) = 0$ , defined on  $[0, \infty)$ , satisfying the following conditions:

(F.1)  $F$  is increasing and convex.

(F.2) There exists some  $\alpha \in \left[0, \frac{\beta}{m}\right]$  and  $c' \in (0, \infty)$ , such that

$$\lim_{u \downarrow 0} \frac{F(u)}{u^{1+\alpha}} = c'.$$

(F.3)  $\int_1^\infty \frac{du}{F(u)} < \infty$ .

It is obvious that, for  $0 < m\alpha \leq \beta$ ,  $u^{1+\alpha}$  satisfies the above conditions.

Here and hereafter,  $u$  denotes a single variable as well as function in obvious

contexts.

For  $0 < \beta \leq 2$ , let  $\left(-\frac{\Delta}{2}\right)^{\beta/2}$  denote the fractional power of the operator  $-\frac{\Delta}{2}$ . As a generalization of (1.1), we consider the equation

$$(1.2) \quad \begin{cases} \frac{\partial u}{\partial t} = -\left(-\frac{\Delta}{2}\right)^{\beta/2} u + F(u), \\ u(0, x) = a(x). \end{cases}$$

Let  $p(t, x)$  be the fundamental solution of (1.2) for  $F(u) \equiv 0$ , i.e., the density of the semigroup of ( $m$ -dimensional) symmetric stable process with index  $\beta$ . It is well known that  $p(t, x)$  is given by

$$(1.3) \quad \int_{R^m} e^{iz \cdot x} p(t, x) dx = e^{-t/2|z|^\beta} \quad 0 < \beta \leq 2.$$

Using this  $p(t, x)$ , we can transform (1.2) into the integral equation

$$(A) \quad u(t, x) = \int_{R^m} p(t, x-y) a(y) dy + \int_0^t \int_{R^m} p(t-s, x-y) F[u(s, y)] dy, \\ t > 0, \quad x \in R^m.$$

What we are going to prove is the following.

**Theorem.** *Let  $0 < \beta \leq 2$ . Suppose that  $a(x)$  is a nontrivial ( $\neq 0$ ), non-negative, and continuous function on  $R^m$ , that  $F(u)$  satisfies (F.1), (F.2), (F.3), and that  $p(t, x)$  is defined by (1.3). Then the nonnegative solution  $u(t, x)$  of the integral equation (A) blows up, i.e., there exists some  $t_0 > 0$  such that  $u(t, x) = \infty$  for every  $t \geq t_0$  and  $x \in R^m$ .*

## 2. Some properties of $p(t, x)$

We here collect some properties of  $p(t, x)$  which are required to show our Theorem. By (1.3), we have

$$(2.1) \quad p(t, x) = t^{-m/\beta} p(1, t^{-1/\beta} x),$$

$$(2.2) \quad p(ts, x) = t^{-m/\beta} p(s, t^{-1/\beta} x).$$

Note that  $p(t, 0)$  is a decreasing function of  $t$ . It is known (see [3; pp. 259–268.]) that

$$p(t, x) = \int_0^\infty f_{t, \beta/2}(s) T(s, x) ds \quad \text{for } 0 < \beta < 2, \\ = T(t, x) \quad \text{for } \beta = 2,$$

where  $f_{t, \beta/2}(s) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{zs - tz^{\beta/2}} dz \geq 0$ ,  $\sigma > 0$ ,  $s > 0$ ,

$$T(s, x) = \left(\frac{1}{2\pi s}\right)^{m/2} \exp\left(-\frac{|x|^2}{2s}\right).$$

The above relation implies that  $p(t, x)$  is a decreasing function of  $|x|$ , i.e.,

$$(2.3) \quad p(t, x) \leq p(t, y) \quad \text{whenever } |x| \geq |y|.$$

We sometimes write  $p(t, |x|)$  for  $p(t, x)$ . Combining (2.1) and (2.3),

$$(2.4) \quad p(t, x) \geq \left(\frac{s}{t}\right)^{m/\beta} p(s, x) \quad \text{for } t \geq s.$$

Finally, it follows that

$$(2.5) \quad \text{if } p(t, 0) \leq 1 \text{ and } \tau \geq 2, \text{ then } p\left(t, \frac{1}{\tau}(x-y)\right) \geq p(t, x)p(t, y).$$

Because  $\frac{1}{\tau}|x-y| \leq \frac{2}{\tau}|x| \vee \frac{2}{\tau}|y| \leq |x| \vee |y|$ , and hence  $p\left(t, \frac{1}{\tau}(x-y)\right) \geq p(t, |x| \vee |y|) \geq p(t, |x|) \wedge p(t, |y|) \geq p(t, x)p(t, y)$ .

### 3. Preliminary lemmas

**Lemma 1.** *If  $F$  satisfies (F.1) and (F.3), then*

$$(3.1) \quad \lim_{u \rightarrow \infty} \frac{1}{u} F(u) = \infty.$$

*Proof.* Since  $F$  is convex, it is obvious that  $\frac{1}{u}(F(u)-F(0))$  is a monotone increasing function. If  $\lim_{u \rightarrow \infty} \frac{1}{u}(F(u)-F(0)) = M < \infty$ , then  $\frac{1}{u}(F(u)-F(0)) \leq M$  for all  $u > 0$ , i.e.,  $\frac{1}{Mu} \leq \frac{1}{F(u)-F(0)}$ . This contradicts assumption (F.3).

If  $F(u)$  is increasing,  $F(\infty)$  is defined by

$$(3.2) \quad F(\infty) = \lim_{u \rightarrow \infty} F(u) \leq \infty.$$

**Lemma 2.** (Jensen's inequality) *Let  $\rho$  be a probability measure on  $R^m$  and  $u(x)$  a nonnegative function. Suppose that  $F(u)$  satisfies (F.1). Then we have*

$$(3.3) \quad F\left(\int_{R^m} u d\rho\right) \leq \int_{R^m} F \circ u d\rho.$$

Note that this inequality is valid even when  $\int_{R^m} u d\rho = \infty$ .

**Lemma 3.** *Suppose that  $F(u)$  ( $\neq 0$ ) satisfies (F.1). Let  $u(t, x)$  be a nonnegative solution of (A) and let*

$$(3.4) \quad f(t) = \int_{R^m} p(t, x) u(t, x) dx.$$

Then the following two conditions are equivalent:

- (a)  $u(t, x)$  blows up.  
 (b)  $f(t)$  blows up, i.e., there exists some  $t_1 > 0$  such that  $f(t) = \infty$  whenever  $t \geq t_1$ .

Proof. It is enough to show that (b) implies (a). We may assume  $p(t_1, 0) \leq 1$ , so that  $p(t, 0) \leq 1$  for any  $t \geq t_1$ . If  $t_1 \leq t$ ,  $t \leq s \leq \frac{8}{2^\beta + 1}t$ , then

$$\begin{aligned} p(8t-s, x-y) &= p\left(s\left(\frac{8t-s}{s}\right), x-y\right) \\ &= \left(\frac{s}{8t-s}\right)^{m/\beta} p\left(s, \left(\frac{s}{8t-s}\right)^{1/\beta}(x-y)\right) \quad \text{by (2.1)} \\ &\geq \left(\frac{s}{8t-s}\right)^{m/\beta} p(s, x)p(s, y) \quad \text{by (2.5)}. \end{aligned}$$

Therefore,

$$\int_{R^m} p(8t-s, x-y)u(s, y)dy \geq \left(\frac{s}{8t-s}\right)^{m/\beta} p(s, x)f(s) = \infty.$$

Finally, applying Jensen's inequality to (A) and noting that  $F(\infty) = \infty$ , we have  $u(8t, x) \geq \int_t^{(8/2^{\beta+1})t} ds F\left[\int_{R^m} p(8t-s, x-y)u(s, y)dy\right] = \infty$ , so that  $u(t, x) = \infty$  for any  $t \geq 8t_1$  and  $x \in R^m$ .

#### 4. Proof of the theorem

Let  $u(t, x)$  be a nonnegative solution of (A), then we can find  $t_0 > 0$ ,  $c > 0$ ,  $\gamma > 0$  such that  $u(t_0, x) \geq cp(\gamma, x)$ . In fact, if we choose  $t_0 > 0$  such that  $p(t_0, 0) \leq 1$ , we have

$$\begin{aligned} p(t_0, x-y) &= p\left(t_0, \frac{1}{2}(2x-2y)\right) \\ &\geq p(t_0, 2x)p(t_0, 2y) \quad \text{by (2.5)} \\ &= 2^{-m}p\left(\frac{t_0}{2^\beta}, x\right)p(t_0, 2y) \quad \text{by (2.2)}. \end{aligned}$$

Therefore,  $u(t_0, x) \geq \int_{R^m} p(t_0, 2y)a(y)dy \cdot 2^{-m} \cdot p\left(\frac{t_0}{2^\beta}, x\right)$ . But  $u(t+t_0, x)$  satisfies

$$\begin{aligned} (4.1) \quad u(t+t_0, x) &= \int_{R^m} p(t, x-y)u(t_0, y)dy \\ &\quad + \int_0^t ds \int_{R^m} p(t-s, x-y)F[u(s+t_0, y)]dy \\ &\quad t > 0, x \in R^m, \end{aligned}$$

so that

$$(4.2) \quad u(t+t_0, x) \geq cp(t+\gamma, x) + \int_0^t ds \int_{R^m} p(t-s, x-y) F[u(s+t_0, y)] dy.$$

Hence, by the comparison theorem, it is enough to show that the solution  $v(t, x)$  of the equation

$$(B) \quad v(t, x) = cp(t+\gamma, x) + \int_0^t ds \int_{R^m} p(t-s, x-y) F[v(s, y)] dy$$

blows up, or by virtue of Lemma 3, that

$$(4.3) \quad f(t) = \int_{R^m} p(t, x) v(t, x) dx$$

blows up. Multiplying both sides of (B) by  $p(t, x)$ , and integrating, we have

$$(4.4) \quad \begin{aligned} f(t) &= cp(2t+\gamma, 0) + \int_0^t ds \int_{R^m} p(2t-s, y) F[v(s, y)] dy \\ &\geq cp(1, 0)(2t+\gamma)^{-m/\beta} + \int_0^t ds \left(\frac{s}{2t-s}\right)^{m/\beta} \int_{R^m} p(s, y) F[v(s, y)] dy \\ &\quad \text{(by (2.1), (2.4))} \\ &\geq cp(1, 0)(2t+\gamma)^{-m/\beta} + \int_0^t ds \left(\frac{s}{2t-s}\right)^{m/\beta} F\left[\int_{R^m} p(s, y) v(s, y) dy\right] \\ &\quad \text{(by Jensen's inequality)} \\ &\geq cp(1, 0)(2t+\gamma)^{-m/\beta} + \int_0^t ds \left(\frac{s}{2t}\right)^{m/\beta} F[f(s)]. \end{aligned}$$

Let  $\delta > 0$  be a fixed positive constant. Hereafter we always assume  $t \geq \delta$ . Put  $f_1(t) = t^{m/\beta} f(t)$ , then by (4.4),

$$(4.5) \quad f_1(t) \geq cp(1, 0) \left(\frac{\delta}{2\delta+\gamma}\right)^{m/\beta} + \int_\delta^t ds \left(\frac{s}{2}\right)^{m/\beta} F[f_1(s) s^{-m/\beta}].$$

Let  $f_2(t)$  be the solution of

$$(4.6) \quad f_2(t) = cp(1, 0) \left(\frac{\delta}{2\delta+\gamma}\right)^{m/\beta} + \int_\delta^t ds \left(\frac{s}{2}\right)^{m/\beta} F[f_2(s) s^{-m/\beta}].$$

By assumption (F.2) and Lemma 1, there exists  $a > 0$  such that

$\max\left(\frac{F(u)}{u}, \frac{F(u)}{u^{1+\alpha}}\right) \geq a$  for all  $u > 0$ . Since

$$s^{m/\beta} F(f_2(s) s^{-m/\beta}) = \frac{F(f_2(s) s^{-m/\beta})}{f_2(s) s^{-m/\beta}} \cdot f_2(s) = \frac{F(f_2(s) s^{-m/\beta})}{(f_2(s) s^{-m/\beta})^{1+\alpha}} \cdot f_2(s)^{1+\alpha} s^{-m/\beta \alpha}$$

it follows that

$$s^{m/\beta} F(f_2(s) s^{-m/\beta}) \geq a \cdot \min(f_2(s), f_2(s)^{1+\alpha} s^{-m/\beta \alpha}).$$

Therefore,

$$f_2(t) \geq cp(1, 0) \left( \frac{\delta}{2\delta + \gamma} \right)^{m/\beta} + \int_{\delta}^t ds \left( \frac{1}{2} \right)^{m/\beta} \cdot a \cdot \min (f_2(s), f_2(s)^{1+\alpha} s^{-m/\beta \alpha}).$$

Let  $f_3(t)$  be the solution of the integral equation

$$(4.7) \quad f_3(t) = cp(1, 0) \left( \frac{\delta}{2\delta + \gamma} \right)^{m/\beta} + \int_{\delta}^t ds \left( \frac{1}{2} \right)^{m/\beta} a \cdot \min (f_3(s), f_3(s)^{1+\alpha} s^{-m/\beta \alpha}),$$

or, equivalently, the ordinary differential equation

$$(4.8) \quad \begin{cases} \frac{df_3(t)}{dt} = \left( \frac{1}{2} \right)^{m/\beta} a \cdot \min (f_3(t), f_3(t)^{1+\alpha} t^{-m/\beta \alpha}), \\ f_3(\delta) = cp(1, 0) \left( \frac{\delta}{2\delta + \gamma} \right)^{m/\beta}. \end{cases}$$

We shall show that  $f_3(t)$  increases exponentially fast. This is obvious if  $\alpha=0$ . Next we consider the case  $\alpha>0$ . By the comparison theorem,  $c$  can be chosen arbitrarily small. We choose  $c$ , if necessary, satisfying the following three conditions (4.9), (4.10) and (4.11).

$$(4.9) \quad f_3(\delta) < \delta^{m/\beta}.$$

Put  $\theta(c) = \inf \{t \geq \delta; f_3(t) = t^{m/\beta}\}$ . For  $t \in [\delta, \theta(c)]$ ,  $\min \{f_3(t), f_3(t)^{1+\alpha} t^{-m/\beta \alpha}\} = f_3(t)^{1+\alpha} t^{-m/\beta \alpha}$  by (4.9). Therefore,  $f_3(t)$  satisfies the equation  $\frac{df_3(t)}{dt} = \left( \frac{1}{2} \right)^{m/\beta} a f_3(t)^{1+\alpha} t^{-(m/\beta \alpha)}$ , which implies that  $\theta(c) < \infty$ . (We here use the condition  $m\alpha \leq \beta$  in (F.2)). On the other hand  $\lim_{c \downarrow 0} \theta(c) = \infty$ . Hence, if  $c$  is small enough, we have

$$(4.10) \quad \frac{\exp \left[ \left( \frac{1}{2} \right)^{m/\beta} at \right]}{t^{m/\beta}} \geq \frac{\exp \left[ \left( \frac{1}{2} \right)^{m/\beta} a\theta(c) \right]}{\theta(c)^{m/\beta}} \quad t \geq \theta(c),$$

$$(4.11) \quad \theta(c)^{-m/\beta \alpha} \leq \alpha \left( \frac{1}{2} \right)^{m/\beta} a \int_{\theta(c)}^t s^{-m/\beta \alpha} ds + t^{-m/\beta \alpha} \quad t \geq \theta(c).$$

For  $t \geq \theta(c)$ ,

$$\begin{cases} f_3(\theta(c)) = \theta(c)^{m/\beta}, \\ \frac{df_3(t)}{dt} = \left( \frac{1}{2} \right)^{m/\beta} a \cdot \min (f_3(t), f_3(t)^{1+\alpha} t^{-m/\beta \alpha}). \end{cases}$$

Let  $x_1(t)$  and  $x_2(t)$  be the solutions of the following equations;

$$(4.12) \quad \begin{cases} x_1(\theta(c)) = \theta(c)^{m/\beta}, \\ \frac{dx_1}{dt} = \left( \frac{1}{2} \right)^{m/\beta} a x_1, \end{cases}$$

$$(4.13) \quad \begin{cases} x_2(\theta(c)) = \theta(c)^{m/\beta}, \\ \frac{dx_2}{dt} = \left(\frac{1}{2}\right)^{m/\beta} a x_2^{1+\alpha} t^{-m/\beta}. \end{cases}$$

Then it follows that, for  $t \geq \theta(c)$ ,  $x_1(t) \geq t^{m/\beta}$  by (4.10) and  $x_2(t) \geq t^{m/\beta}$  by (4.11). From this, it is not difficult to see that  $f_3(t) = x_1(t)$  for  $t \geq \theta(c)$ . Thus  $f_3(t)$  increases exponentially fast. Hence there exists  $b > 0$  such that

$$(4.14) \quad f_3(t) \geq b e^{bt}.$$

By the comparison theorem,  $f_1 \geq f_2 \geq f_3 \geq b e^{bt}$ . Put  $h(t) = t^{-m/\beta} f_2(t)$ . Then, since  $f(t) \geq h(t)$ , it is sufficient to show that  $h(t) = \infty$  if  $t$  is large enough. Suppose that  $h(t) < \infty$  for every  $t > \delta$ . Noting that  $h(t) \rightarrow \infty$  as  $t \rightarrow \infty$  and using Lemma 1, we have

$$(4.15) \quad \sup_{t \geq t'} \frac{m}{\beta t} \frac{h(t)}{F(h(t))} \leq \left(\frac{1}{2}\right)^{m/\beta+1} \quad \text{for some } t' > 0.$$

By (4.6), (4.15), we have for  $t \geq t'$

$$\begin{aligned} \frac{dh(t)}{dt} &= -\frac{m}{\beta t} t^{-m/\beta} f_2(t) + t^{-m/\beta} \frac{df_2(t)}{dt} \\ &= -\frac{m}{\beta t} t^{-m/\beta} f_2(t) + \left(\frac{1}{2}\right)^{m/\beta} F(f_2(t) t^{-m/\beta}) \\ &= \left(\frac{1}{2}\right)^{m/\beta} F(h(t)) - \frac{m}{\beta t} h(t) \\ &\geq \left(\frac{1}{2}\right)^{m/\beta+1} F(h(t)). \end{aligned}$$

It then follows that

$$\left(\frac{1}{2}\right)^{m/\beta+1} (t-t') \leq \int_{h(t')}^{h(t)} \frac{dx}{F(x)} \leq \int_{h(t')}^{\infty} \frac{dx}{F(x)} < \infty$$

for any  $t \geq t'$ , which is a contradiction.

OSAKA UNIVERSITY

### References

- [1] H. Fujita: *On the blowing up of solutions of the Cauchy problem for  $u_t = \Delta u + u^{1+\alpha}$* , J. Fac. Sci. Univ. Tokyo Sect I, **13** (1966), 109–124.
- [2] K. Hayakawa: *On nonexistence of global solutions of some semilinear parabolic differential equations*, Proc. Japan Acad. **49** (1973), 503–505.
- [3] K. Yosida: *Functional Analysis*, Springer, 1966.



