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AN INTEGRAL REPRESENTATION ON THE PATH SPACE FOR SCATTERING LENGTH

Dedicated to Professor N. Ikeda on the occasion of his sixtieth birthday

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0. The *scattering length* Γ is the limit of the scattering amplitude $f_k(e, e')$ as the wave number k tends to 0. It is independent of the choice of unit 3-vectors e and e' . The scattering amplitude is defined as the unique constant $f_k(e, e')$ such that there holds the asymptotics

$$\phi_k(x) \sim e^{ik\langle u, e \rangle} + f_k(e, e') e^{ik\langle u, e' \rangle} / |x| \quad \text{as } |x| \rightarrow \infty \quad \left(e' = \frac{x}{|x|} \right)$$

for a solution ϕ_k , called the *scattering solution*, of the equation

$$\Delta \phi_k - v \phi_k = -k^2 \phi_k,$$

where v is a given potential which is assumed to be nonnegative and integrable. As M. Kac proved,

$$(1) \quad \Gamma = \frac{1}{2\pi} \int_{\mathbb{R}^3} v(x) \phi_0(x) dx$$

where $\phi_0(x)$ is the solution of

$$(2) \quad \phi_0(x) = 1 - \frac{1}{2\pi} \int_{\mathbb{R}^3} \frac{v(y) \phi_0(y)}{|x-y|} dy.$$

In [4], M. Kac gave the formula

$$(3) \quad \Gamma = \frac{1}{2\pi} \lim_{t \rightarrow \infty} \frac{1}{t} \int_{\mathbb{R}^3} E_x \left[1 - \exp \left(- \int_0^t v(w(s)) ds \right) \right] dx$$

where E_x denotes the expectation with respect to the three dimensional Brownian motion starting at x . He conjectured that

(C1) the scattering length $\Gamma = \Gamma(\alpha v)$ for the potential αv has limit as α goes to infinity and

(C2) the limit, say γ_v , is independent of the choice of potential v and depends only on the support $U = \{x; v(x) > 0\}$.

The purpose of the present note is to prove the conjecture C1-2 by giving an integral representation of the scattering length $\Gamma(v)$ on the path space W ,

where $W = \mathcal{C}((-\infty, +\infty), \mathbf{R}^3)$ for the above case.

1. Let us state the result in a little more general setup. Consider a transient Markov process with state space R which admits a reversible invariant measure λ . Assume that R is a Polish space, λ is a Radon measure on R and the path is continuous. Now we define the *scattering length* by the formula

$$(3)' \quad \Gamma(v) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_R E_x [1 - \exp(-\int_0^t v(w(s)) ds)] \lambda(dx)$$

for continuous functions v with compact support on R , where E_x denotes the expectation with respect to the Markov process starting at x . A proof of the existence is given in Lemma 2 below.

By the reversibility the path may be considered to be defined for both positive and negative time and then, given a starting point $x = w(0)$ at time 0, the process $w(-t)$, $t \geq 0$, is an independent copy of $w(t)$, $t \geq 0$. So we take

$$W = \mathcal{C}((-\infty, +\infty), R)$$

and define a measure Λ on the path space W by

$$\int_W \Lambda(dw) \Phi(w) = \int_R \lambda(dx) \int_W P_x(dw) \Phi(w)$$

for bounded Borel function Φ on W , where P_x denotes the law of the Markov process starting at x at the initial time 0.

Theorem. *Let v be a nonnegative continuous function with compact support on R . Then,*

$$(4) \quad \Gamma(v) = \int_{S(v)} \Lambda(dw) \frac{1 - \exp(-\int_{-\infty}^{+\infty} v(w(t)) dt)}{\int_{-\infty}^{+\infty} v(w(t)) dt} v(w(0))$$

where

$$(5) \quad S = S(v) = \{w; \int_{-\infty}^{+\infty} v(w(t)) dt > 0\}.$$

REMARK 1. It is known [4] that $\Gamma(v) \leq C(K)$, where $C(K)$ is the electrostatic capacity of the closure K of the set

$$U = \{x; v(x) > 0\}$$

for the 3-dimensional Brownian motion. Similar bounds hold in general cases. Hence, $\Gamma(v)$ is finite if so is the capacity $C(K)$.

Corollary. *Let u and v be two nonnegative continuous function with common support U . Then the limit $\lim_{\alpha \rightarrow \infty} \Gamma(\alpha u)$ exists and is equal to $\lim_{\alpha \rightarrow \infty} \Gamma(\alpha v)$.*

Consequently, the conjecture (C1-2) is true.

REMARK 2. The proof given below works for certain nonnegative Borel functions v , such as the indicator of a compact set which is the closure of its interior. Thus Corollary is also valid for such functions.

Proof of Corollary. From the formula (4) it follows that the monotone limit $\lim_{\alpha \rightarrow \infty} \Gamma(\alpha v)$ exists and is equal to

$$(6) \quad \int_S \Lambda(dw) \frac{v(w(0))}{\int_{-\infty}^{+\infty} v(w(t)) dt} .$$

Since λ is an invariant measure for the Markov process (W, P_x) , the measure Λ is invariant under the time shift $w(t) \rightarrow w(t+s)$ for any s . Furthermore, it is invariant under the time reversion $w(t) \rightarrow w(-t)$ by the reversibility of λ .

Keeping in mind these properties and the facts that S is common for u and v and that the functions

$$\int_{-\infty}^{+\infty} v(w(t)) dt \quad \text{and} \quad \int_{-\infty}^{+\infty} u(w(t)) dt$$

are invariant under either of the shift and the reversion, we obtain

$$\begin{aligned} & \lim \Gamma(\alpha v) \\ &= \int_S \Lambda(dw) \frac{v(w(0))}{\int_{-\infty}^{+\infty} v(w(t)) dt} \\ &= \int_S \Lambda(dw) \frac{v(w(0))}{\int_{-\infty}^{+\infty} v(w(t)) dt} \frac{\int_{-\infty}^{+\infty} u(w(s)) ds}{\int_{-\infty}^{+\infty} u(w(t)) dt} \\ &= \int_{-\infty}^{+\infty} ds \int_S \Lambda(dw) \frac{v(w(0))}{\int_{-\infty}^{+\infty} v(w(t)) dt} \frac{u(w(s))}{\int_{-\infty}^{+\infty} u(w(t)) dt} \\ &= \int_{-\infty}^{+\infty} ds \int_S \Lambda(dw) \frac{v(w(-s))}{\int_{-\infty}^{+\infty} v(w(t)) dt} \frac{u(w(0))}{\int_{-\infty}^{+\infty} u(w(t)) dt} \\ & \hspace{15em} \text{(by shift invariance)} \\ &= \int_{-\infty}^{+\infty} ds \int_S \Lambda(dw) \frac{v(w(s))}{\int_{-\infty}^{+\infty} v(w(t)) dt} \frac{u(w(0))}{\int_{-\infty}^{+\infty} u(w(t)) dt} \\ & \hspace{15em} \text{(by the reversibility)} \\ &= \int_S \Lambda(dw) \frac{u(w(0))}{\int_{-\infty}^{+\infty} u(w(t)) dt} \frac{\int_{-\infty}^{+\infty} v(w(s)) ds}{\int_{-\infty}^{+\infty} v(w(t)) dt} \end{aligned}$$

$$= \int_S \Lambda(dw) \frac{u(w(0))}{\int_{-\infty}^{+\infty} u(w(t)) dt} = \lim \Gamma(\alpha u).$$

Consequently, we obtain Corollary.

2. Now let us proceed to the proof of Theorem. At first we give another expression for the scattering length Γ . For this sake we prepare the following:

Lemma 1. *For a continuous function v with compact support on R ,*

$$(7) \quad \int_0^T dt E_x[v(w(t)) \exp \{-\int_0^t v(w(s)) ds\}] \\ = 1 - E_x[\exp \{-\int_0^T v(w(t)) dt\}] \quad (0 \leqq T \leqq \infty).$$

Proof. For $T < \infty$ the left hand side is equal to

$$\int_0^T dt E_x[-\frac{d}{dt} \exp \{-\int_0^t v(w(s)) ds\}],$$

which is equal to the right hand side. For $T = \infty$ the convergence is assured by the monotonicity, or, directly, by the transience:

$$\int_0^\infty E_x[v(w(t))] dt < \infty.$$

Lemma 2. *Let v be a nonnegative continuous function with compact support on R . Then,*

$$(8) \quad \Gamma(v) = \int_W \Lambda(dw) v(w(0)) \exp \{-\int_0^\infty v(w(t)) dt\} \\ = \int_R \lambda(dx) v(x) E_x[\exp \{-\int_0^\infty v(w(t)) dt\}].$$

Proof. Note that

$$E_x[1 - \exp \{-\int_0^t v(w(s)) ds\}] \\ = E_x[\int_0^t ds v(w(s)) \exp \{-\int_0^s v(w(r)) dr\}].$$

Integrating this against λ , we obtain

$$\frac{1}{t} \int_W \Lambda(dw) [1 - \exp \{-\int_0^t v(w(s)) ds\}] \\ = \frac{1}{t} \int_0^t ds \int_W \Lambda(dw) v(w(s)) \exp \{-\int_0^s v(w(r)) dr\} \\ = \frac{1}{t} \int_0^t ds \int_W \Lambda(dw) v(w(0)) \exp \{-\int_{-s}^0 v(w(r)) dr\} \\ \text{(by the shift invariance)}$$

$$\begin{aligned}
 &= \frac{1}{t} \int_0^t ds \int_W \Lambda(dw) v(w(0)) \exp \left\{ - \int_0^s v(w(r)) dr \right\} \\
 &\hspace{15em} \text{(by the reversibility)} \\
 &\rightarrow \int_W \Lambda(dw) v(w(0)) \exp \left\{ - \int_0^\infty v(w(t)) dt \right\}
 \end{aligned}$$

as $t \rightarrow \infty$, as is desired.

The formula (8) together with (7) enables us to compute the derivative of the functional Γ at v , which will be denoted by $D\Gamma(v)$:

$$(9) \quad D\Gamma(v)f = \lim_{t \downarrow 0} \frac{1}{t} \{ \Gamma(v+tf) - \Gamma(v) \}$$

for nonnegative continuous functions f with compact support on R . One can remove the restriction that f is nonnegative and may prove that $D\Gamma(v)$ is the Fréchet derivative. But here we only need the Gateaux derivative from the right, whose existence is obvious from the formula (7) by virtue of the transience.

Lemma 3. *The following formula holds for $D\Gamma(v)$:*

$$(10) \quad \begin{aligned}
 D\Gamma(v)f &= \int_R \lambda(dx) f(x) (E_x[\exp \{ - \int_0^\infty v(w(t)) dt \}])^2 \\
 &= \int_W \Lambda(dw) f(w(0)) \exp \left\{ - \int_{-\infty}^\infty v(w(t)) dt \right\} .
 \end{aligned}$$

Proof. Let us differentiate the second expression for Γ in (7). Let us write

$$g(x) = E_x[\exp \{ - \int_0^\infty v(w(s)) ds \}] .$$

Then,

$$\begin{aligned}
 &\frac{d}{dt} \Big|_{t=0+} \Gamma(v+tf) \\
 &= \int_R \lambda(dx) f(x) g(x) \\
 &\quad + \int_R \lambda(dx) v(x) E_x \left[- \int_0^\infty f(w(s)) ds \exp \left\{ - \int_0^\infty v(w(t)) dt \right\} \right] \\
 &= \int_R \lambda(dx) f(x) g(x) \\
 &\quad - \int_0^\infty ds \int_W \Lambda(dw) v(w(0)) f(w(s)) \exp \left\{ - \int_0^\infty v(w(t)) dt \right\} .
 \end{aligned}$$

Now the second term can be written as

$$\begin{aligned}
 & - \int_0^\infty ds \int_W \Lambda(dw) v(w(-s)) \exp \left\{ - \int_{-s}^0 v(w(t)) dt \right\} f(w(0)) \exp \left\{ - \int_0^\infty v(w(s)) ds \right\} \\
 &\hspace{15em} \text{(by the shift invariance)}
 \end{aligned}$$

$$\begin{aligned}
&= -\int_0^\infty ds \int_W \Lambda(dw) v(w(-s)) \exp \left\{ -\int_{-s}^0 v(w(t)) dt \right\} f(w(0)) g(w(0)) \\
&\hspace{20em} \text{(by the Markov property)} \\
&= -\int_0^\infty ds \int_R \lambda(dx) f(x) g(x) E_x[v(w(s)) \exp \left\{ -\int_0^s v(w(t)) dt \right\}] \\
&\hspace{20em} \text{(by the reversibility)} \\
&= -\int_R \lambda(dx) f(x) g(x) [1-g(x)]
\end{aligned}$$

by virtue of Lemma 2. Consequently,

$$D\Gamma(v)f = \int_R \lambda(dx) f(x) g(x)^2.$$

Finally, by the reversibility we obtain the expression

$$g(x)^2 = (E_x[\exp \left\{ -\int_0^\infty v(w(s)) ds \right\}])^2 = E_x[\exp \left\{ -\int_{-\infty}^\infty v(w(s)) ds \right\}].$$

The proof is completed.

Proof of Theorem. From Lemma 3 it follows that

$$\begin{aligned}
\frac{d}{d\alpha} \Gamma(\alpha v) &= \int_S \Lambda(dw) v(w(0)) \exp \left\{ -\alpha \int_{-\infty}^\infty v(w(s)) ds \right\} \\
&\quad + \int_{S^c} \Lambda(dw) v(w(0)).
\end{aligned}$$

Note that $\Gamma(\alpha v) \rightarrow 0$ as $\alpha \rightarrow 0$ and that the second term in the right hand side vanishes because of the definition of the set S and the continuity of the path. Consequently, we obtain

$$\begin{aligned}
\Gamma(v) &= \int_0^1 d\alpha \int_S \Lambda(dw) v(w(0)) \exp \left\{ -\alpha \int_{-\infty}^\infty v(w(s)) ds \right\} \\
&= \int_S \Lambda(dw) v(w(0)) \frac{1 - \exp \left\{ -\int_{-\infty}^\infty v(w(s)) ds \right\}}{\int_{-\infty}^\infty v(w(s)) ds}.
\end{aligned}$$

Hence the proof is completed.

REMARK 3. In the case of three dimensional Brownian motion the constant γ_U with $U = \text{int } K$ coincides for “nice” compacts K called *semiclassical* by Kac [2] (cf. [3] for counter-example) with the electrostatic capacity $C(K)$, for which a similar result to (3) (and more) was obtained earlier by F. Spitzer [7]. A further historical remark can be found in [6].

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