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<td>Author(s)</td>
<td>Bannai, Eiichi</td>
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<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 11(2) P.401–P.411</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1974</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/6086">https://doi.org/10.18910/6086</a></td>
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<td>DOI</td>
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Introduction

In [5], M. Hall determined 4-ply transitive permutation groups whose stabilizer of 4 points is of odd order. (See also Nagao [11].) On the other hand, in Bannai [1] and Miyamoto [9], t-ply transitive finite permutation groups in which the stabilizer of t points is of order prime to an odd prime \( p \) have been determined for \( t = p^2 + p \) and \( 3p \) respectively. The purpose of this series of notes is to strengthen those results. In this first note, we will improve Lemma 2.1 in Miyamoto [9]. Namely, we will prove the following result.

**Theorem 1.** Let \( p \) be an odd prime. Then there exists no permutation group \( G \) on a set \( \Omega = \{1, 2, \ldots, n\} \) which satisfies the following three conditions:

(i) \( G \) is \((p^2 + 2)\)-ply transitive, and \( n \equiv 2 \pmod{p} \),

(ii) a Sylow \( p \) subgroup \( P_0 \) of \( G_{1,2,\ldots,p^2+2} \) is semiregular on \( \Omega - \{1, 2, \ldots, p+2\} \), and

(iii) \( |P_0| \geq p^5 \).

**Corollary to Theorem 1.** Let \( p \) be an odd prime. Let \( G \) be a \((2p+2)\)-ply transitive permutation group on a set \( \Omega = \{1, 2, \ldots, n\} \). If the order of \( G_{1,2,\ldots,p^2+2} \) is not divisible by \( p \), then \( G \) must be \( S_n(2p+2 \leq n \leq 3p+1) \) or \( A_n(2p+4 \leq n \leq 3p+1) \).

This corollary is immediately proved by combining Theorem 1 with a result of Miyamoto [9]. To be more precise, if the order of \( G_{1,2,\ldots,p^2+2} \) is not divisible by \( p^5 \), then the \( 2p \)-ply transitive group \( G_{1,2} \) on \( \Omega - \{1, 2\} \) must contain \( A_{\Omega - \{1,2\}} \) by the result of Miyamoto [9, §1], and so \( G \) must be one of the groups listed in the conclusion of the corollary. If the order of \( G_{1,2,\ldots,p^2+2} \) is divisible by \( p^5 \), then the \((p+2)\)-ply transitive group \( G_{1,2,\ldots,i} \) on \( \Omega - \{1, 2, \ldots, i\} \) (if \( n \equiv i+2 \pmod{p} \) with \( 0 \leq i \leq p-1 \)) satisfies the three conditions of Theorem 1, and we have a contradiction.

In our proof of Theorem 1, the following result is very important. This result is a kind of generalization of a result of Jordan [8, Chap. IV], and will be of independent interest.

*) Supported in part by the Sakkokai Foundation.

Present address: The Ohio State University.
**Theorem A.** Let \( p \) be an odd prime. Then \( A_{p+2} \) (hence \( S_{p+2} \)) is not involved in \( GL(p, p) \).

Theorem A will be proved in §1 by exploiting the theory of modular representations of the symmetric groups due to Nakayama [12] together with some other results (theory of projective representations of the symmetric groups due to Schur [13], theory of \( p \) groups and so on).

In Appendix, we will discuss some partial generalization of Theorem 1.

**Notation.** Our notation will be standard. \( S^\Delta \) and \( A^\Delta \) denote the symmetric and alternating groups on a set \( \Delta \). If \( |\Delta| \), the cardinality of \( \Delta \), is \( m \), we denote them by \( S_m \) and \( A_m \) instead of \( S^\Delta \) and \( A^\Delta \). If \( X \) is a permutation group on a set \( \Omega \), and if \( \Delta \) is a subset of \( \Omega \) which is fixed as a whole by \( X \), we denote by \( X^\Delta \) the restriction of \( X \) to \( \Delta \). For a subset \( \Delta = \{1, 2, \ldots, i\} \) of \( \Omega \), we denote by \( X_{1,2,\ldots,i} \) the pointwise stabilizer of \( \Delta \) in \( X \). \( GL(m, K) \) denotes the general linear group of dimension \( m \) over a field \( K \). \( PGL(m, K) \) denotes the projective linear group of dimension \( m \) over \( K \), \( PGL(m, K) = GL(m, K)/Z(GL(m, K)) \), where \( Z(GL(m, K)) \) denotes the center of \( GL(m, K) \). When \( K \) is of cardinality \( p \), we denote \( GL(m, K) \) by \( GL(m, p) \). For a group \( X \), \( Aut(X) \) denotes the automorphism group of \( X \).

1. **\( A_{p+2} \) is not involved in \( GL(p, p) \)**

The purpose of this section is to prove Theorem A that \( A_{p+2} \) is not involved in \( GL(p, p) \).

We first remark the following lemma.

**Lemma 1.** Theorem A is true for \( p = 3 \) and 5.

**Proof.** \( A_3 \) is not involved in \( GL(3, 3) \), because the order of \( GL(3, 3) \) is not divisible by 5. Similarly, \( A_5 \) is not involved in \( GL(5, 5) \), because the order of \( GL(5, 5) \) is not divisible by 7.

From now on, we always assume that \( p \geq 7 \). In case of \( p \geq 7 \), we can prove Theorem A in a little stronger form as in Lemma 4 mentioned later.

**Lemma 2.** Let \( p \geq 7 \). Then \( S_{p+2} \) is not a subgroup of \( GL(p, K) \), where \( K \) is an algebraically closed field of characteristic \( p \).

**Proof.** We have only to prove that \( S_{p+2} \) has no faithful \( p \)-modular (absolutely) irreducible representation of degree \( \leq p \) over \( K \). Lemma 2 will be proved through the following steps (1) and (2).

1. The degree of any not 1 dimensional ordinary irreducible representation of degree \( \leq p \) over \( K \). Lemma 2 will be proved through the following steps (1) and (2).

   (1) The degree of any not 1 dimensional ordinary irreducible representation of \( S_k(k \geq 5) \) is \( \geq k-1 \). Therefore, the degree of any irreducible \( p \)-modular representation of \( S_{p+2} \) over \( K \) which is contained in a \( p \)-block of defect 0 is more than \( p \).
The first assertion is immediately proved by using the Schur’s recursive formula (a special case of Murnaghan-Nakayama’s recursive formula) (see [13, §44]). The last assertion is obvious from an elementary properties of a $p$-block of defect 0.

(2) The degree of any not 1 dimensional $p$-modular irreducible representation of $S_{p+2}$ over $K$ which is contained in a $p$-block of defect 1 is more than $p$.

By Nakayama [12], we obtain that there exist just two $p$-blocks of defect 1 for $S_{p+2}$. Moreover, one block (say $B_0$) with $p$-core of type [2] consists of $p$ ordinary irreducible representations $T_{0,r}$ with $0 \leq r \leq p-1$, where $T_{0,r}$ is the representation associated with the Young diagram of type $[p+2]$ (for $r=0$), $[p-r, 3, 1^{r-2}]$ (for $1 \leq r \leq p-3$), $[2, 2, 1^{r-2}]$ (for $r=p-2$) and $[2, 1^r]$ (for $r=p-1$). While, the other block (say $B_1$) with $p$-core of type $[^17]$ consists of $p$ ordinary irreducible representations $T_{1,r}$ with $0 \leq r \leq p-1$, where $T_{1,r}$ is the representation associated with the Young diagram which is obtained by transposing that of $T_{0,p-1-r}$. Also by a result of Nakayama [12], $T_{i,r}$ and $T_{i,r+1}$ ($i=0,1, r=0, 1, \cdots, p-1$) have just one $p$-modular irreducible representation (over $K$) in common, say, let us denote it by $\phi_{i,r}$ ($0 \leq r \leq p-2$), and $T_{i,r}$ and $T_{i,s}$ with $s>r+1$ have no $p$-modular irreducible representation in common. That is to say, the Brauer graphs associated with the $p$-blocks $B_i$ ($i=0, 1$) are trees without branches and their nodes are arranged on natural order on $r$.

(For the definition of Brauer graphs, see, e.g., [3, §68].) Therefore, we can calculate the degree $|\phi_{i,r}|$ of $\phi_{i,r}$ inductively for $r=0, 1, 2, \cdots$ (and for $r=p-2$, $p-3, p-4, \cdots$), because the degree $|T_{i,r}|$ of $T_{i,r}$ is given explicitly by the following formula:

$$|T_{i,r}| = (p+2)! / \text{the product of all hook lengths of the Young diagram of } T_{i,r}.$$  

In the case of $p=7$, we can immediately calculate all the values of $|\phi_{i,r}|$ ($i=0, 1, r=0, 1, \cdots, 5$), and we obtain that they are all $\geq 8>7$ except $|\phi_{0,0}|$ and $|\phi_{1,1}|$ which are equal to 1. Thus, in the following we may assume that $p \geq 11$.

Now, we obtain that $|T_{0,0}| = 1$, $|T_{0,1}| = \frac{(p+2)(p+1)(p-3)}{6} > p$, $|T_{0,p-3}| = \frac{(p+2)(p+1)(p-3)}{12} > p^2$, $|T_{0,p-2}| = \frac{(p+1)(p-1)}{2}$ and $|T_{0,p-1}| = p+1$.

Therefore, we obtain $|\phi_{0,0}| = 1, |\phi_{0,p-3}| > p, |\phi_{0,p-2}| = p+1 > p$. Moreover, when $1 \leq r \leq p-4$, we obtain that $|T_{0,r}| / |T_{0,r+1}| = \frac{r(r+3)(p-r-2)}{(r+2)(p-r)(p-3-r)}$. Now, we obtain that $\frac{1}{p} < \frac{r(r+3)(p-r-2)}{(r+2)(p-r)(p-3-r)} < p$ for any $r = 1, 2, \cdots, p-2$, and $\frac{r(r+3)(p-r-2)}{(r+2)(p-r)(p-3-r)} < 1$ when $r \leq p-1$, and $> 1$ when $r \geq p+1$. Therefore, we obtain that $|T_{0,1}| < |T_{0,2}| < \cdots < |T_{0,\lfloor p+1/2 \rfloor}|$, and $|T_{0,\lfloor p+1/2 \rfloor}| > \cdots > |T_{0,p-2}| > |T_{0,p-1}|$. Hence, we obtain that $|\phi_{0,r}| > p$ (for $r=1, 2, \cdots, p-4$), because of the
fact that the Brauer graph of the block $B_o$ is a tree without branches and of natural order on $r$. Therefore, we obtain that $|\phi_{0,r}| > p$ for any $r \neq 0$ $(1 \leq r \leq p-2)$. Since $|T_{0,r}| = |T_{1,p-1-r}|$ and $|\phi_{0,r}| = |\phi_{1,p-2-r}|$ for any $r$, we also obtain that $|\phi_{1,r}| > p$ for any $r \neq p-2$ $(0 \leq r \leq p-3)$ and $|\phi_{1,p-2}| = 1$. Thus, we have proved the assertion of (2).

Since any $p$-block of $S_{p+2}$ is either of defect 0 or 1, we have completed the proof of Lemma 2 by (1) and (2).

We also have

**Lemma 2'.** Let $p \geq 7$. Then $A_{p+2}$ is not a subgroup of $GL(p, K)$, where $K$ is an algebraically closed field of characteristic $p$.

Proof. The assertion corresponding to step (1) in Lemma 2 is easily obtained similarly by using the Schur’s recursive formula for the characters of the symmetric groups. Namely, we have

(1') The degree of any ($p$-modular) irreducible representation of $A_{p+2}$ over $K$ which is contained in a $p$-block of defect 0 is more than $p$.

Since $T_{i,r}$ ($i=0, 1, r=0, 1, \cdots, p-1$) are all irreducible representations of $A_{p+2}$, we immediately obtain the following assertion.

(2') The degree of not 1 dimensional ($p$-modular) irreducible representation of $A_{p+2}$ over $K$ which is contained in a $p$-block of defect 1 is more than $p$.

Thus, we have proved Lemma 2'.

**Lemma 3.** Let $p$ be an odd prime $\geq 7$. Then $S_{p+2}$ is not a subgroup of $PGL(p, K)$, where $K$ is an algebraically closed field of characteristic $p$.

Proof. We have only to prove that $S_{p+2}$ has no not 1 dimensional projective irreducible representation of degree $\leq p$ over $K$. Since we have already proved in Lemma 2 that $S_{p+2}$ is not a subgroup of $GL(p, K)$, we have only to prove that $S_{p+2}$ has no projective representation of degree $\leq p$ over $K$ which is not a linear representation. As is easily seen from a result of Schur (and a slight extension of it) (cf. Yamazaki [15, §3.3, Corollary 1]), there is a finite group (which is a central extension of $S_{p+2}$ and is called a representation group of $S_{p+2}$ over $K$) such that any projective representation of $S_{p+2}$ is induced by a linear representation of the representation group. Moreover, by Yamazaki [15, §3, e.g., Proposition 3.3, 2) and Proposition 3.5], we may take as a representation group of $S_{p+2}$ over $K$ the following group $T_{p+2}$ defined by the generators

$$\{J, X_i \ (i = 1, 2, \cdots, p+1)\}$$

with the defining relations

$$J^2 = 1,$$

$$X_{\alpha}^2 = J (\alpha = 1, 2, \cdots, p+1),$$
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\[ (X_{\beta}X_{\beta+1})^\gamma = J (\beta = 1, 2, \ldots, p) \quad \text{and} \]
\[ X_{\gamma}X_{\delta} = jX_{\gamma}X_{\delta} (\gamma = 1, 2, \ldots, p-1, \delta = \gamma+2, \ldots, p+1). \]

(Note that \( Z(T_{p+2}) = \langle J \rangle \) (which is contained in the commutator subgroup of \( T_{p+2} \)) is a cyclic group of order 2, and \( T_{p+2}/Z(T_{p+2}) = S_{p+1} \). \( T_{p+2} \) is the group denoted \( \mathfrak{T}_{p+2} \) in Schur \([13]\). Also note that \( H^2(S_{p+2}, K^*) = H^2(S_{p+2}, C^*) = Z_2 \).)

The ordinary irreducible representations of \( T_{p+2} \) were completely determined by Schur \([13]\). As in \([13]\), let us call an ordinary irreducible representation of \( T_{p+2} \) is of the first kind (resp. of the second kind) if the kernel of the representation contains \( Z(T_{p+2}) \) (resp. does not contain \( Z(T_{p+2}) \)). The proof of Lemma 3 will be done through the following steps (1), (2) and (3).

(1) The degree of any ordinary irreducible representation of \( T_{p+2} \) of the second kind is more than \( 2^{(p+1)/2} \). Moreover \( 2^{(p+1)/2} > p \).

The degree of any ordinary irreducible representations of \( T_{p+2} \) of the second kind is given as follows (Schur \([13]\)):
\[
 f_{v_1, v_2, \ldots, v_m} = 2^{(p+2-m)/2} g_{v_1, v_2, \ldots, v_m},
\]
with
\[
g_{v_1, v_2, \ldots, v_m} = \frac{(p+2)!}{\nu_1! \nu_2! \cdots \nu_m! \prod_{a<p} (\nu_a + \nu_b)},
\]
where \( \nu_1 + \nu_2 + \cdots + \nu_m = p+2 \) and \( \nu_1 > \nu_2 > \cdots > \nu_m > 0 \). Moreover, by Schur \([13, \S 44]\), it is proved that
\[
f_{v_1, v_2, \ldots, v_m} \geq 2^{(p+2-m)/2} = 2^{(p+1)/2}
\]
for any \( f_{v_1, v_2, \ldots, v_m} \). Thus we obtain the first assertion. The last assertion is clear, because \( p \geq 7 \).

(2) The degree of any ordinary irreducible representation of \( T_{p+2} \) of the second kind which is not divisible by \( p \) is divisible by \( 2^{(p+1)/2} \). Moreover, \( 2^{(p+1)/2} > p \).

Since \( f_{v_1, v_2, \ldots, v_m} \) is not divisible by \( p \), we obtain that \( m \leq 3 \), by noticing the formula of \( f_{v_1, v_2, \ldots, v_m} \). Since \( f_{v_1, v_2, \ldots, v_m} = 2^{(p+2-m)/2} g_{v_1, v_2, \ldots, v_m} \) and \( g_{v_1, v_2, \ldots, v_m} \) is an integer (Schur \([13, \S 40]\), we obtain the first assertion. The last assertion is clear, because \( p \geq 7 \).

(3) The degree of any not 1 dimensional (\( p \)-modular) irreducible representation of \( T_{p+2} \) over \( K \) is more than \( p \).

Let \( \phi \) be an irreducible representation of \( T_{p+2} \) over \( K \) of degree \( > 1 \). If \( \phi \) is contained in a \( p \)-block of defect 0 of \( T_{p+2} \), then by step (1) and the step (1) in Lemma 2, we obtain that the degree of \( \phi \) is more than \( p \). Now, let us assume that \( \phi \) is contained in a \( p \)-block of defect 1. Since any block of defect 1 contains at most \( p \) ordinary irreducible representations in general (and in this case) (cf. [3, \S 68]), \( B_0 \) and \( B_1 \) (\( p \)-blocks of \( S_{p+2} \)) themselves also become \( p \)-blocks.
of $T_{p+2}$ of defect 1 (all representation of $S_{p+2}$ are naturally regarded as representations of $T_{p+2}$). Therefore, any ordinary irreducible representation of $T_{p+2}$ which is contained in a $p$-block of defect 1 and not contained in $B_0$ and $B_1$ (as blocks of $T_{p+2}$) must be of the second kind. Therefore, the degree of any ordinary irreducible representation of $T_{p+2}$ contained in a $p$-block of defect 1 and not contained in $B_0$ and $B_1$ must be divisible by $2^{(p-1)/2}$ by step (2). Since $p$ is to the first power in the order of $T_{p+2}$, the Brauer graph of any $p$-block of defect 1 of $T_{p+2}$ must be a tree (cf. [3, §68]), and so the degree of any irreducible representation of $T_{p+2}$ over $K$ is divisible by $2^{(p-1)/2}$, $p$. Thus, we obtain the assertion of (3).

Thus, we have completed the proof of Lemma 3.
We also have

**Lemma 3'.** Let $p \geq 7$. Then $A_{p+2}$ is not a subgroup of $PGL(p, K)$, where $K$ is an algebraically closed field of characteristic $p$.

**Proof.** The commutator subgroup $T_{p+2}'$ of $T_{p+2}$ with index 2 becomes a representation group of $A_{p+2}$ over $K$.

1. The degree of any ordinary irreducible representation of $T_{p+2}'$ of the second kind is more than $2^{(p+1)/2-1} > p$.

   Proof is clear.

2. The degree of any ordinary irreducible representation of $T_{p+2}'$ of the second kind which is not divisible by $p$ is divisible by $2^{(p-1)/2-1}$ and divisible by 8 if $p = 7$. Moreover, $2^{(p-1)/2-1} > p$ when $p \geq 11$.

   Proof of the first assertion is clear. The second assertion for $p = 7$ is proved directly and easily.

3. The degree of not 1 dimensional ($p$-modular) irreducible representation of $T_{p+2}'$ over $K$ is more than $p$.

   The proof is quite the same as that of step (3) in Lemma 3.

Thus, we have proved Lemma 3'.

**Lemma 4.** $A_{p+2}$ is not involved in a finite subgroup of $GL(p, K)$, where $K$ is an algebraically closed field of characteristic $p$.

**Proof.** Let us assume that $l$ is the smallest integer $\leq p$ such that $A_{p+2}$ is involved in a finite subgroup $X$ of $GL(l, K)$. Moreover, let us take $X$ being of the least order among them, then $X$ contains a normal subgroup $Y$ such that $X/Y = A_{p+2}$. Now, we will derive a contradiction. By the assumption, we may assume that $X$ is an irreducible subgroup of $GL(l, K)$, and moreover that $X$ is a primitive subgroup of $GL(l, K)$, because $A_{p+2}$ is obviously not involved in $S_l$. (Cf. Dixon [2, §4], see also [2] for some fundamental properties of (finite) linear groups). By Lemma 2 and Lemma 3, we may assume that $Y$ is not contained in $Z(GL(l, K))$. Thus, there exists a Sylow $q$ subgroup $Q$ (for some prime
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$q$ of $Y$ such that $Q$ is not contained in $Z(GL(l, K))$. By the theorem of Sylow (Frattini argument), and since $A_{p+2}$ is not involved in $Y$ by the minimality of the order of $X$, we obtain that $X$ normalizes the Sylow $q$ subgroup $Q$ which is not contained in $Z(GL(l, K))$. The proof of Lemma 4 will be completed through the following steps (1) to (6).

(1) $p \neq q$.
Otherwise, $X$ becomes not irreducible as a subgroup of $GL(l, K)$, and this contradicts the minimality of $l$. (Cf. Dixon [2, §§2.2 and 2.8, or §4.2]).

(2) $Q$ does not contain any characteristic abelian subgroup of rank $\geq 2$.
Otherwise, $X$ becomes imprimitive or not irreducible as a subgroup of $GL(l, K)$, and this contradicts the minimality of $l$. (Cf. Dixon [2, §4.2].)

(3) $Q$ is a central product of groups $Q_1$ and $Q_2$, where $Q_1$ is either 1 or extraspecial $q$ group, say of order $q^{2r+1}$, and $Q_2$ is either cyclic or $q=2$ and isomorphic to one of dihedral, generalized quaternion and semidihedral groups of order $\geq 2^4$.

Since $Q$ contains no characteristic abelian subgroup of rank $\geq 2$, we obtain the assertion by a result of P. Hall (cf. Gorenstein [4, Theorem 5.4.9]).

Next, we utilize the following important result of Jordan.

**Lemma of Jordan** ([8, Chap. (V, page 56, (3)]). Let $q$ be a prime. If $r$ is a prime such that $r \neq q$ and $r \leq k-2$, then $A_k$ is not involved in $GL(r-2, q)$.

As a special case of Lemma of Jordan, we obtain the following assertion.

(4) $A_{p+2}$ is not involved in $GL(p-2, q)$, where $q$ is a prime different from $p$.

(5) Let $x$ be an element of $GL(l, K)$ which is of order prime to $p$ and not lying in $Z(GL(l, K))$. Then $A_{p+2}$ is not involved in $C_{GL(l, K)}(x)$.

This assertion is well known and immediately proved, e.g., by Dixon [2, §4.2], because $C_{GL(l, K)}(x)$ becomes either not irreducible or imprimitive as a subgroup of $GL(l, K)$.

(6) $A_{p+2}$ is not involved in $Aut(Q)$.

We obtain that all irreducible components of the natural representation of $Q$ in $GL(l, K)$ are equivalent (cf. [2, §4.2]), and so it is a faithful representation of $Q$. Now, any faithful ordinary absolutely irreducible representation of $Q$ (and hence any faithful absolutely irreducible representation of $Q$ over a field of characteristic $p \neq q$ (cf. Dixon [2, §3.8]) is either of degree $q^r$ (when $Q_1$ is extraspecial of order $q^{2r+1}$ and $Q_2$ is cyclic) or $q^{r+1}$ (when $Q_1$ is extraspecial of order $q^{2r+1}$ and $Q_2$ is one of dihedral, generalized quaternion and semidihedral and $q=2$), or $<2$ (when $Q_1=1$) (cf. Gorenstein [4, Theorem 5.5.5 and Theorem 3.7.2]). If $Q_1=1$, then we easily have that $A_{p+2}$ is not involved in $Aut(Q)$, and so in the following we assume that $Q_1 \neq 1$. Thus, we obtain in every case that $q^r \leq l$ ($p$) or $q^{r+1} \leq l$ ($p$). Now, investigating the structures of the group
in every possible case, we obtain that $Q$ contains a series of characteristic subgroups $Q_{iO}$ such that

$$Q = Q_{(0)} > Q_{(1)} > \cdots > Q_{(k)} = 1,$$

and $Q_{(i)}/Q_{(i+1)}$ are elementary abelian $q$ subgroups of rank $\leq 2r$. Here, note that in every case $Q/Z(Q)$ is a direct product of $Q_i/Z(Q)$ (an elementary abelian group of order $q^{r^i}$) and a group $Q_r/Z(Q)$ which is either trivial or one of cyclic subgroups of order $> q^r$ (since, if of order $q$ then $Q$ becomes an extraspecial $q$ group of order $q^{r+2}$, and this is a contradiction) or $q=2$ and dihedral group of order $\geq 2^r$. Therefore, in any way, since $q^r \leq p$ or $q^{r+1} \leq p$, we obtain that $p-2 \geq 2r$ whenever $p \geq 7$. Therefore, in order that $A_{p+2}$ is involved in $\text{Aut}(Q)$, $A_{p+2}$ must be involved in $GL(2r, q)$, because $\text{Aut}(Q)/(\text{the stabilizer group of the above chain of characteristic subgroups})$ is a subgroup of the direct product of $GL(l_i, q)$'s with $l_i \leq 2r$, and the stabilizer group of the chain is a $q$ group (cf. Gorenstein [4, §5.3]). But, since $p-2 \geq 2r$, this contradicts the assertion of (4). Thus, we have obtained the assertion of (6).

Now, we will complete the proof of Lemma 4. Since $A_{p+2}$ is not involved in $C_{GL(4, k)}(Q)$ by step (5), and since $\text{Aut}(Q)$ is a subgroup of $N_{GL(4, k)}(Q)$, we obtain that $A_{p+2}$ is not involved in $N_{GL(4, k)}(Q)$. But this is a contradiction, and we have completed the proof of Lemma 4.

Thus, we have completed the proof of Theorem A.

REMARK 1. Theorem A improves Lemma of Jordan (stated preceding step (4) in Lemma 4) a little. That is, we can omit the assumption that $r=2$ in Lemma of Jordan.

REMARK 2. Since it will be not easy for us to follow the proof of Lemma of Jordan along the original paper [8] of Jordan, because of its old-fashionedness of its way of description and its terminologies (but not of its context), we give a sketch of an alternative proof.

(a) Let $p$ be a prime $\neq p$. Then $A_{p+2}$ is not a subgroup of $GL(p-2, F)$, where $F$ is an algebraically closed field of characteristic $q$.

$A_{p+2}$ contains a Frobenius group $H$ of order $p(p-1)$ whose any Sylow subgroups are cyclic. Since the Schur multipliers of any cyclic subgroups are trivial, $H^2(H, K^*)$ also becomes trivial (cf. Yamazaki [15, §3]). Therefore, we obtain the assertion by Lemma 1.4 in Harris and Hering [6].

The next assertion will be of independent interest.

(b) Let $G$ be a finite simple group which is not involved in $A_8 \simeq GL(4, 2)^2$. If the degree of any not 1 dimensional projective (including linear) irreducible

1) The assumption that $G$ is not involved in $A_8$ is unnecessary in practice, as we can easily see by the case by case considerations of such simple groups.
representation over any (algebraically closed) field of any characteristic is more than \(t\), then \(G\) is not involved in a finite subgroup of \(GL(t, K)\), where \(K\) is any (algebraically closed) field of any characteristic.

Proof. Let \(l\) \((<t)\) be the smallest integer such that \(G\) is contained in a finite subgroup \(X\) of \(GL(l, K)\) with some algebraically closed field \(K\) of characteristic, say \(s\). Among them, let us take \(X\) to be of the least order. Because of the assumption, we obtain by quite the same argument as used in the proof of Lemma 3, that \(X\) contains a nontrivial normal Sylow \(q\) \((\neq s)\) subgroup \(Q\) which is not contained in \(Z(GL(l, K))\), and that \(X\) is not involved in \(C_{GL(l, K)}(Q)\). Moreover, since \(G\) must be involved in \(Aut(Q)\), \(G\) must be involved in \(GL(2r, q)\), where \(l \geq q^r\) (or \(q^{r+1}\)) holds. From the minimality of \(l\), \(q^r \leq 2r\). This asserts that \(q=2\) and \(r=2\) and \(l=4\). Hence \(G\) must be involved in \(GL(4, 2)\).

Proof of Lemma of Jordan follows immediately from steps (a) and (b) together with Lemma 3′ and Lemma 1.

2. **Proof of Theorem 1**

Let us assume that \(G\) satisfies the three conditions of Theorem 1. Now, we will derive a contradiction.

There is an element \(a\) of \(G\) of order \(p\) such that

\[
a = (1) (2) (3, \ldots, p+2)(p+3)\cdots(p+2)\cdots,
\]

i.e., \(a\) fixes \(p+2\) points. Then there exists a Sylow \(p\) subgroup of \(G_{1,2,\ldots,p+2}\) which is normalized by the element \(a\). We may denote it by \(P_0\) without loss of generality. Now, let us set \(P\) be the subgroup generated by \(a\) and \(P_0\). Then \(P\) is a Sylow \(p\) subgroup of \(G\).

(1) \(P\) is of maximal class (in the sense of Blackburn). Therefore, \(|Z(P)|=p\).

Since we obtain that \(|C_{P_0}(a)|=p\) from the semiregularity of \(P_0\) on \(\Omega-\{1, 2, \ldots, p+2\}\) (cf. Lemma of Nagao [11]), we have \(|C_P(a)|=p^2\), and so we have the first assertion (cf. [7, Kapitel III, Satz 14.23]). The last assertion is immediate from the assumption that \(|P_0| \geq p^3\).

(2) \(N_G(P_0)^{(1,2,\ldots,p+2)}=S^{(1,2,\ldots,p+2)}\).

This assertion is an immediate consequence of Lemma of Witt (cf. [14, Theorem 9.3]).

(3) \(C_G(P_0)^{(1,2,\ldots,p+2)} \geq A^{(1,2,\ldots,p+2)}\).

Otherwise, \(C_G(P_0)^{(1,2,\ldots,p+2)}=1\) (because \(p+2 \geq 5\)), and \(S_{p+2}\) must be involved in \(Aut(P_0)\), because \(N_G(P_0)/C_G(P_0)\) is a subgroup of \(Aut(P_0)\). Now, \(P_0\) has an automorphism \(\sigma\) (induced from the element \(a\)) such that the following condition (*) is satisfied:

\[
(*) \quad \sigma \text{ is of order } p \text{ and } |C_{P_0}(\sigma)|=p.
\]
If a $p$ group $X$ has an automorphism $\sigma$ satisfying the condition $(*)$, then any $\sigma$-invariant subgroup of $X$ and any factor group $X/Y$ for a $\sigma$-invariant normal subgroup $Y$ of $X$ have the automorphism (naturally induced by $\sigma$) satisfying the condition $(*)$ provided $\sigma$ acts nontrivially on them (cf. Huppert [7, Kapital III, §14], or the argument in Zassenhaus [16, pp. 18-19]), because the map $\tau$ of $X$ to $X$ defined by $\tau(x)=x^{-1}x^\sigma$ is $p$ to 1, and if $(xY)^\sigma=xY$ then $\tau(x)$ is contained in $Y$. Moreover, by a lemma of Ito in Nagao [10], an elementary abelian $p$ group which has an automorphism with the property $(*)$ is of rank $\leq p$. Thus, if we take a chain of Frattini subgroups $\Phi^{(i)}(P_0)$ of $P_0$:

$$P_0 > \Phi^{(1)}(P_0) > \Phi^{(2)}(P_0) > \cdots > \Phi^{(k)}(P_0) = 1,$$

where $P_0 = \Phi^{(0)}(P_0)$ and $\Phi^{(i)}(P_0)$ is the Frattini subgroup of $P_0$ and $\Phi^{(i+1)}(P_0) = \Phi^{(i)}(\Phi^{(i)}(P_0))$ for $i \geq 2$, then $\Phi^{(i)}(P_0)/\Phi^{(i+1)}(P_0)$ is an elementary abelian $p$ group of rank $r_i \leq p$ ($i=0, 1, \ldots, k-1$). Therefore, we obtain that

$$\text{Aut}(P_0)/\text{(the stabilizer group of the above chain)}$$

is a subgroup of the direct product of the groups $GL(r_i, p)$ with $r_i \leq p$ ($i=0, 1, \ldots, k-1$), and the stabilizer group of the chain is a $p$ group. Therefore, since $S_{p+2}$ is not involved in $GL(p, p)$ by Theorem A, we obtain that $S_{p+2}$ is not involved in $\text{Aut}(P_0)$. But, this is a contradiction.

Since $C_G(P_0)^{(1,2,\ldots,p+2)} \geq A^{(1,2,\ldots,p+2)}$ we obtain that $|Z(P)| \geq p^2$. But, this contradict the fact (1) that $P$ is of maximal class.

Thus, we have completed the proof of Theorem 1.

**Appendix**

In this appendix, we will prove the following result.

**Theorem 2.** Let $p$ be an odd prime $\geq 11$. Let $G$ be a permutation group on a set $\Omega=\{1, 2, \ldots, n\}$ which satisfies the following conditions:

(i) $G$ is $(p+1)$-ply transitive, and $n \equiv 1 \pmod p$,

(ii) a Sylow $p$ subgroup $P_0$ of $G_{1,2,\ldots,p+1}$ is semiregular on $\Omega=\{1, 2, \ldots, p+1\}$, and

(iii) $|P_0| \geq p^2$.

Then we obtain that $P_0$ is an elementary abelian $p$ group of order $p^p$ and that a Sylow $p$ subgroup $P$ of $G$ is isomorphic to $Z_p \wr Z_p$ (wreathed product).

The next Theorem B is proved by quite the same argument as in Theorem A, and so we omit the proof.

**Theorem B.** Let $p$ be an odd prime $\geq 11$. Then $S_{p+1}$ is not involved in $GL(p-1, p)$. 
**Proof of Theorem 2.** Let $P$ be a Sylow subgroup of $G$ which contains $P_0$. Then $P$ is of maximal class. We obtain that $|P_0/\Phi(P_0)| \leq p^p$, because of Lemma of Ito in Nagao [10]. Since $S_{p+1}$ must be involved in $\text{Aut}(P_0)$ (cf. the proof of Theorem 1) and since $S_{p+1}$ is not involved in $GL(p-1, p)$ by Theorem B, we obtain that $|P_0/\Phi(P_0)| = p^p$, because of a result of Burnside (cf. Gorenstein 4, Theorem 5.1.4.) (The use of the result of Burnside simplifies the argument of the proof of Theorem 1 a little, i.e., in step (3) we have only to show that $S_{p+1}$ is not involved in $\text{Aut}(P_0/\Phi(P_0))$.) Now, $P/\Phi(P_0)$ is a homomorphic image of $P$ and is isomorphic to $Z_p \times Z_p$. Therefore, by a result of Blackburn (cf. Huppert [7, Kapitel III, Satz 14.20]) we obtain that $\Phi(P_0) = 1$, and so we obtain the assertion of Theorem 2.

**University of Tokyo**

References
