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ON FINITE GALOIS COVERING GERMS

Dedicated to Professor Shingo Murakami on his sixtieth birthday

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Introduction. We denote by $\boldsymbol{C}^n$ the n-th Cartesian product of the complex plane $\boldsymbol{C}$. Let $W=(W,O)$ be the germ of open balls in $\boldsymbol{C}^n$ with the center $O=(0,\cdots,0)$. A *finite covering germ* is, by definition, a germ $\pi:X\to W$ of surjective proper finite holomorphic mappings, where $X=(X,p)$ is a germ of irreducible normal complex spaces.

Every normal singularity (X,p) has the structure of a finite covering germ $\pi:X\to W$, (see Gunning-Rossi [4]).

Finite covering germs were discussed in Gunning [3] from the ring theoretic point of view.

In this paper, we introduce the notion of finite Galois covering germs and prove two basic theorems (Theorems 2 and 3 below) on it.

1. Some definitions. Let M be an n-dimensional (connected) complex manifold. A *finite covering* of M is, by definition, a surjective proper finite holomorphic mapping $\pi:X\to M$, where X is an irreducible normal complex space. Let $\pi:X\to M$ and $\mu:Y\to M$ be finite coverings of M. A *morphism* (resp. an *isomorphism*) *of* π *to* μ is, by definition, a surjective holomorphic (resp. biholomorphic) mapping φ: $X\to Y$ such that $\mu\varphi=\pi$. We denote by G_π the group of all *automorphisms* of π and call it the *automorphism group of* π. G_π acts on each fiber of π.

A finite covering π: $X\to M$ is called a *finite Galois covering* if G_π acts transitively on every fiber of π. In this case, the quotient complex space X/G_π (see Cartan[1]) is biholomrophic to M.

For a finite covering π: $X\to M$, put

$$R_\pi = \{p\in X \mid \pi \text{ is not biholomorphic around } p\},$$
$$B_\pi = \pi(R_\pi).$$

They are hypersurfaces (i.e. codimension 1 at every point) of X and M, respectively and are called the *ramification locus* and the *branch locus of* π, respectively.

Let B be a hypersurface of M. A finite covering $\pi: X\to M$ is said to *branch at most at* B if the branch locus B_π of π is contained in B. In this case, the restriction

$$\pi': X-\pi^{-1}(B)\to M-B$$

of π is an unbranched covering. The mapping degree of π' is called the *degree of* π and is denoted by $\deg \pi$.

By a property of normal complex spaces, we have easily (see Namba[5])

Proposition 1. (1) *$G_\pi\simeq G_{\pi'}$ naturally.* (2) *π is a Galois covering if and only if π' is a Galois covering.*

Corollary. *$\sharp G_\pi\leqslant \deg \pi$, where $\sharp G_\pi$ is the order of the group G_π. Moreover, the equality holds if and only if π is a Galois covering.*

The following theorem is a deep one.

Theorem 1 (Grauert-Remmert [2]). *If π': $X'\to M-B$ is an unbranched finite covering, then there exists a unique* (*up to isomorphisms*) *finite covering $\pi: X\to M$ which extends π'.*

Take a point $q_0\in M-B$ and fix it. We denote by $\pi_1(M-B, q_0)$ the fundamental group of $M-B$ with the reference point q_0.

Corollary. *There is a one-to-one correspondence between isomorphism classes of finite* (*resp. Galois*) *coverings $\pi: X\to M$ which branches at most at B and the set of all conjugacy classes of subgroups* (*resp. normal subgroups*) *H of $\pi_1(M-B, q_0)$ of finite index. If H is normal, then π corresponding to H satisfies*

$$G_\pi\simeq\pi_1(M-B, q_0)/H.$$

EXAMPLE 1. Put $X = \boldsymbol{C}^n$, $M = \boldsymbol{C}^n$ and

$$\pi: (x_1, \cdots, x_n)\in\boldsymbol{C}^n\mapsto(a_1, \cdots, a_n)\in\boldsymbol{C}^n,$$

where

$$\begin{aligned} a_1 &= -(x_1+\cdots+x_n), \\ a_2 &= x_1x_2+x_1x_3+\cdots+x_{n-1}x_n \\ &\cdots\cdots\cdots \\ a_n &= (-1)^n x_1\cdots x_n. \end{aligned}$$

In other words, x_j $(1\leqslant j\leqslant n)$ are the roots of the equation

$$x^n+a_1x^{n-1}+\cdots+a_n = 0.$$

Then π is a Galois covering of $M=\boldsymbol{C}^n$ such that (i) $B_\pi=\Delta$ is the discriminant locus and (ii) $G_\pi \simeq S_n$ (the n-th symmetric group).

We may identify G_π and S_n through the isomorphism. S_n is then regarded as a finite subgroup of the general linear group $GL(n,\boldsymbol{C})$.

EXAMPLE 2. We regard S_n as a finite subgroup of $GL(n,\boldsymbol{C})$ as in Example 1. Put $Y=\boldsymbol{C}^n$. Let G be a subgroup of S_n The quotient space Y/G is an irreducible normal comqlex space and the canonical projection

$$\mu\colon Y \to Y/G = N$$

is a holomorphic mapping. Let

$$\alpha\colon M \to N$$

be a resolution of singularity of N. Then the finite Galois covering $\pi\colon X \to M$ of M, defined by the following diagram, satisfies $G_\pi \simeq G$:

$$\begin{tikzcd} X \arrow[r, "\rho"'] \arrow[d, "\pi"'] & M\times_N Y \arrow[r] \arrow[d] & Y \arrow[d, "\mu"] \\ M \arrow[r, "id"'] & M \arrow[r, "\alpha"'] & N. \end{tikzcd}$$

Here, $M\times_N Y$ is the fiber product, ρ is the normalization and id is the identity mapping.

2. Finite Galois covering germs. Now, let $W=(W,O)$ be the germ of open balls in $\boldsymbol{C}^n$ with the center $O=(0, \cdots, 0)$. Let $\pi\colon X\to W$ be a finite coving germ (see Introduction). Every notion in §1 can be easily extended to finite covering germs. In particular, a finite covering germ $\pi\colon X\to W$ is called a *finite Galois covering germ* if G_π acts transitively on every fiber of π. Also, a similar assertion to Corollary to Theorem 1 holds in the case of finite covering germs, if $\pi_1(M-B, q_0)$ is replaced by the local fundamental group $\pi_{1,\mathrm{loc},0}(W-B)$ of $W-B$ at O.

EXAMPLE 3. Let $\pi_0\colon X\to W$ be the restriction of the covering $\pi\colon \boldsymbol{C}^n \to \boldsymbol{C}^n$ in Example 1 to $W=(W,O)$ and $\mathrm{X}=(\mathrm{X},\mathrm{O})=\pi^{-1}(W)$. Then π_0 is an a finite Galois covering germ such that $G_{\pi_0}\simeq S_n$.
There exist a lot of finite Galois covering germs in the following sense:

Theorem 2. *For $n\geqslant 2$, let $W=(W,O)$ be the germ of balls in $\boldsymbol{C}^n$ with the center O. For every finite group G, there exists a finite Galois covering germ $\pi : X\to W$ such that $G_\pi \simeq G$.*

Proof. *Case* 1. We first prove the theorem for the case $n=2$. Let W

be a ball in $\boldsymbol{C}^2$ with the center O. Let L_j $(1 \leqslant j \leqslant s)$ be mutually distinct (complex) lines in $\boldsymbol{C}^2$ passing through O. Put $D_j = L_j \cap W$ $(1 \leqslant j \leqslant s)$ and

$$B = D_1 \cap \cdots \cap D_s,$$

(see Figure 1).

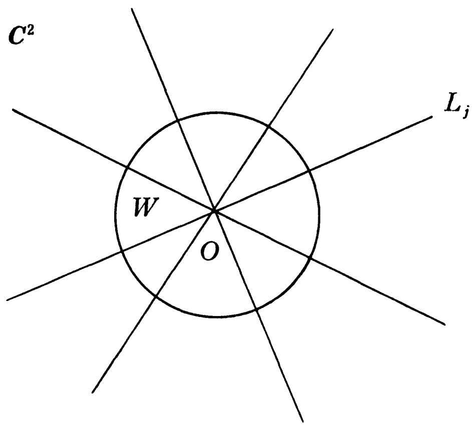

Figure 1

Take a point $q_0 \in M - B$ and fix it. Let γ_j be a loop in $M - B$ starting from q_0 and rounding $D_j - O$ once counterclockwisely as in Figure 2. We identify γ_j with its homotopy class.

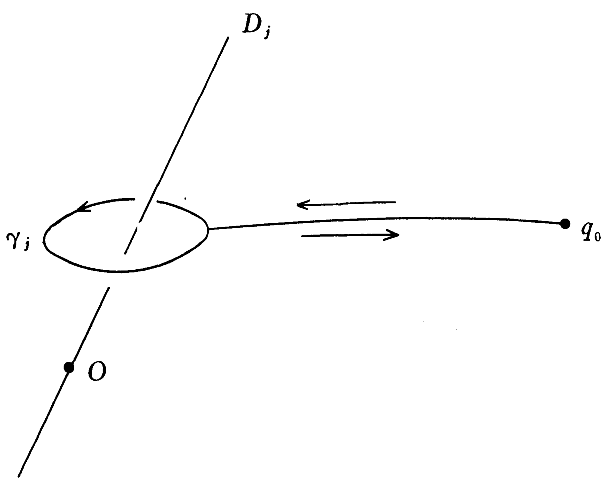

Figure 2

Then, as is well known, $\pi_1(W - B, q_0)$ is a group generated by $\gamma_1, \cdots, \gamma_s$ with the generating relations

$$\gamma_j\delta = \delta\gamma_j \qquad (1 \leqslant j \leqslant s),$$

where $\delta = \gamma_1 \cdots \gamma_s$.

Let F_{s-1} be the free group of $(s-1)$-letters $b_1, \cdots, b_{s-1}$. Put $b_s = (b_1 \cdots b_{s-1})^{-1}$. Then there is the surjective homomorphism

$$\Phi\colon \pi_1(W-B,\, q_0) \to F_{s-1}$$

defined by $\Phi(\gamma_j) = b_j$ $(1 \leqslant j \leqslant s)$.

For any finite group G, there is a surjective homomorphism

$$\Psi\colon F_{s-1} \to G$$

for a sufficiently large s.

Now, the kernel K of the surjective homomorphism

$$\Psi\Phi\colon \pi_1(W-B,\, q_0) \to G$$

has a finite index such that

$$\pi_1(W-B,\, q_0)/K \simeq G.$$

The finite Galois covering $\pi\colon X \to W$ corresponding to K in Corollary to Theorem 1 satisfies $G_\pi \simeq G$.

The finite Galois covering germ determined by π is a desired one.

Case 2. Next, we prove the theorem for the case $n \geqslant 3$. Let W be a ball in $\boldsymbol{C}^n$ with the center O. Let P and Q be a 2-plane and an $(n-2)$-plane in $\boldsymbol{C}^n$, respectively, passing through O such that $P \cap Q = \{O\}$. Let H_j $(1 \leqslant j \leqslant s)$ be mutually distinct hyperplanes in $\boldsymbol{C}^n$ passing through O and containing Q (see Figure 3). Put

$$\begin{aligned} D_j &= H_j \cap W \qquad (1 \leqslant j \leqslant s) \text{ and} \\ B &= D_1 \cap \cdots \cap D_s. \end{aligned}$$

Then $W-B$ and $W \cap P - B \cap P$ are homotopic. Hence, by Case 1, taking sufficiently large s, there exists a normal subgroup K of $\pi_1(W-B,\, q_0)$ of finite index such that

$$\pi_1(W-B,\, q_0)/K \simeq G.$$

The rest of the proof is similar to Case 1. q.e.d.

Now, we give a method of concrete constructions of every finite Galois covering germ. Our method is suggested by Professor Enoki and is different from and simpler than Namba[6] in which finite Galois coverings of projective manifolds were treated.

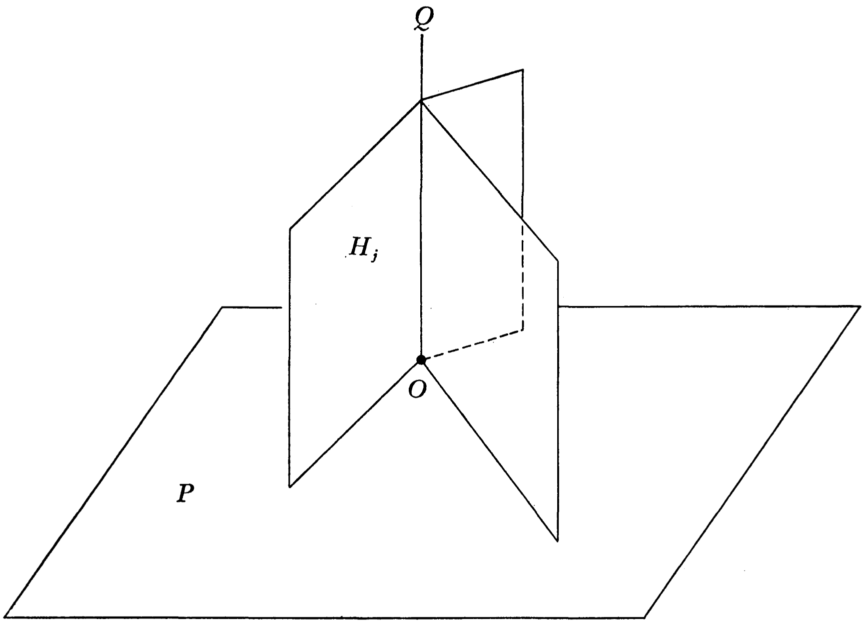

Figure 3

Theorem 3. *Let $\pi: X \to W$ be a finite Galois covering germ. Put $m = \deg \pi$. Then there exists a germ $f: W \to \boldsymbol{C}^m$ of holomorphic mappings and a finite subgroup G of S_m with $G \simeq G_\pi$ such that π is obtained by the following commutative diagram:*

$$\begin{tikzcd} X \arrow[r, "\rho"] \arrow[d, "\pi"'] & W \times_N Y \arrow[r] \arrow[d] & \boldsymbol{C}^m = Y \arrow[d, "\mu"] \\ W \arrow[r, "id"'] & W \arrow[r, "f"'] & \boldsymbol{C}^m/G = N, \end{tikzcd}$$

where $W \times_N Y$ is the fiber product, ρ is the normalization and id is the identity mapping. Here S_m is regarded as a finite subgroup of $GL(m, \boldsymbol{C})$ as in Example 1.

Proof. We may assume that W is a small ball in $\boldsymbol{C}^n$ with the center O. Take a point $q_0 \in W - B$ and put

$$\pi^{-1}(q_0) = \{p_1, \cdots, p_m\}.$$

Put

$$G_\pi = \{\sigma_1 = 1, \sigma_2, \cdots, \sigma_m\}.$$

Note that X is a Stein space. Let h be a holomorphic function on X such that

$$h(p_j) \neq h(p_k) \qquad \text{for } j \neq k \tag{1}$$

Put

$$h_j = \sigma_j^* h = h \cdot \sigma_j \qquad (1 \leqslant j \leqslant m).$$

Let F: $X \to \boldsymbol{C}^m$ be the holomorphic mapping defined by

$$F(p) = (h_1(p), \cdots, h_m(p)).$$

Then, for $\sigma \in G$,

$$\begin{aligned}(\sigma^* F)(p) = F(\sigma(p)) &= (h_1(\sigma(p)), \cdots, h_m(\sigma(p)) \\ &= (h(\sigma(p)), h(\sigma_2\sigma(p)), \cdots, h(\sigma_m\sigma(p)) \\ &= (h_{k(1)}(p), h_{k(2)}(p), \cdots, h_{k(m)}(p))\end{aligned} \tag{2}$$

Thus σ gives the permutation

$$R(\sigma) = \begin{pmatrix} 1 & 2 \cdots & m \\ k(1) & k(2) \cdots & k(m) \end{pmatrix}.$$

The corrwspondence

$$R\colon \sigma \mapsto R(\sigma)$$

is then an isomorphism of G_π onto a subgroup G of S_m. (2) can be rewritten as

$$\sigma^* F = R(\sigma) F \qquad \text{for all } \sigma \in G. \tag{3}$$

Hence F induces a holomorphic mapping $f\colon W \to \boldsymbol{C}^m/G = N$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{F} & C^m = Y \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ W & \xrightarrow[f]{} & C^m/G = N. \end{array}$$

By the assumption (1), we can easily show that f has the following two properties:

(i) $f(W) \not\subset \text{Fix } G$, where Fix G is the union of the fixed points of all elements of G except the identity and

(ii) f is not decomposed as follows:

$$\begin{tikzcd} W \arrow[r, "f'"] \arrow[d, "id"'] & C^m/H \arrow[d, "\nu"] \\ W \arrow[r, "f"'] & C^m/G, \end{tikzcd}$$

where $H(\neq G)$ is a subgroup of G, ν is the canonical projection and f' is a holomorphic mapping.

A holomorphic mapping f with the properties (i) and (ii) is said to be *G-indecomposable* (see Namba[6]). For such a mapping f, the fiber product $W\times_N Y$ is irreducible and the finite Galois covering π_0: $X_0 \to W$ defined by the commutative diagram

$$\begin{tikzcd} X_0 \arrow[r, "\rho"'] \arrow[d, "\pi_0"'] & W\times_N Y \arrow[r] \arrow[d] & C^m=Y \arrow[d, "\mu"] \\ W \arrow[r, "id"'] & W \arrow[r, "f"'] & C^m/G=N \end{tikzcd}$$

satisfies $G_{\pi_0}\simeq G$. Now, we can easily show that π is isomorphic to π_0, (see Namba[6]). q.e.d.

REMARK. (1) $f(O)$ is not necessarily equal to $\mu(O)$, where O is the origin of $\boldsymbol{C}^m$. (2) A similar theorem to Theorem 3 holds for finite Galois coverings of a Stein manifold.

PROBLEM. Characterize normal singularities (X,p) which has the structure of a finite Galois covering germs π: $X\to W$.

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