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ON FINITE GALOIS COVERING GERMS

Dedicated to Professor Shingo Murakami on his sixtieth birthday

MAKOTO NAMBA

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Introduction. We denote by \mathbf{C}^n the n -th Cartesian product of the complex plane \mathbf{C} . Let $W=(W,O)$ be the germ of open balls in \mathbf{C}^n with the center $O=(0,\dots,0)$. A *finite covering germ* is, by definition, a germ $\pi:X\rightarrow W$ of surjective proper finite holomorphic mappings, where $X=(X,p)$ is a germ of irreducible normal complex spaces.

Every normal singularity (X,p) has the structure of a finite covering germ $\pi:X\rightarrow W$, (see Gunning-Rossi [4]).

Finite covering germs were discussed in Gunning [3] from the ring theoretic point of view.

In this paper, we introduce the notion of finite Galois covering germs and prove two basic theorems (Theorems 2 and 3 below) on it.

1. Some definitions. Let M be an n -dimensional (connected) complex manifold. A *finite covering* of M is, by definition, a surjective proper finite holomorphic mapping $\pi:X\rightarrow M$, where X is an irreducible normal complex space. Let $\pi:X\rightarrow M$ and $\mu:Y\rightarrow M$ be finite coverings of M . A *morphism* (resp. an *isomorphism*) of π to μ is, by definition, a surjective holomorphic (resp. biholomorphic) mapping $\varphi:X\rightarrow Y$ such that $\mu\varphi=\pi$. We denote by G_π the group of all *automorphisms* of π and call it the *automorphism group* of π . G_π acts on each fiber of π .

A finite covering $\pi:X\rightarrow M$ is called a *finite Galois covering* if G_π acts transitively on every fiber of π . In this case, the quotient complex space X/G_π (see Cartan[1]) is biholomorphic to M .

For a finite covering $\pi:X\rightarrow M$, put

$$R_\pi = \{p \in X \mid \pi \text{ is not biholomorphic around } p\},$$
$$B_\pi = \pi(R_\pi).$$

They are hypersurfaces (i.e. codimension 1 at every point) of X and M , respectively and are called the *ramification locus* and the *branch locus* of π , respectively.

Let B be a hypersurface of M . A finite covering $\pi: X \rightarrow M$ is said to *branch at most at B* if the branch locus B_π of π is contained in B . In this case, the restriction

$$\pi': X - \pi^{-1}(B) \rightarrow M - B$$

of π is an unbranched covering. The mapping degree of π' is called the *degree of π* and is denoted by $\deg \pi$.

By a property of normal complex spaces, we have easily (see Namba[5])

Proposition 1. (1) $G_\pi \simeq G_{\pi'}$ naturally. (2) π is a Galois covering if and only if π' is a Galois covering.

Corollary. $\#G_\pi \leq \deg \pi$, where $\#G_\pi$ is the order of the group G_π . Moreover, the equality holds if and only if π is a Galois covering.

The following theorem is a deep one.

Theorem 1 (Grauert-Remmert [2]). *If $\pi': X' \rightarrow M - B$ is an unbranched finite covering, then there exists a unique (up to isomorphisms) finite covering $\pi: X \rightarrow M$ which extends π' .*

Take a point $q_0 \in M - B$ and fix it. We denote by $\pi_1(M - B, q_0)$ the fundamental group of $M - B$ with the reference point q_0 .

Corollary. *There is a one-to-one correspondence between isomorphism classes of finite (resp. Galois) coverings $\pi: X \rightarrow M$ which branches at most at B and the set of all conjugacy classes of subgroups (resp. normal subgroups) H of $\pi_1(M - B, q_0)$ of finite index. If H is normal, then π corresponding to H satisfies*

$$G_\pi \simeq \pi_1(M - B, q_0)/H.$$

EXAMPLE 1. Put $X = \mathbf{C}^n$, $M = \mathbf{C}^n$ and

$$\pi: (x_1, \dots, x_n) \in \mathbf{C}^n \mapsto (a_1, \dots, a_n) \in \mathbf{C}^n,$$

where

$$\begin{aligned} a_1 &= -(x_1 + \dots + x_n), \\ a_2 &= x_1x_2 + x_1x_3 + \dots + x_{n-1}x_n \\ &\dots\dots\dots \\ a_n &= (-1)^n x_1 \dots x_n. \end{aligned}$$

In other words, x_j ($1 \leq j \leq n$) are the roots of the equation

$$x^n + a_1x^{n-1} + \dots + a_n = 0.$$

Then π is a Galois covering of $M = \mathbf{C}^n$ such that (i) $B_\pi = \Delta$ is the discriminant

locus and (ii) $G_\pi \simeq S_n$ (the n -th symmetric group).

We may identify G_π and S_n through the isomorphism. S_n is then regarded as a finite subgroup of the general linear group $GL(n, \mathbf{C})$.

EXAMPLE 2. We regard S_n as a finite subgroup of $GL(n, \mathbf{C})$ as in Example 1. Put $Y = \mathbf{C}^n$. Let G be a subgroup of S_n . The quotient space Y/G is an irreducible normal complex space and the canonical projection

$$\mu: Y \rightarrow Y/G = N$$

is a holomorphic mapping. Let

$$\alpha: M \rightarrow N$$

be a resolution of singularity of N . Then the finite Galois covering $\pi: X \rightarrow M$ of M , defined by the following diagram, satisfies $G_\pi \simeq G$:

$$\begin{array}{ccccc} X & \xrightarrow{\quad \rho \quad} & M \times_N Y & \longrightarrow & Y \\ \pi \downarrow & & \downarrow & & \downarrow \mu \\ M & \xrightarrow{\quad id \quad} & M & \xrightarrow{\quad \alpha \quad} & N. \end{array}$$

Here, $M \times_N Y$ is the fiber product, ρ is the normalization and id is the identity mapping.

2. Finite Galois covering germs. Now, let $W = (W, O)$ be the germ of open balls in \mathbf{C}^n with the center $O = (0, \dots, 0)$. Let $\pi: X \rightarrow W$ be a finite covering germ (see Introduction). Every notion in §1 can be easily extended to finite covering germs. In particular, a finite covering germ $\pi: X \rightarrow W$ is called a *finite Galois covering germ* if G_π acts transitively on every fiber of π . Also, a similar assertion to Corollary to Theorem 1 holds in the case of finite covering germs, if $\pi_1(M - B, q_0)$ is replaced by the local fundamental group $\pi_{1, \text{loc}, 0}(W - B)$ of $W - B$ at O .

EXAMPLE 3. Let $\pi_0: X \rightarrow W$ be the restriction of the covering $\pi: \mathbf{C}^n \rightarrow \mathbf{C}^n$ in Example 1 to $W = (W, O)$ and $X = (X, O) = \pi^{-1}(W)$. Then π_0 is a finite Galois covering germ such that $G_{\pi_0} \simeq S_n$.

There exist a lot of finite Galois covering germs in the following sense:

Theorem 2. For $n \geq 2$, let $W = (W, O)$ be the germ of balls in \mathbf{C}^n with the center O . For every finite group G , there exists a finite Galois covering germ $\pi: X \rightarrow W$ such that $G_\pi \simeq G$.

Proof. Case 1. We first prove the theorem for the case $n = 2$. Let W

be a ball in \mathbf{C}^2 with the center O . Let L_j ($1 \leq j \leq s$) be mutually distinct (complex) lines in \mathbf{C}^2 passing through O . Put $D_j = L_j \cap W$ ($1 \leq j \leq s$) and

$$B = D_1 \cap \cdots \cap D_s,$$

(see Figure 1).

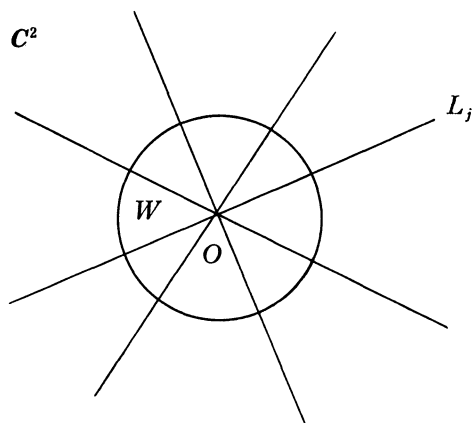


Figure 1

Take a point $q_0 \in M - B$ and fix it. Let γ_j be a loop in $M - B$ starting from q_0 and rounding $D_j - O$ once counterclockwise as in Figure 2. We identify γ_j with its homotopy class.

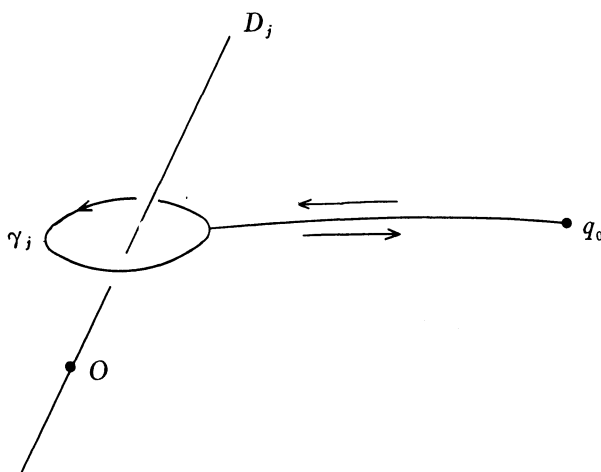


Figure 2

Then, as is well known, $\pi_1(W - B, q_0)$ is a group generated by $\gamma_1, \dots, \gamma_s$ with the generating relations

$$\gamma_j \delta = \delta \gamma_j \quad (1 \leq j \leq s),$$

where $\delta = \gamma_1 \cdots \gamma_s$.

Let F_{s-1} be the free group of $(s-1)$ -letters b_1, \dots, b_{s-1} . Put $b_s = (b_1 \cdots b_{s-1})^{-1}$. Then there is the surjective homomorphism

$$\Phi: \pi_1(W-B, q_0) \rightarrow F_{s-1}$$

defined by $\Phi(\gamma_j) = b_j$ ($1 \leq j \leq s$).

For any finite group G , there is a surjective homomorphism

$$\Psi: F_{s-1} \rightarrow G$$

for a sufficiently large s .

Now, the kernel K of the surjective homomorphism

$$\Psi\Phi: \pi_1(W-B, q_0) \rightarrow G$$

has a finite index such that

$$\pi_1(W-B, q_0)/K \simeq G.$$

The finite Galois covering $\pi: X \rightarrow W$ corresponding to K in Corollary to Theorem 1 satisfies $G_\pi \simeq G$.

The finite Galois covering germ determined by π is a desired one.

Case 2. Next, we prove the theorem for the case $n \geq 3$. Let W be a ball in \mathbf{C}^n with the center O . Let P and Q be a 2-plane and an $(n-2)$ -plane in \mathbf{C}^n , respectively, passing through O such that $P \cap Q = \{O\}$. Let H_j ($1 \leq j \leq s$) be mutually distinct hyperplanes in \mathbf{C}^n passing through O and containing Q (see Figure 3). Put

$$\begin{aligned} D_j &= H_j \cap W \quad (1 \leq j \leq s) \text{ and} \\ B &= D_1 \cap \cdots \cap D_s. \end{aligned}$$

Then $W-B$ and $W \cap P - B \cap P$ are homotopic. Hence, by Case 1, taking sufficiently large s , there exists a normal subgroup K of $\pi_1(W-B, q_0)$ of finite index such that

$$\pi_1(W-B, q_0)/K \simeq G.$$

The rest of the proof is similar to Case 1.

q.e.d.

Now, we give a method of concrete constructions of every finite Galois covering germ. Our method is suggested by Professor Enoki and is different from and simpler than Namba[6] in which finite Galois coverings of projective manifolds were treated.

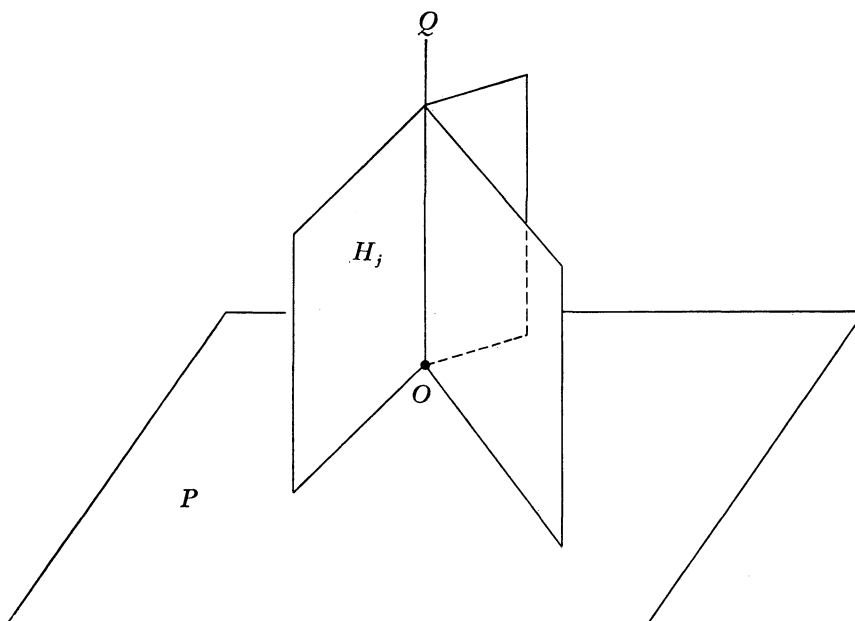


Figure 3

Theorem 3. *Let $\pi : X \rightarrow W$ be a finite Galois covering germ. Put $m = \deg \pi$. Then there exists a germ $f : W \rightarrow \mathbf{C}^m$ of holomorphic mappings and a finite subgroup G of S_m with $G \simeq G_\pi$ such that π is obtained by the following commutative diagram :*

$$\begin{array}{ccccc}
 X & \xrightarrow{\quad} & W \times_N Y & \xrightarrow{\quad} & \mathbf{C}^m = Y \\
 \downarrow \pi & \searrow \rho & \downarrow & & \downarrow \mu \\
 W & \xrightarrow{id} & W & \xrightarrow{f} & \mathbf{C}^m / G = N,
 \end{array}$$

where $W \times_N Y$ is the fiber product, ρ is the normalization and id is the identity mapping. Here S_m is regarded as a finite subgroup of $GL(m, \mathbf{C})$ as in Example 1.

Proof. We may assume that W is a small ball in \mathbf{C}^n with the center O . Take a point $q_0 \in W - B$ and put

$$\pi^{-1}(q_0) = \{p_1, \dots, p_m\}.$$

Put

$$G_\pi = \{\sigma_1 = 1, \sigma_2, \dots, \sigma_m\}.$$

Note that X is a Stein space. Let h be a holomorphic function on X such that

$$h(p_j) \neq h(p_k) \quad \text{for } j \neq k \quad (1)$$

Put

$$h_j = \sigma_j^* h = h \cdot \sigma_j \quad (1 \leq j \leq m).$$

Let $F: X \rightarrow \mathbf{C}^m$ be the holomorphic mapping defined by

$$F(p) = (h_1(p), \dots, h_m(p)).$$

Then, for $\sigma \in G$,

$$\begin{aligned} (\sigma^* F)(p) &= F(\sigma(p)) = (h_1(\sigma(p)), \dots, h_m(\sigma(p))) \\ &= (h(\sigma(p)), h(\sigma_2 \sigma(p)), \dots, h(\sigma_m \sigma(p))) \\ &= (h_{k(1)}(p), h_{k(2)}(p), \dots, h_{k(m)}(p)) \end{aligned} \quad (2)$$

Thus σ gives the permutation

$$R(\sigma) = \begin{pmatrix} 1 & 2 & \dots & m \\ k(1) & k(2) & \dots & k(m) \end{pmatrix}.$$

The correspondence

$$R: \sigma \mapsto R(\sigma)$$

is then an isomorphism of G_π onto a subgroup G of S_m . (2) can be rewritten as

$$\sigma^* F = R(\sigma) F \quad \text{for all } \sigma \in G. \quad (3)$$

Hence F induces a holomorphic mapping $f: W \rightarrow \mathbf{C}^m/G = N$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{F} & \mathbf{C}^m = Y \\ \pi \downarrow & & \downarrow \\ W & \xrightarrow{f} & \mathbf{C}^m/G = N. \end{array}$$

By the assumption (1), we can easily show that f has the following two properties:

- (i) $f(W) \not\subset \text{Fix } G$, where $\text{Fix } G$ is the union of the fixed points of all elements of G except the identity and
- (ii) f is not decomposed as follows:

$$\begin{array}{ccc}
 W & \xrightarrow{f'} & \mathbf{C}^m/H \\
 \text{id} \downarrow & & \downarrow \nu \\
 W & \xrightarrow{f} & \mathbf{C}^m/G,
 \end{array}$$

where $H(\neq G)$ is a subgroup of G , ν is the canonical projection and f' is a holomorphic mapping.

A holomorphic mapping f with the properties (i) and (ii) is said to be *G-indecomposable* (see Namba[6]). For such a mapping f , the fiber product $W \times_N Y$ is irreducible and the finite Galois covering $\pi_0: X_0 \rightarrow W$ defined by the commutative diagram

$$\begin{array}{ccccc}
 X_0 & \xrightarrow{\quad} & W \times_N Y & \xrightarrow{\quad} & \mathbf{C}^m = Y \\
 \pi_0 \downarrow & & \downarrow & & \downarrow \mu \\
 W & \xrightarrow{\quad} & W & \xrightarrow{f} & \mathbf{C}^m/G = N
 \end{array}$$

satisfies $G_{\pi_0} \simeq G$. Now, we can easily show that π is isomorphic to π_0 , (see Namba[6]). q.e.d.

REMARK. (1) $f(O)$ is not necessarily equal to $\mu(O)$, where O is the origin of \mathbf{C}^m . (2) A similar theorem to Theorem 3 holds for finite Galois coverings of a Stein manifold.

PROBLEM. Characterize normal singularities (X, \mathfrak{p}) which has the structure of a finite Galois covering germs $\pi: X \rightarrow W$.

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