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ACTIONS OF HERMANN TYPE AND PROPER COMPLEX EQUIFOCAL SUBMANIFOLDS

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Abstract

In this paper, we mainly prove that principal orbits of an action of Hermann type on a symmetric space of non-compact type are curvature adapted and proper complex equifocal. The proof is performed by showing that principal orbits of the action are partial tubes over a totally geodesic singular orbit and investingating the shape operatores of the partial tubes over a submanifold in a symmetric space of non-compact type.

1. Introduction

Let G/K be a symmetric space of compact type and H be a symmetric subgroup of G (i.e., the group of all fixed points of an involution of G). The H-action on G/K is called a *Hermann action*. Recently, A. Kollross has classified hyperpolar actions on irreducible symmetric spaces of compact type. According to the classification, a hyperpolar action on the symmetric space is a Hermann action or a cohomogeneity one action. It is known that each principal orbit of a hyperpolar action on a symmetric space of compact type is an equifocal submanifold and that conversely any homogeneous equifocal submanifold is catched as a principal orbit of a hyperpolar action (see [13]). Thus any homogeneous equifocal submanifold of codimension bigger than one in a symmetric space of compact type is catched as a principal orbit of a Hermann action. For a submanifold in a symmetric space of non-compact type, the equifocality is not a rigid property. So we [17] have recently defined a rigid property of the *complex equifocality* for a submanifold in the symmetric space. Let G/K be a symmetric space of non-compact type and H be a symmetric subgroup of G. In this paper, we call the H-action on G/K an *action of Hermann type*. We [19] showed that principal orbits of the action are complex equifocal. A pseudo-Riemannian submersion $\widetilde{\phi}$ of a pseudo-Hilbert space onto G/K is defined in a natural manner (see [17]). For each complex equifocal submanifold M in G/K, the inverse image $\widetilde{\phi}^{-1}(M)$ is complex isoparametric in the sense of [17]. In this paper, if the inverse image $\widetilde{\phi}^{-1}(M)$ is proper complex isoparametric in the sense of [17], then we call M a *proper complex equifocal submanifold*. In [19], we introduced the notion of a complex hyperpolar action on a symmetric space of non-compact type. In this paper, we first prove the following fact.

Theorem A. *Actions of Hermann type on a symmetric space of non-compact type are complex hyperpolar.*

The main theorem of this paper is as follows.

Theorem B. *Principal orbits of an action of Hermann type on a symmetric space of non-compact type are curvature adapted and proper complex equifocal.*

REMARK 1.1. By imitating the proof of Theorem B in this paper, we can show that principal orbits of a Hermann action on a symmetric space of compact type are curvature adapted.

Here we propose one problem.

Problem. Is any homogeneous complex equifocal submanifold of codimension bigger than one catched as a principal orbit of an action of Hermann type?

If this problem is solved positively, then it is shown that any homogeneous complex equifocal submanifold of codimension bigger than one is proper complex equifocal.

In Section 2, we recall basic notions and facts. In Section 3, we investigate the shape operators of partial tubes over submanifolds with Lie triple systematic normal bundle in a symmetric space of non-compact type. In Section 4, we prove Theorems A and B.

2. Basic notions and facts

In this section, we recall some notions introduced in [17] and some facts related to them. First we recall the notion of a complex equifocal submanifold. Let $N = G/K$ be a symmetric space, $(\mathfrak{g}, \sigma)$ be its orthogonal symmetric Lie algebra and $\mathfrak{p}$ be the eigenspace for -1 of σ. The subspace $\mathfrak{p}$ is identified with the tangent space $T_{eK}N$ of N at eK, where e is the identity element of G. Let M be an immersed submanifold in N and $T^{\perp}M$ be its normal bundle. If, for each $x(= gK) \in M$, $g_*^{-1}T_x^{\perp}M$ is an abelian subspace in $\mathfrak{p}$, then M is said to *have abelian normal bundle*. Also, if the normal connection of M is flat and has trivial holonomy, then M is said to *have globally flat normal bundle*. In [17], we defined the notion of complex focal radii as imaginary focal radii of submanifolds in a symmetric space of non-compact type as follows. Let M be an immersed submanifold with abelian normal bundle in a symmetric space $N = G/K$ of non-compact type. Denote by A the shape tensor of M. Let $v \in T_x^{\perp}M$ and $X \in T_xM$ $(x = gK)$. Denote by γ_v the geodesic in N with $\dot{\gamma}_v(0) = v$. The Jacobi field Y along γ_v with $Y(0) = X$ and $Y'(0) = -A_v X$ is given by

$$Y(s) = \left(P_{\gamma_v|_{[0,s]}} \circ (D^{co}_{sv} - sD^{si}_{sv} \circ A_v)\right)(X),$$

where $Y'(0) = \widetilde{\nabla}_v Y$, $P_{\gamma_v|_{[0,s]}}$ is the parallel translation along $\gamma_v|_{[0,s]}$,

$$D^{co}_{sv} = g_* \circ \cos\left(\sqrt{-1}\ \mathrm{ad}(sg_*^{-1}v)\right) \circ g_*^{-1}$$

and

$$D^{si}_{sv} = g_* \circ \frac{\sin(\sqrt{-1}\ \mathrm{ad}(sg_*^{-1}v))}{\sqrt{-1}\ \mathrm{ad}(sg_*^{-1}v)} \circ g_*^{-1}$$

(see [25] or [16] in detail). Here ad is the adjoint representation of the Lie algebra $\mathfrak{g}$ of G. Since M has abelian normal bundle, all focal radii (other than conjugate radii) of M are strong focal radii in the sense of [18] (see the proof of Theorem 2 in [18]). Hence all focal radii (other than conjugate radii) of M along γ_v are catched as real numbers s_0 with $\mathrm{Ker}(D^{co}_{s_0 v} - s_0 D^{si}_{s_0 v} \circ A_v) \neq \{0\}$. So, we call a complex number z_0 with $\mathrm{Ker}(D^{co}_{z_0 v} - z_0 D^{si}_{z_0 v} \circ A^{\mathbf{c}}_v) \neq \{0\}$ a *complex focal radius of M along γ_v* and call $\dim \mathrm{Ker}(D^{co}_{z_0 v} - z_0 D^{si}_{z_0 v} \circ A^{\mathbf{c}}_v)$ the *multiplicity* of the complex focal radius z_0, where $D^{co}_{z_0 v}$ (resp. $D^{si}_{z_0 v}$) implies the complexification of a map $(g_* \circ \cos(\sqrt{-1}\, z_0\ \mathrm{ad}(g_*^{-1}v)) \circ g_*^{-1})|_{T_x M}$ (resp. $(g_* \circ \left[\{\sin(\sqrt{-1}\, z_0\ \mathrm{ad}(g_*^{-1}v))\}/\{\sqrt{-1}\, z_0\ \mathrm{ad}(g_*^{-1}v)\}\right] \circ g_*^{-1})|_{T_x M})$ from $T_x M$ to $T_x N^{\mathbf{c}}$. Also, for a complex focal radius z_0 of M along γ_v, we call $z_0 v$ ($\in T^{\perp}_x M^{\mathbf{c}}$) a *complex focal normal vector of M at x*. Furthermore, assume that M has globally flat normal bundle. Let $\tilde{v}$ be a parallel unit normal vector field of M. Assume that the number (which may be 0 and ∞) of distinct complex focal radii along $\gamma_{\tilde{v}_x}$ is independent of the choice of $x \in M$. Furthermore assume that the number is not equal to 0. Let $\{r_{i,x} \mid i = 1, 2, \ldots\}$ be the set of all complex focal radii along $\gamma_{\tilde{v}_x}$, where $|r_{i,x}| < |r_{i+1,x}|$ or "$|r_{i,x}| = |r_{i+1,x}|$ & $\mathrm{Re}\, r_{i,x} > \mathrm{Re}\, r_{i+1,x}$" or "$|r_{i,x}| = |r_{i+1,x}|$ & $\mathrm{Re}\, r_{i,x} = \mathrm{Re}\, r_{i+1,x}$ & $\mathrm{Im}\, r_{i,x} = -\mathrm{Im}\, r_{i+1,x} > 0$." Let r_i ($i = 1, 2, \ldots$) be complex valued functions on M defined by assigning $r_{i,x}$ to each $x \in M$. We call these functions r_i ($i = 1, 2, \ldots$) *complex focal radius functions for $\tilde{v}$*. We call $r_i \tilde{v}$ a *complex focal normal vector field for $\tilde{v}$*. If, for each parallel unit normal vector field $\tilde{v}$ of M, the number of distinct complex focal radii along $\gamma_{\tilde{v}_x}$ is independent of the choice of $x \in M$, each complex focal radius function for $\tilde{v}$ is constant on M and it has constant multiplicity, then we call M a *complex equifocal submanifold*. Let $\phi\colon H^0([0,1], \mathfrak{g}) \to G$ be the parallel transport map for G (see [17] about this definition) and $\pi : G \to G/K$ be the natural projection. It is shown in [17] that M is complex equifocal if and only if each component of $(\pi \circ \phi)^{-1}(M)$ is complex isoparametric. In particular, if each component of $(\pi \circ \phi)^{-1}(M)$ is proper complex isoparametric, then we call M a *proper complex equifocal*. See [17] about the definitions of the complex isoparametricness and the proper complex isoparametricness. In this paper, we assume that all complex equifocal submanifolds are properly immersed complete ones.

Next we recall the definition of a complex hyperpolar action on a symmetric space of non-compact type defined in [19]. Let G/K be a symmetric space of non-compact type and H be a closed subgroup of G. If there exists a complete flat totally geodesic submanifold Σ meeting all H-orbits orthogonally, then we call the H-action on G/K a *complex hyperpolar action*. It is known that all principal orbits of the complex hyperpolar action are complex equifocal (see Theorem 12 of [19]). If H is a symmetric subgroup of G (i.e., a group of all fixed points of an involution σ of G), then we call the H-action an *action of Hermann type*. Let $\theta\colon G \to G$ be the Cartan involution associated with G/K, $\mathfrak{k}$ (resp. $\mathfrak{h}$) be the Lie algebra of K (resp. H) and $\mathfrak{p}$ be the eigenspace for -1 of θ_{*e}. Also, let G^*/K be the compact dual of G/K. Assume that $\sigma\circ\theta = \theta\circ\sigma$. Then we set $\mathfrak{h}^* := \mathfrak{h}\cap\mathfrak{k} + \sqrt{-1}\,\mathfrak{h}\cap\mathfrak{p}$. Let H^* be the connected subgroup of G^* whose Lie algebra is equal to $\mathfrak{h}^*$. It is clear that H^* is a symmetric subgroup of G^*, that is, the H^*-action on G^*/K is a Hermann action. Thus the H-action is the dual action of the Hermann action. On the other hand, in case of $\sigma\circ\theta \neq \theta\circ\sigma$, there exists an automorphism ρ of G with $(\rho\circ\sigma\circ\rho^{-1})\circ\theta = \theta\circ(\rho\circ\sigma\circ\rho^{-1})$ in terms of Lemma 10.2 of [1]. From these facts, it follows that the H-action is conjugate to the dual action of a Hermann action.

At the end of this section, we recall the notion of a curvature adapted submanifold. Let M be a submanifold in a symmetric space G/K and A be the shape tensor of M. Also, let R be the curvature tensor of G/K. If, for each normal vector v of M, the operator $R(\cdot, v)v$ preserves T_xM (x: the base point of v) invariant and it commutes with A_v, then M is called a *curvature adapted submanifold*. Regrettably, examples of a curvature adapted submanifold have not been known very much.

3. Shape operators of partial tubes

In this section, we investigate the shape operators of partial tubes over a submanifold with Lie triple systematic normal bundle in a symmetric space of non-compact type. See [16] about the notion of a submanifold with Lie triple systematic normal bundle. Let M be a submanifold with Lie triple systematic normal bundle in a symmetric space G/K of non-compact type. Let $t(M)$ be a connected submanifold in the normal bundle $T^\perp M$ of M such that, for any curve $\beta\colon [0,1] \to M$, $P^\perp_\beta(t(M)\cap T^\perp_{\beta(0)}M) = t(M)\cap T^\perp_{\beta(1)}M$ holds, where $P^\perp_\beta$ is the parallel transport along β with respect to the normal connection. Denote by F the set of all critical points of the normal exponential map $\exp^\perp$ of M. Assume that $t(M)\cap F = \emptyset$. Then the restriction $\exp^\perp|_{t(M)}$ of $\exp^\perp$ to $t(M)$ is an immersion of $t(M)$ into G/K. Give the metric induced from that of G/K to $t(M)$. Thus $t(M)$ is a (Riemannian) submanifold in G/K isometrically immersed by $\exp^\perp|_{t(M)}$. We call such a submanifold $t(M)$ a *partial tube over M*. This terminology of partial tube was first used for submanifolds in a Euclidean space by Carter-West ([9]). Define a distribution V on $t(M)$ by $V_\xi = T_\xi(t(M)\cap T^\perp_{\pi(\xi)}M)$ ($\xi\in t(M)$), where π is the bundle projection of $T^\perp M$. We call this distribution a *vertical distribution* on $t(M)$. Let $X\in T_{\pi(\xi)}M$. Take a curve α in M with $\dot\alpha(0) = X$. Let $\widetilde{\xi}$

be a parallel normal vector field along α with $\widetilde{\xi}(0)=\xi$. We denote $\dot{\widetilde{\xi}}(0)$ by $\widetilde{X}_\xi$ and call it the *horizontal lift of X to ξ*. Define a distribution H on $t(M)$ by $H_\xi = \{\widetilde{X}_\xi \mid X \in T_{\pi(\xi)}M\}$ ($\xi \in t(M)$). We call this distribution a *horizontal distribution* on $t(M)$. Assume that $t(M)$ is contained in an ε-tube $t_\varepsilon(M) := \{\xi \in T^\perp M \mid \|\xi\| = \varepsilon\}$. Define a subbundle N of $T^\perp t(M)$ by $N_\xi := T^\perp_\xi t(M) \cap \exp^\perp_{*\xi}(T_\xi(t_\varepsilon(M)\cap T^\perp_{\pi(\xi)}M))$ ($\xi \in t(M)$). Clearly we have $T_\xi t(M) = H_\xi \oplus V_\xi$ (orthogonal direct sum) and $T^\perp_\xi t(M) = N_\xi \oplus \mathrm{Span}\{\dot{\gamma}_\xi(1)\}$ (orthogonal direct sum), where γ_ξ is the geodesic in G/K with $\dot{\gamma}_\xi(0)=\xi$. Denote by A (resp. A^t) the shape tensor of M (resp. $t(M)$). Also, denote by A^x that of a submanifold $t(M)\cap T^\perp_x M$ in $\exp^\perp(T^\perp_x M)$ immersed by $\exp^\perp|_{t(M)\cap T^\perp_x M}$. In the sequel, we omit $\exp^\perp_*$ unless otherwise mentioned. For a real analytic function F and $\xi \in T_{gK}G/K$, we denote the operator $g_*\circ F(\mathrm{ad}(g_*^{-1}\xi))\circ g_*^{-1}$ by $F(\mathrm{ad}(\xi))$ for simplicity. Then we can obtain the following relations.

Proposition 3.1. *Let $\xi \in t(M)$ and $\eta \in N_\xi$.*

(i) *For $v \in V_\xi$, we have*

$$A^t_{\dot{\gamma}_\xi(1)}v = A^{\pi(\xi)}_{\dot{\gamma}_\xi(1)}v, \quad A^t_\eta v = A^{\pi(\xi)}_\eta v.$$

(ii) *For $X \in T_{\pi(\xi)}M$, we have*

$$A^t_{\dot{\gamma}_\xi(1)}\widetilde{X}_\xi = P_{\gamma_\xi}\left(-\mathrm{ad}(\xi)\sinh(\mathrm{ad}(\xi))X + \cosh(\mathrm{ad}(\xi))A_\xi X\right).$$

(iii) *Assume that $g_*^{-1}\,\mathrm{Span}\{\xi,\bar{\eta}\}$ is abelian, where $\bar{\eta}$ is the element of $T^\perp_{\pi(\xi)}M$ satisfying $\exp^\perp_{*\xi}(\bar{\eta}) = \eta$ (we regard $\bar{\eta}$ as an element of $T_\xi(T^\perp_{\pi(\xi)}M)$ under the natural identification of $T^\perp_{\pi(\xi)}M$ with $T_\xi(T^\perp_{\pi(\xi)}M)$). Then, for $X \in T_{\pi(\xi)}M$, we have*

$$\begin{aligned} A^t_\eta \widetilde{X}_\xi = P_{\gamma_\xi}\Bigg(&-\mathrm{ad}(\bar{\eta})\sinh(\mathrm{ad}(\xi))X + \frac{\sinh(\mathrm{ad}(\xi))}{\mathrm{ad}(\xi)}A_{\bar{\eta}}X \\ &+\left(\frac{\cosh(\mathrm{ad}(\xi))-\mathrm{id}}{\mathrm{ad}(\xi)} - \frac{\sinh(\mathrm{ad}(\xi))-\mathrm{ad}(\xi)}{\mathrm{ad}(\xi)^2}\right)\mathrm{ad}(\bar{\eta})A_\xi X\Bigg). \end{aligned}$$

Proof. In this proof, we omit $\exp^\perp_*$. Since M has Lie triple systematic normal bundle, the submanifold $\exp^\perp(T^\perp_{\pi(\xi)}M)$ is totally geodesic. From this fact, we can easily show the relations in (i). Now we shall show the relation in (ii). Let $\beta\colon [0,1]\to M$ be a curve with $\dot{\beta}(0)=X$ and $\widetilde{\xi}$ be a parallel normal vector field along β with $\widetilde{\xi}(0)=\xi$. Define a two-parameter map $\delta_1\colon [0,1]\times[0,1]\to G/K$ by $\delta_1(t,s) := \exp^\perp(s\widetilde{\xi}(t))$ ($(t,s)\in[0,1]\times[0,1]$). Define a vector field J along γ_ξ by $J(s) := (\partial\delta_1/\partial t)(0,s)$ ($s\in[0,1]$). Since J is a Jacobi field along γ_ξ with $J(0)=X$ and $J'(0) = -A_\xi X$, it is described as

$$J(s) = P_{\gamma_\xi|_{[0,s]}}\left(D^{co}_{s\xi}X - sD^{si}_{s\xi}(A_\xi X)\right).$$

Hence we have

$$\widetilde{X}_\xi = J(1) = P_{\gamma_\xi}\left(D^{co}_\xi X - D^{si}_\xi(A_\xi X)\right) \in P_{\gamma_\xi}(T_{\pi(\xi)}M), \tag{3.1}$$

which implies $H_\xi = P_{\gamma_\xi}(T_{\pi(\xi)}M)$. Furthermore we have

$$\begin{aligned}\widetilde{\nabla}_{\widetilde{X}_\xi}\frac{\partial\delta_1}{\partial s}\bigg|_{s=1} &= J'(1) = P_{\gamma_\xi}\left(\left(\frac{d}{ds}\bigg|_{s=1} D^{co}_{s\xi}\right)X - \left(\frac{d}{ds}\bigg|_{s=1} sD^{si}_{s\xi}\right)(A_\xi X)\right)\\ &= P_{\gamma_\xi}\left(\mathrm{ad}(\xi)\sinh(\mathrm{ad}(\xi))X - \cosh(\mathrm{ad}(\xi))A_\xi X\right),\end{aligned}$$

which belongs to $T_\xi(t(M))$ because of $H_\xi = P_{\gamma_\xi}(T_{\pi(\xi)}M)$. Hence we have

$$\widetilde{\nabla}_{\widetilde{X}_\xi}\frac{\partial\delta_1}{\partial s}\bigg|_{s=1} = -A^t_{\dot\gamma_\xi(1)}\widetilde{X}_\xi.$$

After all we obtain the relation in (ii). Next we shall show the relation in (iii). Let β and $\widetilde{\xi}$ be as above and $\widetilde{\eta}$ be the parallel normal vector field along β with $\widetilde{\eta}(0) = \bar{\eta}$, where $\bar{\eta}$ is the element of $T^\perp_{\pi(\xi)}M$ as in the statement (iii). Define a three parameter map $\delta_2\colon [0,1]^2\times[0,\pi/2] \to G/K$ by $\delta_2(t,s,u) := \exp^\perp(s(\cos u\widetilde{\xi}(t) + (\varepsilon/\|\bar{\eta}\|)\sin u\widetilde{\eta}(t)))$ $((t,s,u) \in [0,1]^2\times[0,\pi/2])$. For simplicity, set $\eta^\varepsilon := (\varepsilon/\|\eta\|)\eta$, $\bar{\eta}^\varepsilon := (\varepsilon/\|\bar{\eta}\|)\bar{\eta}$ and $\xi(\bar{\eta},u) := \cos u\xi + \sin u(\varepsilon/\|\bar{\eta}\|)\bar{\eta}$. Define a vector field J_{u_0} $(u_0 \in [0,\pi/2])$ along $\gamma_{\xi(\bar{\eta},u_0)}$ by $J_{u_0}(s) = (\partial\delta/\partial t)|_{t=0,u=u_0}$ $(s\in[0,1])$. Since J_{u_0} is a Jacobi field along $\gamma_{\xi(\bar{\eta},u_0)}$ with $J_{u_0}(0) = X$ and $J'_{u_0}(0) = -A_{\xi(\bar{\eta},u_0)}X$, it is described as

$$J_{u_0}(s) = P_{\gamma_{\xi(\bar{\eta},u_0)}|_{[0,s]}}\left(D^{co}_{s\xi(\bar{\eta},u_0)}X - sD^{si}_{s\xi(\bar{\eta},u_0)}(A_{\xi(\bar{\eta},u_0)}X)\right).$$

In particular, we have $J_{u_0}(1) = P_{\xi(\bar{\eta},u_0)}B_{\xi(\bar{\eta},u_0)}X$, where $B_{\xi(\bar{\eta},u_0)} := D^{co}_{\xi(\bar{\eta},u_0)}X - D^{si}_{\xi(\bar{\eta},u_0)}\circ A_{\xi(\bar{\eta},u_0)}$. Furthermore, we have

$$\begin{aligned}\widetilde{\nabla}_{\widetilde{X}_\xi}\frac{\partial\delta}{\partial u}\bigg|_{s=1,u=0} &= \widetilde{\nabla}_{\eta^\varepsilon}\frac{\partial\delta}{\partial t}\bigg|_{t=0,s=1} = \widetilde{\nabla}_{\eta^\varepsilon}J_u(1)\\ &= \lim_{u\to0}\frac{1}{u}\left(P^{-1}_{\delta_{01}\cdot|_{[0,u]}}J_u(1) - J_0(1)\right)\\ &= \lim_{u\to0}\frac{1}{u}\left((P^{-1}_{\delta_{01}\cdot|_{[0,u]}}\circ P_{\gamma_{\xi(\bar{\eta},u)}})(B_{\xi(\bar{\eta},u)}X) - P_{\gamma_\xi}B_\xi X\right),\end{aligned}$$

where $B_\xi := D^{co}_\xi - D^{si}_\xi\circ A_\xi$. Since $g_*^{-1}\,\mathrm{Span}\{\xi,\bar{\eta}\}$ is abelian, we have $P^{-1}_{\delta_{01}\cdot|_{[0,u]}}\circ P_{\gamma_{\xi(\bar{\eta},u)}} = P_{\gamma_\xi}$. Hence we have

$$\begin{aligned}\widetilde{\nabla}_{\widetilde{X}_\xi}\frac{\partial\delta}{\partial u}\bigg|_{s=1,u=0} &= P_{\gamma_\xi}\left(\lim_{u\to0}\frac{1}{u}(B_{\xi(\bar{\eta},u)}X - B_\xi X)\right)\\ &= P_{\gamma_\xi}\left(\left(\frac{d}{du}\bigg|_{u=0}B_{\xi(\bar{\eta},u)}\right)X\right)\end{aligned}$$

$$= P_{\gamma_\xi}\left(\mathrm{ad}(\overline{\eta}^\varepsilon)\sinh(\mathrm{ad}(\xi))X - \frac{\sinh(\mathrm{ad}(\xi))}{\mathrm{ad}(\xi)}A_{\overline{\eta}^\varepsilon}X\right.$$
$$\left. - \left(\frac{\cosh(\mathrm{ad}(\xi)) - \mathrm{id}}{\mathrm{ad}(\xi)} - \frac{\sinh(\mathrm{ad}(\xi)) - \mathrm{ad}(\xi)}{\mathrm{ad}(\xi)^2}\right)\mathrm{ad}(\overline{\eta}^\varepsilon)A_\xi X\right),$$

which belongs to $P_{\gamma_\xi}(T_{\pi(\xi)}M) = H_\xi(\subset T_\xi(t(M)))$. Hence we have $\widetilde{\nabla}_{\widetilde{X}_\xi}(\partial\delta/\partial u)|_{s=1,u=0} = -A^t_{\eta^\varepsilon}\widetilde{X}_\xi$. After all we obtain the relation in (iii) by noticing $\eta^\varepsilon = (\varepsilon/\|\eta\|)\eta$ and $\overline{\eta}^\varepsilon = (\varepsilon/\|\eta\|)\overline{\eta}$. □

As a corollary of this proposition, we have the following facts.

Corollary 3.2. *Let $\xi \in t(M)$ and $\eta \in N_\xi$. Also, let $\pi(\xi) = gK$.*
(i) *Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ containing $g_*^{-1}\xi$ and $\mathfrak{p} = \mathfrak{a} + \sum_{\alpha\in\triangle_+}\mathfrak{p}_\alpha$ be the root space decomposition with respect to $\mathfrak{a}$. If $A_\xi X = \lambda X$ and $g_*^{-1}X \in \mathfrak{p}_\alpha$, then we have*

$$A^t_{\dot{\gamma}_\xi(1)}\widetilde{X}_\xi = \frac{-\alpha(g_*^{-1}\xi)^2\tanh\alpha(g_*^{-1}\xi) + \lambda\alpha(g_*^{-1}\xi)}{\alpha(g_*^{-1}\xi) - \lambda\tanh\alpha(g_*^{-1}\xi)}\widetilde{X}_\xi.$$

(ii) *Assume that $g_*^{-1}\,\mathrm{Span}\{\xi,\overline{\eta}\}$ is abelian. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ containing $g_*^{-1}\,\mathrm{Span}\{\xi,\overline{\eta}\}$ and $\mathfrak{p} = \mathfrak{a} + \sum_{\alpha\in\triangle_+}\mathfrak{p}_\alpha$ be the root space decomposition with respect to $\mathfrak{a}$. If $A_\xi X = \lambda X$, $A_{\overline{\eta}}X = \mu X$ and $X \in \mathfrak{p}_\alpha$, then we have*

$$A^t_\eta\widetilde{X}_\xi = \frac{1}{\alpha(g_*^{-1}\xi) - \lambda\tanh\alpha(g_*^{-1}\xi)}\left\{-\alpha(g_*^{-1}\xi)\alpha(g_*^{-1}\overline{\eta})\tanh\alpha(g_*^{-1}\xi)\right.$$
$$\left. + \left(1 - \frac{\tanh\alpha(g_*^{-1}\xi)}{\alpha(g_*^{-1}\xi)}\right)\alpha(g_*^{-1}\overline{\eta})\lambda + \tanh\alpha(g_*^{-1}\xi)\mu\right\}\widetilde{X}_\xi.$$

Proof. These relations follow from the relations in (ii) and (iii) of Proposition 3.1 and (3.1). □

4. Proofs of Theorems A and B

In this section, we first prove Theorem A.

Proof of Theorem A. Let G/K be a symmetric space of non-compact type and H be a symmetric subgroup of G. Let r be the cohomogeneity of the H-action. According to the proof of Theorem 3 of [19], there exists a r-dimensional abelian subspace $\mathfrak{t}$ of $T^\perp_{eK}H(eK)(\subset \mathfrak{p} = T_{eK}G/K)$ and $\Sigma := \exp^\perp\mathfrak{t}$ is a flat totally geodesic submanifold in G/K meeting orthogonally to all H-orbits through Σ, where $\exp^\perp$ is the normal exponential map of $H(eK)$. We have only to show that all H-orbits meet Σ. Take an arbitrary point $p \in G/K$. Let γ be a curve in G/K with $\gamma(0) = eK$ and $\gamma(1) = p$. The H-orbits give a Riemannian foliation (with singular leaves) and G/K

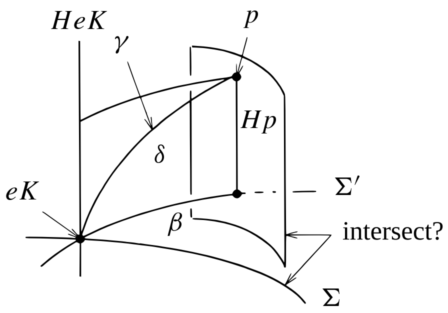

Fig. 1.

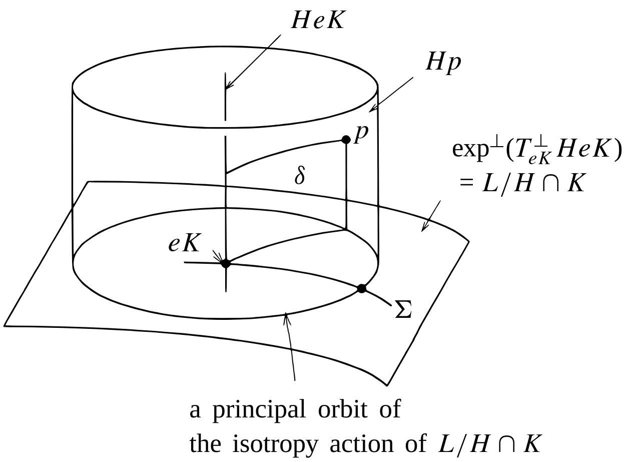

Fig. 2.

is complete. Hence, by imitating the proof of Lemma 2.1 of [7], we can show that there exists a rectangle $\delta\colon [0,1]\times[0,1] \to G/K$ such that the curve $t \to \delta(t,s)$ is orthogonal to the H-orbits for every $s \in [0,1]$, the curve $s \to \delta(t,s)$ is contained in a H-orbit for each $t \in [0,1]$ and $\delta(t,t) = \gamma(t)$ for each $t \in [0,1]$. Let $\beta(t) := \delta(t,0)$. Since the curve β is orthogonal to H-orbits, we see that β is contained in another r-dimensional flat totally geodesic submanifold Σ' through eK (see Fig. 1). Thus Hp meets $\exp^\perp(T^\perp_{eK}H(eK))$. Furthermore, since the intersections of H-orbits with $\exp^\perp(T^\perp_{eK}H(eK))$ are regarded as the orbits of the isotropy action of a symmetric space $L/H\cap K$ (see the proof of Lemma 4.2) and Σ is regarded as a section of the isotropy action, it follows that Hp meets Σ (see Fig. 2). This completes the proof. □

Next we prove Theorem B. For its purpose, we shall prepare two lemmas. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition associated with a symmetric space G/K of non-compact type, where $\mathfrak{g}$ (resp. $\mathfrak{k}$) is the Lie algebra of G (resp. K). Take $Z \in \mathfrak{p}$. Let $\mathfrak{h}$ be the Lie algebra of a symmetric subgroup H of G. Set $N_H^Z[K] := \{h \in H \mid \exp(-Z)h \exp Z \in K\}$, which is the isotropy group of the H-orbit $H((\exp Z)K)$ at $(\exp Z)K$. Denote by $\mathfrak{n}_{\mathfrak{h}}^Z[\mathfrak{k}]$ the Lie algebra of $N_H^Z[K]$. Then we have

$$\mathfrak{n}_{\mathfrak{h}}^Z[\mathfrak{k}] = \{X \in \mathfrak{h} \mid \mathrm{Ad}(\exp(-Z))X \in \mathfrak{k}\},$$

where Ad is the adjoint representation of G on $\mathfrak{g}$. Denote by $\mathfrak{n}_{\mathfrak{h}}^Z[\mathfrak{k}]^\perp$ the orthogonal complement of $\mathfrak{n}_{\mathfrak{h}}^Z[\mathfrak{k}]$ in $\mathfrak{h}$. We have

$$\exp Z_*^{-1}\left(T_{(\exp Z)K}(H((\exp Z)K))\right) = \{(\mathrm{Ad}(\exp(-Z))Y)_{\mathfrak{p}} \mid Y \in \mathfrak{n}_{\mathfrak{h}}^Z[\mathfrak{k}]^\perp\}.$$

Take $X \in \mathfrak{n}_{\mathfrak{h}}^Z[\mathfrak{k}]^\perp$ and $\xi \in \exp Z_*^{-1}(T^\perp_{(\exp Z)K} H((\exp Z)K))$.

Lemma 4.1. *Let A be the shape tensor of the orbit $H((\exp Z)K)$. Then we have*

$$A_{(\exp Z)_*\xi}(\exp Z)_*(\mathrm{Ad}(\exp(-Z))X)_{\mathfrak{p}} = -\exp Z_*[\mathrm{Ad}(\exp(-Z)X)_{\mathfrak{k}}, \xi].$$

Proof. Define a curve $c(t)$ in $H((\exp Z)K)$ by $c(t) := (\exp tX \exp Z)K$ and define a normal vector field $\xi(t)$ of $H((\exp Z)K)$ along $c(t)$ by $\xi(t) := (\exp tX)_*(\exp Z)_*\xi$, which is parallel with respect to the normal connection of the orbit by Theorem 2.1 of [15]. Also, define a curve $g(t)$ in G by $g(t) := \exp tX \exp Z$. Easily we can show $\dot c(t) = g(t)_*(\mathrm{Ad}(\exp(-Z))X)_{\mathfrak{p}}$. Set $\alpha(s) := g(t)^{-1}g(t+s)$. By using a relation in the proof of Lemma 2.2 of [15], we have

$$\begin{aligned}
\widetilde\nabla_{\dot c(t)}\xi(t) &= \widetilde\nabla_{g(t)_*(\mathrm{Ad}(\exp(-Z))X)_{\mathfrak{p}}}\xi(t)\\
&= g(t)_*\left(\frac{d}{ds}\Big|_{s=0}\alpha(s)_*^{-1}g(t)_*^{-1}\xi(t+s) + [(\mathrm{Ad}(\exp(-Z))X)_{\mathfrak{k}}, \xi]\right)\\
&= g(t)_*[(\mathrm{Ad}(\exp(-Z))X)_{\mathfrak{k}}, \xi],
\end{aligned}$$

where $\widetilde\nabla$ is the Levi-Civita connection of G/K. On the other hand, we have $\nabla^\perp_{\dot c(t)}\xi(t) = 0$, where $\nabla^\perp$ is the normal connection of $H((\exp Z)K)$. Hence we have $A_{\xi(t)}\dot c(t) = -g(t)_*[(\mathrm{Ad}(\exp(-Z))X)_{\mathfrak{k}}, \xi]$. Substituting $t = 0$ into this relation, we can obtain the desired relation. □

Next we show the following fact in terms of this lemma.

Lemma 4.2. *There exists a symmetric subgroup H' conjugate to H satisfying the following conditions* (i) *and* (ii):
(i) *The orbit $H'(eK)$ is totally geodesic,*
(ii) *Each H'-orbit is a partial tube over $H'(eK)$ contained in $t_\varepsilon(H'(eK))$ for some positive number ε.*

Proof. Let θ be the Cartan involution of G associated with G/K and σ be an involution of G with $\mathrm{Fix}(\sigma) = H$. According to Lemma 10.2 of [1], there exists an inner automorphism ρ of G with $(\rho\circ\sigma\circ\rho^{-1})\circ\theta = \theta\circ(\rho\circ\sigma\circ\rho^{-1})$. Set $\sigma' := \rho\circ\sigma\circ\rho^{-1}$ and $H' := \mathrm{Fix}(\sigma')$, which is conjugate to H. First we show that H' satisfies the condition (i). Let $\mathfrak{h}'$ be the Lie algebra of H'. Clearly we have $\mathfrak{n}^0_{\mathfrak{h}'}[\mathfrak{k}] = \mathfrak{k}\cap\mathfrak{h}'$, where 0 is the zero element of $\mathfrak{p}$. Hence the tangent space $T_{eK}(H'(eK))$ is identified with the orthogonal complement $(\mathfrak{k}\cap\mathfrak{h}')^{\perp}$ of $\mathfrak{k}\cap\mathfrak{h}'$ in $\mathfrak{h}'$. From $\sigma'\circ\theta = \theta\circ\sigma'$, we have $\mathfrak{h}' = \mathfrak{k}\cap\mathfrak{h}' + \mathfrak{p}\cap\mathfrak{h}'$ and hence $(\mathfrak{k}\cap\mathfrak{h}')^{\perp} = \mathfrak{p}\cap\mathfrak{h}'$. Let A be the shape tensor of $H'(eK)$. By using Lemma 4.1, we have $A_{\xi}X_{\mathfrak{p}} = -[X_{\mathfrak{k}}, \xi]$ $(X \in \mathfrak{n}^0_{\mathfrak{h}'}[\mathfrak{k}]^{\perp})$. Since $X \in \mathfrak{n}^0_{\mathfrak{h}'}[\mathfrak{k}]^{\perp} = \mathfrak{p}\cap\mathfrak{h}' \subset \mathfrak{p}$, we have $X_{\mathfrak{k}} = 0$ and hence $A_{\xi}X_{\mathfrak{p}} = 0$. This implies that $H'(eK)$ is totally geodesic. Next we show that H' satisfies the condition (ii). Note that the orbit $H'(eK)$ has Lie triple systematic normal bundle by Lemma 2.4 of [15]. Denote by $\exp^{\perp}\colon T^{\perp}H'(eK) \to G/K$ the normal exponential map of $H'(eK)$. Since G/K is of non-compact type and $H'(eK)$ is totally geodesic, $\exp^{\perp}$ is a diffeomorphism of $T^{\perp}H'(eK)$ into G/K. Take an arbitrary H'-orbit $H'(gK)$ and set $\mathfrak{T} := \exp^{\perp -1}(H'(gK))$. Take an arbitrary $\xi \in \mathfrak{T}\cap T^{\perp}_{eK}H'(eK)$. Also, take an arbitrary $(\exp Y)K \in H'(eK)$, where exp is the exponential map of G and $Y \in \mathfrak{n}^0_{\mathfrak{h}'}[\mathfrak{k}]^{\perp} = (\mathfrak{k}\cap\mathfrak{h}')^{\perp}$. Define a curve $c\colon [0,1] \to H'(eK)$ by $c(t) := (\exp tY)K$. From $(\mathfrak{k}\cap\mathfrak{h}')^{\perp} = \mathfrak{p}\cap\mathfrak{h}' \subset \mathfrak{p}$, we have $Y \in \mathfrak{p}$. Hence the curve c is a geodesic in G/K. According to Theorem 2.1 of [15], $(\exp Y)_*|_{T^{\perp}_{eK}H'(eK)}$ coincides with the parallel translation $P^{\perp}_c$ along c with respect to the normal connection of $H'(eK)$. Hence we have $\exp^{\perp}(P^{\perp}_c\xi) = \gamma_{P^{\perp}_c\xi}(1) = \gamma_{(\exp Y)_*\xi}(1) = (\exp Y\circ\gamma_{\xi})(1) = \exp Y(\exp^{\perp}\xi) \in H'(gK)$, that is, $P^{\perp}_c\xi \in \mathfrak{T}$. It follows from this fact that $P^{\perp}_{\gamma}(\mathfrak{T}\cap T^{\perp}_{\gamma(0)}H'(eK)) = \mathfrak{T}\cap T^{\perp}_{\gamma(1)}H'(eK)$ holds for every geodesic $\gamma\colon [0,1] \to H'(eK)$. Furthermore, it follows from this fact that $P^{\perp}_{\beta}(\mathfrak{T}\cap T^{\perp}_{\beta(0)}H'(eK)) = \mathfrak{T}\cap T^{\perp}_{\beta(1)}H'(eK)$ holds for every curve $\beta\colon [0,1] \to H'(eK)$. That is, $H'(gK)$ is a partial tube over $H'(eK)$. Set $L := \mathrm{Fix}(\sigma'\circ\theta)$. A pair $(L, H'\cap K)$ is a symmetric pair and a submanifold $\mathfrak{T}\cap T^{\perp}_{eK}H'(eK)$ in $T^{\perp}_{eK}H'(eK)$ is regarded as a principal orbit of the linear isotropy representation of the symmetric space $L/H'\cap K$ (see [13]). Hence $\mathfrak{T}\cap T^{\perp}_{eK}H'(eK)$ is contained in a hypersphere in $T^{\perp}_{eK}H'(eK)$ centered 0 (0: the zero element of $T^{\perp}_{eK}H'(eK)$). Let ε be a radius of the hypersphere. Then we have $\mathfrak{T} \subset t_{\varepsilon}(H'(eK))$. This completes the proof. $\square$

Next, by using Theorem A, Corollary 3.2 and Lemma 4.2, we prove Theorem B.

Proof of Theorem B. Let H be a symmetric subgroup of G. Take a principal orbit M of the H-action on G/K. According to Lemma 4.2, we may assume that the orbit $H(eK)$ is totally geodesic and that M is a partial tube over $H(eK)$ contained in $t_{\varepsilon}(H(eK))$ for some positive number ε. Let $\mathfrak{t}$ be a r-dimensional abelian subspace of $T^{\perp}_{eK}H(eK)$ and $\Sigma := \exp^{\perp}\mathfrak{t}$, where r is the cohomogeneity of the H-action. As stated in the proof of Theorem A, the orbit M meet Σ. Let $\exp^{\perp}\xi$ $(\xi \in \mathfrak{t})$ be a intersection point of M and Σ. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p} = T_{eK}G/K$ containing $\mathfrak{t}$ and $\mathfrak{p} = \mathfrak{a} + \sum_{\alpha\in\triangle_+}\mathfrak{p}_{\alpha}$ be the root space decomposition with respect to $\mathfrak{a}$. For simplicity, denote $\mathfrak{a}$ by $\mathfrak{p}_0$. Since $H(eK)$ has Lie triple systematic normal bundle, we have $T_{eK}H(eK)=\sum_{\alpha\in\triangle_+\cup\{0\}}(\mathfrak{p}_\alpha\cap T_{eK}H(eK))$. Denote by A the shape tensor of M. Since $H(eK)$ is totally geodesic and M is a partial tube with abelian normal bundle over $H(eK)$, we can show

$$
\begin{aligned}
&A_{\dot\gamma_\xi(1)}\widetilde{X}_\xi=-\alpha(\xi)\tanh\alpha(\xi)\widetilde{X}_\xi \quad \big(X\in\mathfrak{p}_\alpha\cap T_{eK}H(eK)\big),\\
&A_\eta\widetilde{X}_\xi=-\alpha(\bar\eta)\tanh\alpha(\xi)\widetilde{X}_\xi\\
&\qquad\big(X\in\mathfrak{p}_\alpha\cap T_{eK}H(eK),\ \ \eta\in T^\perp_{\exp^\perp\xi}M\cap\mathrm{Span}\{\dot\gamma_\xi(1)\}^\perp\big)
\end{aligned}
\tag{4.1}
$$

in terms of Corollary 3.2, where $\alpha\in\triangle_+\cup\{0\}$ and $\bar\eta$ is the element of $T^\perp_{eK}H(eK)$ corresponding to $\exp^{\perp-1}_*\eta\in T_\xi(T^\perp_{eK}H(eK))$ under the natural identification of $T_\xi(T^\perp_{eK}H(eK))$ with $T^\perp_{eK}H(eK)$. Let $\mathfrak{p}':=T^\perp_{eK}H(eK)$. Since $\mathfrak{p}'$ is a Lie triple system and $\mathfrak{t}$ is a maximal abelian subspace of $\mathfrak{p}'$, we have the root space decomposition $\mathfrak{p}'=\mathfrak{t}+\sum_{\beta\in\triangle'_+}\mathfrak{p}'_\beta$ with respect to $\mathfrak{t}$. On the other hand, we have $\mathfrak{p}'=\mathfrak{t}+\sum_{\alpha\in\triangle_+}(\mathfrak{p}_\alpha\cap\mathfrak{p}')$, where we again use the fact that $\mathfrak{p}'$ is a Lie triple system. It is clear that $\mathfrak{p}'_\beta=\sum_{\alpha\in\triangle_\beta}(\mathfrak{p}_\alpha\cap\mathfrak{p}')$, where $\triangle_\beta:=\{\alpha\in\triangle_+|\alpha|_{\mathfrak{t}}=\beta\}$. Hence we have $\triangle'_+=\{\alpha|_{\mathfrak{t}}\mid\alpha\in\triangle_+\}$. Denote by $\bar{A}$ the shape tensor of $M\cap\exp^\perp(T^\perp_{eK}H(eK))$ in $\exp^\perp(T^\perp_{eK}H(eK))$. Since $M\cap\exp^\perp(T^\perp_{eK}H(eK))$ is regarded as a principal orbit of the isotropy action of a symmetric space of non-compact type, we have $T_{\exp^\perp\xi}(M\cap\exp^\perp(T^\perp_{eK}H(eK)))=\sum_{\beta\in\triangle'_+}(\exp\xi)_*\mathfrak{p}'_\beta=\sum_{\alpha\in\triangle_+}(\exp\xi)_*(\mathfrak{p}_\alpha\cap\mathfrak{p}')$. Also, we have

$$
\begin{aligned}
&\bar{A}_{\dot\gamma_\xi(1)}v=\frac{\alpha(\xi)}{\tanh\alpha(\xi)}v \quad \big(v\in(\exp\xi)_*(\mathfrak{p}_\alpha\cap\mathfrak{p}')\big),\\
&\bar{A}_\eta v=\frac{\alpha(\bar\eta)}{\tanh\alpha(\xi)}v \quad \big((v\in(\exp\xi)_*(\mathfrak{p}_\alpha\cap\mathfrak{p}')\big),
\end{aligned}
\tag{4.2}
$$

where η and $\bar\eta$ are as above. On the other hand, according to Proposition 3.1, we have

$$
A_{\dot\gamma_\xi(1)}v=\bar{A}_{\dot\gamma_\xi(1)}v,\quad A_\eta v=\bar{A}_\eta v,
\tag{4.3}
$$

where $v\in T_{\exp^\perp\xi}(M\cap\exp^\perp(T^\perp_{eK}H(eK)))$. Also, it is easy to show that

$$
\begin{aligned}
&(\exp\xi)^{-1}_*\dot\gamma_\xi(1)=\xi\in\mathfrak{a},\\
&(\exp\xi)^{-1}_*\eta=\bar\eta\in\mathfrak{a} \qquad \big(\text{by the abelianity of }\mathrm{Span}\{\xi,\eta\}\big),\\
&(\exp\xi)^{-1}_*\widetilde{X}_\xi=\cosh\alpha(\xi)X\in\mathfrak{p}_\alpha\big(X\in\mathfrak{p}_\alpha\cap T_{eK}H(eK)\big).
\end{aligned}
\tag{4.4}
$$

From (4.1)–(4.4) and the homogeneity of M, it follows that M is curvature adapted. Furthermore, by noticing $\tanh\alpha(\xi)\neq\pm1$, we see that M is proper complex equifocal in terms of (ii) of Theorem A in [17]. $\square$

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