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NUMERICAL BOUNDS OF CANONICAL VARIETIES

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0. Introduction

Let X be a minimal, complex, projective, Gorenstein variety of dimension n. We say that X is *canonical* if for some (any) desingularization $\sigma : Y \longrightarrow X$, the map associated to the canonical linear series $|K_Y|$ is birational.

We note K_X for the canonical divisor of X and $\omega_X = \mathcal{O}_X(K_X)$ the canonical sheaf. Let $p_g = h^0(X, \omega_X)$, $q = h^1(X, \mathcal{O}_X)$. There are several known bounds for K_X^n depending on p_g, the most general one being the bound $K_X^n \geq (n+1)p_g + d_n$ (d_n constant) given by Harris ([9]). Bounds including other invariants are known for canonical surfaces, $K_S^2 \geq 3p_g + q - 7$ ([12], [7]), and for surfaces and threefolds fibred over curves ([20], [25]).

In this paper we prove some results for canonical surfaces and threefolds. In the case of canonical surfaces there are some known results which show that under some additional hypotheses, the bound $K_S^2 \geq 3p_g + q - 7$ can be considerably improved (see Remark 2.2). We give here some other special cases (Remark 2.2) for which is not sharp and prove (Theorem 2.1) that, in fact, $K_S^2 = 3p_g + q - 7$ only if $q = 0$ whenever $p_g(S) \geq 6$.

Canonical surfaces with $K_S^2 = 3p_g - 7$ are known to exist and classified ([1]). Then we can hope that a good bound for canonical surfaces including the irregularity should be of type $K_S^2 \geq 3p_g + aq - 7$, $a > 1$. Since for $q = 1$ it is known ([16]) that $K_S^2 \geq 3p_g$, a should be 7, although unfortunately examples of low K_S^2 (with $q \geq 2$) are not known.

In the case of canonical threefolds we prove that $K_X^3 \geq 4p_g + 6q - 32$. In particular, we prove that the results of Ohno for canonical fibred threefolds are not sharp.

We use basically a result on quadrics containing irreducible varieties due to Reid ([21]) and several techniques originated in [24] and developed by Konno ([16], [17], [18]) for the study of the slope of fibred surfaces. In particular we include in an Appendix the dimension 3 version of the relative hyperquadrics method used by Konno in [18].

After this manuscript was written, the author was informed that Theorem 2.1 was known yet to K. Konno (unpublished).

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1. A general inequality

We need the following result due to Reid ([21], p. 195).

Lemma 1.1. *Let $\Sigma \subseteq \mathbb{P}^N$ be an irreducible variety spanning $\mathbb{P}^N$ of dimension w. Then*

$$h^0\mathcal{J}_{\Sigma,\mathbb{P}^N}(2) \leq \binom{N-w+2}{2} - \min\{\deg \Sigma, 2(N-w)+1\}.$$

Then we have an immediate consequence.

Proposition 1.2. *Let X be a normal projective variety of general type and dimension n. Let $L \in \mathrm{Div}(X)$, $\mathcal{L} = \mathcal{O}_X(L) \in \mathrm{Pic}\, X$ and φ the rational map associated to $\mathcal{L}$. Assume φ is birational; then*

(a) $h^0(X, \mathcal{O}_X(2L)) \geq (n+2)[h^0(X, \mathcal{O}_X(L)) - (n+1)/2]$

(b) *If equality holds in* (a) *then*

(i) $\Sigma := \varphi(X)$ *is contained in a minimal degree variety of* $\mathbb{P}^{h^0(X,\mathcal{L})-1}$ *of dimension* $n+1$ *obtained as the intersection of quadrics containing* Σ.

(ii) $\Sigma \subseteq \mathbb{P}^{h^0(X,\mathcal{L})-1}$ *is linearly and quadratically normal.*

(iii) $\mathrm{Bs}\,|L| = \mathrm{Bs}\,|2L|$.

(iv) *If* $\mathrm{Bs}\,|L| = \emptyset$ *and* $p, q \in X$ *then* $|L|$ *separates* p *and* q *if and only if so does* $|2L|$.

Proof. We can always consider

$$\begin{array}{ccc} \bar{X} & & \\ {\scriptstyle\sigma}\downarrow & \searrow^{\bar{\varphi}} & \\ X & \overset{\varphi}{\dashrightarrow} & \Sigma \subseteq \mathbb{P}^r \end{array}$$

where $r = h^0(\mathcal{L}) - 1$, $\bar{X}$ is smooth, σ is birational and $\bar{\varphi}$ is defined by the moving part M of the linear system $|\sigma^*(L)|$, which has no base point.

By construction we have $\bar{\varphi}^*\mathcal{O}_{\mathbb{P}^r}(1) = \mathcal{O}_{\bar{X}}(M)$ and $2M \leq$ moving part of $|\sigma^*(2L)|$. Then, since X is normal and σ has connected fibres

$$\begin{aligned} h^0(X, \mathcal{O}_X(2L)) &= h^0(X, \sigma_*\sigma^*\mathcal{O}_X(2L)) \\ &= h^0(\bar{X}, \sigma^*\mathcal{O}_X(2L)) \geq h^0(\bar{X}, \mathcal{O}_{\bar{X}}(2M)) \\ &= h^0(\bar{X}, \bar{\varphi}^*\mathcal{O}_{\mathbb{P}^r}(2)) \end{aligned} \tag{1}$$

$$= h^0(\Sigma, \bar{\varphi}_*\bar{\varphi}^*\mathcal{O}_{\mathbb{P}^r}(2)) \geq h^0(\Sigma, \mathcal{O}_\Sigma(2)).$$

Now if we consider

$$0 \longrightarrow H^0\mathcal{J}_{\Sigma,\mathbb{P}^r}(2) \longrightarrow H^0\mathcal{O}_{\mathbb{P}^r}(2) \xrightarrow{f} H^0\mathcal{O}_\Sigma(2).$$

Lemma 1.1 gives

$$(2)\quad h^0(\Sigma, \mathcal{O}_\Sigma(2)) \geq \dim \operatorname{Im} f \geq \binom{r+2}{2} - \binom{r+2-n}{2} + \min\{\deg \Sigma, 2(r-n)+1\}$$
$$= (n+2)\left[r - \frac{n-1}{2}\right] = (n+2)\left[h^0(\mathcal{O}_X(L)) - \frac{n+1}{2}\right]$$

if $\deg \Sigma \geq 2(r-n)+1$. If H_i $(i = 1, \ldots, n)$ are general hyperplanes in $\mathbb{P}^r$ and $\Sigma_k = \Sigma \cap H_1 \cap \cdots \cap H_{n-k}$ is a general section of Σ of dimension k we have that Σ_2 is an irreducible surface of general type and then ([2], p. 115):

$$\deg \Sigma = \deg \Sigma_2 \geq 2(r-n+2) - 1 > 2(r-n)+1.$$

This proves (a).

Assume from now on that equality holds in (a). In particular equality must hold at every step of (1) and (2). Then f is an epimorphism and $h^1\mathcal{J}_{\Sigma,\mathbb{P}^r}(2) = 0$. Since $h^1\mathcal{J}_{\Sigma,\mathbb{P}^r}(1)$ is always zero we have (ii). Moreover we have

$$\begin{array}{ccccccc} S^2H^0\mathcal{O}_{\mathbb{P}^r}(1) & \longrightarrow & S^2H^0\mathcal{O}_\Sigma(1) & \cong & S^2H^0(\bar{X}, \sigma^*\mathcal{L}) & \cong & S^2H^0(X, \mathcal{L}) \\ \downarrow & & & & & & \downarrow \alpha \\ H^0\mathcal{O}_{\mathbb{P}^r}(2) & \xrightarrow{f} & H^0\mathcal{O}_\Sigma(2) & \cong & H^0(\bar{X}, \sigma^*\mathcal{O}_X(2L)) & \cong & H^0(X, \mathcal{O}_X(2L)) \end{array}$$

and hence α is an epimorphism and (iii) follows immediately.

In order to prove (iv), consider local trivializations of $\mathcal{L}$ at p and q. For $\alpha, \beta \in H^0(\mathcal{L})$ we confuse α, β with their local expressions at these trivializations.

We need

CLAIM. If $\operatorname{Bs}|L| = \emptyset$ then $|L|$ does not separate p and q if and only if for all α, $\beta \in H^0(L)$, $\begin{vmatrix} \alpha(p) & \beta(p) \\ \alpha(q) & \beta(q) \end{vmatrix} = 0$.

Proof of Claim. Let $\beta \in H^0(\mathcal{L})$ be such that $\beta(p) = 0$. Since p is not a base point of $|L|$ there exists $\alpha \in H^0(\mathcal{L})$ such that $\alpha(p) \neq 0$. Then, from $\begin{vmatrix} \alpha(p) & 0 \\ \alpha(q) & \beta(q) \end{vmatrix} = 0$ we get $\beta(q) = 0$ and then β does not separate p and q.

Assume there exist α, $\beta \in H^0(\mathcal{L})$ such that $\alpha(p) = a$, $\alpha(q) = b$, $\beta(p) = \bar{a}$, $\beta(q) = \bar{b}$ and $a\bar{b} - b\bar{a} \neq 0$. Let $\sigma = a\beta - \bar{a}\alpha \in H^0(\mathcal{L})$. Then clearly σ separates p and q. □

If $|2L|$ does not separate p and q then trivially so does not $|L|$.

Assume $|L|$ does not separate p and q. Since $S^2H^0(\mathcal{L}) \longrightarrow H^0(\mathcal{L}^{\otimes 2})$ is surjective for every α, $\beta \in H^0(2L)$, $\alpha = \sum a_{ij}s_is_j$, $\beta = \sum b_{ij}s_is_j$, $s_i \in H^0(\mathcal{L})$. Since $|L|$ has no base point and does not separate p and q we can take $\bar{s} \in H^0(\mathcal{L})$ such that $\bar{s}(p) = a \neq 0$, $\bar{s}(q) = b \neq 0$. Since by the claim we have $\begin{vmatrix} s_i(p) & a \\ s_i(q) & b \end{vmatrix} = 0$ for every s_i we can define $\lambda_i = (s_i(p))/a = (s_i(q))/b$. Then $\alpha(p) = \sum a_{ij}\lambda_i\lambda_j a^2$, $\alpha(q) = \sum a_{ij}\lambda_i\lambda_j b^2$, $\beta(p) = \sum b_{ij}\lambda_i\lambda_j a^2$, $\beta(q) = \sum b_{ij}\lambda_i\lambda_j b^2$, and then

$$\begin{vmatrix} \alpha(p) & \beta(p) \\ \alpha(q) & \beta(q) \end{vmatrix} = a^2b^2 \begin{vmatrix} \sum a_{ij}\lambda_i\lambda_j & \sum b_{ij}\lambda_i\lambda_j \\ \sum a_{ij}\lambda_i\lambda_j & \sum b_{ij}\lambda_i\lambda_j \end{vmatrix} = 0$$

and hence, by the claim, $|2L|$ does not separate p and q.

For the proof of (i) we refer again to [21] p. 195. If we call $\Sigma_0 = \Sigma \cap H_1 \cap \cdots \cap H_n$ we have that Σ_0 is a set of $d = \deg \Sigma \geq 2(r-n)+3$ points in $\mathbb{P}^{r-n}$. Proof of Lemma 1.1 (cf. [21] p. 195) shows that if we consider

$$H^0\mathcal{O}_{\mathbb{P}^r}(2) \xrightarrow{f} H^0\mathcal{O}_{\Sigma}(2)$$
$$H^0\mathcal{O}_{\mathbb{P}^{r-n}}(2) \xrightarrow{f_0} H^0\mathcal{O}_{\Sigma_0}(2)$$

then

$$\begin{aligned} \dim \operatorname{Im} f &\geq \binom{r+2}{2} - \binom{r+2-n}{2} + \dim \operatorname{Im} f_0 \\ &\geq \binom{r+2}{2} - \binom{r+2-n}{2} + \min\{d, 2(r-n)+1\}. \end{aligned}$$

Under our hypothesis equality holds and then we have that Σ_0 is a set of d points in $\mathbb{P}^{r-n}$ imposing exactly $2(r-n)+1$ conditions on quadrics. Then Σ_0 is contained in a rational normal curve Γ intersection of the quadrics containing Σ_0. Let T_k be the intersection of quadrics of $\mathbb{P}^{r-n+k}$ containing Σ_k. We have $T_k \subseteq T_{k+1} \cap H_{n-k}$ and hence $\Gamma = T_0 \subseteq T_n \cap H_1 \cap \cdots \cap H_n$. Then T_n has an irreducible component W containing Σ of dimension at least $n+1$. But then

$$h^0\mathcal{J}_{W,\mathbb{P}^r}(2) = h^0\mathcal{J}_{\Sigma,\mathbb{P}^r}(2) = \binom{r-n}{2}$$

since $\Sigma \subseteq W \subseteq T_n$. Again applying Lemma 1.1 to W, if $w = \dim W \geq n+2$:

$$h^0\mathcal{J}_{W,\mathbb{P}^r}(2) \leq \binom{r-n}{2} - 1.$$

So $\dim W = n+1$ and, since $W \cap H_1 \cap \cdots \cap H_n = \Gamma$, W is a variety of minimal degree in $\mathbb{P}^r$. Since such varieties are always intersection of quadrics we have in particular $W = T_n$. □

2. Canonical surfaces

As a consequence of Proposition 1.1 we get the following result for minimal canonical surfaces. The first part is a well known fact (cf. [7], [12]).

Theorem 2.1. *Let S be a minimal canonical surface. Then*

(a) $K_S^2 \geq 3p_g + q - 7$.

(b) *Assume* $p_g(S) \geq 6$. *If* $K_S^2 = 3p_g + q - 7$, *then* $q = 0$.

Proof. (a) By the pluri-genus formula, we have $h^0(2K_S) = K_S^2 + \chi(\mathcal{O}_S)$. Hence the inequality $K_S^2 \geq 3p_g + q - 7$ follows immediatly from Proposition 1.2 (see [7], [12]).

(b) In order to prove the statement we need first some properties of surfaces lying on the border line; let $\Sigma = \varphi(S) \subseteq \mathbb{P}^{p_g - 1}$.

CLAIM 1. If $K_S^2 = 3p_g + q - 7$ then

(i) Σ lies in a threefold Z of minimal degree.

(ii) $|K_S|$ is base point free.

(iii) $|K_S|$ does not separate $p, q \in S$ (possibly infinitely near) if and only if so does not $|2K_S|$.

(iv) $q = 0$ or $q \geq 3$.

(v) If $\dim \operatorname{Sing} \Sigma = 1$ and $K_S^2 \geq 13$ then the one dimensional components of $\operatorname{Sing} \Sigma$ are double lines.

Proof of Claim 1. (i), (ii) and (iii) are direct consequence of Proposition 1.2 and the fact that $|2K_S|$ has no base points if $p_g \geq 4$ ([5]).

(iv) If $q \neq 0$ and $q \leq 2$, $K_S = 3p_g + q - 7$, then $K_S^2 < 3\chi\mathcal{O}_S$ and the canonical map of S can not be birational (cf. [11]).

(v) Assume $\dim \operatorname{Sing} \Sigma = 1$. Let D be a one dimensional component of $\operatorname{Sing} \Sigma$. The canonical map φ is not an embedding over D. Since $K_S^2 \geq 10$ and since, by (iii) points which are not separated by $|K_S|$ are those which are not separated by $|2K_S|$ we can apply Reider's Theorem (see [23]). Let $q \in D$ be a general point of D and let $p_1, p_2 \in S$ (possibly infinitely near) such that $\varphi(p_1) = \varphi(p_2) = q$. By Reider's Theorem we have that there exists an effective divisor E passing through p_1, p_2 and verifying

$$(K_S E, E^2) = (0, -2), (1, -1), (2, 0).$$

Since $q \in D$ is general, E can not be contracted by φ. Since $\varphi(p_1) = \varphi(p_2)$, we have that $\deg(\varphi_{|E}) \geq 2$ and hence that $(K_S E, E^2) = (2, 0)$ holds.

Note that moving $q \in D$ the curve E can not move because then we would have the surface S covered by curves of genus at most two and this is impossible since S is canonical. So we must have $\varphi(E) = D$ (set-theoretically), $\deg \varphi_{|E} = 2$ and that D is a

line $\mathbb{P}^{p_g-1}$. In order to prove that it is a double line, assume that, for $q \in D$ general, there are 3 points p_1, p_2, p_3 (possibly infinitely near) such that $\varphi(p_i) = q$. Consider the 0-cycle $Z = p_1 + p_2 + p_3$. Then Z is clearly in special position with respect to $|2K_S|$ (following Reider's notation in [23]). Then if $K_S^2 \geq 4 \deg Z + 1 = 13$, we have a decomposition $K_S = M + F$ as in [23], where F is an effective curve passing through Z (and so that $K_S F \geq 3$) with the following numerical conditions:

$$M^2 \geq F^2, \quad MF \geq 0, \quad M^2 > 0$$
$$F^2 \leq \frac{(MF)^2}{(M^2)}$$
$$K_S^2 = M^2 + 2MF + F^2 \geq 13$$
$$MF \text{ is even.}$$

An easy computation show that these conditions have no solution. Then a general point $q \in D$ has two pre-images through φ. □

It is a well known fact that the only possibilities for a threefold Z of minimal degree in $\mathbb{P}^{p_g-1}$ are

(A) $Z = \mathbb{P}^3$ $(p_g = 4)$.
(B) Z is a cone over the Veronese surface $(p_g = 7)$.
(C) Z is a smooth quadric in $\mathbb{P}^4$ $(p_g = 5)$.
(D) Z is a scroll of type $\mathbb{P}_{a,b,c}$, $0 \leq a \leq b \leq c$, $2 \leq a + b + c = p_g - 3$.

CLAIM 2. If $p_g(S) \geq 6$ and $K_S^2 = 3p_g + q - 7$, then $q = 0$.

Proof of Claim 2. We will often use the following results on the irregularity of a surface of general type:

(∗) (Beauville, [4]): $p_g \geq 2q - 4$. If S is canonical, then $p_g(S) \geq 2q - 3$.
(∗∗) (Xiao, [25]): if $f : S \longrightarrow \mathbb{P}^1$ is a linear pencil of general fibre F, then $q \leq (1/2)(g(F) + 1)$.
(∗∗∗) (Konno, [19]): with the same notation as in (∗∗), is r is the rank of the subsheaf of $f_*\omega_S$ generically generated by its global sections, then $q \leq \min\{g(F) - r, r + 1\}$.

Assume first that $p_g(S) = 7$ and that Z is a cone over the Veronese surface. Choosing hyperplane sections of Z passing through the vertex and decomposing as two components as sections of the Veronese surface, we have a decomposition:

$$K_S \sim 2D + D'$$

where $|D|$ is a linear pencil on S, $DD' \geq 1$ (by the connectedness of canonical divisors) and that $K_S D' = 0$ (D' is contracted by φ to the vertex of Z; possibly $D' = 0$). Then, the only possibility is that $(K_S^2, q) = (18, 4)$ and hence that $g(D) \leq 7$. But using

$(***)$ we obtain than $r \leq 3$ wich is impossible since $\varphi(D)$ is not a plane curve.

Assume Z is a scroll. Consider

$$\begin{array}{ccccc} \bar{S} & \longrightarrow & \bar{\Sigma} \subseteq \bar{Z} & \longrightarrow & \mathbb{P}^1 \\ {\scriptstyle\sigma}\downarrow & & \downarrow & & \\ S & \longrightarrow & \Sigma \subseteq Z & & \end{array}$$

where $\bar{Z}$ is the desingularization of Z. Let $\bar{\alpha} : \bar{S} \longrightarrow \mathbb{P}^1$ be the induced fibration and $\bar{G}$ be a general fibre. Note that, by construction $(\varphi \circ \sigma)_{|\bar{G}} : \bar{G} \longrightarrow \mathbb{P}^{p_g-1}$ induces on $\bar{G}$ a base point free sublinear system of $|K_{\bar{G}}|$ and that $(\varphi \circ \sigma)(\bar{G}) \subseteq \mathbb{P}^2 \cong T$, where T is a general ruling of Z.

Note that the singularities of $(\varphi \circ \sigma)(\overline{G})$, for $\bar{G}$ general, lie on $\operatorname{Sing} Z$ (produced by the base points of $|\sigma(\bar{G})|$ on S) or on $\operatorname{Sing} \Sigma \cap T$. If $a+b+c \geq 2$ (we only exclude the case $Z = \mathbb{P}^3$ which is Case (A)) then $p_g \geq 6$ and $K_S^2 \geq 11 + q \geq 14$ if $q \neq 0$. Then, if $\operatorname{Sing} \Sigma$ has one dimensional components, they must be double lines by Claim 1. Moreover we can assume that they are transversal to the general ruling. Since any such line in Z corresponds to an epimorphism $\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b) \oplus \mathcal{O}_{\mathbb{P}^1}(c) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1)$, under the assumption $a + b + c \geq 4$ ($p_g \geq 7$) we have that the lines transversal to the ruling cut a general plane T in points which are on a line $\ell \subseteq T$. Then we can proceed as follows.

Assume first Z is smooth, i.e. $1 \leq a \leq b \leq c$. We have then that $S = \bar{S}$, $\sigma(\bar{G}) = G$ and $\varphi(G)$ is a plane curve of degree $d = 2g(G) - 2$ with only double points as singularities, lying all of them on a line if $p_g \geq 7$. Let m be the number of such double points (possibly $m = 0$). Then we have:

$$\begin{gathered} d \geq 2m, \\ d = \bar{G}\mathcal{O}(1) = GK_S = GK_S + G^2 = 2g - 2, \\ 2g - 2 = \frac{1}{2}(d-1)(d-2) - m, \end{gathered}$$

which forces $d \leq 5$ and hence $g = g(G) \leq 3$. By $(**)$ we get a contradiction.

If $p_g(S) = 6$ and Z is a smooth scroll of $\mathbb{P}^5$ we can proceed as follows. Since $\bar{G} \subset \mathbb{P}^2$ we have that $r = 3$ and hence that $g(G) \geq 6$ by $(***)$.

Take a general hyperplane section of Σ and Z. We get an irreducible curve $\bar{C}$ lying on a smooth ruled surface V of minimal degree in $\mathbb{P}^4$. Let h, f be the hyperplane divisor class and the fibre divisor class in V. We have that $h^2 = \deg V = p_g - 3 = 3$ and that $\bar{C} = \alpha h + \beta f$ with $\alpha \geq 1$, $\beta \geq -\alpha$ (obeserve that V is $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ embeded in $\mathbb{P}^4$ by $C_0 + 2f$ ($C_0^2 = -e$), and hence $e = 1$, $\bar{C} \sim \alpha C_0 + (\beta + 2\alpha)f$ and so $\beta + 2\alpha \geq e\alpha = \alpha$, following [2] page 69 and [10] Corollary V.2.18). Let $C \in |K_S|$ be the smooth curve lying over $\bar{C}$. Using that $K_V = -2h + (p_g - 5)f$ we get

$$d = \deg \bar{G} = \bar{G}H = f\bar{C} = \alpha$$

$$\begin{aligned} d &= GK_S = K_SG + G^2 = 2g(G) - 2 \geq 10 \\ K_S^2 &= \deg(\bar{C}) = \bar{C}h = \alpha(p_g - 3) + \beta \geq 2\alpha \\ K_S^2 &= 3p_g + q - 7 = 11 + q \end{aligned}$$

which is impossible.

Assume $\dim \operatorname{Sing} Z = 1$, i.e. $0 = a = b < c$. Take a general section Γ of Σ containing $\operatorname{Sing} Z$. Γ corresponds to a section $|K_S| \ni C = cG + L$ where L is the component of the sublinear system containing $\operatorname{Sing} Z$ (possibly $L = 0$).

We have then, since $p_g \geq 2q - 3$

$$\frac{7}{2}c + 5 \geq 3p_g + q - 7 = K_S^2 = cKG + KL \geq cKG.$$

Then, using $c \geq 3$, $cG^2 \leq K_SG$ and evenness of $K_SG + G^2$ we get that, in any case $2p_a(G) - 2 = K_SG + G^2 \leq 6$. Then $g(\bar{G}) \leq p_a(G) \leq 4$. Again by $(**)$ we get $q \leq 2$ and hence $q = 0$.

Finally assume $\dim \operatorname{Sing} Z = 0$, i.e. $0 = a < b \leq c$. We have a Hirzebruch surfce $\Sigma_{c-b} \subset \mathbb{P}^{p_g-2}$ as a smooth hyperplane section of Z embedded by $|C_0 + cf|$ ($C_0^2 = -e = -(c - b)$). If we take now hyperplane sections of Z passing through its vertex and having exactly the above decomposition when meeting Σ_{c-b}, then we produce a decomposition

$$\begin{aligned} K_S &= cD + D' + D'' \\ K_SD &\geq 1 \\ K_SD' &\geq 1 \\ K_SD &= 0 \end{aligned}$$

such that $|D|$ is a linear pencil on S with $\varphi(D)$ a plane curve. Note that if $b = c$ (then $e = 0$), then $|D'|$ also moves on S. A case by case computation of the cases $c \geq 4$, $c = 3$, $c = 2, b = 1$, and $c = b = 2$ shows that we always get a contradiction by using $(**)$ and $(***)$.

We get then that the only possibilities for S with $q \neq 0$ occur when $p_g \leq 5$. □

REMARK 2.2. Part (b) of Theorem 2.1 shows that inequality $K_S^2 \geq 3p_g + q - 7$ is not sharp if $p_g \gg 0$. Since surfaces with $K_S^2 = 3p_g - 7$ are known to exist (and are completely understood, see [1]), it seems that a sharp bound should look like $K_S^2 \geq 3p_g + aq - 7$, with $a > 1$. There are several partial results in this direction:

(i) Let $alb : S \longrightarrow alb(S)$ be the Albanese map of S. As a direct consequence of the study of the slope of fibrations, Konno ([16]) shows that, if $\dim alb(S) = 1$ then $K_S^2 \geq 3p_g + 7q - 7$.

(ii) In the same paper Konno proves that if the cotangent sheaf of S is nef then $K_S^2 \geq 6\chi\mathcal{O}_S = 6p_g - 6q + 6$ which is better than $K_S^2 \geq 3p_g + q - 7$ if $p_g \gg q$.
(iii) Note that even if $\dim alb(S) = 2$ but there exists a fibration $\pi : S \longrightarrow B$ with $b = g(B) \geq 2$ we have $K_S^2 \geq 3p_g + 2q - 7$. Indeed, for a general fibration we have $K_S^2 \geq \lambda\chi\mathcal{O}_S + (8 - \lambda)(b - 1)(g - 1)$ ($g = g(F)$, F smooth fibre of π). Note that if S is canonical $g \geq 3$. Under our hypothesis $\pi \neq alb$ and then Xiao ([24]) proves that $\lambda \geq 4$. Finally note that since $b + g \geq q$ ([4]) we have

$$(b-1)(g-1) \geq (b-1) + (g-1) \geq q - 2 \quad \text{if } b \geq 3$$
$$\text{and } (b-1)(g-1) \geq (b-1) + (g-1) - 1 \geq q - 3 \quad \text{if } b = 2.$$

But if $b = 2$ and $(b - 1)(g - 1) = q - 3$ we have $q = b + g$. Again by [4] we can say that $S = B \times F$ with $b = g(B) = 2$. This is not possible if S is canonical. Finally we can apply (*) and we get the desired bound.
(iv) Let $C \in |K_S|$; then we have

$$0 \to H^0(\mathcal{O}_S) \to H^0(\omega_S) \to H^0(C, \omega_{S|C}) \to H^1(\mathcal{O}_S) \xrightarrow{\rho_C} H^1(\omega_S) \to \cdots .$$

Note that the above sequence is self-dual and then we can consider $\rho_C \in \operatorname{Sym} \mathbb{C}^q$. The correspondence $H^0(S, \omega_S) \xrightarrow{\Phi} \operatorname{Sym} \mathbb{C}^q$ is clearly linear since it is induced by the natural map $H^0(S, \omega_S) \otimes H^1(\mathcal{O}_S) \longrightarrow H^1(S, \omega_S)$. Then, if $p_g \geq \binom{q+1}{2}$ there must exist $C \in |K_S|$ such that $\rho_C = 0$. For such C we have $h^0(C, \omega_{S|C}) = p_g + q - 1$.

Assume C to be irreducible. Since the linear system $|K_S|_{|C}$ is birational we can apply "Clifford plus" ([21] p. 195) and get

$$p_g + q - 1 = h^0(C, \omega_{S|C}) \leq \frac{1}{3}(K_S^2 + 4),$$

and hence $K_S^2 \geq 3p_g + 3q - 7$.

3. Canonical threefolds

Theorem 3.1. *Let T be a canonical threefold. Then*

$$K_T^3 \geq 4p_g + 6q - 32$$

Proof. Since T is canonical (in particular, T is minimal), K_T is nef and big and hence by the general Kawamata-Viehweg Theorem ([14] Thm. 2.17) and Proposition 1.2 we get

$$\frac{1}{2}K_T^3 - 3\chi\mathcal{O}_T = \chi_T(\omega_T^{\otimes 2}) = h^0(T, \omega_T^{\otimes 2}) \geq 5(h^0(T, \omega_T) - 2) \tag{1}$$

and hence

$$K_T^3 \geq 4p_g + 6(h^2(\mathcal{O}_T) - h^1(\mathcal{O}_T)) - 14.$$

Assume $h^2(\mathcal{O}_T) \geq 2h^1(\mathcal{O}_T) - 3$; then we get

$$K_T^3 \geq 4p_g + 6q - 32, \tag{2}$$

and then the Theorem is proved under this hypothesis.

From now on we assume $h^2(\mathcal{O}_T) \leq 2h^1(\mathcal{O}_T) - 4$; then by [2] Lemma X.7 and [3] Proposition 1 we obtain the existence of a fibration $\pi : T \longrightarrow B$ where B is a smooth curve of genus $b \geq 2$.

Let F be a general fibre of π. Since $K_T + F_{|F} = K_F$ we have that the general fibre is a smooth canonical minimal surface (note that K_T is nef so in particular it is π-nef).

Then we can apply the results of Ohno ([20]) and state that (Main Theorem 2):

$$K_T^3 - 6(b-1)K_F^2 = K_{T/B}^3 \geq 4(\chi\mathcal{O}_B\chi\mathcal{O}_F - \chi\mathcal{O}_T) \tag{3}$$

except for a finite number of exceptions. We have

$$\begin{aligned} K_T^3 &\geq 2(b-1)[3K_F^2 - 2\chi\mathcal{O}_F] + 4p_g - 4(h^2(\mathcal{O}_T) - h^1(\mathcal{O}_T)) - 4 \\ &\geq 2(b-1)[3K_F^2 - 2\chi\mathcal{O}_F] + 4p_g - 4q + 12 \end{aligned}$$

since we are assuming $h^2\mathcal{O}_T \leq 2h^1(\mathcal{O}_T) - 4$. Note that since F is canonical

$$3K_F^2 - 2\chi\mathcal{O}_F \geq 7p_g(F) + 5q(F) - 23 \geq 5(q(F) - 1)$$

and

$$2(b-1)[3K_F^2 - 2\chi\mathcal{O}_F] \geq 10(b-1)(q(F)+1) \geq 10(q(F)+b) - 10$$

since $b \geq 2$, $q(F) \geq 0$.

Note also that from the Albanese maps associated to $F \hookrightarrow T \longrightarrow B$ we get $q(F) + b \geq q(T) = q$ and so

$$K_T^3 \geq 4p_g + 6q + 2$$

which is stronger than we wanted.

Finally we must deal with the exceptions of Main Theorem 2 in [20]. Notice that since F is a canonical surface we must have, by Section 2, $K_F^2 \geq 3p_g(F) + q(F) - 7$. From this, only a few exceptional cases hold. We divide them in three cases (following [15] a canonical surface verifying $p_g(F) = 6$, $q(F) = 0$, $K_F^2 = 3p_g(F) - 6 = 12$ is

classified in two types according its canonical image is contained in a threefold of Δ-genus 0 or 1). In all of them we will prove $K^3_{T/B} \geq 4(\chi\mathcal{O}_B\chi\mathcal{O}_F - \chi\mathcal{O}_T)$. Then the same argument as above works.

CASE 1. $p_g(F) = 4, 5$.

We use the results of the relative hyperquadrics method of the Appendix. If $\mathcal{E} = \pi_*\omega_{T/B}$ and we consider the relative canonical image of T:

$$\begin{array}{ccc} T & \overset{\psi}{\dashrightarrow} & Y \subseteq \mathbb{P}_B(\mathcal{E}) = Z. \\ \downarrow & \swarrow_{\varphi} & \\ B & & \end{array}$$

Then formula (A.2) gives

$$K^3_{T/B} \geq (2p_g(F) - 4)(\chi\mathcal{O}_B\chi\mathcal{O}_F - \chi\mathcal{O}_T) - 2\deg K - 2\ell(2)$$

where $K = \varphi_*\mathcal{J}_{Y,Z}(2)$. Note that since T is Gorenstein, $\ell(2) = 0$ ([8]).

If $p_g(F) = 4$, $K = 0$ and

$$K^3_{T/B} \geq 4(\chi\mathcal{O}_B\chi\mathcal{O}_F - \chi\mathcal{O}_T)$$

which produces, as in (3)

$$K^3_T \geq 4p_g + 6q + 2. \tag{4}$$

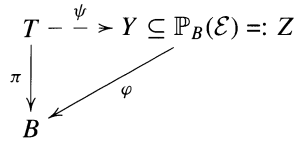
If $p_g(F) = 5$ then $\operatorname{rk} K = 1$ and $\deg K = x$ for some relative hyperquadric $Q \equiv 2L_{\mathcal{E}} - xF$ containing Y (see proof of Lemma A.4). Lemma A.5 of the Appendix gives that $\deg K = x \leq (2/3)\deg\mathcal{E}$ since $\operatorname{rk} Q \geq 3$. Then from the proof of Corollary A.2 we get

$$\begin{aligned} K^3_{T/B} &\geq 2(p_g(F) + 1)\deg\mathcal{E} - 6(\chi\mathcal{O}_B\chi\mathcal{O}_F - \chi\mathcal{O}_T) - 2\deg K \\ &\geq \frac{32}{3}\deg\mathcal{E} - 6(\chi\mathcal{O}_B\chi\mathcal{O}_F - \chi\mathcal{O}_T) \\ &\geq \frac{14}{3}(\chi\mathcal{O}_B\chi\mathcal{O}_F - \chi\mathcal{O}_T) \\ &\geq 4(\chi\mathcal{O}_B\chi\mathcal{O}_F - \chi\mathcal{O}_T) \end{aligned}$$

which gives again (4).

CASE 2. $p_g(F) = 6, 7$, $q(F) = 0$ and $K^2_F = 3p_g - 7$ or $p_g(F) = 6$, $q(F) = 0$, $K^2_F = 3p_g - 6$ and the canonical image of F is contained in a threefold of Δ-genus 0, intersection of the quadrics containing it.

Consider again the relative canonical image of T.



If $A \in \operatorname{Pic} B$ is ample enough we have an epimorphism

$$H^0(\mathcal{J}_{Y,Z}(2L_{\mathcal{E}} \otimes \varphi^*(A)) \longrightarrow H^0(\mathcal{J}_{F,\mathbb{P}^{p_g-1}}(2)).$$

Let W be the horizontal irreducible component of the base locus of the linear system given by the sections of $H^0(\mathcal{J}_{Y,Z}(2L_{\mathcal{E}} \otimes \varphi^*(A)))$. Since under our hypothesis intersections of quadrics containing F is a threefold of minimal degree (see [1] and [15]) W is a fourfold fibred over B by threefolds of minimal degree. Let $\widetilde{W}$ be a desingularization of W.

We want to relate the invariants of $\pi : T \longrightarrow B$ with those of $\Phi : \widetilde{W} \longrightarrow B$. In [17], Konno gives a general method for this. We refer there for details. Let H be the pull-back of the tautological divisor of Z to $\widetilde{W}$.

Lemma 3.2. (a) $\Phi_* \mathcal{O}_{\widetilde{W}}(H) = \pi_* \omega_{T/B}$.

(b) $\deg \Phi_* \mathcal{O}_{\widetilde{W}}(2H) = H^4 + 4 \deg \pi_* \omega_{T/B}$.

(c) $K^3_{T/B} \geq 2H^4 + 2(\chi \mathcal{O}_B \chi \mathcal{O}_F - \chi \mathcal{O}_T)$.

Proof. (a) Follows directly from the construction of $\widetilde{W}$ and H.

(b) Note that the formula we want to prove is invariant under the change of H by $H + \Phi^*(A)$, $A \in \operatorname{Pic} B$. So we can assume $|H|$ is base point free and hence get a smooth ladder $\widetilde{W} = W_4 \supseteq W_3 \supseteq W_2 \supseteq W_1 \supseteq W_0$ (i.e., W_i is smooth and $W_i \in |H_{|W_{i+1}}|$). Notice that W_2 is a ruled surface over B. By induction one easily proves that

$$\forall i \geq 0 \ \forall m \geq 1 \ \forall n \geq 0 \quad R^m \Phi_* \mathcal{O}_{W_i}(nH_{|W_i}) = 0$$

and hence that

$$\deg \Phi_* \mathcal{O}_{W_i}(2H_{|W_i}) = \deg \Phi_* \mathcal{O}_{\widetilde{W}}(H) + \deg \Phi_* \mathcal{O}_{W_{i-1}}(2H_{i-1}).$$

Finally note that $\deg \Phi_* \mathcal{O}_{W_0}(2H) = H^4$.

(c) The natural map $0 \longrightarrow \Phi_* \mathcal{O}_{\widetilde{W}}(2H) \longrightarrow \pi_* \omega^{\otimes 2}_{T/B}$ has a torsion cokernel since it is an isomorphism at a general fibre. Then the result follows calculating $\deg \pi_* w^{\otimes 2}_{T/B}$ as in proof of Corollary A.2 and applying (b). □

In order to finish *Case 2* note that, since part (c) of Lemma holds, it is enough to prove that $H^4 \geq \deg \Phi_* \mathcal{O}_{\widetilde{W}}(H)$.

CLAIM. Let X be a smooth variety and $f : X \longrightarrow B$ a filtration onto a smooth curve. Let $D \in \mathrm{Div}(X)$ be a nef divisor and let $\mathcal{E} = f_*\mathcal{O}_X(D)$. Then $D^n \geq \deg f_*\mathcal{O}_X(D)$.

Proof of Claim. It follows easily by induction from [17] Lemma 2.1. □

CASE 3. $p_g(F) = 6$, $q(F) = 0$, $K_F^2 = 12$ and the canonical image of F is contained in a threefold of Δ-genus 1, intersection of quadrics containing it.

In this case (see [15]) the canonical image of F is a complete intersection of two quadrics and a cubic. We follow the notations of Case 2. Denote $H_i = H_{|W_i}$. Now $\widetilde{W} = W_4$ is fibred over B by threefolds of degree four in $\mathbb{P}^5$, complete intersections of two quadrics, and $W_2 \longrightarrow B$ is an elliptic surface over B.

Then we have

Lemma 3.3. (a) $\Phi_*\mathcal{O}_{\widetilde{W}}(H) = \pi_*\omega_{T/B}$.

(b) $\deg \Phi_*\mathcal{O}_{\widetilde{W}}(2H) \geq (1/2)H^4 + 5 \deg \Phi_*\mathcal{O}_{\widetilde{W}}(H)$.

(c) $K_{T/B}^3 \geq H^4 + 4(\chi\mathcal{O}_B\chi\mathcal{O}_F - \chi\mathcal{O}_T)$.

Proof. (a) Follows as in Case 2.

(b) Note that, as in Case 2, formula (b) is invariant under changing H by $H + \Phi^*(A)$ so we can construct a smooth ladder of $(\widetilde{W}, H)$. For $i \geq 2$ and $t \in B$ general $(W_i)_t \subseteq \mathbb{P}^{i+1}$ is a complete intersection so it is projectively normal. On the other side $R^1\Phi_*\mathcal{O}_{W_i}$ is locally free for $i \geq 1$ (see [14]) and in fact $R^1\Phi_*\mathcal{O}_{W_i} = 0$ except for $i = 2$, for which it is a line bundle of degree $-\chi\mathcal{O}_{W_2}$. Let $E = (W_2)_t$ any fibre of $\Phi : W_2 \longrightarrow B$. Since $H_2 + E$ is nef and big on W_2 we have from Kawamata-Viehweg vanishing and the exact sequence

$$0 \to H^0(W_2, -E - H_2) \to H^0(W_2, -H_2) \to H^0(E, \mathcal{O}_E(-1)) \to H^1(W_2, -E - H_2)$$

that $h^1(E, \mathcal{O}_E(1)) = h^0(E, \mathcal{O}_E(-1)) = 0$ (recall that $K_E = \mathcal{O}_E$ since W_2 is elliptic) and hence that $R^1\Phi_*\mathcal{O}_{W_2}(H) = 0$. Then again by induction we have

$$\begin{aligned} \forall i \geq 0 \quad \forall n \geq 1 \qquad & R^1\Phi_*\mathcal{O}_{W_i}(nH) = 0, \\ \forall i \neq 2 \qquad & R^1\Phi_*\mathcal{O}_{W_i} = 0. \end{aligned}$$

Therefore we have exact sequences

$$\begin{aligned} &0 \longrightarrow \Phi_*\mathcal{O}_{W_{i+1}}(H) \longrightarrow \Phi_*\mathcal{O}_{W_{i+1}}(2H) \longrightarrow \Phi_*\mathcal{O}_{W_i}(2H) \longrightarrow 0 \quad \text{for } i \geq 0 \\ &0 \longrightarrow \Phi_*\mathcal{O}_{W_{i+1}} \longrightarrow \Phi_*\mathcal{O}_{W_{i+1}}(H) \longrightarrow \Phi_*\mathcal{O}_{W_i}(H) \longrightarrow 0 \quad \text{for } i \neq 1 \\ &0 \longrightarrow \Phi_*\mathcal{O}_{W_2} \longrightarrow \Phi_*\mathcal{O}_{W_2}(H) \longrightarrow \Phi_*\mathcal{O}_{W_1}(H) \longrightarrow R^1\Phi_*\mathcal{O}_{W_2} \longrightarrow 0. \end{aligned}$$

Denote $d = \deg \Phi_* \mathcal{O}_{\widetilde{W}}(H) = \deg \pi_* \omega_{T/C}$. Then we have

$$\begin{aligned} d &= \deg \Phi_* \mathcal{O}_{W_4}(H) = \deg \Phi_* \mathcal{O}_{W_3}(H) = \deg \Phi_* \mathcal{O}_{W_2}(H) \\ &= \deg \Phi_* \mathcal{O}_{W_1}(H) - \deg R^1 \Phi_* \mathcal{O}_{W_2} \end{aligned}$$

and then

$$\deg \Phi_* \mathcal{O}_{\widetilde{W}}(2H) = 4d + H^4 + \deg R^1 \Phi_* \mathcal{O}_{W_2} = 4d + H^4 - \chi \mathcal{O}_{W_2}.$$

Note that, since $\Phi : W_2 \longrightarrow B$ is an elliptic fibration, we have that $K_{W_2} \cong \Phi^*(L) + M$, where $M \geq 0$ and contained in fibres and $\deg L = \chi \mathcal{O}_{W_2} + 2(b-1)$. So, Riemann-Roch on W_2 and Leray spectral sequence yields

$$\begin{aligned} d - 4(b-1) &= \chi \Phi_* \mathcal{O}_{W_2}(H) = \chi \mathcal{O}_{W_2}(H) = \chi \mathcal{O}_{W_2} + \frac{1}{2} H_2^2 - \frac{1}{2} H_2 K_{W_2} \\ &\leq -\chi \mathcal{O}_{W_2} - 4(b-1) + \frac{1}{2} H^4 \end{aligned}$$

since $H_2^2 = H^4$, H_2 is nef and $H\Phi^{-1}(t) = 4$ for $t \in B$. Then $-\chi \mathcal{O}_{W_2} \geq d - (1/2)H^4$ and hence $\deg \Phi_* \mathcal{O}_{\widetilde{W}}(2H) \geq 5d + (1/2)H^4$.

(c) The same argument as in Case 2 works. □

Now we only have to use $H^4 \geq 0$ and get $K_{T/B}^3 \geq 4(\chi \mathcal{O}_B \chi \mathcal{O}_F - \chi \mathcal{O}_T)$ as needed. Note that using good lower bounds for H^4 as in Case 2 we can obtain stronger bounds for $K_{T/B}^3$ in this case. □

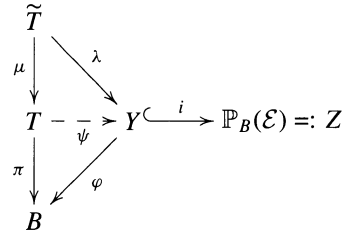
REMARK 3.4. The bounds obtained in Theorem 3.1 for fibred canonical threefolds hold when simply $|K_F|$ induces a birational map (it is not necessary that T be canonical).

Appendix. The relative hyperquadrics method for threefolds

The method of counting relative hyperquadrics, originated in [22] and [6] was successfully applied by Konno in [18] to study the slope of fibred surfaces with small fibre genus. Here we construct the fundamental sequence and prove the first elementary conclusions which are needed in the previous Section.

Let T be a normal, $\mathbb{Q}$-factorial, projective threefold with only terminal singularities, and let $\pi : T \longrightarrow B$ be a relatively minimal fibration onto a smooth curve of genus b. Following Ohno ([20]), if D is a Weil divisor on T and $\mathcal{E} = \pi_* \mathcal{O}_T(D)$ we

have



where (1) ψ is induced by $\pi^*\pi_*\mathcal{O}_T(D) \longrightarrow \mathcal{O}_T(D)$ and $Y = \overline{\mathrm{Im}}\psi$.
(2) $\mu : \widetilde{T} \longrightarrow T$ is a desingularization of T such that $\lambda = \psi \circ \mu$ is everywhere defined.
(3) $(\lambda^* \circ i^*)L_{\mathcal{E}} \sim_{\mathbb{Q}} \mu^*(D - D_1) - E$, being $L_{\mathcal{E}}$ the tautological divisor on Z, D_1 the codimension one base Weil divisor of $\mathcal{O}_T(D)$ and E is an effective $\mathbb{Q}$-divisor μ-exceptional.

Proposition A.1. *Under the above hypothesis we have an exact sequence*

$$0 \longrightarrow \varphi_*\mathcal{J}_{Y,Z}(2L_{\mathcal{E}}) \longrightarrow S^2\pi_*\mathcal{O}_T(D) \longrightarrow \pi_*\mathcal{O}_T(D)^{[2]}$$

(the generalized Max-Noether sequence associated to π).

Proof. From the exact sequence

$$0 \longrightarrow \mathcal{J}_{Y,Z}(2L_{\mathcal{E}}) \longrightarrow \mathcal{O}_Z(2L_{\mathcal{E}}) \longrightarrow i_*\mathcal{O}_Y \otimes \mathcal{O}_Z(2L_{\mathcal{E}}) \longrightarrow 0$$

we have

$$0 \longrightarrow \varphi_*\mathcal{J}_{Y,Z}(2L_{\mathcal{E}}) \longrightarrow S^2\pi_*\mathcal{O}_T(D) \longrightarrow \varphi_*(i_*\mathcal{O}_Y \otimes \mathcal{O}_Z(2L_{\mathcal{E}})).$$

Now the natural map $\pi^*\pi_*\mathcal{O}_T(D) \longrightarrow \mathcal{O}_T(D)$ induces a map $\pi^*(S^2\pi_*\mathcal{O}_T(D)) = S^2\pi^*\pi_*\mathcal{O}_T(D) \to (\mathcal{O}_T(D) \otimes \mathcal{O}_T(D))^{**} = \mathcal{O}_T(D)^{[2]}$ and hence $\delta : S^2\pi_*\mathcal{O}_T(D) \to \pi_*\mathcal{O}_T(D)^{[2]}$. Let $K = \ker\delta$. For general $t \in B$, $K_t = (\varphi_*\mathcal{J}_{Y,Z}(2L_{\mathcal{E}}))_t$, and hence $K = \varphi_*\mathcal{J}_{Y,Z}(2L_{\mathcal{E}})$ since $\pi_*\mathcal{O}_T(D)^{[2]}$ is locally free. □

Corollary A.2. *Under the same hypothesis we have*

$$K^3_{T/B} \geq (2p_g(F) - 4)(\chi\mathcal{O}_B\chi\mathcal{O}_F - \chi\mathcal{O}_T) - 2\deg K - 2\ell(2) \tag{1}$$

where $K = \varphi_*\mathcal{J}_{Y,Z}(2L_{\mathcal{E}})$ $F = \pi^{-1}(t)$ for $t \in B$, and $l(2)$ is the second order correction term of Reid-Fletcher to the plurigenera of T (cf. [8]).

Proof. Let $D = K_{T/B}$ (which is in general a Weil divisor) and take degrees in the generalized Max-Noether sequence. Use

$$d = \deg \pi_* \omega_{T/B} \geq (\chi \mathcal{O}_B \chi \mathcal{O}_F - \chi \mathcal{O}_T) \qquad ([20] \text{ p. } 656)$$

$$\deg \pi_* \omega_{T/B}^{[2]} = \frac{1}{2} K_{T/B}^3 + 3(\chi \mathcal{O}_B \chi \mathcal{O}_F - \chi \mathcal{O}_T) + \ell(2) \qquad ([20] \text{ Lemma } 2.8)$$

$$\deg S^2 \pi_* \omega_{T/B} = (p_g(F) + 1)d$$

$$\operatorname{rk} S^2 \pi_* \omega_{T/B} = \binom{p_g(F) + 1}{2}$$

and that if $\mathcal{C} = \operatorname{coker}(S^2 \pi_* \omega_{T/B} \to \pi_* \omega_{T/B}^{[2]})$, $\deg \mathcal{C} \geq 0$ since $\pi_* \omega_{T/B}^{[2]}$ is semipositive ([20]). □

REMARK A.3. For small values of the invariants $p_g(F)$, $q(F)$, K_F^2 it could be interesting to consider $D = mK_{T/B}$ for $m \geq 1$. We obtain then bounds for $K_{T/B}^3$ which are better than (1).

In general $\deg K$ is difficult to be computed or bounded. There are some special cases where this is easier. Notice that $\operatorname{rk} K = h^0(I_{\Sigma,\mathbb{P}^r}(2))$ where Σ is the canonical image of F and $r = p_g(F) - 1$. Then following Lemma 1.1 we have

$$h^0(\mathcal{J}_{\Sigma,\mathbb{P}^r}(2)) \leq \frac{(r-2)(r-3)}{2} \qquad \text{if } \Sigma \text{ is a non ruled surface}$$

$$h^0(\mathcal{J}_{\Sigma,\mathbb{P}^r}(2)) \leq \frac{(r-1)(r-2)}{2} - q(\Sigma) \qquad \text{if } \Sigma \text{ is a ruled surface}$$

$$h^0(\mathcal{J}_{\Sigma,\mathbb{P}^r}(2)) \leq \frac{r(r-1)}{2} \qquad \text{if } \Sigma \text{ is a curve.}$$

Lemma A.4. (a) *If $p_g(F) \geq 2$ and $\mathcal{E} = \pi_* \omega_{T/B}$ is semistable then* $\deg K \leq 2(\operatorname{rk} K/(p_g(F)))d$.

(b) *If $K = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_s$ ($s = \operatorname{rk} K$) then*

$$\deg K \leq (\operatorname{rk} K)\frac{2}{3}d$$

(*in particular this happens if $s \leq 1$ or $b = 0$*).

Proof. (a) If $\mathcal{E}$ is semistable then so it is $S^2\mathcal{E}$. Then we use the natural inclusion $K \hookrightarrow S^2\mathcal{E}$.

(b) If $x_i = \deg \mathcal{L}_i$ then there exists a section $s \in H^0(K \otimes \mathcal{L}_i^{-1}) \cong H^0(\mathcal{J}_{Y,Z}(2L_\mathcal{E}) \otimes \mathcal{O}_Z(\varphi^*(\mathcal{L}_i^{-1})) \hookrightarrow H^0(Z, \mathcal{O}(2L_\mathcal{E}) \otimes \varphi^*(\mathcal{L}_i^{-1}))$ so there exists a relative hyperquadric $Q_i \equiv 2L_\mathcal{E} - x_i \varphi^{-1}(t)$ (numerical equivalence). The result follows then from the following Lemma which is a slight refinement of [18] Remark 1.7, and the fact that for

every i, $\operatorname{rk} Q_i \geq 3$. □

Lemma A.5. *Let $Q \equiv 2L_{\mathcal{E}} - x\varphi^{-1}(t)$ be a relative hyperquadric. Let $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_k$ the virtual slopes of the Harder-Narashiman filtration of $\mathcal{E}$ ($k = \operatorname{rk}\mathcal{E}$). Let $p = \operatorname{rk} Q$; then*

$$x \leq \min_{1\leq i\leq p}\{\nu_i + \nu_{p-i}\} \leq \frac{2}{p}\deg \mathcal{E}.$$

Corollary A.6. *With the same notations as above, assume $p_g(F) \geq 2$.*

(a) *If $\mathcal{E} = \pi_*\omega_{T/B}$ is semistable then*

$$K^3_{T/B} \geq \Big(10 - \frac{24}{p_g(F)}\Big)(\chi\mathcal{O}_B\chi\mathcal{O}_F - \chi\mathcal{O}_T) - 2\ell(2) \quad \text{if } \Sigma \text{ is a non-ruled surface}$$

$$K^3_{T/B} \geq \Big(6 - \frac{12}{p_g(F)}\Big)(\chi\mathcal{O}_B\chi\mathcal{O}_F - \chi\mathcal{O}_T) - 2\ell(2) \quad \text{if } \Sigma \text{ is a ruled surface}$$

$$K^3_{T/B} \geq \Big(2 - \frac{4}{p_g(F)}\Big)(\chi\mathcal{O}_B\chi\mathcal{O}_F - \chi\mathcal{O}_T) - 2\ell(2) \quad \text{if } \Sigma \text{ is a curve,}$$

(b) *If $h^0(\mathcal{J}_{\Sigma,\mathbb{P}^r}(2)) = 0$ then*

$$K^3_{T/B} \geq (2p_g(F) - 4)(\chi\mathcal{O}_F\chi\mathcal{O}_B - \chi\mathcal{O}_T) - 2\ell(2),$$

(c) *If $h^0(\mathcal{J}_{\Sigma,\mathbb{P}^r})(2) = 1$ then*

$$K^3_{T/B} \geq \Big(2p_g(F) - \frac{16}{3}\Big)(\chi\mathcal{O}_F\chi\mathcal{O}_B - \chi\mathcal{O}_T) - 2\ell(2).$$

Proof. Take degrees at the generalized Max-Noether sequence and use Remark A.3 and Lemma A.4. □

References

[1] T. Ashikaga and K. Konno: *Algebraic surfaces of general type with $c_1^2 = 3p_g - 7$*, Tohoku Math. J. **42** (1990), 517–536.

[2] A. Beauville: *Surfaces algébriques complexes*, Société Mathématique de France. Astérisque **54**, 1978.

[3] A. Beauville: *Annullation du H^1 et systèmes paracanoniques sur les surfaces*, J. reine angew. Math. **388** (1988), 149–157.

[4] A. Beauville: *L'inégalité $p_g \geq 2q-4$ pour les surfaces de type général*, Bull. Soc. Math. France, **110** (1982), 343–346.

[5] E. Bombieri: *Canonical models of surfaces of general type*, Publ. Math. IHES, **42** (1973), 171–219.

[6] F. Catanese and C. Ciliberto: *Surfaces with $p_g = q = 1$. Problems in the Theory of Surfaces and their Classification*, Symposia Mathematica vol. **XXXII**, ed. Catanese et al., Academic Press (1991), 49–79.
[7] O. Debarre: *Inégalités numériques pur les surfaces de type général*, Bull. Soc. Math. France, **110** (1982), 319–342.
[8] A.R. Fletcher: *Contributions to Riemann-Roch on Projective threefolds with only canonical singularities and Applications*, Proc. Symp. in Pure Math. **46** (1987), 221–231.
[9] J. Harris: *A bound on the geometric genus of projective varieties*, Ann. Sc. Norm. Sup. Pisa Ser. **IV 8** (1981), 35–68.
[10] R. Hartshorne: *Algebraic Geometry*, Graduate Texts in Mathematics 52, Springer-Verlag, 1977.
[11] E. Horikawa: *Algebraic surfaces of general type with small c_1^2, V*, J. Fac. Sci. Univ. Tokyo. Sect. **A 28** (1981), 745–755.
[12] F. Jongmans: *Contributions a l'étude des variétés algébriques*, Mém. Soc. R. Sc. Liège, **7** (1947), 367–468.
[13] J. Kollár: *Higher direct images of dualizing sheaves I.*, Ann. of Math. **123** (1986), 11–42.
[14] J. Kollár: *Singularities of pairs*, preprint.
[15] K. Konno: *Algebraic surfaces of general type with $c_1^2 = 3p_g - 6$*, Math. Ann. **290** (1991), 77–107.
[16] K. Konno: *A note on surfaces with pencils of non-hyperelliptic curves of genus 3*, Osaka J. Math. **28** (1991), 737–745.
[17] K. Konno: *A lower bound of the slope of trigonal fibrations*. Internat. J. Math. **7(1)** (1996), 19–27.
[18] K. Konno: *Non-hyperelliptic fibrations of small genus and certain irregular canonical surfaces*, Ann. Sc. Norm. Sup. Pisa, **IV 20** (1993), 575–595.
[19] K. Konno: *On the irregularity of special non-canonical surfaces*, Publ. RIMS Kyoto Univ. **30** (1994), 671–688.
[20] K. Ohno: *Some inequalities for minimal fibrations of surfaces of general type over curves*. J. Math. Soc. Japan, **44** (1992), 643–666.
[21] M. Reid: *Quadrics through a canonical surface*, Algebraic geometry (L'Aquila 1988). Lecture Notes in Math. **1417**, 191–213.
[22] M. Reid: *Problems on pencils of small genus*, (1990), preprint.
[23] I. Reider: *Vector bundles of rank 2 and linear systems on algebraic surfaces*, Ann. of Math. **127** (1988), 309–316.
[24] G. Xiao: *Fibred algebraic surfaces with low slope*, Math. Ann. **276** (1987), 449–466.
[25] G. Xiao: *Irregularity of surfaces with a linear pencil*, Duke Math. J. **55(3)** (1987), 597–602.

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