Ishii, T.
Osaka J. Math.
12 (1975), 473–482

ON LOCALLY DIRECT SUMMANDS OF MODULES

Dedicated to Professor Kiiti Morita on his 60th birthday

TADAMASA ISHII

(Received July 12, 1974)

Throughout R will represent a ring with unit element 1, and all modules will be unitary R-modules. We call a module M a *completely indecomposable module* if the endomorphism ring of M is a local ring. Let $\mathfrak{M}=\{M_\alpha\}_I$ be a set of completely indecomposable right R-modules, and $\mathfrak{A}$ the full subadditive category of the category of all right R-modules, whose objects consist of all R-modules which are isomorphic to direct sums of M_α's in $\mathfrak{M}$. We define the subclass $\mathfrak{J}'$ of the morphisms in $\mathfrak{A}$ as follows: for any objects $M=\sum_{\alpha\in K}\oplus M_\alpha'$, $N=\sum_{\beta\in L}\oplus N_\beta$ in $\mathfrak{A}$, $\mathfrak{J}'\cap \mathrm{Hom}_R(M, N)=\{f\in\mathrm{Hom}_R(M,N) \setminus p_\beta f i_\alpha$ is not isomorphic, for all $\alpha\in K, \beta\in L$, where $i_\alpha\colon M_\alpha'\to M$ is the inclusion and $p_\beta\colon N\to N_\beta$ is the projection$\}$. Then, $\mathfrak{J}'$ does not depend on the decompositions of M and N (see Corollary to Lemma 5 in [5]).

M. Harada and Y. Sai [4], [5] gave several equivalent conditions for $S_M\cap\mathfrak{J}'$ to be equal to the Jacobson radical $J(S_M)$ of S_M, where $M\in\mathfrak{A}$ and $S_M=\mathrm{Hom}_R$ (M, N). Among those conditions, they made great use of structures of the factor cagegory $\mathfrak{A}/\mathfrak{J}'$ in order to show the following fact: if $J(S_M)=S_M\cap\mathfrak{J}'$, then for any two decompositions $M=\sum_{\alpha\in K}\oplus M_\alpha=\sum_{\beta\in L}\oplus N_\beta$ and any subset K' of K, there exists a one-to-one mapping φ of K' into L such that $M_\alpha\approx N_{\varphi(\alpha)}$ for all $\alpha\in K'$ and $M=\sum_{\alpha\in K'}\oplus N_{\varphi(\alpha)}\oplus\sum_{\alpha'\in K-K'}\oplus M_{\alpha'}$.

The purpose of this note is to give a ring-theoretical proof of the above fact by using a few structure of $\mathfrak{A}/\mathfrak{J}'$. We shall define a concept of locally direct summands of M in $\mathfrak{A}$ for this purpose. Let $N=\sum_{\gamma\in L}\oplus N_\gamma$ be a submodule of M in $\mathfrak{A}$. If $\sum_{\gamma'\in L'}\oplus N_{\gamma'}$ is a direct summand of M for every finite subset L' of L, we call a *locally direct summand* of M (with respect to the decomposition $N=\sum_{\gamma\in L}\oplus N_\gamma$) We shall give a relation between some locally direct summands of M and dense submodules of M defined in [4], and using this relation we shall give a proof of the statement above.

The author would like to express his hearty thanks to the referee and Prof. M. Harada for their advices and suggestions.

We begin with preliminary definitions and results on $\mathfrak{J}'$ and S_M. From now, we understand that a module M is in $\mathfrak{A}$ and that M_α's are completely indecomposable, if there are no confusions.

Let M, N be in $\mathfrak{A}$, and $f \in \mathrm{Hom}_R(M, N)$. f is said to be *left regular* modulo $\mathfrak{J}'$ if, for any homomorphism g of any L in $\mathfrak{A}$ to M, fg in $\mathfrak{J}'$ implies g in $\mathfrak{J}'$. The right regularity of / modulo $\mathfrak{J}'$ is defined similarly. / is said to be an *isomorphism* modulo $\mathfrak{J}'$ if there exists some g: $N \rightarrow M$ such that $gf = 1_M \mathrm{mod.} \mathfrak{J}'$ and $fg = 1_N$ mod. $\mathfrak{J}'$.

REMARK 1. Let $M = \sum_{\beta \in K} \oplus M_\beta$, $N = 2_{\beta' \in K'} \oplus M_{\beta'}$ be in $\mathfrak{A}$ where K' is a subset of K, i the inclusion of N to M and p the projection of M onto N. Then, i is left regular mod. $\mathfrak{J}'$ and p is right regular mod. $\mathfrak{J}'$.

Lemma 1. *For any morphismf in* SI *and any g in $\mathfrak{J}'$, fg and gf are in $\mathfrak{J}'$.* See Lemma 5 in [5].

Lemma *2. Let $M = \sum_{\alpha \in K} \oplus M_\alpha$ be in $\mathfrak{A}$, and S_M the endomorphismring of M. Then,*

(1) *$S_M / S_M \Pi \mathfrak{J}'$ is a regular ring (in the sense of von Neumann), moreover*

(2) for any fin S_M with $f = f^2$ modulo $\mathfrak{J}' \cap S_M$, there exist some elements a and e in S_M such that a is regular in $S_M / S_M \cap \mathfrak{J}'$, e is a projection of M to $\sum_{\alpha' \in K'} \oplus M_{\alpha'}$ for some subset K' of K and $f = aea'$ modulo $\mathfrak{J}'$, where $aa' = a'a = 1$ modulo $\mathfrak{J}'$ and $a' \in S_M$.

See [1], Lemma 6 and Theorem 7 in [5] and [6].

Corollary 1. *Let M, N be in $\mathfrak{A}$, and $f: M \rightarrow N$. Then,*

(1) *f is left (resp. right) regular mod. $\mathfrak{J}'$ if and only if there exists some $g: N \rightarrow M$ such that $gf = 1_M$ (resp. $fg = 1_N$) mod. $\mathfrak{J}'$, and*

(2) f is an isomorphism mod. $\mathfrak{J}'$ if and only iff is left and right regular mod. $\mathfrak{J}'$.

Proof. (1) "If" part is trivial. Conversely, we assume that f is left regular mod. $\mathfrak{J}'$ Since $S_M / S_M \cap \mathfrak{J}'$ is a regular ring by the lemma, there exists some g: $N \rightarrow M$ such that $fgf = f$ mod. $\mathfrak{J}'$. The left regularity of / mod. $\mathfrak{J}'$ implies that $gf = 1_M \mathrm{mod.} \mathfrak{J}'$. The right regularity is similar. *(2)* is clear.

Corollary *2. If f: $M \rightarrow N$ is left regular mod. $\mathfrak{J}'$ for M, N in $\mathfrak{A}$ and $S_M \Pi \mathfrak{J}'$ is equal to the Jacobson radical $J(S_M)$ of S_M, then f is an R-monomorphism and M is R-isomorphic to a direct summand of N.*

Proof. By Corollary 1(1), there exists some g: $N \rightarrow M$ such that $gf = 1_M$ mod. $\mathfrak{J}'$, since f is left regular mod. $\mathfrak{J}'$. Hence, $1_M - gf \in S_M \cap \mathfrak{J}' = J(S_M)$ and so gf is an R-isomorphism. Therefore, / is an R-monomorphism and M is R-isomorphic to a direct summand of N.

Let U, V be right R-modules, /: $U \to V$, and $U = \sum_{\gamma \in K} \theta\, U_\gamma$ a direct sum of right R-submodules of U. Then, we consider the following condition:

/ is an R-monomorphism and $\cdots(*)$

for any finite subset K' of K, $f(\sum_{\gamma' \in K'} \Phi\, U_{\gamma'})$ is a direct summand of V.

If f satisfies the above (*)-condition, we call f a (*)-monomorphism (with respect to this decomposition of U).

For example, let f, U and V be as above. If f is an R-monomorphism and each U_γ is injective, then / is a (*)-monomorphism (with respect to the decomposition $U = \sum_{\gamma \in K} \oplus U_\gamma$).

From now on, (*)-monomorphisms will be considered in $\mathfrak{A}$.

The following lemma on (*)-monomorphisms is essential in this note.

Lemma 3. *Let $M = \sum_{\alpha \in K} \oplus M_\alpha$, N be in $\mathfrak{A}$ and f: $M \to N$. Then, f is left regular* **mod.**$\mathfrak{J}'$ *if and only iff is a* **(*)-monomorphism** *(w.r.t. the* **decomposition** $M = \sum_{\alpha \in K} \oplus M_\alpha$).

Proof. First, we assume that f is left regular mod.$\mathfrak{J}'$. Put $M_0 = \sum_{\alpha' \in K'} \oplus M_{\alpha'}$ for any finite subset K' of K. Let i be the inclusion of M_0 to M. Then, fi is left regular mod.$\mathfrak{J}'$ and $S_{M_0} \cap \mathfrak{J}' = (S_{M_0})$ by Lemma 8 in [5], because K' is a finite set. Hence, fi is an R-monomorphism and $fi(M_0)$ is a direct summand of N by Corollary 2 to Lemma 2. Therefore, / is an R-monomorphism and $f(M_0)$ is a direct summand of N, i.e. / is a (*)-monomorphism (w.r.t. the decomposition $M = \sum_{\alpha \in K} \oplus M_\alpha$). Conversely, let $g \in \mathrm{Hom}_R(T, M)$ for any module $T = \sum_{\gamma \in L} \theta\, T_\gamma$ in $\mathfrak{A}$ and assume that fg in $\mathfrak{J}'$. Put $g_\gamma = g i_\gamma$, where i_γ is the inclusion of T_γ to T for all $\gamma \in L$. Then, we can express g_γ as a column-summable matrix for all $\gamma \in L$. Hence, g_γ is a column-matrix whose finite components are isomorphic and the others are all non-isomorphisms. We can rearrange g_γ as follows: the first n components are isomorphisms. Put $M_0 = \sum_{i=1}^{n} \oplus M_i$ Let i be the inclusion of M_0 to M, and p the projection of M onto M_0. Then, $fipg_y = fg_y = fgi_{f_y} = Q$ mod. $\mathfrak{J}'$. Since fi is left regular mod.$\mathfrak{J}'$, pg_γ is in $\mathfrak{J}'$ Hence, g_γ and so g are in $\mathfrak{J}'$, because o:$pg_\gamma + (1-p)g_\gamma = g_\gamma$mod.$\mathfrak{J}'$ Therefore, / is left regular mod.$\mathfrak{J}'$.

We note that a (*)-monomorphism does not depend on the decomposition of M from Lemma 3.

Corollary 1 (cf. Lemma 3.2.3 in [3]) (1) *// /: $M \to N$ is left regular* **mod.**$\mathfrak{J}'$*, then f is an R-monomorphism. (2) For any f* **in** $S_M \cap \mathfrak{J}'$, $1_M - f$ *is an R-monomorphism.*

Proof. (1) is clear by the lemma. (2) Since / is in $S_M \cap \mathfrak{J}'$, $1_M - f$ is left regular mod.$\mathfrak{Y}'$ and hence an R-monomorphism by (1).

Corollary 2. *Let M, N be in $\mathfrak{A}$, and f: $M \rightarrow N$ an isomorphism mod.$\mathfrak{Y}'$. Then f is an R-isomorphism provided either $S_M \cap \mathfrak{Y}' = J(S_M)$ or $S_N \cap \mathfrak{Y}' = J(S_N)$. Especially, if M is a finite direct mm of M_α's in $\mathfrak{M}$, then an isomorphism mod. $\mathfrak{Y}'$ means an R-isomorphism.*

Proof. Since / is isomorphic mod.$\mathfrak{Y}'$, there exists some g: $N \rightarrow M$ such that $gf = 1_M$ mod.$\mathfrak{Y}'$ and $fg = 1_N$ mod.$\mathfrak{Y}'$. Hence, / and g are left regular mod. $\mathfrak{Y}'$, that is, both are R-monomorphisms by Corollary 1. In case $S_N \cap \mathfrak{Y}' = J(S_N)$, $1_N - fg \in J(S_N)$ Hence, fg is an R-isomorphism and so is /. On the other hand, if $S_M \cap \mathfrak{Y}'$is equal to $J(S_M)$, then $1_M - gf \in J(S_M)$and hence gf is an R-isomorphism. Therefore, / is an R-isomorphism. The latter assertion is clear by Lemma 8 in [5].

We define here an important concept as follows (see [3]): let M, N be in $\mathfrak{A}$, and $N = \sum_{\beta \in K} \oplus N_\beta$ a submodule of M. Then, N is said to be a *locally direct summand* of M (with respect to the decomposition $N = 2_{\beta \in K} \oplus N_\beta$) if the inclusion i: $N \rightarrow M$ is a (*)-monomorphism (with respect to this decomposition of N).

In the following lemma, we consider the existence of locally direct summands of a module M in SI.

We remark that in the above definition, the concept of locally direct summands of M in $\mathfrak{A}$ does not depend on the decomposition of M, since (*)-mononorphisms do not depend on the decomposition of M.

Lemma 4. *Let M, N be in $\mathfrak{A}$, and f: $M \rightarrow N$. Then, there exist a locally direct summand N' of N in $\mathfrak{A}$ via the inclusion i: $N' \rightarrow N$ and some f': $M \rightarrow N'$ such that $f = if'$mod.$\mathfrak{Y}'$, i is left regular mod.$\mathfrak{Y}'$ and f is right regular mod.$\mathfrak{Y}'$.*

Proof. We begin with the case $M = N$ and $f = f^2$ mod.$\mathfrak{Y}'$. There exist a projection e of $M = \sum_{\alpha \in K} \oplus M_\alpha$ onto $\sum_{\alpha' \in K'} \oplus M_{\alpha'}$ for some subset K' of K and elements a, a' in S_M such that $f = aea'$ mod.$\mathfrak{Y}'$ and $aa' = a'a = 1$ mod.$\mathfrak{Y}'$, by Lemma 2(2). Put $N' = aeM$, $N'' = eM$, and consider the inclusions i: $N' \rightarrow M$, i': $N'' \rightarrow M$. Then, by Lemma 3, N' is a locally direct summand of M and i is left regular mod.$\mathfrak{Y}'$, since N' is isomorphic to N'' under ai' that is left regular mod.$\mathfrak{Y}'$ Moreover, ea' is right regular mod.$\mathfrak{Y}'$, and hence so is $f' = aea'$: $M \rightarrow N'$. Thus, our lemma holds. In the general case, for /: $M \rightarrow$, there exist some homomorphisms g: $N \rightarrow N$ and k: $N \rightarrow M$ such that $g = g^2$ mod.$\mathfrak{Y}'$, $f = gf$ mod.$\mathfrak{Y}'$ and $g = fk$mod.$\mathfrak{Y}'$, by Lemma 2. For g: $N \rightarrow N$with $g = g^2$ mod.$\mathfrak{Y}'$, there exist a locally direct summand N' of N in SI, some g': $N \rightarrow N'$ and the inclusion i: $N' \rightarrow N$ such that $g = ig'$ mod.$\mathfrak{Y}'$, g' and i are right and left regular mod.$\mathfrak{Y}'$, respectively, by the above argument. We can easily show that $g'f$is right regular mod.$\mathfrak{J}'$ since $g'i=1_{N'}$ mod.$\mathfrak{J}'$, and $f=ig'f$mod.$\mathfrak{J}'$.

Lemma 5. *For M and N in $\mathfrak{A}$, a homomorphism f: $M\to N$ is right regular mod. $\mathfrak{J}'$ if and only if there exist a locally direct summand M' of M in $\mathfrak{A}$ and some g: $N\to M'$ such that fig— 1_N mod.$\mathfrak{J}'$ and g is an isomorphism mod.$\mathfrak{J}'$, where i is the inclusion of M' to M.*

Proof. "If" part is trivial. Conversely, suppose that / is right regular mod.$\mathfrak{J}'$. Then there exists some g': $N\to M$ such that $fg'=1_N$ mod.$\mathfrak{J}'$, by Corollary 1 to Lemma 2. Since g' is left regular mod.$\mathfrak{J}'$, there exists a locally direct summand $M'$of M in $\mathfrak{A}$ such that $g'=ig$ mod.$\mathfrak{J}'$, where g: $N\to M'$is right regular mod.$\mathfrak{J}'$ and the inclusion i:$M'\to M$ is left regular mod.$\mathfrak{J}'$, by Lemma 4. Therefore, $fig=l_N$ mod.$\mathfrak{J}'$ and g is an isomorphism mod.$\mathfrak{J}'$.

Lemma 6. *Let M, N be in $\mathfrak{A}$, e an idempotent element in S_M where N is contained in eM, and let the inclusion i: $N\to M$ be left regular mod.$\mathfrak{J}'$. Then, there exists a locally direct summand N' of M in $\mathfrak{A}$ such that $e=ip+i'p'$mod.$\mathfrak{J}'$, $pi=1_N$ mod.$\mathfrak{J}'$, $p'i'=1_{N'}$mod.$\mathfrak{J}'$ and $pi'=p'i=0$ mod.$\mathfrak{J}'$, where i' is the inclusion of N' to M and p,p' are homomorphisms of M to N, N' respectively. Furthermore, the formal direct sum $N\oplus N'$ is R-isomorphic to a locally direct summand of eM.*

Proof. For i: $N\to M$, there exists some p_0: $M\to N$such that $ip_0i=i$mod. $\mathfrak{J}'$. Since $ei=i$, $ip_0ei=i$mod.$\mathfrak{J}'$. Put $p=p_0e$, that is, p: $M\to N$and $ipi=i$mod.$\mathfrak{J}'$. Since i is left regular mod.$\mathfrak{J}'$, $pi=1_N$ mod.$\mathfrak{J}'$. Now, we put $f=ip$and $g=e-f$. Then, $ef=fe=f$and $eg=ge=g$. For g: $M\to M$ there exist a locally direct summand N' of M in $\mathfrak{A}$ and some p': $M\to N'$ such that $g=i'p'$ mod.$\mathfrak{J}'$, the inclusion i': $N'\to M$ is left regular mod.$\mathfrak{J}'$ and p' is right regular mod.$\mathfrak{J}'$, by Lemma 4. Therefore, $e=f+g=ip+i'p$mod.$\mathfrak{J}'$. Since $gf=fg=0, ipi'p'=i'p'ip$ $=0$ mod.$\mathfrak{J}'$ implies $pi'=p'i=0$ mod.$\mathfrak{J}'$, because i and i' are left regular mod.$\mathfrak{J}'$ and p, p' are right regular mod.$\mathfrak{J}'$. Moreover, $g=g^2$ mod.$\mathfrak{J}'$ implies $p'i'=1_{N'}$ mod.$\mathfrak{J}'$. Finally, we show that the formal direct sum $N\oplus N'$ is R-isomorphic to a locally direct summand of eM. Let $I=(i, i')$:$N\oplus N'\to M$and $t=\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$: $T\to$ $N\oplus N'$for any Tin $\mathfrak{A}$. Suppose that $It=it_1+i't_2$is in $\mathfrak{J}'$. Then, $pit_1+pi't_2$ is in $\mathfrak{J}'$. Since pi' is in $\mathfrak{J}'$, t_1 is in $\mathfrak{J}'$ and so is t_2. Hence, t is in $\mathfrak{J}'$. It follows that / is a (*)-monomorphism. Therefore, $N\oplus N'$is a locally direct summand of M in $\mathfrak{A}$. On the other hand, $g=eg$ and $g=i'p'$mod.$\mathfrak{J}'$ imply $ei'=i'$mod.$\mathfrak{J}'$, and so we may assume $ei'=i'$ in the above. Since i' is a (*)-monomorphism, Im(i') is contained in eM. Hence, Im(I) is contained in eM, whence $N\oplus N'$is R-isomorphic to a locally direct summand of eM.

Corollary. *Let $N\subset M$ be in $\mathfrak{A}$. If the inclusion i: $N\to M$ is left regular mod. $\mathfrak{J}'$, then there exist a locally direct summand N' of M in $\mathfrak{A}$, the inclusion i':*

$N' \to M$ and some p, p' of M to N, N' respectively such that $1_M = ip + i'p' \bmod. \mathfrak{F}'$, $pi = 1_N \bmod. \mathfrak{F}'$, $p'i' = 1_{N'} \bmod. \mathfrak{F}'$ and $p'i = pi' = 0 \bmod. \mathfrak{F}'$.

Proof. Put $e = 1_M$ in the lemma.

Let N be an R-module, and $\{N_j = \sum_{i \in I_j} \oplus N_i^{(j)}\}_{j \in J}$ the set of submodules of N in $\mathfrak{A}$ which are locally direct summands of N. We define an order $>$ in the set $\{N_j\}$ as follows:

for each locally direct summand N_j of N,

$N_j > N_k$ if and only if $\{N_i^{(j)}\}_{I_j} \supset \{N_i^{(k)}\}_{I_k}$, for any $j \neq k$ in J.

Then, there exists a maximal submodule of N among the set $\{N_j\}$ with respect to this order $>$, by Zorn's lemma. We call it a *maximal* locally direct summand of N.

Proposition 7. *Assume that all M_α in $\mathfrak{M}$ are injective. Let $N \subset M$ be in $\mathfrak{A}$. Then, N is essential in M if and only if N is a maximal locally direct summand of M.*

Proof. "Only if" part is trivial. Conversely, if N is not essential, there exists a cyclic submodule N' of M with $N \cap N' = (0)$. Then, the injective hull $E(N')$ in M is a direct summand of M. On the other hand, $N \cap E(N') = (0)$. Since $E(N')$ contains an injective submodule M_β for some β, this contradicts the maximality of N. Hence, N is an essential submodule of M.

Next, we show that a dense submodule of a module in $\mathfrak{A}$ defined in [4] is equal to a maximal locally direct summand of the module.

Lemma 8. *Let $N \subset M$ be in $\mathfrak{A}$. Then, N is a maximal locally direct summand of M if and only if the inclusion i of N to M is an isomorphism $\bmod. \mathfrak{F}'$.*

Proof. First, we assume that N is a maximal locally direct summand of M. If the inclusion i: $N \to M$ is not isomorphic modulo $\mathfrak{F}'$, there exists a locally direct summand N' of M in $\mathfrak{A}$ such that $1_M = ip + i'p' \bmod. \mathfrak{F}'$, where i' is the inclusion of N' to M, p: $M \to N$ and p': $M \to N'$, by Corollary to Lemma 6. Then, $I = (i, i')$: $N \oplus N' \to M$ is a (*)-monomorphism. Hence, the image of / is equal to a locally direct summand $N \oplus \mathrm{Im}(i')$ of M in $\mathfrak{A}$ which contains N; this contradicts the maximality of N. Hence, $N' = 0$. Therefore, $1_M = ip \bmod. \mathfrak{F}'$ and so i is an isomorphism $\bmod. \mathfrak{F}'$. Conversely, suppose that i is an isomorphism $\bmod. \mathfrak{F}'$ Then, there exists some p: $M \to N$ such that $pi = 1_N \bmod. \mathfrak{F}'$ and $ip = 1_M, \bmod. \mathfrak{F}'$, and also N is a locally direct summand of M. If N is not maximal in M, there exists a locally direct summand N' of M in $\mathfrak{A}$ such that $N \oplus N'$ is a locally direct summand of M in $\mathfrak{A}$. Hence, the inclusion $I = (i, i')$: $N \oplus N' \to M$ is left regular $\bmod. \mathfrak{F}'$, where i' is the inclusion *of* N' to M. Therefore, there exists some g: $M \to N \oplus N'$ such that $gI = 1_{N \oplus N'} \bmod. \mathfrak{F}'$, by Corollary 1(1) to Lemma 2. Let p_1 be the projection of $N\oplus N'$ onto N. Then, $p_1 gi=1_N=pi$ mod. $\mathfrak{J}'$ and so $p\iota g=p$ rn.od.9f', which implies that $pi'=0$ mod.$\mathfrak{J}'$. Hence, $N'=0$; a contradiction. It follows that N is a maximal locally direct summand of M.

REMARK 3. The submodule N in the lemma is called a *dense* submodule of M, in [4]. We note that $N\oplus N'$ in Corollary to Lemma 6 is a dense submodule of M.

Corollary 1. *Let M,N be in $\mathfrak{A}$, and f: $M\to N$. Then, there exist locally direct summands M' and N' of M and N in $\mathfrak{A}$, respectively, such that the restriction $f|_{M'}$ to N' is an R-isomorphism. Especially, f is isomorphic mod.$\mathfrak{J}'$ if and only if M' and N' are dense in M and N, respectively.*

Proof. For f: $M\to N$, there exist a locally direct summand N'' of N in $\mathfrak{A}$, the inclusion i': $N''\to N$ and some f': $M\to N''$ such that $f=i'f'$ mod.$\mathfrak{J}'$, i' is left regular mod.$\mathfrak{J}'$ and f' is right regular mod.$\mathfrak{J}'$, by Lemma 4. Since f' is right regular mod.$\mathfrak{J}'$, there exist a locally direct summand M' of M in $\mathfrak{A}$ and some g: $N''\to M'$ such that $f'ig=1_{N'}$mod.$\mathfrak{J}'$ and g is isomorphic mod.$\mathfrak{J}'$, where i is the inclusion of M' to M, by Lemma 5. Since fig is left regular mod.$\mathfrak{J}'$ and g is isomorphic mod.$\mathfrak{J}'$, fi is left regular mod.$\mathfrak{J}'$ and so R-monomorphic. Let N' be the image of fi in N. Then, fi: $M'\to N'$ is an R-isomorphism, whence it follows that N' is a locally direct summand of N. Particularly, in case / is isomorphic mod.$\mathfrak{J}'$, i' and f' are isomorphic mod.$\mathfrak{J}'$ by Corollary 1(2) to Lemma 2 *and fig:* $N''\to N'$ is isomorphic mod.$\mathfrak{J}'$, so that N' and M' are dense in N and M, respectively, by the lemma. Conversely, if N' and M' are dense in N and M' respectively, i and i' are isomorphic mod.$\mathfrak{J}'$, and hence / is isomorphic mod.$\mathfrak{J}'$ by the lemma.

Corollary 2. *// $S_M\cap\mathfrak{J}'=J(S_M)$ for a module M in $\mathfrak{A}$, then M is the only one dense submodule in M.*

Proof. Let N be a dense submodule of M. Then, the inclusion i: $N\to M$ is isomorphic mod. $\mathfrak{J}'$ by the lemma. Hence, i is an R-isomorphism by Corollary 2 to Lemma 3 and so $N=M$.

Lemma 9. *Let e be an idempotent element in S_M for a module $M=\sum_{\alpha\in K}\oplus M_\alpha$ in* SI. *Then, there exist a submodule N of eM in $\mathfrak{A}$ and p: $M\to N$ such that $e=ip$ mod.$\mathfrak{J}'$ and $pi=1_N$ mod.$\mathfrak{J}'$, where i: $N\to M$ is the inclusion.*

Proof. Since eM is a direct summand of M, eM contains some M_α by [2]. Hence, there exists a maximal locally direct summand of eM in $\mathfrak{A}$. Let N be the maximal one, and i the inclusion of N to M. Since i is left regular mod.$\mathfrak{J}'$, there exists a locally direct summand N' of M in $\mathfrak{A}$ such that $e=ip+i'p'$ mod.$\mathfrak{J}'$, $pi=1_{N'}$mod.$\mathfrak{J}'$ and $N\oplus N'$ is R-isomorphic to a locally direct summand of eM, where p: $M\to N, p'$: $M\to N'$ and i' is the inclusion of N' to M, by Lemma 6. Since N is maximal in eM, $N'=0$ and hence $e=ip$ mod.$\mathfrak{J}'$

Corollary 1 (cf. Theorem 1 in [4]). *Let* $P=\sum_{\alpha\in L}\oplus P_\alpha$ *in* $\mathfrak{A}$ *(not necessarily each* P_α *is in* $\mathfrak{M}$*). Then, there exists a submodule* N_α *of* P_α *in* $\mathfrak{A}$ *such that* $e_\alpha=i_\alpha p_\alpha$ *mod.* $\mathfrak{J}'$, *where* p_α: $P\to N_\alpha$, i_α: $N_\alpha\to P$ *is the inclusion and* e_α: $P\to P_\alpha$ *is the projection, for each* $\alpha\in L$. *Moreover,* $\sum_{\alpha\in L}\oplus N_\alpha$ *is a maximal locally direct summand of* P *in* $\mathfrak{A}$. *(Such* N_α *is called a dense submodule of* P_α, in [4].)

Proof. We can find a maximal locally direct summand N_α of $e_\alpha P=P_\alpha$ such that $e_\alpha=i_\alpha p_\alpha$mod.$\mathfrak{J}'$, where p_α: $P\to N_\alpha$, i_α: $N_\alpha\to P$ is the inclusion and e_α: $P\to P_\alpha$ is the projection, for every $\alpha\in L$, by the lemma. Since a finite direct sum $\sum_{i=1}^{n}\oplus N_{\alpha_i}$ is a direct summand of P, $\sum_{\alpha\in L}\oplus N_\alpha$ is a locally direct summand of P. Hence, the inclusion I: $\sum_{\alpha\in L}\oplus N_\alpha\to P$ is left regular mod.$\mathfrak{J}'$ In order to see that $\sum_{\alpha\in L}\oplus N_\alpha$ is dense in P, we have only to prove that I is right regular mod.$\mathfrak{J}'$. Let t be a homomorphism of P to any module T in $\mathfrak{A}$ and assume that tI is in $\mathfrak{J}'$. If t is not in $\mathfrak{J}'$, there exists some direct summand P_β in P such that the restriction $t\,|_{P_\beta}$ is not in $\mathfrak{J}'$. $\sum_{\alpha\in L}e_\alpha i=i$ and $e_\alpha i$ is non-isomorphic for almost all $\alpha\in L$, where i is the inclusion of P_β to P. Hence, for some integer n, $\sum_{j=1}^{n}e_{\alpha_j}\cdot i=i$ mod.$\mathfrak{J}'$ and so $ti=te_Ni=tI_Np_Ni-tIp_Ni=0$ mod.$\mathfrak{J}'$, where I_N: $\sum_{j=1}^{n}\oplus N_{\alpha_j}\to P$, p_N: $P\to\sum_{j=1}^{n}\oplus N_{\alpha_j}$ and e_N: $P\to\sum_{j=1}^{n}\oplus P_{\alpha_j}$. Therefore, ti is in $\mathfrak{J}'$, which is a contradiction. Thus, t is in $\mathfrak{J}'$ and so I is right regular mod.$\mathfrak{J}'$.

Corollary 2. *Let M be in* $\mathfrak{A}$, *and N a direct summand of M. If* $S_M\cap\mathfrak{J}'$ *is equal to* $J(S_M)$, *then N is in* $\mathfrak{A}$.

Proof. Since N is a direct summand of M, there exists a submodule N' of M such that $M=N\oplus N'$. Hence, there exist dense submodules N_0 and N_0' of N and N' in $\mathfrak{A}$, respectively such that $N_0\oplus N_0'$ is dense in M, by the above corollary. Hence, $N_0\oplus N_0'=M$ by Corollary 2 to Lemma 8, which implies that N is isomorphic to a direct sum of completely indecomposable modules M_α's in $\mathfrak{M}$.

Proposition 10. *Let* M, N *be in* $\mathfrak{A}$, *and* f: $M\to N$. *If either* $S_M\cap\mathfrak{J}'=J(S_M)$ *or* $S_N\cap\mathfrak{J}'=J(S_N)$, *then there exist submodules* M_1 *and* M_2 *of* M *in* $\mathfrak{A}$ *such that* $M=M_1\oplus M_2$ *and the restrictions of f to* M_1 *and* M_2 *are a zero homomorphism mod.*$\mathfrak{J}'$ *and an R-monomorphism, respectively.*

Proof. By Corollary 1(1) to Lemma 2 and Lemma 4, there exist a locally direct summand N' of N in $\mathfrak{A}$, f': $M\rightarrow N'$, g': $N'\rightarrow M$ and the inclusion i: $N'\rightarrow N$ such that $f=if'$ mod. $\mathfrak{J}'$, $f'g'=1_{N'}$ mod. $\mathfrak{J}'$, i is left regular rnod.^7 and f' is right regular mod. $\mathfrak{J}'$. In case $S_N\cap\mathfrak{J}'=J(S_N)$, $S_{N'}\Pi\,\mathfrak{J}'=J(S_{N'})$ and hence $f'g'$ is an R-isomorphism. Therefore, $M=\mathrm{Im}(g')\oplus\mathrm{Ker}(f')$. We put $M_1=\mathrm{Ker}(f')$ and $M_2=\mathrm{Im}(g')$. Then, the restriction $f|_{M_1}$is a zero homomorphism mod. $\mathfrak{J}'$ Since/I $_{M_2}=if'|_{M_2}$ mod. $\mathfrak{J}'$ and $f'|_{M_2}$is an isomorphismrn.od.3i', $f|_{M_2}$ is an R-monomorphism. On the other hand, if $S_M\cap\mathfrak{J}'=J(S_M)$, $S_{M'}\cap\mathfrak{J}'=J(S_{M'})$ where M' is a locally direct summand of M in $\mathfrak{A}$ such that some g: $N'\rightarrow M'$ is isomorphic mod. $\mathfrak{J}'$ (cf. Lemma 5). Since g is an R-isomorphism by Corollary 2 to Lemma 3, $S_{N'}\cap\mathfrak{J}'=J(S_{N'})$and so $M=\mathrm{Im}(g')\oplus\mathrm{Ker}(f')$ as above. We put $M_2=\mathrm{Im}(g')$. Then, M_1 and M_2 satisfy the proposition.

Now, we shall show ring-theoretically the main theorem in this note by $M_1=\mathrm{Ker}(f')$ and only using the concept "modulo $\mathfrak{J}'$".

Theorem 11. *Let* $M=\sum_{\alpha\in K}\oplus M_\alpha=\sum_{\beta\in J}\oplus N_\beta$ *be any two direct sum decompositions of a module M in* $\mathfrak{A}$ *into completely indecomposable modules* M_α*'s and* $N\beta$*'s, respectively and assume that* $S_M\Pi\,\mathfrak{J}'=J(S_M)$. *Then, for any subset* K' *of* K, *there exists a one-to-one mapping* φ *of* K' *into J such that* $M=\sum_{\alpha'\in K'}\oplus N_{\varphi(\alpha')}\oplus\sum_{\alpha''\in K-K'}\oplus M_{\alpha''}$ *and* $M_{\alpha'}\approx N_{\varphi(\alpha')}$ *for* $\alpha\in K'$.

Proof. For any subset K' of K, we put $M_0=\sum_{\alpha''\in K-K'}\oplus M_{\alpha''}$. Then, there exists a maximal member M* in the set $\{M_0\oplus\sum_{k\in J_i}\oplus N_{\gamma}\}_{i\in I}$ of locally direct summands of M with each subset J_i of J, by Zorn's lemma. Since M is the only one dense submodule of M by Corollary 2 to Lemma 8, M* is a direct summand of M, say, $M=M^*\oplus M'$ for some submodule M' of M. By Corollary 2 to Lemma 9, M' is in $\mathfrak{A}$ if $M'\neq 0$. And so by [2] there exists some N_β such that $M^*\oplus N_\beta$ is a direct summand of M. This contradiction shows that $M^*=M$. Since $\sum_{\alpha'\in K'}\oplus M_{\alpha'}\approx M/M_0\approx\sum_{\gamma'\in J'}\oplus N_{\gamma'}$ with some subset J' of J, by [2] we can find a one-to-one mapping 99 of K' onto J' such that $M_{\alpha'}\approx N_{\varphi(\alpha')}$ for $\alpha'\in K'$.

KINKI UNIVERSITY

Refesences

[1] G. Azumaya: *On generalized semi-primary rings and Krull-Remak-Schmidt's theorem,* Japan. J. Math. 19 (1947), 525-547.

[2]————: *Correction and supplementaries to my paper concerning Krull-Remak-Schmidt's theorem,* Nagoya Math. Math. J. 1 (1950), 117-124.

[3] M. Harada: *Applications of Factor categories to completely indecomposable modules*, to appear.
[4] ———: *On categories of indecomposable modules* II, Osaka J. Math. 8 (1971), 309–321.
[5] ——— and Y. Sai: *On categories of indecomposable modules* I, ibid. 7 (1970), 323–344.
[6] Y. Sai: *On regular categories*, ibid. 7 (1970), 301–306.