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ON HAKEN'S THEOREM AND ITS EXTENSION

Dedicated to Professor Minoru Nakaoka on his sixtieth birthday

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1. Introduction

It is an interesting problem to investigate how a surface in a 3-manifold M intersects a fixed Heegaard surface in M. In this direction, the first basic work was done by W. Haken [1], and later W. Jaco [2] proved Haken's theorem in [1] in a complete form by using a theory of hierarchy for planar surfaces. In contrast with their works, the main purpose of this paper is to discuss a relationship of 2-sided projective planes in a 3-manifold M and a fixed Heegaard surface in M. In our discussion, a certain property which planar surfaces and Möbius strips with holes have in common plays an important role and then an observation on such a property enables us to prove the following;

Main Theorem 1. *Let M be a closed connected 3-manifold with a fixed Heegaard splitting (M, F) of genus g. Then the following holds;*

(1) *If there exists a 2-sided projective plane P in M, then there exists a 2-sided projective plane P' in M such that $F \cap P'$ is a single circle. In particular, if M is irreducible, then P' is isotopic in M to P.*

(2) *If M contains an incompressible 2-sphere, then there exists an incompressible 2-sphere S^2 in M such that $F \cap S^2$ is a single circle which is not contractible in F.*

It will be noticed that the second assertion of the Main Theorem 1 is the one of Haken's theorem in [1], [2] and so the above theorem includes Haken's theorem and that recently the author proved in [3] that every closed connected 3-manifold, with a Heegaard splitting of genus 2, which admits a 2-sided embedding of the projective plane P^2, is homeomorphic to $P^2 \times S^1$.

Throughout this paper, spaces and maps will be considered in the piecewise-linear category, unless otherwise specified. S^n, D^n denote n-sphere, n-disk, respectively. Closure, interior, boundary are denoted by $\mathrm{cl}(\cdot)$, $\mathrm{int}(\cdot)$, $\partial(\cdot)$, respectively. If X, Y are spaces with $X \supset Y$, then $N(Y, X)$ denotes a regular neighborhood of Y in X.

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2. Intersections of 2-sided closed surfaces and Heegaard surfaces

Let S be a compact surface with a non-empty boundary and α be an arc in S with $\partial S \cap \alpha = \partial\alpha$. Then α is called an essential arc in S if the closure of each connected component of $S-\alpha$ is not a 2-disk. Let α be an essential arc in S. Then α is said to be of type I (resp. of type II) if it joins a single circle (resp. two different circles) in ∂S. Next let $\alpha_1, \cdots, \alpha_n$ be mutually disjoint essential arcs in S. Then the collection of such the arcs is called a complete system of arcs for S if the closure of each connected component of $S-(\alpha_1 \cup \cdots \cup \alpha_n)$ is a 2-disk. Let $\bar{\alpha} = \alpha_1 \cup \cdots \cup \alpha_n$ be a complete system of arcs for S and C be a circle in ∂S. Then C is called a distinguished circle related to $\bar{\alpha}$ if all arcs, which meet, in $\bar{\alpha}$ are of type II.

For the definitions of handlebody of genus g, complete system of meridian-disks, Heegaard splitting, and Heegaard surface we refer to [3]. For the definitions of irreducible 3-manifold and incompressible surface we refer to [2].

Let S be an orientable or non-orientable closed surface in a 3-manifold M and $(M; H_1, H_2)$ be a Heegaard splitting of genus g for M with a Heegaard surface F. Suppose that S meets F transversely in circles $C_1, \cdots, C_m$. Next let us suppose that Δ is a 2-disk in M such that $\Delta \cap S = \alpha$ is an arc in $\partial\Delta$, $\Delta \cap F = \beta$ is an arc $\partial\Delta$, $\partial\alpha = \partial\beta$ and $\alpha \cup \beta = \partial\Delta$. Let $N(\Delta)$ be a regular (product) neighborhood of Δ in M such that $N(\alpha, S)$ is a 2-disk in $\partial N(\Delta)$. Then Δ is contained in one of H_1 and H_2, say H_1 and we may assume that $H_1 \cap cl(\partial N(\Delta) - N(\alpha, S))$ consists of two disjoint 2-disks Δ_1, Δ_2. Let $N^+(\beta) = H_2 \cap cl(\partial N(\Delta) - N(\alpha, S))$ and $N(\beta)$ be a regular neighborhood of β in F such that $\partial N(\beta)$ is a circle in $\partial N(\Delta)$. It will be noticed that $H_2 \cap N(\alpha, S)$ consists of disjoint 2-disks $\delta_1(\beta)$, $\delta_2(\beta)$ such that $N^+(\beta) \cup N(\beta) \cup \delta_1(\beta) \cup \delta_2(\beta)$ is a 2-sphere in H_2. Let $S' = cl(S - N(\alpha, S)) \cup (\Delta_1 \cup \Delta_2 \cup N^+(\beta))$. Evidently S' is obtained from S by an isotopic deformation. According Jaco [2], such an isotopy is called an isotopy of type A at α through Δ along β.

Next let us consider "an inverse isotopy" to an isotopy of type A. Let θ be an isotpy of type A for S at α through Δ along β such that Δ is contained in H_1. Then there exists a 2-disk Δ' in H_2 such that $\Delta' \cap N^+(\beta) = \alpha'$ is an arc in $\partial\Delta'$, $\Delta' \cap N(\beta) = \beta'$ is an arc in $\partial\Delta'$, $\partial\alpha' = \partial\beta'$, and $\alpha' \cup \beta' = \partial\Delta'$. Let θ' be an isotopy of type A at α' through Δ' along β' and set S' equal to the image of S' after the isotopy θ'. It happens no confusion that S' is identified with S. Thus θ' can be thought of as the inverse isotopy to θ.

Let S be a 2-sided orientable or non-orientable incompressible closed connected surface in a 3-manifold M and $(M; H_1, H_2)$ be a Heegaard splitting of genus g for M with a Heegaard surface F. We may assume that S meets F transversely in circles $C_1, \cdots, C_m$. We denote by $c(F \cap S) = m$ the number of connected component of $F \cap S$. From now on, we suppose that each connected component of $H_2 \cap S$ is a 2-disk and let us consider the case when $m \geqq 2$. In later argument, we will verify that if a certain condition is satisfied, then there exists a 2-sided incompressible surface S' in M with $c(F \cap S') \leqq m-1$ such that S' is homeomorphic to S and that $H_2 \cap S'$ consists of disjoint 2-disks. Let $\bar{D} = D_1 \cup \cdots \cup D_g$ be a complete system of meridian-disks of H_1, $\bar{D}' = D_1' \cup \cdots \cup D_m' = H_2 \cap S$ with $C_i = \partial D_i' (i=1, \cdots, m)$, and $Q = H_1 \cap S$. Then Q is a compact connected surface with $\partial Q = C_1 \cup \cdots \cup C_m$. Here Q may not be incompressible in H_1. However if it is not incompressible, we can compress Q until it becomes incompressible in H_1. This process preserves the homeomorphism type of S because of S being incompressible in M but may not be performed by isotopic deformation. For this reason, the resulting surface S' in Lemma 1 is not necessarily isotopic in M to the original S. It will be noticed that the above process also preserves the isotopy type of S, if M is irreducible. Next we may assume that each connected component of $Q \cap \bar{D}$ is an arc properly embedded in $\bar{D}$ which is essential in Q and that the closure of each connected component of $Q - (Q \cap \bar{D})$ is a 2-disk. Then we have;

Lemma 1. *If at least one of the circles $C_1, \cdots, C_m$ is a distinguished circle related to $Q \cap \bar{D}$, then there exists a 2-sided incompressible surface S' in M with $c(F \cap S') \leqq m-1$ such that S' is homeomorphic to S and that each connected component of $H_2 \cap S'$ is a 2-disk.*

Proof. We may assume that C_1 is a distinguished circle related to $\bar{\alpha} = Q \cap \bar{D}$. Let α be an arc in $\bar{\alpha}$ with $C_1 \cap \alpha \neq \emptyset$ and we may suppose that α is contained in D_1. If α is innermost in D_1, then the lemma holds because α is

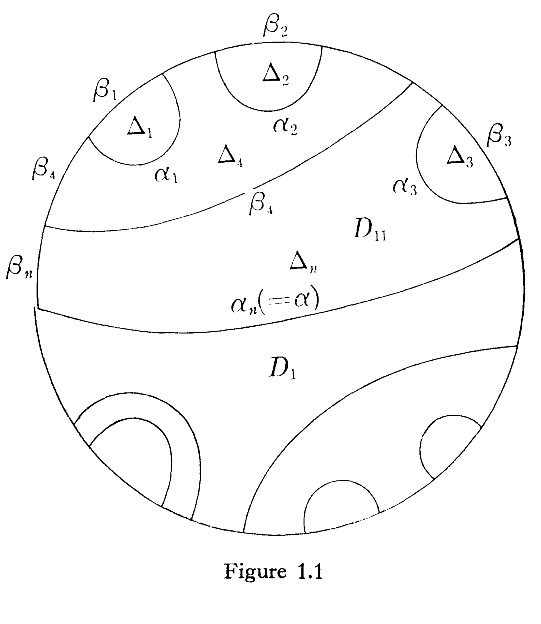

Figure 1.1

of type II. Thus we can assume that α is not innermost in D_1. Let D_{11} be the closure of one of two connected component of $D_1-\alpha$. Then we may assume without loss of generality that $D_{11}-\alpha$ contains nc arcs in $\bar{\alpha}$ which meet C_1, since every arc in $\bar{\alpha}$ which meets C_1 is of type II. Let $\alpha_1, \cdots, \alpha_n$ be all arcs in $Q\cap D_{11}$. See Figure 1.1. Moreover let $\Delta_1, \cdots, \Delta_n$ be 2-disks in D_{11} and $\beta_1, \cdots, \beta_n$ be arcs in $\partial D_{11}-\text{int}(\alpha)$ with $\partial\Delta_i=\alpha_i\cup\beta_i$ $(i=1, \cdots, m)$. See Figure 1.1. Then there exists a sequence of isotopies $\theta_1, \cdots, \theta_n$ of S in M where the first isotopy θ_1 is of type A at α_1 through Δ_1 along β_1, the second θ_2 is of type A at α_2 through Δ_2 along $\beta_2, \cdots$, and the n-th isotopy θ_n is of type A at α_n through θ_n along β_n.

Set S_1 equal tc the image of S after this sequence of isotopies. See Figure 1.2. By the construction of S_1, there exists a homeomorphism Ψ_i from $\delta_1(\beta_i)\times I$

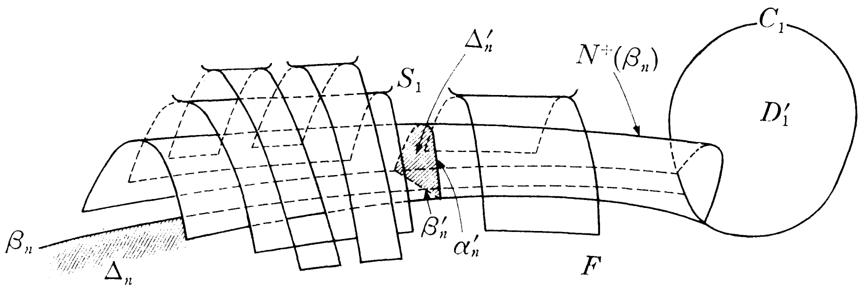

Figure 1.2

into H_2 with $\Psi_i(\delta_1(\beta_i), 0)=\delta_1(\beta_i)$, $\psi_i(\delta_1(\beta_i, 1)=\delta_2(\beta_i)$, and $\Psi_i(\partial(\delta_1(\beta_i)), I)=N^+(\beta_i)\cup N(\beta_i)$ $(i=1, \cdots, n)$ such that $\Psi_n(\delta_1(\beta_n), I)\cap S_1=N^+(\beta_n)$, and that for every $j\neq n$ there exists some integer $k>j$ such that $\Psi_j(\delta_1(\beta_j), I)\cap\Psi_k(\delta_1(\beta_k), I)=\Psi_k(\delta_1(\beta_k), [t_1, t_2])$, $cl(\Psi_j(\delta_1(\beta_j), I)-\Psi_k(\delta_1(\beta_k), [t_1, t_2]))\cap S_1=N^+(\beta_j)\cup(\Psi_k(\partial(\delta_1(\beta_k))\cap N^+(\beta_k), [t_1, t_2])$, and $cl(\Psi_j(\delta_1(\beta_j), I)-\Psi_k(\delta_1(\beta_k), [t_1, t_2]))=N^+(\beta_j)\times I$. Thus there exists a sequence of 2-disks $\Delta'_n, \cdots, \Delta'_1$ in H_2 such that $\Delta'_i\supset\Delta'_n$ $(i=1, \cdots, n-1)$, $\Delta'_i\cap\Delta'_k$ for $i<k$ $(i, k=1, \cdots, n)$ is same Δ'_j $(j>k)$ and that $\Delta'_i\cap N^+(\beta_i)=\alpha'_i$ is an arc in $\partial\Delta'_i$, $\Delta'_i\cap N(\beta_i)=\beta'_i$ is an arc in $\partial\Delta'_i$, $\partial\alpha'_i=\partial\beta'_i$ and $\alpha'_i\cup\beta'_i=\partial\Delta'_i$ $(i=1, \cdots, n)$. See Figure 1.3. Then thcre exists a sequence

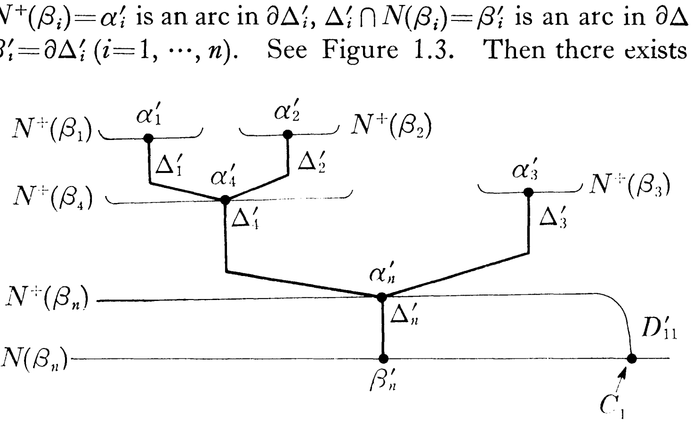

Figure 1.3

of "inverse isotopies" $\theta'_n, \cdots, \theta'_i$ of S_1 in M where the first isotopy θ'_n is of type A at α'_n through Δ'_n along β'_n, the second isotopy θ'_{n-1} is of type A at α'_{n-1} through Δ'_{n-1} along $\beta'_{n-1}, \cdots$, and the n-th isotopy θ'_1 is of type A at α'_1 through Δ'_1 along β'_1. See Figure 1.4. Set S_2 equal to the image of S_1 after this sequence of isotopies. It is easy to see that S_2 is isotopic in M to S with $c(F \cap S_2) = c(F \cap S)$.

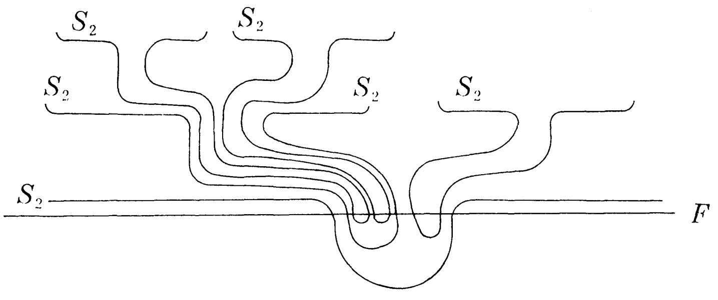

Figure 1.4

Next let us consider a slight modification of these "inverse isotopies". Since S_1 is obtained immediately after the isotopy θ_n and α_n joins C_1 and some C_k $(k \neq 1)$, $(H_2 \cap S_1) - \alpha'_n$ has two connected components and one of them is not disjoint from C_1. Let Δ be the closure of such a connected component. Then Δ is a 2-disk in H_2. Let $\Delta''_i = (\Delta'_i - \Delta'_n) \cup \Delta$ $(i = 1, \cdots, n-1)$. Modify Δ'_i in the part Δ by a slight isotopic deformation, as a result the modified Δ''_i is disjoint from S near Δ $(i = 1, \cdots, n-1)$. Let Δ''_i be the modified Δ''_i $(i = 1, \cdots, n-1)$. Then Δ''_i is a 2-disk in H_2 such that $\Delta''_i \cap S_1 = \alpha'_i = \alpha''_i$ is an arc in $\partial\Delta''_i$, $\Delta''_i \cap F = \beta''_i$ is an arc in $\partial\Delta''_i$, $\partial\alpha''_i = \partial\beta''_i$, and $\alpha''_i \cup \beta''_i = \partial\Delta''_i$ $(i = 1, \cdots, n-1)$. See Figure 1.5. Then there exists a sequence of isotopies of S_1 in M where the first isotopy is of type A at α''_{n-1} through Δ''_{n-1} along β''_{n-1}, the

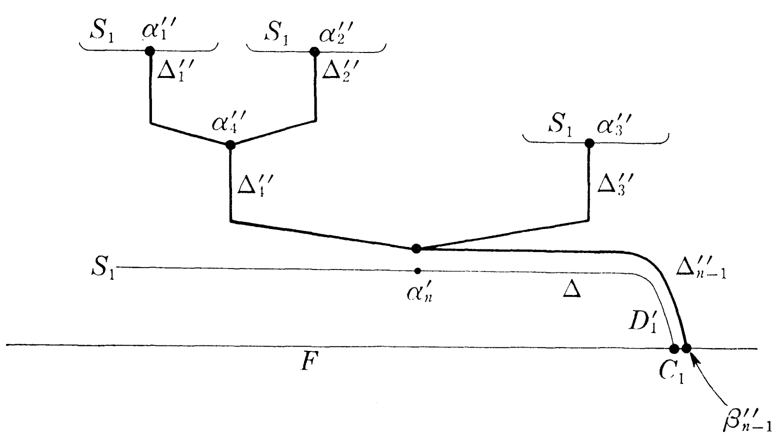

Figure 1.5

second isotopy is of type A at α''_{n-2} through Δ''_{n-2} along $\beta''_{n-2}, \cdots$, and the $(n-1)$-th isotopy is of type A at α''_1 through Δ''_1 along β''_1. See Figure 1.6. Set S' equal to the image of S_1 after this sequence of isotopies. Then S' is a 2-sided

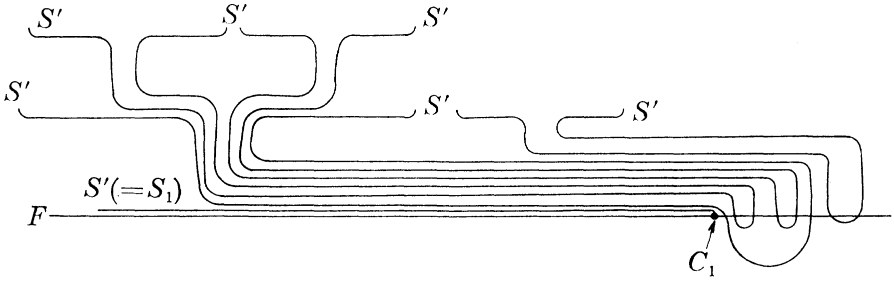

Figure 1.6

incompressible surface in M with $c(F \cap S') \leqq m-1$ such that S' is homeomorphic to S and $H_2 \cap S'$ consists of disjoint 2-disks. This completes the proof of the lemma.

Next let us consider the case when the surface S is either a 2-sphere or a projective plane. In this case, we will prove that if $c(F \cap S) \geqq 2$, then at least one distinguished circle always exists. Let Q be a compact connected surface with a non-empty boundary and $\partial Q = C_1 \cup \cdots \cup C_m$. Let us suppose that $m \geqq 2$. Let $\bar{\alpha} = \alpha_1 \cup \cdots \cup \alpha_n$ be a complete system of arcs for Q. In this context, we have;

Lemma 2. *If Q is either a planar surface or a Möbius strip with holes, then there exists at least one circle C in ∂Q such that every arc in $\bar{\alpha}$ which meets C is of type II.*

Proof. Let us suppose that the lemma is false. That is, we suppose that for every i $(i=1, \cdots, m)$ there exists an arc α'_i of type I in $\bar{\alpha}$ such that it joins two points in C_j. Since Q is either a planar surface or a Möbius strip with holes, we may assume that Q_1 is a connected planar surface with k_1 boundary circles, where Q_1 is one of the connected components of $cl(Q - N(\alpha'_1, S))$ and $k_1 \geqq 2$. Here all circles in ∂Q_1 except one boundary circle can be joined to themselves by the arcs $\alpha'_2, \cdots, \alpha'_m$, which bound no 2-disks in Q_1. If $k_1 = 2$, then Q_1 is an annulus and does not contain such an arc. Thus we have that $k_1 > 2$. But then there exists a connected planar surface Q_2 in Q_1 with k_2 boundary circles, where $k_1 > k_2 \geqq 2$. Here Q_2 has the similar property to Q_1. Repeating this process, at the final step an annulus Q_m is obtained. And Q_m contains an arc which joins points in one component of ∂Q_m but bounds no 2-disks. But it is impossible. Consequently, the first assumption is false. This completes the proof of the lemma.

It will be noticed that the original idea for the proof of the above lemma lies in the last part in the proof of Lemma 5 in [3].

Main Theorem 1. *Let M be a closed connected 3-manifold with a fixed Heegaard splitting (M, F) of genus g. Then the following holds;*

(1) *If there exists a 2-sided projective plane P in M, then there exists a 2-sided projective plane P' in M such that $F \cap P'$ is a single circle. In particular, if M is irreducible, then P' is isotopic in M to P.*

(2) *If M contains an incompressible 2-sphere, then there exists an incompressible 2-sphere S^2 in M such that $F \cap S^2$ is a single circle which is not contractible in F.*

Proof. To avoid complexity, we will prove only the first part of the theorem. Let M be a closed connected non-orientable 3-manifold with a Heegaard splitting $(M; H_1, H_2)$ of genus g with a Heegaard surface F and P be a 2-sided projective plane in M. Then there exists a 2-sided projective plane P_1 in M such that each component of $P_1 \cap H_2$ is a 2-disk. Let $Q = P_1 \cap H_1$ and $\bar{D} = D_1 \cup \cdots \cup D_g$ be a complete system of meridian-disks of H_1. We may assume that Q is incompressible in H_1. It will be noticed that by this assumption the desired surface P' may not be isotopic in M to P except the case when M is irreducible. Moreover, we may assume without loss of generality that $Q \cap \bar{D}$ is a complete system of arcs for Q. In this context, we will prove the theorem by induction of $c(F \cap P_1) = m$.

If $m = 1$, then the assertion (1) in the theorem is valid. Thus we suppose that $m \geqq 2$. Then by Lemma 2 and Lemma 1, there exists a 2-sided projective plane P_2 in M with $c(F \cap P_2) \leqq m-1$ such that each connected component of $H_2 \cap P_2$ is a 2-disk. By induction, there exists a 2-sided projective plane P' in M such that $c(F \cap P') = 1$. This completes the proof of the assertion (1).

3. Z_2-equivariant Haken's theorem

Let M' be a closed connected orientable 3-manifold with a Heegaard splitting $(M'; H_1', H_2')$ of genus g. Let us suppose that M' admits an orientation-reversing fixed point free involution τ with $\tau(H_i) = H_i$ $(i = 1, 2)$ and contains an incompressible 2-sphere. Then by Tollefson [4], there exists an incompressible 2-sphere S^2 in M' such that either $\tau(S^2) = S^2$ or $\tau(S^2) \cap S^2 = \emptyset$. Let M be the orbit space of M' by τ. Then M is a closed connected non-orientable 3-manifold with the Heegaard splitting $(M; H_1, H_2)$ of genus $1 + ((g-1)/2)$ such that H_i' is the orientable double covering of H_i $(i = 1, 2)$. If $\tau(S^2) \cap S^2 = \emptyset$, then S_1^2 is an incompressible 2-sphere in M, where S_1^2 is the orbit space of S^2 by τ. Then by Main Theorem 1 or Haken's theorem in [1], [2], there exists an incompressible 2-sphere S_2^2 in M such that $F \cap S_2^2$ is a single circle which is not contractible in F. Hence there exists an incompressible 2-sphere S' in

M' such that $F' \cap S'$ is a single circle which is not contractible in F' and that $\tau(S') \cap S' = \emptyset$.

Let us consider the case when $\tau(S^2) = S^2$. Let P be the orbit space of S^2 by τ. Then P is a 2-sided projective plane in M, since τ is orientation-reversing. Then by Main Theorem 1, there exists a 2-sided projective plane P' in M such that $F \cap P'$ is a single circle. Hence there exists an incompressible 2-sphere S'' in M' such that $F' \cap S''$ is two circles which are not contractible in F and that $\tau(S'') = S''$. Thus we have;

Theorem 2 (Z_2-equivariant Haken's theorem). *Let M be a closed connected orientable 3-manifold with a fixed Heegaard splitting $(M; H_1, H_2)$ of genus g. If M contains an incompressible 2-sphere and admits an orientation-reversing fixed point free involution τ with $\tau(H_i) = H_i$ $(i = 1, 2)$, then there exists an incompressible 2-sphere S^2 in M such that either $\tau(S^2) \cap S^2 = \emptyset$ and $F \cap S^2$ is a single circle which is not contractible in F or $\tau(S^2) = S^2$ and $F \cap S^2$ consists of two circle which are not contractible in F.*

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