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Author(s)	Ikegami, Teruo
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## A NOTE ON AXIOMATIC DIRICHLET PROBLEM

TERUO IKEGAMI

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### 1. Introduction

Axiomatic Dirichlet problem was first discussed by M. Brelot in connection with a metrizable compactification of Green space  $\Omega$  and a positive harmonic function  $h$  in  $\Omega$ . In his paper [1] the theory was developed under the assumption  $\mathcal{R}_h$ , that is, *all bounded continuous functions on the boundary are  $h$ -resolutive*. In our present paper we call a compactification with this property  *$h$ -resolutive*.

This axiomatic treatment of Dirichlet problem yields some complicated situations. For instance, Brelot gave many definitions for the regularity of boundary points, such as strongly  $h$ -regular,  $h$ -regular, weakly  $h$ -regular. A strongly  $h$ -regular boundary point is  $h$ -regular and weakly  $h$ -regular, but an  $h$ -regular boundary point is not weakly  $h$ -regular in general. It has been asked by M. Brelot [1] and L. Naïm [4] whether the complementary set of all  $h$ -regular boundary points is of  $h$ -harmonic measure zero ( *$h$ -négligeable*) or not. We can not yet give an answer to this question. However we can prove the following theorem:

**Theorem.** *Let  $\hat{\Omega}$  be an arbitrary metrizable  $h$ -resolutive compactification of Green space  $\Omega$ . Then there exists a metrizable  $h$ -resolutive compactification having  $\hat{\Omega}$  as a quotient space and in which the complementary set of all  $h$ -regular and weakly  $h$ -regular boundary points is of  $h$ -harmonic measure zero.*

As a corollary of this theorem we can construct a family of filters  $\{\mathcal{F}_x\}$  converging in  $\hat{\Omega}$  and satisfying axioms

$A_h$ ) *If  $s$  is subharmonic in  $\Omega$ ,  $s/h$  is bounded from above and  $\limsup_{\mathcal{F}} s/h \leq 0$  for every  $\mathcal{F}$  in  $\{\mathcal{F}_x\}$ , then  $s \leq 0$ .*

$B_h'$ ) *Every filter in  $\{\mathcal{F}_x\}$  is  $h$ -regular and weakly  $h$ -regular, where the latter is weaker than that of Brelot-Choquet [2].*

### 2. Preliminaries

Let  $\Omega$  be a Green space in the sense of Brelot-Choquet [2]. For a real valued function  $f$  defined in  $\Omega$  we shall define a family  $\bar{W}_f(\underline{W}_f)$  of superharmonic (subharmonic) functions  $s$  such that  $s \geq f$  ( $s \leq f$ ) on  $\Omega - K$ , where  $K$  is a compact

set depending on  $s$  in general. If  $\bar{W}_f (W_f)$  is not empty its lower (upper) envelope will be denoted by  $\bar{d}_f (d_f)$ .  $\bar{d}_f$  and  $d_f$  are harmonic and  $d_f \leq \bar{d}_f$ . When  $d_f = \bar{d}_f$  they are denoted by  $d_f$  simply.

Throughout this paper we shall take a positive harmonic function  $h$  in  $\Omega$  and fix it.

**DEFINITION 1.** A function  $f$  defined in  $\Omega$  is  *$h$ -harmonizable* if the following conditions are satisfied:

- 1) there exists a superharmonic function  $s$  such that  $|fh| \leq s$ ,
- 2)  $d_{fh} = \bar{d}_{fh}$

If  $f$  is  $h$ -harmonizable and  $d_{fh} = 0$  then  $f$  is termed an  *$h$ -Wiener potential*, and the class of all  $h$ -Wiener potentials is denoted by  $W_{0,h}^{(1)}$ .

**Proposition 2.1.** Every  $f \in W_{0,h}$  has a potential  $p$  such that  $|fh| \leq p$ .

Let  $\hat{\Omega}$  be a compactification of  $\Omega$ , that is  $\hat{\Omega}$  is compact and contains  $\Omega$  as an everywhere dense subspace. Set  $\Delta = \hat{\Omega} - \Omega$ . In this paper it is always assumed that  $\hat{\Omega}$  is *metrizable*.

For an arbitrary real valued function  $\varphi$  on  $\Delta$ , which is permitted to take the values  $\pm\infty$ ,  $\bar{\mathcal{P}}_{\varphi,h}$  denotes the class of all superharmonic functions  $s$  such that

- a)  $s/h$  is bounded from below,
- b)  $\lim_{a \rightarrow x} s(a)/h(a) \geq \varphi(x)$  for every  $x \in \Delta$ .

Similarly we define the class of subharmonic functions  $\underline{\mathcal{P}}_{\varphi,h}$ . When  $\bar{\mathcal{P}}_{\varphi,h}$ ,  $\underline{\mathcal{P}}_{\varphi,h}$  are not empty, we set

$$\begin{aligned} \bar{\mathcal{D}}_{\varphi,h} &= \inf \{s; s \in \bar{\mathcal{P}}_{\varphi,h}\}, \\ \underline{\mathcal{D}}_{\varphi,h} &= \sup \{s; s \in \underline{\mathcal{P}}_{\varphi,h}\}. \end{aligned}$$

$\underline{\mathcal{D}}_{\varphi,h}$  and  $\bar{\mathcal{D}}_{\varphi,h}$  are both harmonic and  $\underline{\mathcal{D}}_{\varphi,h} \leq \bar{\mathcal{D}}_{\varphi,h}$ . When  $\underline{\mathcal{D}}_{\varphi,h} = \bar{\mathcal{D}}_{\varphi,h}$ ,  $\varphi$  is called  *$h$ -resolutive* and the envelopes are denoted by  $\mathcal{D}_{\varphi,h}$  simply.

**DEFINITION 2.** If all bounded continuous functions on  $\Delta$  are  $h$ -resolutive,  $\hat{\Omega}$  is called an  *$h$ -resolutive compactification* of  $\Omega$ .

In the sequel,  $\hat{\Omega}$  always denotes a metrizable  $h$ -resolutive compactification of  $\Omega$ . Then, for  $a \in \Omega$  there exists a Radon measure  $\omega_h^a$  on  $\Delta$  such that

$$\mathcal{D}_{\varphi,h} = \int \varphi d\omega_h^a \quad \text{for every } \varphi \in C(\Delta)^{(2)}.$$

$\omega_h^a$  is called an  *$h$ -harmonic measure* (with respect to  $a$ ).

- 1) In the case that  $h=1$  and  $\Omega$  is a hyperbolic Riemann surface, this definition is slightly different from [3].
- 2)  $C(\Delta)$  denotes the family of all bounded continuous functions on  $\Delta$ .

**Proposition 2.2.** *Let  $F$  be bounded and continuous on  $\hat{\Omega}$  and  $\varphi, f$  be its restrictions on  $\Delta$  and on  $\Omega$  respectively, then  $f$  is  $h$ -harmonizable and  $d_{fh} = \mathcal{D}_{\varphi, h}$ .*

**Proposition 2.3.** *In order that an arbitrary compactification  $\bar{\Omega}$  of  $\Omega$  be  $h$ -resolutive, it is necessary and sufficient that for every bounded continuous function  $F$  on  $\bar{\Omega}$ , its restriction on  $\Omega$  is  $h$ -harmonizable.*

DEFINITION 3. For potential  $p$  we set

$$\Gamma_{p, h} = \{x \in \Delta; \lim_{a \rightarrow x} p(a)/h(a) = 0\},$$

$$\Gamma_h = \bigcap_p \Gamma_{p, h}.$$

$\Gamma_h$  is called an  $h$ -harmonic boundary.

$\Gamma_h$  is non-empty and compact.

**Proposition 2.4.** *If  $s$  is subharmonic in  $\Omega$  such that  $s/h$  is bounded from above and  $\overline{\lim}_{a \rightarrow x} s(a)/h(a) \leq 0$  for all  $x \in \Gamma_h$  then  $s \leq 0$ .*

**Proposition 2.5.** *Let  $F$  be a bounded continuous function on  $\hat{\Omega}$ . The restriction of  $F$  on  $\Omega$  is an  $h$ -Wiener potential if and only if  $F$  vanishes on  $\Gamma_h$ .*

**Proposition 2.6.**  $\Gamma_h$  is the carrier of  $h$ -harmonic measure  $\omega_h$ .

In the case that  $h=1$  and  $\Omega$  is a hyperbolic Riemann surface, Constantinescu-Cornea [3] have given these propositions. Proofs of our propositions will be obtained from them with slight modifications.

### 3. $Q$ -compactification of Green space

1. Let  $h$  be a positive harmonic function on Green space  $\Omega$  and  $\hat{\Omega}$  be an arbitrary metrizable,  $h$ -resolutive compactification of  $\Omega$ . Set  $\Delta = \hat{\Omega} - \Omega$ .

For  $F \in C(\hat{\Omega})$ , its restrictions on  $\Omega$  and on  $\Delta$  are denoted by  $F|_{\Omega}$  and  $F|_{\Delta}$  respectively.

We set  $Q_0' = \{F|_{\Omega}; F \in C(\hat{\Omega})\}$ ,  $Q_0'' = \{d_{fh}/h; f \in Q_0'\}$  and

$$Q_0 = Q_0' \cup Q_0'' \cup \left\{ A \frac{\min(G_{a_0}, h)}{h} + B \right\},$$

where  $G_{a_0}$  is a Green function of  $\Omega$  with pole at  $a_0$  and  $A, B$  are constants. The compactification  $\Omega^{Q_0}$  of  $\Omega$  is the one on which all functions of  $Q_0$  are extended continuously and the boundary  $\Delta^{Q_0} = \Omega^{Q_0} - \Omega$  is separated by functions in  $Q_0^{(3)}$ . We have

**Proposition 3.1.**  $\Omega^{Q_0}$  is a metrizable  $h$ -resolutive compactification of  $\Omega$ .

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3) We say functions in  $Q_0$  separate points of  $\Delta^{Q_0}$  if for every pair of distinct points  $x, y$  of  $\Delta^{Q_0}$  there exists a function  $F$  in  $Q_0$  such that  $F(x) \neq F(y)$ .

$\hat{\Omega}$  is a quotient space of  $\Omega^{\mathcal{Q}_0}$ .

To prove this proposition, we require some lemmas.

In  $C(\hat{\Omega})$  we select a countable subfamily  $\{F_k\}$  which is dense in the topology of uniform norm ( $\|F\| = \sup_{a \in \hat{\Omega}} |F(a)|$ ).

If we set  $f_k = F_k|_{\Omega}$ ,  $f_k$  is  $h$ -harmonizable (Prop. 2.2). We form the family of a countable number of functions

$$Q = \{f_k\} \cup \{d_{f_k}/h\} \cup \left\{ \frac{\min(G_{a_0}, h)}{h} \right\},$$

which is a subfamily of  $Q_0$ .

The  $Q$ -compactification  $\Omega^{\mathcal{Q}}$  of  $\Omega$  is compact and contains  $\Omega$  as an everywhere dense subspace. Functions in  $Q$  are extended continuously on  $\Omega^{\mathcal{Q}}$  and separate two distinct points of  $\Delta^{\mathcal{Q}} = \Omega^{\mathcal{Q}} - \Omega$ .

Theory of general topology tells us  $\Omega^{\mathcal{Q}}$  is metrizable (for instance, N. Bourbaki: Topologie générale, Chap. IX, §2).

**Lemma 3.1.** *For every  $F \in C(\hat{\Omega})$ , if we set  $f = F|_{\Omega}$ , then  $f$  and  $d_{f_h}/h$  are extended continuously on  $\Omega^{\mathcal{Q}}$ .*

*Proof.* (i) Case of  $f$ . It will be sufficient to show that for every  $x \in \Delta^{\mathcal{Q}}$  and for every sequence of points  $\{a_n\}$  in  $\Omega$  converging to  $x$  in the topology of  $\Omega^{\mathcal{Q}}$   $\{f(a_n)\}$  has the unique limit. If it were not, there should exist two sequences  $\{a_n\}, \{b_n\}$  in  $\Omega$  such that  $a_n \rightarrow x, b_n \rightarrow x$  (in the topology of  $\Omega^{\mathcal{Q}}$ ) and  $\alpha = \lim_{n \rightarrow \infty} f(a_n) > \lim_{n \rightarrow \infty} f(b_n) = \beta$ .

We take a positive number  $\varepsilon = (\alpha - \beta)/4$ . For this  $\varepsilon$  and  $F \in C(\hat{\Omega})$  we can find  $F_k$  in our countable family such that

$$\sup_{\hat{\Omega}} |F_k - F| \leq \varepsilon.$$

Then we have

$$\begin{aligned} \alpha &= \lim_{n \rightarrow \infty} f(a_n) \leq \overline{\lim}_{n \rightarrow \infty} f_k(a_n) + \varepsilon, \\ \underline{\lim}_{n \rightarrow \infty} f_k(b_n) - \varepsilon &\leq \lim_{n \rightarrow \infty} f(b_n) = \beta. \end{aligned}$$

where  $f_k = F_k|_{\Omega}$ . Since  $f_k$  is extended continuously on  $\Omega^{\mathcal{Q}}$ ,

$$\alpha - \varepsilon \leq \lim_{n \rightarrow \infty} f_k(a_n) = \lim_{n \rightarrow \infty} f_k(b_n) \leq \beta + \varepsilon,$$

this leads to a contradiction  $4\varepsilon = \alpha - \beta \leq 2\varepsilon$ .

(ii) Case of  $d_{f_h}/h$ . We take  $f_k$  as above. Then we have

$$\frac{d_{f_k h}}{h} - \varepsilon \leq \frac{d_{f_h}}{h} \leq \frac{d_{f_k h}}{h} + \varepsilon$$

and we can proceed quite in the same way as in (i).

**Lemma 3.2.** *Let  $\mathcal{H}$  be a class of all functions  $F'$  each of which is bounded and continuous on  $\Omega^{\mathcal{Q}_0}$  and its restriction on  $\Omega$  is  $h$ -harmonizable. Then  $\mathcal{H}$  is dense in  $C(\Omega^{\mathcal{Q}_0})$  in the topology of uniform norm in  $\Omega^{\mathcal{Q}_0}$ .*

*Proof.* Clearly  $\mathcal{H}$  contains all constant functions and  $\mathcal{H}$  is a linear space. All functions in  $\mathcal{Q}_0$ , are extended continuously on  $\Omega^{\mathcal{Q}_0}$  and these extended functions are contained in  $\mathcal{H}$ , therefore  $\Omega^{\mathcal{Q}_0}$  is separated by functions in  $\mathcal{H}$ . To see  $\mathcal{H}$  is closed under the maximum and minimum operations, that is  $F_1', F_2' \in \mathcal{H}$  implies  $\max(F_1', F_2'), \min(F_1', F_2') \in \mathcal{H}$ , let  $F_1', F_2' \in \mathcal{H}$  and  $f_i = F_i'|_{\Omega}$  ( $i=1,2$ ).  $\min(F_1', F_2')|_{\Omega} = \min(f_1, f_2)$  and  $d_{\min(f_1, f_2)h} = d_{\min(f_1h, f_2h)} = d_{f_1h} \wedge d_{f_2h}$  where  $u \wedge v$  denotes the greatest harmonic function which is dominated by  $u$  and  $v$ . This means  $\min(f_1, f_2)$  is  $h$ -harmonizable. By Stone's theorem<sup>4)</sup>  $\mathcal{H}$  is dense in  $C(\Omega^{\mathcal{Q}_0})$ .

*Proof of Proposition 3.1.* On account of Lemma 3.1 all functions of  $\mathcal{Q}_0$  are extended continuously on  $\Omega^{\mathcal{Q}}$ . Thus  $\Omega^{\mathcal{Q}_0}$  is homeomorphic to  $\Omega^{\mathcal{Q}}$  and therefore  $\Omega^{\mathcal{Q}_0}$  is metrizable. Since  $\hat{\Omega}$  is homeomorphic to  $\Omega^{\mathcal{Q}_0'}$ ,  $\hat{\Omega}$  is a quotient space of  $\Omega^{\mathcal{Q}_0}$ . For arbitrary  $F' \in C(\Omega^{\mathcal{Q}_0})$  and any positive number  $\varepsilon$ , by Lemma 3.2 we can find  $F_0' \in \mathcal{H}$  such that

$$\sup_{\Omega^{\mathcal{Q}_0}} |F' - F_0'| \leq \varepsilon.$$

Setting  $f = F'|_{\Omega}$ ,  $f_0 = F_0'|_{\Omega}$  we have

$$\underline{d}_{f_0h} - \varepsilon h \leq \underline{d}_{fh} \leq \bar{d}_{fh} \leq \bar{d}_{f_0h} + \varepsilon h.$$

Since  $f_0$  is  $h$ -harmonizable we get  $0 \leq \bar{d}_{fh} - \underline{d}_{fh} \leq 2\varepsilon h$ .  $f$  is  $h$ -harmonizable, and by Proposition 2.3  $\Omega^{\mathcal{Q}_0}$  is  $h$ -resolutive.

2. For an arbitrary metrizable  $h$ -resolutive compactification  $\hat{\Omega}$  of  $\Omega$  we have constructed  $\Omega^{\mathcal{Q}_0}$  of the same type which contains  $\hat{\Omega}$  as a quotient space. If we start from  $\Omega^{\mathcal{Q}_0}$  it will be expected that we can arrive at a new larger compactification of the same type, but this is not so, that is

**Proposition 3.2.** *Let  $\Omega^{\mathcal{Q}_0}$  be the compactification of  $\Omega$  constructed in the above paragraph. If we set  $Q_1' = \{f = F|_{\Omega}; F \in C(\Omega^{\mathcal{Q}_0})\}$ ,  $Q_1'' = \left\{ \frac{d_{fh}}{h}; f \in Q_1' \right\}$  and  $Q_1 = Q_1' \cup Q_1''$  the compactification  $\Omega^{\mathcal{Q}_1}$  is homeomorphic to  $\Omega^{\mathcal{Q}_0}$ .*

Before proving this proposition we remark the following:

**Lemma 3.3.** *For every  $f \in Q_1'$ , and for every positive number  $\varepsilon$  there exists  $g \in Q_1''$  such that*

$$\sup_{\Omega} \left| \frac{d_{fh}}{h} - \frac{d_{gh}}{h} \right| \leq \varepsilon.$$

4) Cf. [3], p. 5.

Proof. For arbitrary distinct points  $x_1, x_2$  in  $\Omega^Q_0$  and for any numbers  $\alpha_1, \alpha_2$  there exists a function  $\lambda \in C(\Omega^Q_0)$  which satisfies the following conditions:

- 1)  $\lambda|_{\Omega} \in Q_0$ .
- 2)  $\lambda(x_i) = \alpha_i \quad (i=1,2)$ .

Since continuous extensions of functions in  $Q_0$  separate points of  $\Omega^Q_0$  we can find  $l \in C(\Omega^Q_0)$  with  $l(x_1) \neq l(x_2)$  among these extensions. Thus, either (i)  $l|_{\Omega} = f \in Q_0'$  or (ii)  $l|_{\Omega} = d_{fh}/h$  for some  $f \in Q_0'$  or (iii)  $l|_{\Omega} = A \frac{\min(G_{a_0}, h)}{h} + B$ . In cases (i) and (iii) we have

$$\lambda(x) = \frac{\alpha_1 - \alpha_2}{l(x_1) - l(x_2)} l(x) - \frac{\alpha_1 l(x_2) - \alpha_2 l(x_1)}{l(x_1) - l(x_2)},$$

in the case (ii) we take, as  $\lambda$ , the continuous extension on  $\Omega^Q_0$  of  $d_{f_0h}/h$ , where

$$f_0 = \frac{\alpha_1 - \alpha_2}{l(x_1) - l(x_2)} f - \frac{\alpha_1 l(x_2) - \alpha_2 l(x_1)}{l(x_1) - l(x_2)} \in Q_0'.$$

Let  $F \in C(\Omega^Q_0)$ ,  $f = F|_{\Omega}$ ,  $\varepsilon > 0$ . For arbitrary  $x, y \in \Omega^Q_0$  we can take  $\lambda_{xy} \in C(\Omega^Q_0)$  satisfying the following:

- 1)  $\lambda_{xy}|_{\Omega} \in Q_0$ .
- 2)  $\lambda_{xy}(x) = F(x), \quad \lambda_{xy}(y) = F(y)$ .

$U_{xy} = \{z \in \Omega^Q_0; \lambda_{xy}(z) < F(z) + \varepsilon\}$  is open and contains  $x, y$ . From an open covering  $\{U_{xy}; y \in \Omega^Q_0\}$  of  $\Omega^Q_0$  we select a finite subcovering  $\{U_{xy_j}; j=1, 2, \dots, n\}$ . Set

$$u_x = \min_{1 \leq j \leq n} \lambda_{xy_j},$$

where  $\lambda_{xy_j}$  is a function corresponding to  $U_{xy_j}$  ( $j=1, 2, \dots, n$ ).  $u_x < F + \varepsilon$  on  $\Omega^Q_0$  and  $u_x(x) = F(x)$ . Then, there exists a function  $g_0$  of  $Q_0'$  such that  $d_{u_x h} = d_{g_0 h}$ .

In fact, let  $\lambda_{xy_j}|_{\Omega}$  be  $f_1, f_2, \dots, f_k; \frac{d_{f_{k+1}h}}{h}, \frac{d_{f_{k+2}h}}{h}, \dots, \frac{d_{f_{k+l}h}}{h}; A_{k+l+1} \frac{\min(G_{a_0}, h)}{h} + B_{k+l+1}, \dots, A_n \frac{\min(G_{a_0}, h)}{h} + B_n$ , then

$$\begin{aligned} d_{u_x h} &= d(\min_{1 \leq j \leq n} \lambda_{xy_j} h) = \bigwedge_{j=1}^n d_{\lambda_{xy_j} h} = (\bigwedge_{j=1}^k d_{f_j h}) \wedge (\bigwedge_{j=k+1}^{k+l} d_{f_j h}) \wedge (\min_{k+l+1 \leq j \leq n} B_j) h \\ &= (\bigwedge_{j=1}^{k+l} d_{f_j h}) \wedge (\bigwedge_{j=k+l+1}^n d_{B_j h}) = d_{g_0 h}, \end{aligned}$$

where  $g_0 = \min(\min_{1 \leq j \leq k+l} f_j, \min_{k+l+1 \leq j \leq n} B_j) \in Q_0'$ . Since  $U_x = \{z \in \Omega^Q_0; u_x(z) > F(z) - \varepsilon\}$  is open and contains  $x$ , we can form a finite subcovering  $\{U_{x_j}; j=1, 2, \dots, l'\}$  of  $\Omega^Q_0$ . Setting  $v = \max_{1 \leq j \leq l'} u_{x_j}$ , where  $u_{x_j}$  is a function corresponding to  $U_{x_j}$  ( $j=1, 2, \dots, l'$ ), we have  $|u - F| < \varepsilon$  on  $\Omega^Q_0$  and as above we can find  $g \in Q_0'$  such that  $d_{vh} = d_{gh}$ .

$$d_{vh} - \varepsilon h \leq d_{fh} \leq d_{vh} + \varepsilon h$$

means  $\left| \frac{d_{fh} - d_{gh}}{h} \right| = \left| \frac{d_{fh} - d_{vh}}{h} \right| \leq \varepsilon$ , *q.e.d.*

Proof of Proposition 3.2. Since all functions of  $Q_0''$  are extended continuously on  $\Omega^{\mathcal{Q}_0}$  we have  $Q_0'' \subset Q_1'$ . The closure  $\overline{Q_0''}$  of  $Q_0''$  in the topology of uniform norm ( $\|f\| = \sup_{\Omega} |f|$ ) is contained in  $Q_1'$ . On the other hand, above lemma tells us  $Q_1'' \subset \overline{Q_0''}$ . We have thus  $Q_1'' = \overline{Q_0''} \subset Q_1'$  which implies  $Q_1 = Q_1'$  and the proposition follows.

#### 4. Regularity of boundary points

Let  $\hat{\Omega}$  be an arbitrary metrizable  $h$ -resolutive compactification of  $\Omega$ , and  $\Delta = \hat{\Omega} - \Omega$ .

In this section we give a proof of theorem stated in the introduction. For definiteness we recall the definition of regularity of boundary points.

DEFINITION 4. A filter  $\mathcal{F}$  on  $\Omega$  converging to a boundary point  $x$  is called *strongly  $h$ -regular* if there exists an open neighbourhood  $\delta$  of  $x$  and a positive superharmonic function  $s$  in  $\delta \cap \Omega$  such that  $s/h \xrightarrow{\mathcal{F}} 0$  and the infimum of  $s/h$  outside of arbitrary open neighbourhood of  $x$  contained in  $\delta$  is positive.

A filter  $\mathcal{F}$  on  $\Omega$  converging to a boundary point  $x$  is called  *$h$ -regular* if for every bounded continuous function  $\varphi$  on  $\Delta$  we have  $\frac{1}{h} \mathcal{D}_{\varphi, h} \xrightarrow{\mathcal{F}} \varphi(x)$ .

A filter  $\mathcal{F}$  on  $\Omega$  converging to a boundary point  $x$  is called *weakly  $h$ -regular* if there exists a positive superharmonic function  $s$  such that  $s/h \xrightarrow{\mathcal{F}} 0$ .

A boundary point  $x$  is called *strongly  $h$ -regular*,  *$h$ -regular* and *weakly  $h$ -regular* according as the filter formed by the trace on  $\Omega$  of filter of neighbourhoods of  $x$  is strongly  $h$ -regular,  $h$ -regular and weakly  $h$ -regular respectively.

It is known that a strongly  $h$ -regular filter is  $h$ -regular and weakly  $h$ -regular. However an example of one-point compactification of  $\Omega$  shows us that an  $h$ -regular filter is not necessarily weakly  $h$ -regular.

Since by Proposition 2.6  $\Delta^{\mathcal{Q}_0} - \Gamma_h^{\mathcal{Q}_0}$  is of  $h$ -harmonic measure zero, to prove our theorem it will be sufficient to show the following proposition:

**Proposition 4.1.** *Let  $\Omega^{\mathcal{Q}_0}$  be the compactification constructed in the preceding section and let  $\Delta^{\mathcal{Q}_0} = \Omega^{\mathcal{Q}_0} - \Omega$ . Every point of the  $h$ -harmonic boundary  $\Gamma_h^{\mathcal{Q}_0}$  of  $\Delta^{\mathcal{Q}_0}$  is  $h$ -regular and weakly  $h$ -regular.*

Proof. We use the same notations as in the preceding section. Let  $x \in \Gamma_h^{\mathcal{Q}_0}$  and  $\varphi \in C(\Delta^{\mathcal{Q}_0})$ . Let  $F$  be a bounded continuous extension of  $\varphi$  on  $\Omega^{\mathcal{Q}_0}$  and set  $f = F|_{\Omega}$ .



Since  $f \in Q_1'$ , and  $d_{fh}/h \in Q_1''$ ,  $f$  and  $d_{fh}/h$  can be extended continuously onto  $\Omega^{Q_1}$ . By Proposition 3.2  $\Omega^{Q_1}$  is homeomorphic to  $\Omega^{Q_0}$ , therefore  $f$  and  $d_{fh}/h$  are extended continuously onto  $\Omega^{Q_0}$ . This is also true for  $g=f-d_{fh}/h$ . Since  $d_{gh}=0$ ,  $g$  is an  $h$ -Wiener potential and by Proposition 2.1 there exists potential  $p$  such that  $|gh| \leq p$ . For an arbitrary sequence of points  $\{a_n\}$  in  $\Omega$  converging to  $x$  we have

$$\lim_{n \rightarrow \infty} |g(a_n)| \leq \lim_{n \rightarrow \infty} \frac{p(a_n)}{h(a_n)} = 0.$$

Hence

$$\lim_{n \rightarrow \infty} \left[ f(a_n) - \frac{d_{fh}(a_n)}{h(a_n)} \right] = 0,$$

which means  $\lim_{a \rightarrow x} \frac{D_{\varphi, h}(a)}{h(a)} = \varphi(x)$ . Thus, all points of  $\Gamma_h^{Q_0}$  are  $h$ -regular.

Since  $\min(G_{a_0}, h)/h$  is extended continuously on  $\Omega^{Q_0}$ , this function assumes the value zero on  $\Gamma_h^{Q_0}$ , therefore all points of  $\Gamma_h^{Q_0}$  are weakly  $h$ -regular, *q.e.d.*

If we take at every point  $x \in \Gamma_h^{Q_0}$  the filter formed by the trace on  $\Omega$  of neighbourhoods of  $x$  in  $\Omega^{Q_0}$ , we obtain the family  $\{\mathcal{F}_x\}$  of filters converging in  $\hat{\Omega}$  and satisfying the following axioms:

$A_h$ ) If  $s$  is subharmonic in  $\Omega$ ,  $s/h$  is bounded from above and  $\limsup_{\mathcal{F}} s/h \leq 0$  for every  $\mathcal{F}$  in  $\{\mathcal{F}_x\}$ , then  $s \leq 0$ .

$B_h'$ ) Every filters in  $\{\mathcal{F}_x\}$  is  $h$ -regular and weakly  $h$ -regular.

Indeed,  $A_h$ ) follows from Proposition 2.4 and  $B_h'$ ) is a consequence of the above proposition.

The second axiom  $B_h'$ ) is weaker than the following axiom of Brelot-Choquet [2]:

$B_h$ ) Every filter in  $\{\mathcal{F}_x\}$  is strongly  $h$ -regular.

Thus, we have

**Proposition 4.2.** Let  $\hat{\Omega}$  be an arbitrary metrizable  $h$ -resolutive compactification of  $\Omega$ . Then, there exists a family of filters in  $\Omega$  converging in  $\hat{\Omega}$  and satisfying the axiom  $A_h$ ),  $B_h'$ ).

L. Naïm gave a family of filters satisfying the axiom  $A_h$ ),  $B_h$ ) by using fine neighbourhoods on Martin space. Our filter is quite different from it.

OSAKA CITY UNIVERSITY

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