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1. Introduction

Axiomatic Dirichlet problem was first discussed by M. Brelot in connection with a metrizable compactification of Green space $\Omega$ and a positive harmonic function $h$ in $\Omega$. In his paper [1] the theory was developed under the assumption $G_h$, that is, all bounded continuous functions on the boundary are $h$-resolutive. In our present paper we call a compactification with this property $h$-resolutive.

This axiomatic treatment of Dirichlet problem yields some complicated situations. For instance, Brelot gave many definitions for the regularity of boundary points, such as strongly $h$-regular, $h$-regular, weakly $h$-regular. A strongly $h$-regular boundary point is $h$-regular and weakly $h$-regular, but an $h$-regular boundary point is not weakly $h$-regular in general. It has been asked by M. Brelot [1] and L. Naïm [4] whether the complementary set of all $h$-regular boundary points is of $h$-harmonic measure zero ($h$-négligeable) or not. We can not yet give an answer to this question. However we can prove the following theorem:

**Theorem.** Let $\hat{\Omega}$ be an arbitrary metrizable $h$-resolutive compactification of Green space $\Omega$. Then there exists a metrizable $h$-resolutive compactification having $\hat{\Omega}$ as a quotient space and in which the complementary set of all $h$-regular and weakly $h$-regular boundary points is of $h$-harmonic measure zero.

As a corollary of this theorem we can construct a family of filters $\{\mathcal{F}_x\}$ converging in $\hat{\Omega}$ and satisfying axioms

$A_h$) If $s$ is subharmonic in $\Omega$, $s/h$ is bounded from above and $\limsup_{\mathcal{F}} s/h \leq 0$ for every $\mathcal{F}$ in $\{\mathcal{F}_x\}$, then $s \leq 0$.

$B_h$) Every filter in $\{\mathcal{F}_x\}$ is $h$-regular and weakly $h$-regular, where the latter is weaker than that of Brelot-Choquet [2].

2. Preliminaries

Let $\Omega$ be a Green space in the sense of Brelot-Choquet [2]. For a real valued function $f$ defined in $\Omega$ we shall define a family $\overline{W}_f(W_f)$ of superharmonic (subharmonic) functions $s$ such that $s \geq f$ ($s \leq f$) on $\Omega - K$, where $K$ is a compact
set depending on \( s \) in general. If \( W_f(W_f) \) is not empty its lower (upper) envelope will be denoted by \( \bar{d}_f(d_f) \). \( d_f \) and \( d_f \) are harmonic and \( d_f \leq \bar{d}_f \). When \( d_f = \bar{d}_f \) they are denoted by \( d_f \) simply.

Throughout this paper we shall take a positive harmonic function \( h \) in \( \Omega \) and fix it.

**DEFINITION 1.** A function \( f \) defined in \( \Omega \) is \( h \)-harmonizable if the following conditions are satisfied:
1) there exists a superharmonic function \( s \) such that \( |fh| \leq s \),
2) \( d_{fh} = \bar{d}_{fh} \)

If \( f \) is \( h \)-harmonizable and \( d_{fh} = 0 \) then \( f \) is termed an \( h \)-Wiener potential, and the class of all \( h \)-Wiener potentials is denoted by \( W_{0,h} \).

**Proposition 2.1.** Every \( f \in W_{0,h} \) has a potential \( p \) such that \( |fh| \leq p \).

Let \( \tilde{\Omega} \) be a compactification of \( \Omega \), that is \( \tilde{\Omega} \) is compact and contains \( \Omega \) as an everywhere dense subspace. Set \( \Delta = \tilde{\Omega} - \Omega \). In this paper it is always assumed that \( \tilde{\Omega} \) is metrizable.

For an arbitrary real valued function \( \varphi \) on \( \Delta \), which is permitted to take the values \( \pm \infty \), \( \mathcal{G}_{\varphi,h} \) denotes the class of all superharmonic functions \( s \) such that
a) \( s/h \) is bounded from below,
b) \( \lim_{a \to x} s(a)/h(a) \geq \varphi(x) \) for every \( x \in \Delta \).

Similarly we define the class of subharmonic functions \( \mathcal{L}_{\varphi,h} \). When \( \mathcal{G}_{\varphi,h}, \mathcal{L}_{\varphi,h} \) are not empty, we set
\[
\mathcal{G}_{\varphi,h} = \inf \{ s ; s \in \mathcal{G}_{\varphi,h} \}, \\
\mathcal{L}_{\varphi,h} = \sup \{ s ; s \in \mathcal{L}_{\varphi,h} \}.
\]
\( \mathcal{G}_{\varphi,h} \) and \( \mathcal{L}_{\varphi,h} \) are both harmonic and \( \mathcal{G}_{\varphi,h} \leq \mathcal{L}_{\varphi,h} \). When \( \mathcal{G}_{\varphi,h} = \mathcal{L}_{\varphi,h} \), \( \varphi \) is called \( h \)-resolutive and the envelopes are denoted by \( \mathcal{D}_{\varphi,h} \) simply.

**DEFINITION 2.** If all bounded continuous functions on \( \Delta \) are \( h \)-resolutive, \( \tilde{\Omega} \) is called an \( h \)-resolutive compactification of \( \Omega \).

In the sequel, \( \tilde{\Omega} \) always denotes a metrizable \( h \)-resolutive compactification of \( \Omega \). Then, for \( a \in \Omega \) there exists a Radon measure \( \omega_a^\varphi \) on \( \Delta \) such that
\[
\mathcal{D}_{\varphi,h} = \int \varphi d\omega_a^\varphi \quad \text{for every } \varphi \in C(\Delta)^{\mathbb{R}}.
\]
\( \omega_a^\varphi \) is called an \( h \)-harmonic measure (with respect to \( a \)).

---

1) In the case that \( h=1 \) and \( \Omega \) is a hyperbolic Riemann surface, this definition is slightly different from [3].

2) \( C(\Delta) \) denotes the family of all bounded continuous functions on \( \Delta \).
**Proposition 2.2.** Let \( F \) be bounded and continuous on \( \hat{\Omega} \) and \( \varphi, f \) be its restrictions on \( \Delta \) and on \( \Omega \) respectively, then \( f \) is \( h \)-harmonizable and \( d_{fh} = \Omega_{\varphi,h} \).

**Proposition 2.3.** In order that an arbitrary compactification \( \hat{\Omega} \) of \( \Omega \) be \( h \)-resolutive, it is necessary and sufficient that for every bounded continuous function \( F \) on \( \Omega \), its restriction on \( \Omega \) is \( h \)-harmonizable.

**Definition 3.** For potential \( p \) we set

\[
\Gamma_{p,h} = \{x \in \Delta; \lim_{a \to x} p(a)/h(a) = 0\},
\]

\( \Gamma_h = \bigcap_{p} \Gamma_{p,h} \).

\( \Gamma_h \) is called an \( h \)-harmonic boundary.

\( \Gamma_h \) is non-empty and compact.

**Proposition 2.4.** If \( s \) is subharmonic in \( \Omega \) such that \( s/h \) is bounded from above and \( \lim_{x \to \Gamma_h} s(a)/h(a) \leq 0 \) for all \( x \in \Gamma_h \) then \( s \leq 0 \).

**Proposition 2.5.** Let \( F \) be a bounded continuous function on \( \hat{\Omega} \). The restriction of \( F \) on \( \Omega \) is an \( h \)-Wiener potential if and only if \( F \) vanishes on \( \Gamma_h \).

**Proposition 2.6.** \( \Gamma_h \) is the carrier of \( h \)-harmonic measure \( \omega_h \).

In the case that \( h=1 \) and \( \Omega \) is a hyperbolic Riemann surface, Constantinescu-Cornea [3] have given these propositions. Proofs of our propositions will be obtained from them with slight modifications.

### 3. \( Q \)-compactification of Green space

1. Let \( h \) be a positive harmonic function on Green space \( \Omega \) and \( \hat{\Omega} \) be an arbitrary metrizable, \( h \)-resolutive compactification of \( \Omega \). Set \( \Delta = \hat{\Omega} - \Omega \).

For \( F \in C(\hat{\Omega}) \), its restrictions on \( \Omega \) and on \( \Delta \) are denoted by \( F|_{\Omega} \) and \( F|_{\Delta} \) respectively.

We set \( Q_0' = \{F|_{\Omega}; F \in C(\hat{\Omega})\}, \quad Q_0'' = \{d_{fh}/h; f \in Q_0'\} \) and

\[
Q_0 = Q_0' \cup Q_0'' \cup \left\{ A \frac{\min (G_{a_0} h)}{h} + B \right\},
\]

where \( G_{a_0} \) is a Green function of \( \Omega \) with pole at \( a_0 \) and \( A, B \) are constants. The compactification \( \Omega^{Q_0} \) of \( \Omega \) is the one on which all functions of \( Q_0 \) are extended continuously and the boundary \( \Delta^{Q_0} = \Omega^{Q_0} - \Omega \) is separated by functions in \( Q_0 \). We have

**Proposition 3.1.** \( \Omega^{Q_0} \) is a metrizable \( h \)-resolutive compactification of \( \Omega \).

---

3) We say functions in \( Q_0 \) separate points of \( \Delta^{Q_0} \) if for every pair of distinct points \( x, y \) of \( \Delta^{Q_0} \) there exists a function \( F \) in \( Q_0 \) such that \( F(x) \neq F(y) \).
is a quotient space of $\Omega^o$.

To prove this proposition, we require some lemmas.

In $C(\hat{\Omega})$ we select a countable subfamily $\{F_k\}$ which is dense in the topology of uniform norm ($\|F\| = \sup_{\alpha \in \hat{\Omega}} |F(\alpha)|$).

If we set $f_k=F_k|_{\Omega}$, $f_k$ is $h$-harmonizable (Prop. 2.2). We form the family of a countable number of functions

$$Q = \{f_k\} \cup \{d_{f_k|h}\} \cup \left\{\frac{\min \{G_{a_0^h}|h\}}{h}\right\},$$

which is a subfamily of $Q_o$.

The $Q$-compactification $\Omega^Q$ of $\Omega$ is compact and contains $\Omega$ as an everywhere dense subspace. Functions in $Q$ are extended continuously on $\Omega^Q$ and separate two distinct points of $\Delta^Q=\Omega^Q-\Omega$.

Theory of general topology tells us $\Omega^Q$ is metrizable (for instance, N. Bourbaki: Topologie générale, Chap. IX, §2).

**Lemma 3.1.** For every $F \in C(\hat{\Omega})$, if we set $f=F|_{\Omega}$, then $f$ and $d_{f|h}$ are extended continuously on $\Omega^Q$.

Proof. (i) Case of $f$. It will be sufficient to show that for every $x \in \Delta^Q$ and for every sequence of points $\{a_n\}$ in $\Omega$ converging to $x$ in the topology of $\Omega^Q$ $\{f(a_n)\}$ has the unique limit. If it were not, there should exist two sequences $\{a_n\}$, $\{b_n\}$ in $\Omega$ such that $a_n \rightarrow x$, $b_n \rightarrow x$ (in the topology of $\Omega^Q$) and $\alpha = \lim\limits_{n \to \infty} f(a_n) > \lim\limits_{n \to \infty} f(b_n) = \beta$.

We take a positive number $\varepsilon=(\alpha-\beta)/4$. For this $\varepsilon$ and $F \in C(\hat{\Omega})$ we can find $F_k$ in our countable family such that

$$\sup_{\hat{\Omega}} |F_k-F| \leq \varepsilon.$$ 

Then we have

$$\alpha = \lim\limits_{n \to \infty} f(a_n) \leq \lim\limits_{n \to \infty} f_k(a_n) + \varepsilon, \quad \lim\limits_{n \to \infty} f_k(b_n) - \varepsilon \leq \lim\limits_{n \to \infty} f(b_n) = \beta.$$ 

where $f_k=F_k|_{\Omega}$. Since $f_k$ is extended continuously on $\Omega^Q$,

$$\alpha - \varepsilon \leq \lim\limits_{n \to \infty} f_k(a_n) = \lim\limits_{n \to \infty} f_k(b_n) \leq \beta + \varepsilon,$$

this leads to a contradiction $4\varepsilon = \alpha - \beta \leq 2\varepsilon$.

(ii) Case of $d_{f|h}$. We take $f_k$ as above. Then we have

$$\frac{d_{f|h}}{h} - \varepsilon \leq \frac{d_{f|h}}{h} \leq \frac{d_{f|h}}{h} + \varepsilon$$

and we can proceed quite in the same way as in (i).
Lemma 3.2. Let $\mathcal{H}$ be a class of all functions $F'$ each of which is bounded and continuous on $\Omega^{Q_0}$ and its restriction on $\Omega$ is $h$-harmonizable. Then $\mathcal{H}$ is dense in $C(\Omega^{Q_0})$ in the topology of uniform norm in $\Omega^{Q_0}$.

Proof. Clearly $\mathcal{H}$ contains all constant functions and $\mathcal{H}$ is a linear space. All functions in $Q_o$ are extended continuously on $\Omega^{Q_0}$ and these extended functions are contained in $\mathcal{H}$, therefore $\Omega^{Q_0}$ is separated by functions in $\mathcal{H}$. To see $\mathcal{H}$ is closed under the maximum and minimum operations, that is $F'_1, F'_2 \in \mathcal{H}$ implies $\max(F'_1, F'_2), \min(F'_1, F'_2) \in \mathcal{H}$, let $F'_1, F'_2 \in \mathcal{H}$ and $f_i = F'_i|_{\Omega}$ ($i = 1, 2$). $\min(F'_1, F'_2)|_{\Omega} = \min(f_1, f_2)$ and $d_{\min}(f_1, f_2) = d_{\min}(f'_1, f'_2) = d_{f_1h} \wedge d_{f_2h}$, where $u \wedge v$ denotes the greatest harmonic function which is dominated by $u$ and $v$. This means $\min(f_1, f_2)$ is $h$-harmonizable. By Stone’s theorem $\mathcal{H}$ is dense in $C(\Omega^{Q_0})$.

Proof of Proposition 3.1. On account of Lemma 3.1 all functions of $Q_0$ are extended continuously on $\Omega^{Q_0}$. Thus $\Omega^{Q_0}$ is homeomorphic to $\Omega^{Q}$ and therefore $\Omega^{Q_0}$ is metrizable. Since $\Omega$ is homeomorphic to $\Omega^{Q_0}$, $\Omega$ is a quotient space of $\Omega^{Q_0}$. For arbitrary $F' \in C(\Omega^{Q_0})$ and any positive number $\varepsilon$, by Lemma 3.2 we can find $F'_0 \in \mathcal{H}$ such that

$$\sup_{\Omega^{Q_0}} |F' - F'_0| \leq \varepsilon.$$ 

Setting $f = F'|_{\Omega}$, $f_0 = F'_0|_{\Omega}$ we have

$$d_{f_0h} - \varepsilon h \leq d_{f_0h} \leq d_{f_0h} \leq d_{f_0h} + \varepsilon h.$$ 

Since $f_0$ is $h$-harmonizable we get $0 \leq d_{f_0h} - d_{f_0h} \leq 2\varepsilon h$. $f$ is $h$-harmonizable, and by Proposition 2.3 $\Omega^{Q_0}$ is $h$-resolutive.

2. For an arbitrary metrizable $h$-resolutive compactification $\Omega$ of $\Omega$ we have constructed $\Omega^{Q_0}$ of the same type which contains $\Omega$ as a quotient space. If we start from $\Omega^{Q_0}$ it will be expected that we can arrive at a new larger compactification of the same type, but this is not so, that is

Proposition 3.2. Let $\Omega^{Q_0}$ be the compactification of $\Omega$ constructed in the above paragraph. If we set $Q_i' = \{f = F|_{\Omega}; F \in C(\Omega^{Q_0})\}$, $Q_i'' = \{d_{f_0h}/h; f \in Q_i'\}$ and $Q_i = Q_i' \cup Q_i''$ the compactification $\Omega^{Q_i}$ is homeomorphic to $\Omega^{Q_0}$.

Before proving this proposition we remark the following:

Lemma 3.3. For every $f \in Q_i'$, and for every positive number $\varepsilon$ there exists $g \in Q_i''$ such that

$$\sup_{\Omega} \left| \frac{d_{f_0h}}{h} - \frac{d_{g}h}{h} \right| \leq \varepsilon.$$ 

4) Cf. [3], p. 5.
Proof. For arbitrary distinct points \( x_1, x_2 \in \Omega^0 \) and for any numbers \( \alpha_1, \alpha_2 \) there exists a function \( \lambda \in C(\Omega^0) \) which satisfies the following conditions:

1) \( \lambda \big|_\Omega \in Q_0^\varepsilon \).
2) \( \lambda(x_i) = \alpha_i \) \( (i=1,2) \).

Since continuous extensions of functions in \( Q_0 \) separate points of \( \Omega^0 \), we can find \( l \in C(\Omega^0) \) with \( l(x_1) \neq l(x_2) \) among these extensions. Thus, either (i) \( l|_\Omega = f \in Q_0' \) or (ii) \( l|_\Omega = d_f/h \) for some \( f \in Q_0' \) or (iii) \( l|_\Omega = A \frac{\min(G_{a_0}, h)}{h} + B \). In cases (i) and (iii) we have

\[
\lambda(x) = \frac{\alpha_1 - \alpha_2}{l(x_1) - l(x_2)} l(x) - \frac{\alpha_1 l(x_2) - \alpha_2 l(x_1)}{l(x_1) - l(x_2)},
\]

in the case (ii) we take, as \( \lambda \), the continuous extension on \( \Omega^0 \) of \( d_f/h \), where

\[
f_0 = \frac{\alpha_1 - \alpha_2}{l(x_1) - l(x_2)} f - \frac{\alpha_1 l(x_2) - \alpha_2 l(x_1)}{l(x_1) - l(x_2)} \in Q_0'.
\]

Let \( F \in C(\Omega^0) \), \( f = F|_\Omega \), \( \varepsilon > 0 \). For arbitrary \( x, y \in \Omega^0 \) we can take \( \lambda_{xy} \in C(\Omega^0) \) satisfying the following:

1) \( \lambda_{xy} \big|_\Omega \in Q_0^\varepsilon \).
2) \( \lambda_{xy}(x) = F(x), \lambda_{xy}(y) = F(y) \).

\( U_{xy} = \{ z \in \Omega^0; \lambda_{xy}(z) < F(z) + \varepsilon \} \) is open and contains \( x, y \). From an open covering \( \{U_{xy}; y \in \Omega^0\} \) of \( \Omega^0 \) we select a finite subcovering \( \{U_{xy}; j=1,2,\ldots,n\} \). Set

\[
u_x = \min_{i \leq j \leq n} \lambda_{xy}^i,
\]

where \( \lambda_{xy} \) is a function corresponding to \( U_{xy} \). Let \( u_x < F + \varepsilon \) on \( \Omega^0 \) and \( u_x(x) = F(x) \). Then, there exists a function \( g_0 \) of \( Q_0' \) such that \( u_{xy} = d_g/h \). In fact, let \( \lambda_{xy} \big|_\Omega \to f_1, f_2, \ldots, f_n; d_{f_1} = d_{f_2} = \ldots = d_{f_n}$

\[
\min(G_{a_0}, h)
\]

\[
A_{n-1} \frac{\min(G_{a_0}, h)}{h} + B, \text{ then}
\]

\[
d_{xy} = d(\min_{1 \leq i \leq n} \lambda_{xy}^i) = \frac{\sum_{j=1}^n d_{\lambda_{xy}^j h}}{h} = \left( \bigwedge_{j=1}^n d_{f_j} \right) \wedge \left( \bigwedge_{j=k+1}^{k+l} d_{f_j} \wedge \min_{1 \leq i \leq n} \right) \wedge \left( \min_{1 \leq i \leq n} \right) B_j h \]

where \( g_0 = \min(\min_{k+1 \leq i \leq n} B_j) \in Q_0' \). Since \( U_x = \{ z \in \Omega^0; u_x(z) > F(z) - \varepsilon \} \) is open and contains \( x \), we can form a finite subcovering \( \{U_{xy}^j; j=1,2,\ldots,l\} \) of \( \Omega^0 \). Setting \( \upsilon = \max u_{xy} \), where \( u_{xy} \) is a function corresponding to \( U_{xy}^j \) \( (j=1,2,\ldots,l) \), we have \( |u - F| < \varepsilon \) on \( \Omega^0 \) and as above we can find \( g \in Q_0' \) such that \( d_{vh} = d_{gh} \).
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\[ d_{eh} - \varepsilon h \leq d_{fh} \leq d_{eh} + \varepsilon h \]

means \( \frac{|d_{eh} - d_{fh}|}{h} \leq \frac{|d_{eh} - d_{fh}|}{h} \leq \varepsilon \), q.e.d.

Proof of Proposition 3.2. Since all functions of \( Q_0'' \) are extended continuously on \( \Omega^0 \) we have \( Q_0'' \subset Q_1' \). The closure \( \overline{Q_0''} \) of \( Q_0'' \) in the topology of uniform norm (\( ||f||=\sup \{ |f(\xi)|\} \)) is contained in \( Q_1' \). On the other hand, above lemma tells us \( Q_1'' \subset \overline{Q_0''} \). We have thus \( Q_1'' = \overline{Q_0''} \subset Q_1' \) which implies \( Q_1 = Q_1' \) and the proposition follows.

4. Regularity of boundary points

Let \( \hat{\Omega} \) be an arbitrary metrizable \( h \)-resolutive compactification of \( \Omega \), and \( \Delta = \hat{\Omega} - \Omega \).

In this section we give a proof of theorem stated in the introduction. For definiteness we recall the definition of regularity of boundary points.

**Definition 4.** A filter \( \mathcal{F} \) on \( \Omega \) converging to a boundary point \( x \) is called **strongly \( h \)-regular** if there exists an open neighbourhood \( \delta \) of \( x \) and a positive superharmonic function \( s \) in \( \delta \cap \Omega \) such that \( s|h_\mathcal{F} \rightarrow 0 \) and the infimum of \( s|h \) outside of arbitrary open neighbourhood of \( x \) contained in \( \delta \) is positive.

A filter \( \mathcal{F} \) on \( \Omega \) converging to a boundary point \( x \) is called **\( h \)-regular** if for every bounded continuous function \( \phi \) on \( \Delta \) we have \( -\frac{1}{h} \Delta_{\phi, h} \rightarrow \phi(x) \).

A filter \( \mathcal{F} \) on \( \Omega \) converging to a boundary point \( x \) is called **weakly \( h \)-regular** if there exists a positive superharmonic function \( s \) such that \( s|h_\mathcal{F} \rightarrow 0 \).

A boundary point \( x \) is called **strongly \( h \)-regular, \( h \)-regular** and **weakly \( h \)-regular** according as the filter formed by the trace on \( \Omega \) of filter of neighbourhoods of \( x \) is strongly \( h \)-regular, \( h \)-regular and weakly \( h \)-regular respectively.

It is known that a strongly \( h \)-regular filter is \( h \)-regular and weakly \( h \)-regular. However an example of one-point compactification of \( \Omega \) shows us that an \( h \)-regular filter is not necessarily weakly \( h \)-regular.

Since by Proposition 2.6 \( \Delta^0 = \Gamma^0 \) is of \( h \)-harmonic measure zero, to prove our theorem it will be sufficient to show the following proposition:

**Proposition 4.1.** Let \( \Omega^0 \) be the compactification constructed in the preceding section and let \( \Delta^0 = \Omega^0 - \Omega \). Every point of the \( h \)-harmonic boundary \( \Gamma^0 \) of \( \Delta^0 \) is \( h \)-regular and weakly \( h \)-regular.

Proof. We use the same notations as in the preceding section. Let \( x \in \Gamma^0 \) and \( \phi \in C(\Delta^0) \). Let \( F \) be a bounded continuous extension of \( \phi \) on \( \Omega^0 \) and set \( f = F|_\Omega \).
Since \( f \in Q' \), and \( d_{f_n}/h \in Q'' \), \( f \) and \( d_{f_n}/h \) can be extended continuously onto \( \Omega^{\phi} \). By Proposition 3.2 \( \Omega^{\phi} \) is homeomorphic to \( \Omega^{\phi_0} \), therefore \( f \) and \( d_{f_n}/h \) are extended continuously onto \( \Omega^{\phi_0} \). This is also true for \( g = f - d_{f_n}/h \). Since \( d_{f_n}/h = 0 \), \( g \) is an \( h \)-Wiener potential and by Proposition 2.1 there exists potential \( \varphi \) such that \( |gh| \leq p \). For an arbitrary sequence of points \( \{a_n\} \) in \( \Omega \) converging to \( x \) we have

\[
\lim_{n \to \infty} |g(a_n)| \leq \lim_{n \to \infty} \frac{p(a_n)}{h(a_n)} = 0.
\]

Hence

\[
\lim_{n \to \infty} \left[ f(a_n) - \frac{d_{f_n}(a_n)}{h(a_n)} \right] = 0,
\]

which means \( \lim_{n \to \infty} \frac{D_{f_n,a}(a)}{h(a)} = \varphi(x) \). Thus, all points of \( \Gamma^{\phi_0} \) are \( h \)-regular.

Since \( \min(G_{a_0}, h)/h \) is extended continuously on \( \Omega^{\phi_0} \), this function assumes the value zero on \( \Gamma^{\phi_0} \), therefore all points of \( \Gamma^{\phi_0} \) are weakly \( h \)-regular, \( q.e.d. \)

If we take at every point \( x \in \Gamma^{\phi_0} \) the filter formed by the trace on \( \Omega \) of neighbourhoods of \( x \) in \( \Omega^{\phi_0} \), we obtain the family \( \{\mathcal{F}_x\} \) of filters converging in \( \hat{\Omega} \) and satisfying the following axioms:

- **A_{h}** If \( s \) is subharmonic in \( \Omega \), \( s/h \) is bounded from above and \( \lim_{a \to x} \sup_{\mathcal{F}} s/h \leq 0 \) for every \( \mathcal{F} \) in \( \{\mathcal{F}_x\} \), then \( s \leq 0 \).

- **B_{h}** Every filter in \( \{\mathcal{F}_x\} \) is \( h \)-regular and weakly \( h \)-regular.

Indeed, \( A_{h} \) follows from Proposition 2.4 and \( B_{h} \) is a consequence of the above proposition.

The second axiom \( B_{h} \) is weaker than the following axiom of Brelot-Choquet [2]:

- **B_{h}** Every filter in \( \{\mathcal{F}_x\} \) is strongly \( h \)-regular.

Thus, we have

**Proposition 4.2.** Let \( \hat{\Omega} \) be an arbitrary metrizable \( h \)-resolutive compactification of \( \Omega \). Then, there exists a family of filters in \( \Omega \) converging in \( \hat{\Omega} \) and satisfying the axiom \( A_{h} \), \( B_{h} \).

L. Naïm gave a family of filters satisfying the axiom \( A_{h} \), \( B_{h} \) by using fine neighbourhoods on Martin space. Our filter is quite different from it.

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**References**

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