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A NOTE ON AXIOMATIC DIRICHLET PROBLEM

TERUO IKEGAMI

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1. Introduction

Axiomatic Dirichlet problem was first discussed by M. Brelot in connection with a metrizable compactification of Green space Ω and a positive harmonic function \( h \) in Ω. In his paper [1] the theory was developed under the assumption \( G_h \), that is, all bounded continuous functions on the boundary are \( h \)-resolutive. In our present paper we call a compactification with this property \( h \)-resolutive.

This axiomatic treatment of Dirichlet problem yields some complicated situations. For instance, Brelot gave many definitions for the regularity of boundary points, such as strongly \( h \)-regular, \( h \)-regular, weakly \( h \)-regular. A strongly \( h \)-regular boundary point is \( h \)-regular and weakly \( h \)-regular, but an \( h \)-regular boundary point is not weakly \( h \)-regular in general. It has been asked by M. Brelot [1] and L. Naïm [4] whether the complementary set of all \( h \)-regular boundary points is of \( h \)-harmonic measure zero (\( h \)-négligeable) or not. We can not yet give an answer to this question. However we can prove the following theorem:

**Theorem.** Let \( \hat{\Omega} \) be an arbitrary metrizable \( h \)-resolutive compactification of Green space \( \Omega \). Then there exists a metrizable \( h \)-resolutive compactification having \( \hat{\Omega} \) as a quotient space and in which the complementary set of all \( h \)-regular and weakly \( h \)-regular boundary points is of \( h \)-harmonic measure zero.

As a corollary of this theorem we can construct a family of filters \( \{ \mathcal{F}_x \} \) converging in \( \Omega \) and satisfying axioms

\[ A_h \] If \( s \) is subharmonic in \( \Omega \), \( s/h \) is bounded from above and \( \limsup_{\mathcal{F}} s/h \leq 0 \) for every \( \mathcal{F} \) in \( \{ \mathcal{F}_x \} \), then \( s \leq 0 \).

\[ B'_h \] Every filter in \( \{ \mathcal{F}_x \} \) is \( h \)-regular and weakly \( h \)-regular, where the latter is weaker than that of Brelot-Choquet [2].

2. Preliminaries

Let \( \Omega \) be a Green space in the sense of Brelot-Choquet [2]. For a real valued function \( f \) defined in \( \Omega \) we shall define a family \( \mathcal{W}_f(\mathcal{W}_f) \) of superharmonic (subharmonic) functions \( s \) such that \( s \geq f (s \leq f) \) on \( \Omega - K \), where \( K \) is a compact
set depending on \( s \) in general. If \( \overline{W}_f(W_f) \) is not empty its lower (upper) envelope will be denoted by \( \bar{d}_f(d_f) \). \( d_f \) and \( \bar{d}_f \) are harmonic and \( d_f \leq \bar{d}_f \). When \( d_f = \bar{d}_f \) they are denoted by \( d_f \), simply.

Throughout this paper we shall take a positive harmonic function \( h \) in \( \Omega \) and fix it.

Definition 1. A function \( f \) defined in \( \Omega \) is \( h \)-harmonizable if the following conditions are satisfied:

1) there exists a superharmonic function \( s \) such that \( |fh| \leq s \),
2) \( d_{fh} = \bar{d}_{fh} \)

If \( f \) is \( h \)-harmonizable and \( d_{fh} = 0 \) then \( f \) is termed an \( h \)-Wiener potential, and the class of all \( h \)-Wiener potentials is denoted by \( W_{0,h} \).

Proposition 2.1. Every \( f \in W_{0,h} \) has a potential \( p \) such that \( |fh| \leq p \).

Let \( \hat{\Omega} \) be a compactification of \( \Omega \), that is \( \hat{\Omega} \) is compact and contains \( \Omega \) as an everywhere dense subspace. Set \( \Delta = \hat{\Omega} - \Omega \). In this paper it is always assumed that \( \hat{\Omega} \) is metrizable.

For an arbitrary real valued function \( \varphi \) on \( \Delta \), which is permitted to take the values \( \pm \infty \), \( \mathcal{F}_{\varphi,h} \) denotes the class of all superharmonic functions \( s \) such that

a) \( s/h \) is bounded from below,

b) \( \lim_{a \to x} s(a)/h(a) \geq \varphi(x) \) for every \( x \in \Delta \).

Similarly we define the class of subharmonic functions \( \mathcal{S}_{\varphi,h} \). When \( \mathcal{F}_{\varphi,h}, \mathcal{S}_{\varphi,h} \) are not empty, we set

\[
\mathcal{D}_{\varphi,h} = \inf \{ s; s \in \mathcal{F}_{\varphi,h} \}, \quad \mathcal{S}_{\varphi,h} = \sup \{ s; s \in \mathcal{S}_{\varphi,h} \}.
\]

\( \mathcal{D}_{\varphi,h} \) and \( \mathcal{S}_{\varphi,h} \) are both harmonic and \( \mathcal{D}_{\varphi,h} \leq \mathcal{S}_{\varphi,h} \). When \( \mathcal{D}_{\varphi,h} = \mathcal{S}_{\varphi,h} \), \( \varphi \) is called \( h \)-resolutive and the envelopes are denoted by \( \mathcal{D}_{\varphi,h} \) simply.

Definition 2. If all bounded continuous functions on \( \Delta \) are \( h \)-resolutive, \( \hat{\Omega} \) is called an \( h \)-resolutive compactification of \( \Omega \).

In the sequel, \( \hat{\Omega} \) always denotes a metrizable \( h \)-resolutive compactification of \( \Omega \). Then, for \( a \in \Omega \) there exists a Radon measure \( \omega_a^h \) on \( \Delta \) such that

\[
\mathcal{D}_{\varphi,h} = \int \varphi \, d\omega_a^h \quad \text{for every} \quad \varphi \in C(\Delta)^{\#},
\]

\( \omega_a^h \) is called an \( h \)-harmonic measure (with respect to \( a \)).

---

1) In the case that \( h = 1 \) and \( \Omega \) is a hyperbolic Riemann surface, this definition is slightly different from [3].

2) \( C(\Delta) \) denotes the family of all bounded continuous functions on \( \Delta \).
**Proposition 2.2.** Let $F$ be bounded and continuous on $\hat{\Omega}$ and $\varphi$, $f$ be its restrictions on $\Delta$ and on $\Omega$ respectively, then $f$ is $h$-harmonizable and $d_{fh} = \Omega_{\varphi,h}$.

**Proposition 2.3.** In order that an arbitrary compactification $\hat{\Omega}$ of $\Omega$ be $h$-resolutive, it is necessary and sufficient that for every bounded continuous function $F$ on $\Omega$, its restriction on $\Omega$ is $h$-harmonizable.

**Definition 3.** For potential $p$ we set

$$\Gamma_{p,h} = \{x \in \Delta; \lim_{a \to x} p(a)/h(a) = 0\},$$

$$\Gamma_h = \bigcap_p \Gamma_{p,h}.$$

$\Gamma_h$ is called an $h$-harmonic boundary.

$\Gamma_h$ is non-empty and compact.

**Proposition 2.4.** If $s$ is subharmonic in $\Omega$ such that $s/h$ is bounded from above and $\lim_{a \to x} s(a)/h(a) \leq 0$ for all $x \in \Gamma_h$ then $s \leq 0$.

**Proposition 2.5.** Let $F$ be a bounded continuous function on $\hat{\Omega}$. The restriction of $F$ on $\Omega$ is an $h$-Wiener potential if and only if $F$ vanishes on $\Gamma_h$.

**Proposition 2.6.** $\Gamma_h$ is the carrier of $h$-harmonic measure $\omega_h$.

In the case that $h=1$ and $\Omega$ is a hyperbolic Riemann surface, Constantinescu-Corna [3] have given these propositions. Proofs of our propositions will be obtained from them with slight modifications.

### 3. $Q$-compactification of Green space

1. Let $h$ be a positive harmonic function on Green space $\Omega$ and $\hat{\Omega}$ be an arbitrary metrizable, $h$-resolutive compactification of $\Omega$. Set $\Delta = \hat{\Omega} - \Omega$.

For $F \in C(\hat{\Omega})$, its restrictions on $\Omega$ and on $\Delta$ are denoted by $F|_{\Omega}$ and $F|_{\Delta}$ respectively.

We set $Q_0' = \{F|_{\Omega}; F \in C(\hat{\Omega})\}$, $Q_{0''} = \{d_{fh}/h; f \in Q_0'\}$ and

$$Q_0 = Q_0' \cup Q_{0''} \cup \left\{A \min (G_{a_0} h) + B\right\},$$

where $G_{a_0}$ is a Green function of $\Omega$ with pole at $a_0$ and $A$, $B$ are constants. The compactification $\Omega_{Q_0}$ of $\Omega$ is the one on which all functions of $Q_0$ are extended continuously and the boundary $\Delta_{Q_0} = \Omega_{Q_0} - \Omega$ is separated by functions in $Q_0$.

We have

**Proposition 3.1.** $\Omega_{Q_0}$ is a metrizable $h$-resolutive compactification of $\Omega$.

---

3) We say functions in $Q_0$ separate points of $\Delta_{Q_0}$ if for every pair of distinct points $x, y$ of $\Delta_{Q_0}$ there exists a function $F$ in $Q_0$ such that $F(x) \neq F(y)$.
is a quotient space of $\Omega^\Omega$. To prove this proposition, we require some lemmas.

In $C(\hat{\Omega})$ we select a countable subfamily $\{F_k\}$ which is dense in the topology of uniform norm $||F||=\sup_{a\in\hat{\Omega}} |F(a)|$.

If we set $f_k=F_k|_\Omega$, $f_k$ is $h$-harmonizable (Prop. 2.2). We form the family of a countable number of functions

$$Q = \{f_k\} \cup \{d_{f_k}/h\} \cup \left\{ \min \left\{ \frac{(G_{a_0^\Omega} h)}{h} \right\} \right\},$$

which is a subfamily of $Q_\Omega$.

The $Q$-compactification $\Omega^Q$ of $\Omega$ is compact and contains $\Omega$ as an everywhere dense subspace. Functions in $Q$ are extended continuously on $\Omega^Q$ and separate two distinct points of $\Delta^Q=\Omega^Q-\Omega$.

Theory of general topology tells us $\Omega^Q$ is metrizable (for instance, N. Bourbaki: Topologie générale, Chap. IX, §2).

**Lemma 3.1.** For every $F\in C(\hat{\Omega})$, if we set $f=F|_\Omega$, then $f$ and $d_{f_k}/h$ are extended continuously on $\Omega^Q$.

Proof. (i) Case of $f$. It will be sufficient to show that for every $x\in\Delta^Q$ and for every sequence of points $\{a_n\}$ in $\Omega$ converging to $x$ in the topology of $\Omega^Q$ $\{f(a_n)\}$ has the unique limit. If it were not, there should exist two sequences $\{a_n\}, \{b_n\}$ in $\Omega$ such that $a_n\to x$, $b_n\to x$ (in the topology of $\Omega^Q$) and $a=\lim f(a_n)>\lim f(b_n)=\beta$.

We take a positive number $\varepsilon=(\alpha-\beta)/4$. For this $\varepsilon$ and $F\in C(\hat{\Omega})$ we can find $F_k$ in our countable family such that

$$\sup_{\hat{\Omega}} |F_k-F| \leq \varepsilon.$$ 

Then we have

$$\alpha = \lim_{n\to\infty} f(a_n) \leq \lim_{n\to\infty} f_k(a_n) + \varepsilon,$$

$$\lim_{n\to\infty} f_k(b_n) - \varepsilon \leq \lim_{n\to\infty} f(b_n) = \beta.$$

where $f_k=F_k|_\Omega$. Since $f_k$ is extended continuously on $\Omega^Q$,

$$\alpha-\varepsilon \leq \lim_{n\to\infty} f_k(a_n) = \lim_{n\to\infty} f_k(b_n) \leq \beta + \varepsilon,$$

this leads to a contradiction $4\varepsilon = \alpha-\beta \leq 2\varepsilon$.

(ii) Case of $d_{f_k}/h$. We take $f_k$ as above. Then we have

$$\frac{d_{f_k}}{h} - \varepsilon \leq \frac{d_{f_k}}{h} \leq \frac{d_{f_k}}{h} + \varepsilon$$

and we can proceed quite in the same way as in (i).
Lemma 3.2. Let $\mathcal{H}$ be a class of all functions $F'$ each of which is bounded and continuous on $\Omega^{Q_0}$ and its restriction on $\Omega$ is $h$-harmonizable. Then $\mathcal{H}$ is dense in $C(\Omega^{Q_0})$ in the topology of uniform norm in $\Omega^{Q_0}$.

Proof. Clearly $\mathcal{H}$ contains all constant functions and $\mathcal{H}$ is a linear space. All functions in $Q_0$ are extended continuously on $\Omega^{Q_0}$ and these extended functions are contained in $\mathcal{H}$, therefore $\Omega^{Q_0}$ is separated by functions in $\mathcal{H}$. To see $\mathcal{H}$ is closed under the maximum and minimum operations, that is $F_1', F_2' \in \mathcal{H}$ implies $\max (F_1', F_2') \in \mathcal{H}$, let $F_1', F_2' \in \mathcal{H}$ and $f_i=F_i'|_\Omega$ ($i=1,2$). $\min (F_1', F_2')|_\Omega=\min (f_1, f_2)$ and $d_{\min} (f_1, f_2)^h = d_{\min} (f_1^h, f_2^h)^h = d_{f_1^h} \wedge d_{f_2^h}$, where $u \wedge v$ denotes the greatest harmonic function which is dominated by $u$ and $v$. This means $\min (f_1, f_2)$ is $h$-harmonizable. By Stone's theorem⁴ $\mathcal{H}$ is dense in $C(\Omega^{Q_0})$.

Proof of Proposition 3.1. On account of Lemma 3.1 all functions of $Q_0$ are extended continuously on $\Omega^{Q_0}$. Thus $\Omega^{Q_0}$ is homeomorphic to $\Omega^Q$ and therefore $\Omega^{Q_0}$ is metrizable. Since $\hat{\Omega}$ is homeomorphic to $\Omega^{Q_0}$, $\hat{\Omega}$ is a quotient space of $\Omega^{Q_0}$. For arbitrary $F' \in C(\Omega^{Q_0})$ and any positive number $\varepsilon$, by Lemma 3.2 we can find $F'_0 \in \mathcal{H}$ such that

$$\sup_{\Omega^{Q_0}} |F' - F'_0| \leq \varepsilon.$$ 

Setting $f=F'|_\Omega$, $f_0=F'_0|_\Omega$ we have

$$d_{f_0^h} - \varepsilon h \leq d_{f^h} \leq d_{f_0^h} + \varepsilon h.$$ 

Since $f_0$ is $h$-harmonizable we get $0 \leq d_{f_0^h} - d_{f^h} \leq 2\varepsilon h$. $f$ is $h$-harmonizable, and by Proposition 2.3 $\Omega^{Q_0}$ is $h$-resolutive.

2. For an arbitrary metrizable $h$-resolutive compactification $\hat{\Omega}$ of $\Omega$ we have constructed $\Omega^{Q_0}$ of the same type which contains $\hat{\Omega}$ as a quotient space. If we start from $\Omega^{Q_0}$ it will be expected that we can arrive at a new larger compactification of the same type, but this is not so, that is

Proposition 3.2. Let $\Omega^{Q_0}$ be the compactification of $\Omega$ constructed in the above paragraph. If we set $Q_1'=\{f=F|_\Omega; F \in C(\Omega^{Q_0})\}$, $Q_1''=\left\{\frac{d_{f^h}}{h}; f \in Q', \right\}$ and $Q_1=Q_1' \cup Q_1''$ the compactification $\Omega^{Q_1}$ is homeomorphic to $\Omega^{Q_0}$.

Before proving this proposition we remark the following:

Lemma 3.3. For every $f \in Q_1'$, and for every positive number $\varepsilon$ there exists $g \in Q_0'$ such that

$$\sup_{\Omega} \left| \frac{d_{f^h}}{h} - \frac{d_{g^h}}{h} \right| \leq \varepsilon.$$ 

⁴ Cf. [3], p. 5.
Proof. For arbitrary distinct points \( x_1, x_2 \) in \( \Omega^0 \) and for any numbers \( \alpha_1, \alpha_2 \), there exists a function \( \lambda \in C(\Omega^0) \) which satisfies the following conditions:

1) \( \lambda \mid_\Omega \in Q_0 \).
2) \( \lambda(x_i) = \alpha_i \) \((i=1,2)\).

Since continuous extensions of functions in \( Q_0 \) separate points of \( \Omega^0 \), we can find \( l \in C(\Omega^0) \) with \( l(x_1) \neq l(x_2) \) among these extensions. Thus, either (i) \( l \mid_\Omega = f \in Q_0 \) or (ii) \( l \mid_\Omega = d_{f,h}/h \) for some \( f \in Q_0 \) or (iii) \( l \mid_\Omega = \min (G_{\alpha_0}, h) + B \).

In cases (i) and (iii) we have

\[
\lambda(x) = \frac{\alpha_1 - \alpha_2}{l(x_1) - l(x_2)} \frac{l(x_2) - \alpha_2}{l(x_1) - l(x_2)},
\]

in the case (ii) we take, as \( \lambda \), the continuous extension on \( \Omega^0 \) of \( d_{f,h}/h \), where

\[
f_0 = \frac{\alpha_1 - \alpha_2}{l(x_1) - l(x_2)} f - \frac{\alpha_1}{l(x_1) - l(x_2)} l(x_1) \in Q_0.
\]

Let \( F \in C(\Omega^0), f = F \mid_\Omega, \varepsilon > 0 \). For arbitrary \( x, y \in \Omega^0 \) we can take \( \lambda_{xy} \in C(\Omega^0) \) satisfying the following:

1) \( \lambda_{xy} \mid_\Omega \in Q_0 \).
2) \( \lambda_{xy}(x) = F(x), \lambda_{xy}(y) = F(y) \).

\( U_{xy} = \{ z \in \Omega^0; \lambda_{xy}(z) < F(z) + \varepsilon \} \) is open and contains \( x, y \). From an open covering \( \{ U_{xy}; y \in \Omega^0 \} \) of \( \Omega^0 \) we select a finite subcovering \( \{ U_{xy}; j = 1, 2, \cdots, n \} \)

Set

\[
u = \min_{1 \leq i \leq n} \lambda_{xy}, \]

where \( \lambda_{xy} \) is a function corresponding to \( U_{xy}, (j = 1, 2, \cdots, n). \) \( u_x < F + \varepsilon \) on \( \Omega^0 \).

Then, there exists a function \( g_0 \) of \( Q_0 \) such that \( d_{u,h} = d_{g_0,h} \).

In fact, let \( \lambda_{xy} \mid_\Omega \) be \( f, f_2, \cdots, f_{k+1}, d_{f_{k+1},h}/h, d_{f_{k+2},h}/h, \cdots, d_{f_{k+h},h}/h, \cdots, \min (G_{\alpha_0}, h) \)

where \( g_0 = \min \{ \min_{1 \leq s \leq k+1} f_j, \min_{1 \leq s \leq k+1} B_j \} \in Q_0 \). Since \( U_x = \{ z \in \Omega^0; u_x(z) > F(z) - \varepsilon \} \) is open and contains \( x \), we can form a finite subcovering \( \{ U_{x_j}; j = 1, 2, \cdots, l' \} \) of \( \Omega^0 \).

Setting \( v = \max u_{x_j}, \) where \( u_{x_j} \) is a function corresponding to \( U_{x_j} \), \((j = 1, 2, \cdots, l')\), we have \( |u - F| < \varepsilon \) on \( \Omega^0 \) and as above we can find \( g \in Q_0 \) such that \( d_{v,h} = d_{g,h} \).
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\[ d_{vh} - d_{fh} \leq d_{vh} + \varepsilon h \]

means \[ \frac{|d_{vh} - d_{fh}|}{h} \leq \frac{|d_{vh} - d_{vh}|}{h} \leq \varepsilon, \quad q.e.d. \]

Proof of Proposition 3.2. Since all functions of \( Q_0'' \) are extended continuously on \( \Omega^\varepsilon \) we have \( Q_0'' \subset Q_1' \). The closure \( Q_0'' \) of \( Q_0'' \) in the topology of uniform norm \( (\|f\| = \sup_{\Omega} |f|) \) is contained in \( Q_1' \). On the other hand, above lemma tells us \( Q_1'' \subset Q_0'' \). We have thus \( Q_1'' = Q_0'' \subset Q_1' \) which implies \( Q_1 = Q_1' \) and the proposition follows.

4. Regularity of boundary points

Let \( \hat{\Omega} \) be an arbitrary metrizable \( h \)-resolutive compactification of \( \Omega \), and \( \Delta = \hat{\Omega} - \Omega \).

In this section we give a proof of theorem stated in the introduction. For definiteness we recall the definition of regularity of boundary points.

**Definition 4.** A filter \( \mathcal{F} \) on \( \Omega \) converging to a boundary point \( x \) is called strongly \( h \)-regular if there exists an open neighbourhood \( \delta \) of \( x \) and a positive superharmonic function \( s \) in \( \delta \cap \Omega \) such that \( s/h \to 0 \) and the infimum of \( s/h \) outside of arbitrary open neighbourhood of \( x \) contained in \( \delta \) is positive.

A filter \( \mathcal{F} \) on \( \Omega \) converging to a boundary point \( x \) is called \( h \)-regular if for every bounded continuous function \( \varphi \) on \( \Delta \) we have \( \frac{1}{h} \Omega_{\varphi, h} \to \varphi(x) \).

A filter \( \mathcal{F} \) on \( \Omega \) converging to a boundary point \( x \) is called weakly \( h \)-regular if there exists a positive superharmonic function \( s \) such that \( s/h \to 0 \).

A boundary point \( x \) is called strongly \( h \)-regular, \( h \)-regular and weakly \( h \)-regular according as the filter formed by the trace on \( \Omega \) of filter of neighbourhoods of \( x \) is strongly \( h \)-regular, \( h \)-regular and weakly \( h \)-regular respectively.

It is known that a strongly \( h \)-regular filter is \( h \)-regular and weakly \( h \)-regular. However an example of one-point compactification of \( \Omega \) shows us that an \( h \)-regular filter is not necessarily weakly \( h \)-regular.

Since by Proposition 2.6 \( \Delta_0 = \Gamma_0^\varepsilon \) is of \( h \)-harmonic measure zero, to prove our theorem it will be sufficient to show the following proposition:

**Proposition 4.1.** Let \( \Omega_0^\varepsilon \) be the compactification constructed in the preceding section and let \( \Delta_0 = \Omega_0^\varepsilon - \Omega \). Every point of the \( h \)-harmonic boundary \( \Gamma_0^\varepsilon \) of \( \Delta_0 \) is \( h \)-regular and weakly \( h \)-regular.

Proof. We use the same notations as in the preceding section. Let \( x \in \Gamma_0^\varepsilon \) and \( \varphi \in C(\Delta_0^\varepsilon) \). Let \( F \) be a bounded continuous extension of \( \varphi \) on \( \Omega_0^\varepsilon \) and set \( f = F \chi_\omega \).
Since \( f \in \Omega^1 \), and \( d_{f,h}/h \in \Omega^1_o \), \( f \) and \( d_{f,h}/h \) can be extended continuously onto \( \Omega^0 \). By Proposition 3.2 \( \Omega^1 \) is homeomorphic to \( \Omega^0 \), therefore \( f \) and \( d_{f,h}/h \) are extended continuously onto \( \Omega^0 \). This is also true for \( g = f - d_{f,h}/h \). Since \( d_{f,h} = 0 \), \( g \) is an \( h \)-Wiener potential and by Proposition 2.1 there exists potential \( p \) such that \( |gh| \leq p \). For an arbitrary sequence of points \( \{a_n\} \) in \( \Omega \) converging to \( x \) we have

\[
\lim_{n \to \infty} |g(a_n)| \leq \lim_{n \to \infty} \frac{p(a_n)}{h(a_n)} = 0.
\]

Hence

\[
\lim_{n \to \infty} \left[ f(a_n) - \frac{d_{f,h}(a_n)}{h(a_n)} \right] = 0,
\]

which means \( \lim_{a \to x} \frac{D_{f,h}(a)}{h(a)} = \varphi(x) \). Thus, all points of \( \Gamma^0 \) are \( h \)-regular.

Since \( \min(G_{\Lambda_0}/h) \) is extended continuously on \( \Omega^0 \), this function assumes the value zero on \( \Gamma^0 \), therefore all points of \( \Gamma^0 \) are weakly \( h \)-regular, \( q.e.d. \).

If we take at every point \( x \in \Gamma^0 \) the filter formed by the trace on \( \Omega \) of neighbourhoods of \( x \) in \( \Omega^0 \), we obtain the family \( \{F_x\} \) of filters converging in \( \hat{\Omega} \) and satisfying the following axioms:

\( A_h \) If \( s \) is subharmonic in \( \Omega \), \( s/h \) is bounded from above and \( \lim \sup_s s/h \leq 0 \) for every \( F \) in \( \{F_x\} \), then \( s \leq 0 \).

\( B_h \) Every filter in \( \{F_x\} \) is \( h \)-regular and weakly \( h \)-regular.

Indeed, \( A_h \) follows from Proposition 2.4 and \( B_h \) is a consequence of the above proposition.

The second axiom \( B_h \) is weaker than the following axiom of Brelot-Choquet [2]:

\( B_h \) Every filter in \( \{F_x\} \) is strongly \( h \)-regular.

Thus, we have

**Proposition 4.2.** Let \( \hat{\Omega} \) be an arbitrary metrizable \( h \)-resolutive compactification of \( \Omega \). Then, there exists a family of filters in \( \Omega \) converging in \( \hat{\Omega} \) and satisfying the axiom \( A_h \), \( B_h \).

L. Naïm gave a family of filters satisfying the axiom \( A_h \), \( B_h \) by using fine neighbourhoods on Martin space. Our filter is quite different from it.

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**References**


