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TEST MAPS AND DISCRETE GROUPS IN $SL(2, C)$

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Abstract

In this paper we present several discreteness criterion for a non-elementary group G in $SL(2, C)$ by using a test map which need not to be in G.

1. Introduction

The discreteness of Möbius groups is a fundamental problem, which have been discussed by many authors. In 1976, Jørgensen established the following discreteness criterion by using the well-known Jørgensen's inequality [5]:

Theorem 1.1. *A non-elementary subgroup G of Möbius transformations acting on $\overline{R}^2$ is discrete if and only if for each f and g in G the group $\langle f, g\rangle$ is discrete.*

This important result has become standard in literature and it shows that the discreteness of a non-elementary Möbius group depends on the information of all its rank two subgroups. There are many further discussions in this direction. Gilman [3] and Isochenko [4] showed that the discreteness of all two-generator subgroups, where each generator is loxodromic, is enough to secure the discreteness of the group. This is also a direct consequence of Rosenberger's result [6] about minimal generating system of a non-elementary Möbius group.

In 2002, Tukia and Wang [9] generalized Theorem 1.1 by considering elliptic elements as follows.

Theorem 1.2. *Let G be a non-elementary subgroup of $SL(2, C)$. If G contains an elliptic element of order at least 3, then G is discrete if and only if each non-elementary subgroups generated by two elliptic elements of G is discrete.*

They also asked in [9] that for a non-elementary group G containing parabolic and elliptic elements whether G is discrete if every subgroup of G generated by a parabolic

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and an elliptic is discrete. We gave a positive answer to this question and proved the following three theorems in [10].

Theorem 1.4. *Let G be a non-elementary subgroup of $SL(2, C)$ containing parabolic and elliptic elements. Then G is discrete if and only if for each parabolic f and elliptic g in G the subgroup $\langle f, g\rangle$ is discrete.*

Theorem 1.5. *Let G be a non-elementary subgroup of $SL(2, C)$ containing parabolic (resp. elliptic) elements. Then G is discrete if and only if for each loxodromic f and parabolic (resp. elliptic) g in G the subgroup $\langle f, g\rangle$ is discrete.*

Theorem 1.6. *Let G be a non-elementary subgroup of $SL(2, C)$ containing parabolic elements. Then G is discrete if and only if for each pair of parabolic elements f and g in G the subgroup $\langle f, g\rangle$ is discrete.*

Recently, Chen Min in [2] proposed to use a fixed Möbius transformation as a test map to test the discreteness of a given Möbius group. More precisely, let G be a non-elementary group and let f be a non-trivial Möbius map. If each group generated by f and an element in G is discrete, then G is discrete. A novelty of this discreteness criteria is that the test map f need not be in G, which suggests that the discreteness is not a totally interior affair of the involved group. Following the idea of Theorems 1.2 to 1.6, it is natural to ask whether one can generalize these results by using test maps. There are altogether 9 cases; see the next section for details.

2. Main results

We begin with some elementary notations about Möbius groups. The reader is referred to [1] for more information.

Denote by Möb(2) the group of all (orientation-preserving) Möbius transformations of the extended complex plane $\overline{C} = R^2 \cup \{\infty\}$. Recall that any matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL(2, C)$ induces a Möbius transformation $f_A(z) = (az + b)/(cz + d)$. Then Möb(2) is isomorphic to $SL(2, C)/\{\pm I\}$, where I is the identity matrix.

Let $\mathrm{tr}^2(f_A) = \mathrm{tr}^2(A)$ where $\mathrm{tr}(A)$ denotes the trace of A. Non-trivial elements of $SL(2, C)$ or equivalently Möb(2) can be classified by their traces: if $\mathrm{tr}^2(f)$ is real with $0 \le \mathrm{tr}^2(f) < 4$, f is called elliptic; if $\mathrm{tr}^2(f) = 4$, f is called parabolic; if $\mathrm{tr}^2(f)$ is real and $\mathrm{tr}^2(f) > 4$, f is called hyperbolic and if $\mathrm{tr}^2(f)$ is not in the interval $[0, +\infty)$, f is termed strictly loxodromic. We use the term loxodromic to include both hyperbolic and strictly loxodromic elements. It is easy to see $\mathrm{tr}^2(f_n) \to \mathrm{tr}^2(f)$ when f_n converges to f in $SL(2, C)$. Thus we have

Lemma 2.1. (a) *The set consisting of all loxodromic (resp. strictly loxodromic) elements is open in $SL(2, C)$;*

(b) *The set consisting of all hyperbolic* (*resp. elliptic*) *elements is open in* $SL(2, R)$.

We also need the following lemma, which is a direct consequence of the well-known proposition in [7, §1].

Lemma 2.2. *Let* G *be a non-elementary and non-discrete subgroup of* $SL(2, C)$. *After replacing* G *by its subgroups of index* 2 *if necessary,* G *is* (a) *dense in* $SL(2, C)$, *or* (b) *conjugate to a dense group of* $SL(2, R)$.

The following characterization of uniform convergence is useful for us; see [8, p. 158].

Lemma 2.3. *Let* g_i *and* g *be Möbius transformations. Then* g_i *converges uniformly to* g *if and only if* $g_i(x_i) \to g(x)$ *whenever* x_i *is a sequence such that* $x_i \to x$.

Let G be a subgroup of $SL(2, C)$ and f a non-trivial element in $SL(2, C)$. Denote by $\mathrm{fix}(f)$ the set of all fixed points of f, and $L(G)$ is the limit set of G. Recall that G is discrete if the identity map is isolated in G, and G is elementary if $L(G)$ contains at most two points if G is discrete, and in addition if no $x \in L(G)$ is in $\mathrm{fix}(g)$ for each $g \in G$ if G is non-discrete (cf. [8, p. 165]).

Now we can state our main result.

Theorem 2.4. *Let* G *be a non-elementary subgroup of* $SL(2, C)$ *and* f *a non-trivial Möbius transformation. If for each loxodromic element* g *in* G *the group* $\langle f, g\rangle$ *is discrete, then* G *is discrete.*

Proof. Suppose that G is not discrete. Since the discreteness of a Möbius group and its finite-index subgroup are equivalent, then we may assume that there is a sequence $\{g_n\}$ of distinct loxodromic elements in G such that $g_n \to I$ by Lemmas 2.1 and 2.2. By Jørgensen's inequality we may assume that the group $\langle f, g_n\rangle$ is discrete and elementary for all n. There are three cases:

CASE 1. f is loxodromic. Then f and g_n share the same fixed points. Since G is non-elementary, there is a loxodromic element $g \in G$ which has distinct fixed points from that of f. Note that $gg_ng^{-1} \to I$. Similarly, $\langle f, gg_ng^{-1}\rangle$ is discrete and elementary, and hence f and gg_ng^{-1} have the same fixed points for large n, which means that g either fixes or exchanges two fixed points of f. This is impossible since g is loxodromic.

CASE 2. f is parabolic. But it is known that there exist no discrete and elementary groups which contain both loxodromic and parabolic elements.

CASE 3. f is elliptic.

Since the extended complex plane is compact, we may assume that $\text{fix}(g_n) = \{a_n, b_n\}$ with $a_n \to a$ and $b_n \to b$. Choose g in the non-elementary group G, such that the following holds:

(i) $\text{fix}(g) \cap \{a, b\} = \emptyset$;

(ii) $\text{fix}(g) \cap \text{fix}(f) = \emptyset$.

Let a_g denote the attractive fixed point of g. By (i) and Lemma 2.3, we may assume that both $g^k(a_n)$ and $g^k(b_n)$ converge to a_g uniformly for all n as $k \to \infty$. Then by (ii) we see that there exists an integer k_1, such that $\{g^{k_1}(a_n), g^{k_1}(b_n) : n \geq 1\}$ lies in a neighborhood U of a_g which is disjoint with $f(U)$.

Because $\langle f, g^{k_1} g_n g^{-k_1} \rangle$ is discrete and elementary for large n, f either fixes or exchanges $g^{k_1}(a_n)$ and $g^{k_1}(b_n)$, which means $f(U) \cap U \neq \emptyset$. This is a contradiction. □

Theorem 2.5. *Let G be a non-elementary subgroup of $SL(2, C)$ containing elliptic elements and f a parabolic transformation. If for each elliptic element $g \in G$ the group $\langle f, g \rangle$ is discrete, then G is discraete.*

Proof. Suppose that G is not discrete. By Lemma 2.2 the proof can be divided into two cases:

CASE 1. We may assume that G is dense in $SL(2, R)$. By Lemma 2.1, there exists a sequence $\{g_n\}$ of distinct elliptic elements in G such that $g_n \to I$. Then the group $\langle f, g_n \rangle$ is discrete and elementary for large n by Jørgensen's inequality. Thus g_n stabilizes the fixed point of f. Since G is non-elementary, there is $g \in G$ which has distinct fixed points from that of f. Similarly, we can deduce $g g_n g^{-1}$ stabilizes the fixed point of f for large n, which is a contradiction.

CASE 2. G is dense in $SL(2, C)$. Normalize f such that $f = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Note that the closure of the set of fixed points of all elliptic elements in G contains the limit set of the non-elementary group G. Thus we may suppose that $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ is elliptic, where $b \neq 0$ and $c \neq 0$.

Construct a matrix $h = \begin{pmatrix} 1 & -2\sqrt{b} \\ 1/(2\sqrt{b}) & 0 \end{pmatrix} \in SL(2, C)$. Since G is dense in $SL(2, C)$, there exists a sequence $\{h_n\}$ in G which converges to h. Then $\langle f, h_n g h_n^{-1} \rangle$ is discrete and non-elementary for large n. By computation the third entry of $h_n g h_n^{-1}$ converges to $-1/2$. This contradicts Jørgensen's inequality for $\langle f, h_n g h_n^{-1} \rangle$. □

Theorem 2.6. *Let G be a non-elementary subgroup of $SL(2, C)$ containing parabolic elements and f a parabolic transformation. If for each parabolic element $g \in G$ the group $\langle f, g \rangle$ is discrete, then G is discrete.*

Proof. Suppose that G is not discrete. Then we may assume that G is dense either in $SL(2, R)$ or in $SL(2, C)$. Here we only prove the former case; for the latter case, the proof can use the same construction.

Normalize such that $f = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Note that the closure of the set comprising fixed points of all parabolic elements in G is exactly the limit set of the non-elementary group G. Thus we may suppose that $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ is parabolic with $c \neq 0$.

Construct a matrix $h = \begin{pmatrix} 1 & -m \\ 1/m & 0 \end{pmatrix} \in SL(2, R)$, where m is an positive integer. Since G is dense in $SL(2, R)$, there exists a sequence $\{h_n\}$ in G which converges to h. Then $\langle f, h_n g h_n^{-1} \rangle$ is discrete and non-elementary for large n. By computation the third entry of $h_n g h_n^{-1}$ converges to $-b/m^2$. This contradicts Jørgensen's inequality for large m. □

Theorem 2.7. *Let G be a non-elementary subgroup of $SL(2, C)$ containing parabolic elements and f a loxodromic (resp. an elliptic) transformation. If for each parabolic element $g \in G$ the group $\langle f, g \rangle$ is discrete, then G is discrete.*

Proof. Suppose that G is not discrete. Then we may assume that G is dense either in $SL(2, R)$ or in $SL(2, C)$. Similarly, we only prove the former case.

Normalize such that $f = \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}$. Note that the closure of the set comprising fixed points of all parabolic elements in G is exactly the limit set of the non-elementary group G. Thus we may suppose that $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ is parabolic with $b \neq 0$ and $c \neq 0$.

Construct a matrix $h = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \in SL(2, R)$, where $\beta = (d-a)/(2c)$. Since G is dense in $SL(2, R)$, there exists a sequence $\{h_n\}$ in G which converges to h. Then $h_n g h_n^{-1} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$ converges to $\begin{pmatrix} a + c\beta & -c\beta^2 + (d-a)\beta + b \\ c & -c\beta + d \end{pmatrix}$.

Note that $\langle f, h_n g h_n^{-1} \rangle$ is discrete and non-elementary for large n. Then by Jørgensen's inequality we have

$$|b_n c_n| \left| r - \frac{1}{r} \right|^2 \geq 1.$$

But $b_n c_n$ converges to $c(-c\beta^2 + (d-a)\beta + b)$ which is 0 since $\beta = (d-a)/(2c)$. This is a contradiction. □

For the remaining two cases, we ask the following

Conjecture 2.8. *Let G be a non-elementary subgroup of $SL(2, C)$ containing elliptic elements and f a loxodromic (resp. an elliptic) transformation. If for each elliptic element $g \in G$ the group $\langle f, g \rangle$ is discrete, then G is discrete.*

Now we can prove the following two special cases.

Theorem 2.9. *Let G be a non-elementary subgroup of $SL(2, R)$ containing elliptic elements and f a loxodromic (resp. an elliptic) transformation. If for each elliptic element $g \in G$ the group $\langle f, g\rangle$ is discrete, then G is discrete.*

Proof. Suppose that G is not discrete. Then we can find a sequence $\{g_n\}$ of distinct elliptic elements in G such that $g_n \to I$ and each g_n is not of order 2 by Lemmas 2.1, 2.2 and the following Lemma 2.10. By Jørgensen's inequality we may assume that the subgroup $\langle f, g_n\rangle$ is discrete and elementary for all n, which deduce that f and g_n^2 share the same fixed points if f is loxodromic, and either f and g_n^2 have a common fixed point or f exchanges two fixed points of g_n^2 if f is elliptic. In both cases we can get a contradiction by using the same method as Case 1 and Case 3 in the proof of Theorem 2.4, respectively. □

Lemma 2.10. *If $\{f_i\} \subset SL(2, C)$ is a sequence of elements with order 2, then f_i can not converge to the identity as $i \to \infty$.*

Proof. Note that each f_i can be represented as $f_i(x) = (a_i x + b_i)/(c_i x - a_i)$. It is obvious that $\begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix}$ cannot converge to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. □

Theorem 2.11. *Let G be a non-elementary subgroup of $SL(2,C)$ containing elliptic elements and f a loxodromic (resp. an elliptic) transformation with $|\mathrm{tr}^2(f)-4| < 1$. If for each elliptic element $g \in G$ the group $\langle f, g\rangle$ is discrete, then G is discrete.*

Proof. Suppose that G is not discrete. Then we may assume that G is dense either in $SL(2, R)$ or in $SL(2, C)$. Similarly, we only prove the former case.

Normalize such that $f = \begin{pmatrix} r & 1 \\ 0 & 1/r \end{pmatrix}$, and we suppose $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ is elliptic with $b \neq 0 \neq c$.

Construct a matrix $h = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \in SL(2, R)$, where $\beta = (d - a)/(2c)$. Since G is dense in $SL(2, R)$, there exists a sequence $\{h_n\}$ in G which converges to h. Then $h_n g h_n^{-1} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$ converges to $\begin{pmatrix} a + c\beta & 0 \\ c & -c\beta + d \end{pmatrix}$.

Note that $\langle f, h_n g h_n^{-1}\rangle$ is discrete and non-elementary for large n. Then by Jørgensen's inequality we have

$$(1 + |b_n c_n|)\left|r - \frac{1}{r}\right|^2 \geq 1,$$

that is,

$$|b_n c_n| \geq -1 + \frac{1}{|r - 1/r|^2} = -1 + \frac{1}{|\mathrm{tr}^2(f) - 4|}.$$

This contradicts that $b_n c_n \to 0$. □

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